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Γ-Convergence of Concentration Problems

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Abstract. In this paper, we use Γ-convergence techniques to study the following variational problem

$$S_\varepsilon^F(\Omega) := \sup\left\{\varepsilon^{-2^*}\int_\Omega F(u)\,dx \;:\; \int_\Omega |\nabla u|^2\,dx \le \varepsilon^2\,,\ u=0 \text{ on } \partial\Omega\right\}\,,$$

where $0 \le F(t) \le |t|^{2^*}$, with $2^* = \frac{2n}{n-2}$, and Ω is a bounded domain of $\mathbb{R}^n$, $n \ge 3$. We obtain a Γ-convergence result, on which one can easily read the usual concentration phenomena arising in critical growth problems. We extend the result to a non-homogeneous version of problem $S_\varepsilon^F(\Omega)$. Finally, a second order expansion in Γ-convergence permits to identify the concentration points of the maximizing sequences, also in some non-homogeneous case.

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1. – Introduction

In this paper we propose a description by Γ-convergence of the asymptotic behaviour of problems of the type

$$S_\varepsilon^F(\Omega) := \sup\left\{\varepsilon^{-2^*}\int_\Omega F(u)dx : \int_\Omega |\nabla u|^2\ dx \le \varepsilon^2,\ u=0 \text{ on } \partial\Omega\right\}\,, \tag{1.1}$$

when $\varepsilon \to 0^+$, where Ω is a bounded domain of $\mathbb{R}^n$, $n \ge 3$, and F is a nonnegative upper semicontinuous function bounded from above by $C_F|t|^{2^*}$ (C_F being a constant and $2^* = 2n/(n-2)$ being the critical Sobolev exponent). If F is a smooth function, the maximizers of (1.1) satisfy the Euler equation

$$\begin{cases} -\Delta u = \lambda_\varepsilon f(u) & \text{in } \ \Omega\,, \\ u = 0 & \text{on } \ \partial\Omega\,, \end{cases} \tag{1.2}$$

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where $f = F'$ and λ_ε is a Lagrange multiplier, which tends to $+\infty$, as $\varepsilon \to 0^+$. After the fundamental paper by Brezis and Nirenberg [5], many different variants of this problem have been studied by several authors (for a wide bibliography see, for instance, [11]).

In the generality of problem (1.1) we can consider many interesting cases. For instance, considering discontinuous F, one can recover also free boundary problems as that related to the volume functional, i.e.

$$S_\varepsilon^V(\Omega) := \sup\{\varepsilon^{-2^*}|A| \ : \ A \subset \Omega, \ \operatorname{cap}_\Omega(A) \leq \varepsilon^2\}.$$

In the case $F(t) = |t|^{2^*}$, the behaviour of the maximizing sequences of (1.1) has been characterized by P. L. Lions in [17] by means of the Concentration Compactness Alternative. He proves that maximizing sequences either concentrate at a single point or they are compact. In particular, when $\Omega \neq \mathbb{R}^n$, one deduces that only concentration is allowed. In a recent paper (see [14]), problem (1.1) for general F has been studied by Flucher and Müller. They prove a generalized version of the Concentration Compactness Alternative, which permits to conclude that all maximizing sequences must concentrate at a single point. The approach of Flucher and Müller strongly relies on the Concentration Compactness Principle technique.

Another possible way to describe this asymptotic behaviour is through De Giorgi's Γ-convergence, which is a natural notion of convergence of functionals implying convergence of extrema. The idea of Γ-convergence is to substitute a sequence $\{\mathcal{F}_\varepsilon\}$ by an effective "Γ-limit" functional $\overline{\mathcal{F}}$ which captures the relevant features of the sequence $\{\mathcal{F}_\varepsilon\}$; in particular, sequences of almost maximizers converge to a maximum point of $\overline{\mathcal{F}}$. The description given by Γ-convergence is, in a sense, more complete since the Γ-limit $\overline{\mathcal{F}}$ describes the asymptotic behaviour of $\mathcal{F}_\varepsilon(u_\varepsilon)$ along all converging sequences $\{u_\varepsilon\}$. A key point is to specify in which sense this convergence of sequences must be defined.

We study the Γ-convergence of the following sequence of functionals

$$\mathcal{F}_\varepsilon(u) = \varepsilon^{-2^*} \int_\Omega F(\varepsilon u) \ dx . \tag{1.3}$$

By a simple scaling, problem (1.1) can be rewritten in terms of $\mathcal{F}_\varepsilon$ as follows

$$\sup\{\mathcal{F}_\varepsilon(u) : u \in H_0^1(\Omega), \ \|\nabla u\|_{L^2(\Omega)} \leq 1\}. \tag{1.4}$$

The constraint on the gradient suggests to study all the sequences $\{u_\varepsilon\}$ weakly converging in $H_0^1(\Omega)$ to some function u. However a functional defined only in $H_0^1(\Omega)$ cannot capture the behaviour of $\mathcal{F}_\varepsilon(u_\varepsilon)$. Hence we have to introduce the measure μ which is the limit of $|\nabla u_\varepsilon|^2$ in the sense of measures. Thus the limit functional will be given by two terms: a first term depending on the limit u and a second one depending on μ which takes into account the singularities developed by the weak convergence of the gradients (see Theorem 3.1).

The Γ-limit is then defined on $H_0^1(\Omega) \times \mathcal{M}(\overline{\Omega})$ and given by

$$\overline{\mathcal{F}}(u, \mu) = F_0 \int_\Omega |u|^{2^*} \, dx + S^F \sum_{i=0}^{+\infty} \mu_i^{2^*/2} ,$$

where $\mu(\overline{\Omega}) \leq 1$, $\mu \geq |\nabla u|^2$ and $\mu = \hat{\mu} + \sum_{i=0}^{+\infty} \mu_i \delta_{x_i}$ is the decomposition of μ is its non-atomic and atomic part. From the form of the functional $\overline{\mathcal{F}}$ we immediately deduce that maximizers have $u = 0$ and $\mu = \delta_{x_0}$, which gives the concentration of the maximizing sequences (see Theorem 3.9). This will be done without using the Concentration Compactness Principle of P. L. Lions.

In a second part of this paper, we study a non-homogeneous version of (1.4), maximizing the functional

$$\mathcal{F}_\varepsilon^A(u) = \varepsilon^{-2^*} \int_\Omega A(x) F(\varepsilon u) \, dx . \tag{1.5}$$

As above, if F is a smooth function, the corresponding maximization problem is related to the following Euler equation

$$\begin{cases} -\Delta u = \lambda_\varepsilon A(x) f(u) & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases} \tag{1.6}$$

where $f = F'$ and λ_ε is a Lagrange multiplier, which tends to $+\infty$, as $\varepsilon \to 0^+$.

The approach of Γ-convergence permits to reconstruct the limit functional of (1.5) using the homogeneous result (see Theorem 5.1). This can be done for a large class of coefficients $A(x)$, provided they are not too irregular; this includes, for instance, all piecewise continuous functions which are upper semicontinuous. Moreover, the structure of the Γ-limit immediately implies that the maximizing sequences must concentrate at a single point among the maxima of A (see Theorem 5.3).

Finally, in the last part of the paper, we consider a second-order asymptotic expansion of the Γ-convergence, in order to identify the concentration points. In the homogeneous case, this result was essentially obtained in [12] and [15], where it is proved that the maximizing sequences "prefer" to concentrate in some particular points (the minima of the Robin function, i.e. the diagonal of the regular part of the Green function of the domain), so that the choice of the concentration points depends on the geometry of the domain. Here the Robin function plays the role of the renormalized energy introduced by Bethuel, Brézis and Heléin in the context of Ginzburg-Landau functionals (see [3]). The role of the Robin function in concentration problems involving the critical exponent was pointed out by Brézis and Peletier [6], Schoen [19], Bahri [1] (see also [16], [18]). On the other hand, the relevance of the result in [12] and [15] is that concentration at the critical points of the Robin function is a more general phenomenon. In fact, it is not restricted to integrands whose leading term is the critical power (for a complete review of these phenomena see [11] and the references therein). We interpret the result of [12] and [15] in terms of Γ-convergence, in order to apply it to the x-dependent case. Indeed, although the influence of the Robin function on the location of the concentration points is weaker than any kind of inhomogeneity $A(x)$, nevertheless, when the set of maxima of A is not a singleton, the geometry of Ω still plays a role in the choice of the concentration points. This can be seen by means of a second-order expansion. More precisely, we prove that the maximizing sequences concentrates at the minima of the Robin function among the maximizers of A. This result will be proved under a "flatness" assumption around the maxima of A (see Theorem 6.8, Example 6.6 and Remark 6.7), that simply assures that the optimal profile for the maximizing sequences is not modified by the presence of the coefficient.

2. – Notation and known results

The set Ω will be always a bounded open subset of $\mathbb{R}^n$, with $n \geq 3$. We denote by $\overline{\Omega}$ the closure of Ω and by $\partial\Omega$ the boundary of Ω. Given $x \in \mathbb{R}^n$ and $r > 0$, the set $B_r(x) \subset \mathbb{R}^n$ denotes the ball of radius r centered in x.

We denote by $\mathcal{M}(\overline{\Omega})$ the set of all nonnegative Borel measures on $\overline{\Omega}$ of bounded total variation. The space $\mathcal{M}(\overline{\Omega})$ can be identified with the nonnegative elements of the topological dual space of $\mathcal{C}^0(\overline{\Omega})$. We say that a sequence $\{\mu_h\} \subseteq \mathcal{M}(\overline{\Omega})$ weakly* converges to a measure $\mu \in \mathcal{M}(\overline{\Omega})$, if and only if

$$\int_{\overline{\Omega}} \psi \; d\mu_h \to \int_{\overline{\Omega}} \psi \; d\mu \qquad \text{for every } \psi \in \mathcal{C}^0(\overline{\Omega}) \tag{2.1}$$

and we write $\mu_h \rightharpoonup \mu \quad w^*\text{-}\mathcal{M}(\overline{\Omega}.)$

As usual, for $p > 1$, $L^p(\Omega)$ denotes the Lebesgue space, as well as $H_0^1(\Omega)$ denotes the Sobolev space, while $D^{1,2}(\mathbb{R}^n)$ is defined as the closure of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|\nabla v\|_{L^2(\mathbb{R}^n)}$.

We denote by 2^* the Sobolev critical exponent, i.e. $2^* = 2n/(n-2)$. We recall the well-known Sobolev inequality

$$\|u\|_{L^{2^*}(\Omega)} \leq S^* \|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in H_0^1(\Omega)\,,$$

where S^* is the so-called Sobolev constant, i.e. the best possible constant for which the previous inequality holds.

Throughout this paper, the letter C denotes a strictly positive constant, independent of the parameters of the problem, whose value may vary each time.

In [14], Flucher and Müller studied the following problem

$$\sup\left\{\varepsilon^{-2^*}\int_\Omega F(\varepsilon u)dx : u \in H_0^1(\Omega),\, \|\nabla u\|_{L^2(\Omega)} \leq 1\right\} =: S_\varepsilon^F(\Omega) \text{ as } \varepsilon \to 0^+ \tag{2.2}$$

where

$$
(2.3)\qquad \begin{aligned} &\text{(i)} \quad F : \mathbb{R} \to [0, +\infty) \quad \text{is an upper semicontinuous function}\,, \\ &\text{(ii)} \quad 0 \le F(t) \le C_F |t|^{2^*}\,, \quad F \neq 0 \quad \text{in the } L^1\text{-sense}\,. \end{aligned}
$$

In the sequel we will also assume that the following limit exists

$$
(2.4)\qquad \lim_{t\to 0^+} \frac{F(t)}{t^{2^*}} =: F_0
$$

A function that clearly satisfies (2.3) and (2.4) is the critical power integrand, i.e. $F(t) = |t|^{2^*}$, which is an important starting point to understand the properties of this problem. Another important example to highlight the geometrical aspect of this problem is given by the so called *volume problem*

$$
(2.5)\qquad S_\varepsilon^V(\Omega) := \sup\{\varepsilon^{-2^*} |\{\varepsilon u \ge 1\}| \ : \ u \in H_0^1(\Omega),\ \|\nabla u\|_{L^2(\Omega)} \le 1\}\,,
$$

corresponding to the choice $F(t) = \chi_{\{t\ge 1\}}(t)$. It is easy to see that this problem can be equivalently written as

$$
S_\varepsilon^V = \max\{|A| : A \subset \Omega,\ \mathrm{cap}_\Omega(A) \le \varepsilon^2\}\,,
$$

where $\mathrm{cap}_\Omega(A)$ denotes the *harmonic capacity* of the set $A \subset \Omega$ with respect to Ω, defined for any open set A by

$$
\mathrm{cap}_\Omega(A) = \inf\left\{ \int_\Omega |\nabla u|^2 \, dx \ : \ u \in H_0^1(\Omega)\ ,\ u \ge 1 \text{ a.e. on } A \right\}
$$

and then extended by approximation to any subset of Ω. It is well known that, for any subset A of Ω with finite capacity, there exists a function $u \in H_0^1(\Omega)$, satisfying the constraint $u \ge 1$ on A in a suitable sense, such that $\int_\Omega |\nabla u|^2\, dx = \mathrm{cap}_\Omega(A)$. This function is called the *capacitary potential* of A in Ω.

Using a scaling argument, in [14, Lemma 2] the following lemma is proved.

LEMMA 2.1. *Let $S^F := S_1^F(\mathbb{R}^n)$. For every domain $\Omega \subseteq \mathbb{R}^n$*

1. $S_\varepsilon^F(\Omega) \le S^F$ *for every* $\varepsilon > 0$;
2. $S_\varepsilon^F(\Omega) \to S^F$ *for* $\varepsilon \to 0^+$;
3. $F_0 S^* \le S^F$;
4. *the following Generalized Sobolev Inequality holds*

$$
\int_\Omega F(u)\, dx \le S^F \|\nabla u\|_{L^2(\Omega)}^{2^*} \qquad \forall\, u \in H_0^1(\Omega)\,.
$$

Note that, in particular, if $F(t) = |t|^{2^*}$, we have $S_\varepsilon^F(\Omega) = S^F = S^*$ for every $\varepsilon > 0$.

Flucher and Müller also proved a generalized version of the Concentration Compactness Alternative of P. L. Lions, which permits to obtain the following result, concerning the asymptotic behaviour and the profile of sequences of extremals.

THEOREM 2.2 (see [14, Theorem 3]). *If $\{u_\varepsilon\}$ satisfies $\|\nabla u_\varepsilon\| \leq 1$ and $\varepsilon^{-2^*}\int_\Omega F(\varepsilon u_\varepsilon)\,dx \to S^F$ as $\varepsilon \to 0^+$, then a subsequence of $\{u_\varepsilon\}$ concentrates at a single point $x_0 \in \overline{\Omega}$, i.e.*

$$|\nabla u_\varepsilon|^2 \rightharpoonup \delta_{x_0}, \qquad \frac{F(\varepsilon u_\varepsilon)}{\varepsilon^{2^*}} \rightharpoonup S^F \delta_{x_0}$$

weakly in the sense of measures. If in addition*

$$\max(F_0, F_\infty^+) < S^F/S^*, \qquad \text{where} \qquad F_\infty^+ := \limsup_{|t|\to+\infty} \frac{F(t)}{|t|^{2^*}},$$

then there exists a sequence $x_\varepsilon \to x_0$ such that a subsequence of the rescaled functions

$$w_\varepsilon(y) := u_\varepsilon\left(x_\varepsilon + \varepsilon^{\frac{2}{n-2}} y\right)$$

tends to an extremal for S^F, i.e. $w_\varepsilon \to w$ strongly in $D^{1,2}(\mathbb{R}^n)$, $\|\nabla w\|_{L^2(\Omega)} = 1$, and $\int_{\mathbb{R}^n} F(w)\,dx = S^F$.

It is our purpose to study problem (2.2) from the point of view of Γ-convergence, introduced by E. De Giorgi. More precisely, this problem will be studied using the notion of Γ^+-convergence, which is a variational convergence that assures the convergence of maxima. This convergence is defined symmetrically with respect to the well-known notion of Γ^--convergence, usually adopted in the framework of variational minimum problems (see Definition 2.4 below). For every $\varepsilon > 0$, we define the functional

$$\mathcal{F}_\varepsilon(u) = \varepsilon^{-2^*} \int_\Omega F(\varepsilon u)\; dx$$

for every $u \in H_0^1(\Omega)$, such that $\int_\Omega |\nabla u|^2\; dx \leq 1$.

In order to apply Γ-convergence we have to study the behaviour of the sequence $\{\mathcal{F}_\varepsilon(u_\varepsilon)\}$, for any given sequence $\{u_\varepsilon\}$ satisfying the constraint $\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq 1$. This implies, up to a subsequence, that there exists $\mu \in \mathcal{M}(\overline{\Omega})$ such that $|\nabla u_\varepsilon|^2 \rightharpoonup \mu$ w^*-$\mathcal{M}(\overline{\Omega})$ and, by Sobolev Embedding Theorem, there exists $u \in H_0^1(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ w-$L^{2^*}(\Omega)$. Moreover, $\mu(\overline{\Omega}) \leq 1$.

By the lower semicontinuity of the L^2-norm, it is easy to see that $\mu \geq |\nabla u|^2$. Hence, we can always decompose μ as follows

$$\mu = |\nabla u|^2 + \widetilde{\mu} + \sum_{i=0}^{+\infty} \mu_i \delta_{x_i}$$

where $\widetilde{\mu}$ is non atomic, $\mu_i \in [0,1]$, $x_i \in \overline{\Omega}$, $x_i \neq x_j$ for $i \neq j$. A subtle application of the Sobolev inequality permits to prove that

$$\mu = |\nabla u|^2 + \widetilde{\mu} \quad \Longrightarrow \quad u_\varepsilon \to u \quad \text{strongly in} \quad L^{2^*}(\Omega) \tag{2.6}$$

(see [17], Lemma I.1 and Remark I.3). Hence, in general, one can deduce that $u_\varepsilon \to u$ strongly in $L^{2^*}_{\mathrm{loc}}(\Omega \setminus \{x_i : i \in \mathbb{N}\})$.

This suggests the natural setting for the limit functional, which is the space $X(\Omega)$, i.e. the subspace of $H^1_0(\Omega) \times \mathcal{M}(\overline{\Omega})$, defined by

$$X(\Omega) = \{(u, \mu) \in H^1_0(\Omega) \times \mathcal{M}(\overline{\Omega}) \ : \ \mu \geq |\nabla u|^2 \ , \ \mu(\overline{\Omega}) \leq 1\} .$$

For the sake of simplicity, in the sequel, we will write X, instead of $X(\Omega)$, if no confusion is possible. We endow X with a topology τ, defined by

$$(u_\varepsilon, \mu_\varepsilon) \overset{\tau}{\to} (u, \mu) \quad \Longleftrightarrow \quad \begin{cases} u_\varepsilon \rightharpoonup u & w\text{-}L^{2^*}(\Omega) \\ \mu_\varepsilon \rightharpoonup \mu & w^*\text{-}\mathcal{M}(\overline{\Omega}) . \end{cases}$$

Using the capacitary potentials, we can prove that every pair $(u, \mu) \in X$ can be obtained as a τ-limit of a sequence of pairs $(u_\varepsilon, |\nabla u_\varepsilon|^2)$, with $u_\varepsilon \in H^1_0(\Omega)$. This is shown in the following proposition, which clarifies that X is the smallest space where we have to set our problem.

Proposition 2.3. *Let $(u, \mu) \in X$, then there exists a sequence $\{u_\varepsilon\} \subset H^1_0(\Omega)$ such that, for every $\varepsilon > 0$, $\int_\Omega |\nabla u_\varepsilon|^2 \, dx \leq 1$ and $(u_\varepsilon, |\nabla u_\varepsilon|^2) \overset{\tau}{\to} (u, \mu)$, when $\varepsilon \to 0^+$.*

Proof. Clearly, without loss of generality, we may assume $\mu(\overline{\Omega}) < 1$. The main point is to approximate pairs of the type $(0, \mu) \in X$. This can be done for instance using the construction in [8]: for every $\varepsilon > 0$, let us cover Ω with a grid of cubes Q^i_ε with side length ε and such that $\{Q^i_\varepsilon \cap \overline{\Omega}\}$ is a partition of $\overline{\Omega}$. The idea is to choose in any cube Q^i_ε a suitable capacitary potential $u^i_\varepsilon \in H^1_0(Q^i_\varepsilon \cap \Omega)$ such that

$$\int_{Q^i_\varepsilon} |\nabla u^i_\varepsilon|^2 \ dx = \mu(Q^i_\varepsilon \cap \overline{\Omega}) .$$

Moreover, the capacitary potentials can be chosen in a way that the sequence $u_\varepsilon = \sum_i u^i_\varepsilon$, where u^i_ε are extended to zero outside $Q^i_\varepsilon \cap \overline{\Omega}$, converges strongly to zero in $L^2(\Omega)$ and then weakly in $L^{2^*}(\Omega)$, as $\varepsilon \to 0^+$. Finally, it is not difficult to prove that $|\nabla u_\varepsilon|^2 \rightharpoonup \mu \ w^*\text{-}\mathcal{M}(\overline{\Omega})$.

To recover the general case, it is enough to decompose the pair (u, μ) as $(0, \widetilde{\mu} + \sum_i \mu_i \delta_{x_i})$ and $(u, |\nabla u|^2)$, to consider an approximation $(\widetilde{u}_\varepsilon, |\nabla \widetilde{u}_\varepsilon|^2)$ of $(0, \widetilde{\mu} + \sum_i \mu_i \delta_{x_i})$, and hence to set $u_\varepsilon = u + \widetilde{u}_\varepsilon$. □

Let us now recall the notion of Γ^+-convergence, introduced in [9]. For more details see [7] and [4].

Definition 2.4. We shall say that the sequence $\{\mathcal{F}_\varepsilon\}$ Γ^+-*converges* to a functional $\overline{\mathcal{F}} : X \to [0, +\infty)$, as $\varepsilon \to 0^+$, if for every $(u, \mu) \in X$ the following two conditions are satisfied

(i) for all sequences $\{u_\varepsilon\}$, with $u_\varepsilon \rightharpoonup u$ in w-$L^{2^*}(\Omega)$ and $|\nabla u_\varepsilon|^2 \rightharpoonup \mu$ in w^*-$\mathcal{M}(\overline{\Omega})$, we have

$$\overline{\mathcal{F}}(u,\mu) \geq \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon)\,;$$

(ii) there exists a sequence $\{\widetilde{u}_\varepsilon\}$, with $\widetilde{u}_\varepsilon \rightharpoonup u$ in w-$L^{2^*}(\Omega)$ and $|\nabla \widetilde{u}_\varepsilon|^2 \rightharpoonup \mu$ in w^*-$\mathcal{M}(\overline{\Omega})$, such that

$$\overline{\mathcal{F}}(u,\mu) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\widetilde{u}_\varepsilon)\,.$$

We will write

$$\overline{\mathcal{F}}(u,\mu) = \left(\Gamma^+\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon\right)(u,\mu)\,.$$

A sequence as in (ii) will be called an optimal or recovery sequence.

As already pointed out, this variational convergence is a proper tool to study the convergence of maximizers; in fact it is easy to check that the Γ^+-convergence of $\{\mathcal{F}_\varepsilon\}$ to $\overline{\mathcal{F}}$ implies the convergence of maximizing sequences to maximizers of $\overline{\mathcal{F}}$ and also the convergence of maxima. More precisely, if $\mathcal{F}_\varepsilon \stackrel{\Gamma^+}{\to} \overline{\mathcal{F}}$ and $\{u_\varepsilon\}$ is a sequence of maximizers of $\mathcal{F}_\varepsilon$ (i.e. $\mathcal{F}_\varepsilon(u_\varepsilon) = S^F_\varepsilon(\Omega)$, or more generally, $\mathcal{F}_\varepsilon(u_\varepsilon) = S^F_\varepsilon(\Omega)(1+o(1))$ for $\varepsilon \to 0^+$), up to a subsequence, $u_\varepsilon \rightharpoonup u$ in w-$L^{2^*}(\Omega)$ and $|\nabla u_\varepsilon|^2 \rightharpoonup \mu$ in w^*-$\mathcal{M}(\overline{\Omega})$, where $(u,\mu) \in X$ and it is a maximizer of $\overline{\mathcal{F}}$; moreover,

$$S^F_\varepsilon(\Omega) \to \max_{X(\Omega)} \overline{\mathcal{F}}\,.$$

Note that, by Lemma 2.1, it follows that $\max \overline{\mathcal{F}} = S^F$.

Finally, since the τ-topology is metrizable on X, it is well known that there always exists a functional $\mathcal{F} : X \to [0,+\infty)$ such that, up to a subsequence, $\mathcal{F}_{\varepsilon_h} \stackrel{\Gamma^+}{\to} \mathcal{F}$.

3. – The Γ^+-Convergence result

In the present section, we will prove the existence of the Γ^+-limit of the sequence $\{\mathcal{F}_\varepsilon\}$ and we will characterize it.

As already said, in [14] Flucher and Müller prove a generalized version of the Concentration Compactness Alternative of P. L. Lions for the sequence $\{\varepsilon^{-2^*}\mathcal{F}(\varepsilon u_\varepsilon)\}$, where $u_\varepsilon \rightharpoonup u$ in w-$H^1_0(\Omega)$, from which they deduce the concentration of the sequences of extremals. Among other things, they prove that if $|\nabla u_\varepsilon|^2 \rightharpoonup \mu = |\nabla u|^2 + \widetilde{\mu} + \sum_{i=0}^{+\infty} \mu_i \delta_{x_i}$ in w^*-$\mathcal{M}(\overline{\Omega})$ then

$$\varepsilon^{-2^*} F(\varepsilon u_\varepsilon) \rightharpoonup \nu = g + \sum_{i=0}^{+\infty} \nu_i \delta_{x_i} \qquad w^*\text{-}\mathcal{M}(\overline{\Omega}) \tag{3.1}$$

and

$$\nu_i \le S^F \mu_i^{2^*/2} \quad \text{and} \quad g \le F_0 |u|^{2^*} \quad \text{a.e. in } \Omega\,. \tag{3.2}$$

This suggests the structure we may expect for the limit functional, as shown in the following Γ^+-convergence result.

Properties (3.1) and (3.2), together with Lemma 2.1, are the only two results that we borrow from [14], in order to prove the Γ-convergence Theorem. Those results are proved by Flucher and Müller (in [14], Theorem 12, part 1, and Lemma 2), using a localized version of the Sobolev inequality and represent a starting point for their analysis.

THEOREM 3.1. *There exists the Γ^+-limit $\overline{\mathcal{F}}$ of the sequence of functionals $\{\mathcal{F}_\varepsilon\}$ and*

$$\overline{\mathcal{F}}(u,\mu) = F_0 \int_\Omega |u|^{2^*}\,dx + S^F \sum_{i=0}^{+\infty} \mu_i^{2^*/2}\,,$$

for every $(u,\mu) \in X$.

REMARK 3.2. As a consequence of the previous theorem, we obtain that (3.2) is optimal. Indeed, by the definition of Γ^+-convergence, for any pair $(u,\mu) \in X$, there exists a sequence $\{u_\varepsilon\} \subseteq H_0^1(\Omega)$ such that $(u_\varepsilon, |\nabla u_\varepsilon|^2) \xrightarrow{\tau} (u,\mu)$ and

$$\varepsilon^{-2^*} \int_\Omega F(\varepsilon u_\varepsilon) \to F_0 \int_\Omega |u|^{2^*}\ dx + S^F \sum_{i=0}^{+\infty} \mu_i^{2^*/2}$$

where $\mu = |\nabla u|^2 + \widetilde{\mu} + \sum_{i=0}^{+\infty} \mu_i \delta_{x_i}$. This together with (3.1) and (3.2) implies that actually

$$\varepsilon^{-2^*} F(\varepsilon u_\varepsilon) \rightharpoonup F_0 |u|^{2^*} + S^F \sum_{i=0}^{+\infty} \mu_i^{2^*/2} \delta_{x_i} \quad w^*\text{-}\mathcal{M}(\overline{\Omega})\,.$$

To prove Theorem 3.1, we have to verify (i) and (ii) of Definition 2.4. The crucial point will be the construction of the sequence $\{\widetilde{u}_\varepsilon\}$ in (ii) of Definition 2.4. The basic idea consists in obtaining a sort of localization of the functional, which actually is not a local functional. More precisely, we will prove that every pair $(u,\mu) \in X$ can be decomposed into the sum of two pairs $(u, |\nabla u|^2 + \widetilde{\mu})$ and $(0, \sum_i \mu_i \delta_{x_i})$, which can be approximated disjointly. This decomposition is essential in the proof of the theorem, since on pairs of the first type the sequence $\{\mathcal{F}_\varepsilon\}$ is continuous, while on pairs of the second type the functionals $\mathcal{F}_\varepsilon$ are local and their limit can be explicitly computed on each single Dirac mass (see Proposition 3.7).

In spite of a bit heavier notation, in the sequel, it will be useful to extend the functional $\mathcal{F}_\varepsilon$ to the whole space X (keeping the same symbol), in the following way:

$$\mathcal{F}_\varepsilon(u,\mu) = \begin{cases} \varepsilon^{-2^*} \displaystyle\int_\Omega F(\varepsilon u)\ dx & \text{if } (u,\mu) \in X \text{ and } \mu = |\nabla u|^2 \\ 0 & \text{otherwise in } X \end{cases}.$$

Moreover, we need to introduce also the functionals defined by

$$\mathcal{F}^{+}(u,\mu) = \sup\{\limsup_{\varepsilon\to 0} \mathcal{F}_{\varepsilon}(u_{\varepsilon},\mu_{\varepsilon}) \quad \text{where} \quad (u_{\varepsilon},\mu_{\varepsilon}) \xrightarrow{\tau} (u,\mu)\}$$

and

$$\mathcal{F}^{-}(u,\mu) = \sup\{\liminf_{\varepsilon\to 0} \mathcal{F}_{\varepsilon}(u_{\varepsilon},\mu_{\varepsilon}) \quad \text{where} \quad (u_{\varepsilon},\mu_{\varepsilon}) \xrightarrow{\tau} (u,\mu)\},$$

for every $(u,\mu)\in X$. They are usually called the Γ^{+}- lim sup and the Γ^{+}- lim inf of $\{\mathcal{F}_{\varepsilon}\}$ respectively. Note that $\mathcal{F}^{-}\le\mathcal{F}^{+}$. Moreover, we can rewrite (i) and (ii) of Definition 2.4 in terms of the Γ^{+}- lim sup and Γ^{+}- lim inf, i.e.

$$\text{(i)} \quad \overline{\mathcal{F}}\ge\mathcal{F}^{+} \qquad \text{and} \qquad \text{(ii)} \quad \overline{\mathcal{F}}\le\mathcal{F}^{-},$$

and this easily implies that, the Γ^{+}-limit $\overline{\mathcal{F}}$ exists if and only if $\mathcal{F}^{-}=\mathcal{F}^{+}$ and, in this case, $\overline{\mathcal{F}}=\mathcal{F}^{-}=\mathcal{F}^{+}$.

We point out that, when it is necessary in the context, we will add the dependence on the space in the notation used for the functionals (for instance, it will be used $\mathcal{F}_{\varepsilon}(u_{\varepsilon},\mu_{\varepsilon};\Omega)$ instead of $\mathcal{F}_{\varepsilon}(u_{\varepsilon},\mu_{\varepsilon})$ and so on).

For the sake of simplicity, we also set

$$\widetilde{\mathcal{F}}(u,\mu) = F_0\int_{\Omega}|u|^{2^*}\,dx + S^F\sum_{i=0}^{+\infty}\mu_i^{2^*/2}.$$

With this notation, Theorem 3.1 states that there exists the Γ^{+}-limit $\overline{\mathcal{F}}$ of the sequence $\{\mathcal{F}_{\varepsilon}\}$ and $\overline{\mathcal{F}}=\widetilde{\mathcal{F}}$. Since $\mathcal{F}^{-}\le\mathcal{F}^{+}$ always holds true, in order to obtain this result it is enough to prove that

$$\mathcal{F}^{+}\le\widetilde{\mathcal{F}}\le\mathcal{F}^{-}. \tag{3.3}$$

By (3.1) and (3.2), it is easy to obtain the first inequality, as stated in the following proposition.

Proposition 3.3. *For every $(u,\mu)\in X$, we have*

$$\widetilde{\mathcal{F}}(u,\mu) = F_0\int_{\Omega}|u|^{2^*}\,dx + S^F\sum_{i=0}^{+\infty}\mu_i^{2^*/2} \ge \limsup_{\varepsilon\to 0}\mathcal{F}_{\varepsilon}(u_{\varepsilon},\mu_{\varepsilon})$$

for every sequence $\{(u_{\varepsilon},\mu_{\varepsilon})\}\subset X$ such that $(u_{\varepsilon},\mu_{\varepsilon})\xrightarrow{\tau}(u,\mu)$.

The remaining part of this section will be devoted to obtain the second inequality of (3.3), i.e. $\mathcal{F}^{-}\ge\widetilde{\mathcal{F}}$. In order to do this, we firstly prove the theorem in the case $(u,\mu)=(u,|\nabla u|^2+\widetilde{\mu})$ (see Proposition 3.4 and Corollary 3.5) and in the case $(u,\mu)=(0,\sum_i\mu_i\delta_{x_i})$ (see Proposition 3.7 and Proposition 3.8), disjointly. Finally, the result will be achieved unifying these two cases using a technical lemma in the following section (see Lemma 4.1).

PROPOSITION 3.4. *Let* $(u, \mu) \in X$ *be such that* $\mu = |\nabla u|^2 + \widetilde{\mu}$; *then*

$$\mathcal{F}^-(u, \mu) = F_0 \int_\Omega |u|^{2^*} dx\,,$$

i.e. $\mathcal{F}^-(u, \mu) = \widetilde{\mathcal{F}}(u, \mu)$.

PROOF. Let us take $\{(u_\varepsilon, \mu_\varepsilon)\} \subseteq X$ such that $\mu_\varepsilon = |\nabla u_\varepsilon|^2$ and $(u_\varepsilon, \mu_\varepsilon) \xrightarrow{\tau} (u, \mu)$. Since the atomic part of μ is zero, by (2.6), $u_\varepsilon \to u$ strongly in $L^{2^*}(\Omega)$. Let us take a subsequence, still denoted by $\{(u_\varepsilon, \mu_\varepsilon)\}$, such that $\{u_\varepsilon\}$ converges to u a.e. in Ω and $\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon) = \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon)$. By (2.3), we have

$$\frac{F(\varepsilon u_\varepsilon)}{\varepsilon^{2^*}} \le C_F |u_\varepsilon|^{2^*}\,. \tag{3.4}$$

Let us fix $x \in \Omega$ such that $u_\varepsilon(x) \to u(x)$. If $u(x) = 0$, by (3.4) it follows that also $\varepsilon^{-2^*} F(\varepsilon u_\varepsilon(x)) \to 0 = F_0 |u(x)|^{2^*}$; if $u(x) \neq 0$, then for ε sufficiently small, also $u_\varepsilon(x) \neq 0$, hence

$$\frac{F(\varepsilon u_\varepsilon(x))}{\varepsilon^{2^*}} = \frac{F(\varepsilon u_\varepsilon(x))}{(\varepsilon |u_\varepsilon(x)|)^{2^*}} |u_\varepsilon(x)|^{2^*} \to F_0 |u(x)|^{2^*}\,.$$

Hence, we have that $\varepsilon^{-2^*} F(\varepsilon u_\varepsilon) \to F_0 |u|^{2^*}$ a.e. in Ω. Then, by Lebesgue Convergence Theorem, we obtain

$$\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon) = F_0 \int_\Omega |u|^{2^*}\ dx = \widetilde{\mathcal{F}}(u, \mu) \tag{3.5}$$

which concludes the proof. □

COROLLARY 3.5. *Let* $(u, \mu) \in X$ *with* $\mu = |\nabla u|^2 + \widetilde{\mu}$. *Then, there exists the* Γ^+*-limit* $\overline{\mathcal{F}}$ *and* $\overline{\mathcal{F}}(u, \mu) = \widetilde{\mathcal{F}}(u, \mu)$.

PROOF. It follows from Proposition 3.4 and Proposition 3.3. □

In order to study the case of a purely atomic measure, we preliminarily need the following lemma.

LEMMA 3.6. *For every* $(u, \mu) \in X$, *we have*

$$\widetilde{\mathcal{F}}(u, \mu) \le S^F$$

and the equality holds if and only if $(u, \mu) = (0, \delta_{x_0})$, *for some* $x_0 \in \overline{\Omega}$.

Proof. Using the Sobolev inequality, Lemma 2.1, and the convexity of $t^{2^*/2}$, we obtain

$$
\begin{aligned}
\widetilde{\mathcal{F}}(u,\mu) &= F_0 \int_\Omega |u|^{2^*}\ dx + S^F \sum_{i=0}^{+\infty} \mu_i^{2^*/2} \\
&\le F_0 S^* \left(\int_\Omega |\nabla u|^2\ dx \right)^{2^*/2} + S^F \sum_{i=0}^{+\infty} \mu_i^{2^*/2} \\
&\le S^F \left[\left(\int_\Omega |\nabla u|^2\ dx \right)^{2^*/2} + \sum_{i=0}^{+\infty} \mu_i^{2^*/2} \right] \\
&\le S^F \left(\int_\Omega |\nabla u|^2\ dx + \sum_{i=0}^{+\infty} \mu_i \right) \le S^F \mu(\overline{\Omega}) \le S^F .
\end{aligned}
\tag{3.6}
$$

In particular for any $x_0 \in \overline{\Omega}$, $\widetilde{\mathcal{F}}(0,\delta_{x_0}) = S^F$. For all the remaining pairs, inequality (3.6) is strict. Indeed, let $(u,\mu) \in X$, with $\mu = |\nabla u|^2 + \widetilde{\mu} + \sum_i \mu_i \delta_{x_i}$. If $\mu = \widetilde{\mu}$, the inequality is trivially strict, since $u = 0$ and $\widetilde{\mathcal{F}}(u,\mu) = 0 < S^F$. If $|\nabla u|^2 \neq 0$, since Ω is bounded, the Sobolev inequality is strict and the inequality in (3.6), too. Finally if $u = 0$ and $\mu = \widetilde{\mu} + \sum_i \mu_i \delta_{x_i}$ there exists at least one coefficient $\mu_i \in (0,1)$, which implies $\mu_i^{2^*/2} < \mu_i$, and again inequality (3.6) is strict. □

Proposition 3.7. *For every open set $\Omega' \subseteq \Omega$, for every $x \in \overline{\Omega}'$ and for every $(u,\mu) \in X$ with $(u,\mu) = (0,\delta_x)$, there exists the Γ^+-limit of the sequence $\{\mathcal{F}_\varepsilon\}$ restricted to Ω' in $(0,\delta_x)$ and the equality*

$$
\left(\Gamma^+\text{-} \lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon \right)(0,\delta_x;\Omega') = S^F \qquad \forall x \in \overline{\Omega}'
$$

holds.

Proof. Let us fix $\Omega' \subseteq \Omega$ and let $\varepsilon_h \to 0$ as $h \to +\infty$. By the compactness property of Γ^+-convergence, there exists a subsequence, still denoted by $\{\varepsilon_h\}$, and a functional $\mathcal{F} : X(\Omega') \to [0,+\infty)$, such that $\mathcal{F}_{\varepsilon_h}(\cdot;\Omega') \overset{\Gamma^+}{\to} \mathcal{F}(\cdot;\Omega')$. Moreover, by Lemma 2.1, we have that

$$
\sup_{X(\Omega')} \mathcal{F}_{\varepsilon_h} = S^F_{\varepsilon_h}(\Omega') \to S^F = \max_{X(\Omega')} \mathcal{F} ,
$$

when $h \to +\infty$, while by Proposition3.3 and Lemma 3.6, it follows that $\mathcal{F}(u,\mu;\overline{\Omega}') \le \mathcal{F}^+(u,\mu;\overline{\Omega}') \le \widetilde{\mathcal{F}}(u,\mu;\overline{\Omega}') < S^F$, if $(u,\mu) \neq (0,\delta_{\overline{x}})$, for any $\overline{x} \in \overline{\Omega}'$. Hence, there exists $\overline{x} \in \overline{\Omega}'$ such that $\mathcal{F}(0,\delta_{\overline{x}};\overline{\Omega}') = S^F$.

Now, let us consider the countable family of spheres $B_r(q) \subset \Omega'$, centered in $q \in \mathbf{Q}^n \cap \Omega'$ with radii $r \in \mathbf{Q}$. By a diagonalization procedure, we may find a subsequence, still denoted by $\{\varepsilon_h\}$, and a functional, still denoted by $\mathcal{F}$, such that $\mathcal{F}_{\varepsilon_h}(\cdot; B_r(q)) \overset{\Gamma^+}{\to} \mathcal{F}(\cdot; \overline{B}_r(q))$, for every $r \in \mathbf{Q}$ and every $q \in \mathbf{Q}^n \cap \Omega'$, and $\mathcal{F}_{\varepsilon_h}(\cdot; \Omega') \overset{\Gamma^+}{\to} \mathcal{F}(\cdot; \overline{\Omega}')$. Reasoning as above, for every sphere $B_r(q)$ we may find a point $x_r^q \in \overline{B}_r(q)$ which satisfies

$$\mathcal{F}(0, \delta_{x_r^q}; \overline{B}_r(q)) = S^F .$$

Letting $r \to 0^+$, it follows that $x_r^q \to q$ and hence by the upper semicontinuity of the Γ^+-limit we have

$$S^F = \lim_{r\to 0^+} \mathcal{F}(0, \delta_{x_r^q}; \overline{B}_r(q)) \le \limsup_{r\to 0^+} \mathcal{F}(0, \delta_{x_r^q}; \overline{\Omega}') \le \mathcal{F}(0, \delta_q; \overline{\Omega}')$$

which implies that

$$\mathcal{F}(0, \delta_q; \overline{\Omega}') = S^F \qquad \forall q \in \mathbf{Q}^n \cap \Omega' .$$

Using the density of $\mathbf{Q}^n \cap \Omega'$ in $\overline{\Omega}'$ and the upper semicontinuity of the Γ^+-limit, we obtain that, for every $x \in \overline{\Omega}'$ and every pair $(u, \mu) = (0, \delta_x)$,

$$\mathcal{F}(0, \delta_x; \overline{\Omega}') = S^F .$$

Since this is independent of the choice of the sequence, it follows that, for every $(u, \mu) = (0, \delta_x)$, with $x \in \overline{\Omega}'$, there exists the Γ^+-limit $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}(0, \delta_x; \overline{\Omega}') = \mathcal{F}(0, \delta_x; \overline{\Omega}') = S^F$.

Finally, this holds for every open set $\Omega' \subseteq \Omega$, and this concludes the proof. □

PROPOSITION 3.8. *Let $(u, \mu) \in X$ be such that $u = 0$ and $\mu = \sum_{i=0}^N \mu_i \delta_{x_i}$. Then there exists the Γ^+-limit $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}(u, \mu) = \widetilde{\mathcal{F}}(u, \mu)$.*

PROOF. Let us assume now that $N = 1$, i.e. $\mu = \mu_0 \delta_{x_0} + \mu_1 \delta_{x_1}$ with $\mu_0, \mu_1 \in (0, 1)$ and $\mu_0 + \mu_1 \le 1$. The general case $N \ge 1$, being analogous.

Set $B_{r_i}(x_i) \cap \Omega = V_i$, for $i = 0, 1$, with $\mathrm{dist}(V_0, V_1) > 0$. By Proposition 3.7, for $i = 0, 1$, we may choose a sequence $(u_\varepsilon^i, \mu_\varepsilon^i) \in X(V_i)$ such that $\mu_\varepsilon^i = |\nabla u_\varepsilon^i|^2$, $(u_\varepsilon^i, \mu_\varepsilon^i) \overset{\tau}{\to} (0, \delta_{x_i})$ and

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon^i, \mu_\varepsilon^i; V_i) \ge \overline{\mathcal{F}}(0, \delta_{x_i}; \overline{V}_i) = S^F ,$$

more precisely,

$$\lim_{\varepsilon\to 0^+} \int_{V_i} \frac{F(\varepsilon u_\varepsilon^i)}{\varepsilon^{2^*}}\, dx = S^F , \qquad i = 0, 1 . \tag{3.7}$$

Let us define $u_\varepsilon = \sqrt{\mu_0} u^0_{\sqrt{\mu_0}\varepsilon} + \sqrt{\mu_1} u^1_{\sqrt{\mu_1}\varepsilon}$ and $\mu_\varepsilon = |\nabla u_\varepsilon|^2$. Clearly,

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx = \mu_0 \int_{V_0} |\nabla u^0_{\sqrt{\mu_0}\varepsilon}|^2 \, dx + \mu_1 \int_{V_1} |\nabla u^1_{\sqrt{\mu_1}\varepsilon}|^2 \, dx \le \mu_0 + \mu_1 \le 1$$

hence, $(u_\varepsilon, \mu_\varepsilon) \in X(\Omega)$; moreover,

$$\begin{aligned} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon; \Omega) &= \int_\Omega \frac{F(\varepsilon u_\varepsilon)}{\varepsilon^{2^*}}, dx \\ &= \mu_0^{2^*/2} \int_{V_0} \frac{F(\varepsilon \sqrt{\mu_0} u^0_{\sqrt{\mu_0}\varepsilon})}{(\sqrt{\mu_0}\varepsilon)^{2^*}} \, dx + \mu_1^{2^*/2} \int_{V_1} \frac{F(\varepsilon \sqrt{\mu_1} u^1_{\sqrt{\mu_1}\varepsilon})}{(\mu_1^{1/2}\varepsilon)^{2^*}} \, dx \end{aligned}$$

and the by (3.7) we have

$$\lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon; \Omega) = S^F(\mu_0^{2^*/2} + \mu_1^{2^*/2}) = \widetilde{\mathcal{F}}(0, \mu_0\delta_{x_0} + \mu_1\delta_{x_1}; \overline{\Omega}) .$$

The theorem is then accomplished. □

Besides for a technical lemma (see Lemma 4.1) proved in the following section, we are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. In view of Proposition 3.3 it remains to prove that $\mathcal{F}^- \ge \widetilde{\mathcal{F}}$. Let $(u, \mu) \in X$. By the technical Lemma 4.1 in the next section, we may assume that

$$\mu = |\nabla u|^2 + \widetilde{\mu} + \sum_{i=0}^N \mu_i \delta_{x_i} , \quad \mu(\overline{\Omega}) < 1 \text{ and dist } \left(\overline{\operatorname{supp}(|u| + \widetilde{\mu})}, \bigcup_{i=0}^N \{x_i\} \right) > 0 .$$

Set $\mu^1 = |\nabla u|^2 + \widetilde{\mu}$ and $\mu^2 = \mu - \mu^1 = \sum_{i=0}^N \mu_i \delta_{x_i}$; let A_1 and A_2 be two open subsets of Ω, such that $\overline{\operatorname{supp}(|u| + \widetilde{\mu})} \subset \overline{A_1}$, $\overline{\operatorname{supp}(\mu^2)} \subset \overline{A_2}$ and $\overline{A_1} \cap \overline{A_2} = \emptyset$. Since $u \in H^1_0(A_1)$, by Corollary 3.5 (with Ω replaced by A_1), we obtain that there exists a sequence $\{(u^1_\varepsilon, \mu^1_\varepsilon)\}$ such that $u^1_\varepsilon \in H^1_0(A_1)$, $\mu^1_\varepsilon = |\nabla u^1_\varepsilon|^2$, $\mu^1_\varepsilon(\overline{\Omega}) < 1$ for ε sufficiently small, $(u^1_\varepsilon, \mu^1_\varepsilon) \xrightarrow{\tau} (u, \mu^1)$ and

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u^1_\varepsilon, \mu^1_\varepsilon) &= \lim_{\varepsilon \to 0^+} \int_{A_1} \frac{F(\varepsilon u^1_\varepsilon)}{\varepsilon^{2^*}} \, dx \\ &= F_0 \int_{A_1} |u|^{2^*} \, dx = F_0 \int_\Omega |u|^{2^*} \, dx = \widetilde{\mathcal{F}}(u, \mu^1) . \end{aligned} \tag{3.8}$$

Moreover, by Proposition 3.8, there exists another sequence $\{(u^2_\varepsilon, \mu^2_\varepsilon)\}$ such that $u^2_\varepsilon \in H^1_0(A_2)$, $\mu^2_\varepsilon = |\nabla u^2_\varepsilon|^2$, $\mu^2_\varepsilon(\overline{\Omega}) < 1$ for ε sufficiently small, $(u^2_\varepsilon, \mu^2_\varepsilon) \xrightarrow{\tau} (0, \mu^2)$ and

$$\lim_{\varepsilon \to 0^+} \int_{A_2} \frac{F(\varepsilon u^2_\varepsilon)}{\varepsilon^{2^*}} \, dx = S^F \sum_{i=0}^N \mu_i^{2^*/2} = \widetilde{\mathcal{F}}(0, \mu^2) . \tag{3.9}$$

Let us define $u_\varepsilon = u_\varepsilon^1 + u_\varepsilon^2$ and $\mu_\varepsilon = \mu_\varepsilon^1 + \mu_\varepsilon^2$. Clearly, since the supports of u_ε^1 and u_ε^2 are disjoint, $u_\varepsilon \in H_0^1(\Omega)$, $\mu_\varepsilon = |\nabla u_\varepsilon|^2 = |\nabla u_\varepsilon^1|^2 + |\nabla u_\varepsilon^2|^2$. Moreover

$$\mu_\varepsilon(\overline{\Omega}) = \int_\Omega |\nabla u_\varepsilon^1|^2 \, dx + \int_\Omega |\nabla u_\varepsilon^2|^2 \, dx \to \mu^1(\overline{\Omega}) + \mu^2(\overline{\Omega}) = \mu(\overline{\Omega}) < 1$$

and $\mu_\varepsilon(\overline{\Omega}) < 1$ for ε small enough. Finally, by (3.8) and (3.9), it follows that

$$\begin{aligned}\mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon) &= \int_\Omega \frac{F(\varepsilon u_\varepsilon)}{\varepsilon^{2^*}} \, dx = \int_{A_1} \frac{F(\varepsilon u_\varepsilon^1)}{\varepsilon^{2^*}} \, dx + \int_{A_2} \frac{F(\varepsilon u_\varepsilon^2)}{\varepsilon^{2^*}} \, dx \\ &\to F_0 \int_\Omega |u|^{2^*} \, dx + S^F \sum_{i=0}^N \mu_i^{2^*/2} = \widetilde{\mathcal{F}}(u, \mu) \,.\end{aligned}$$

This proves (ii) of Definition 2.4 and, jointly with Proposition 3.3, concludes the proof of the theorem. □

As a consequence of the Γ^+-convergence result, we easily obtain the following concentration theorem.

THEOREM 3.9. *Let* $\{(u_\varepsilon, \mu_\varepsilon)\} \subseteq X$, *with* $\mu_\varepsilon = |\nabla u_\varepsilon|^2$, *be a maximizing sequence for* $\mathcal{F}_\varepsilon$. *Then, for* $\varepsilon \to 0^+$, *it concentrates at a single point, i.e.* $(u_\varepsilon, \mu_\varepsilon) \stackrel{\tau}{\to} (0, \delta_{x_0})$, *with* $x_0 \in \overline{\Omega}$. *Moreover, for any* $x_0 \in \overline{\Omega}$, *there exists a maximizing sequence concentrating at* x_0.

PROOF. By Theorem 3.1, the properties of Γ^+-convergence and Lemma 3.6, it follows that every maximizing sequence $\{(u_\varepsilon, \mu_\varepsilon)\}$ must converge to a pair $(0, \delta_{x_0})$, with $x_0 \in \overline{\Omega}$ and every pair $(0, \delta_{x_0})$ is a maximizer for $\overline{\mathcal{F}}$. The statement follows immediately by Proposition 2.3. □

4. – Technical lemma

This section is devoted to a technical lemma which, in the proof of the Γ^+-convergence result, permitted to restrict our attention to a subclass of pairs in $X(\Omega)$.

LEMMA 4.1. *Assume that* $\mathcal{F}^-(u, \mu) \geq \widetilde{\mathcal{F}}(u, \mu)$ *for every pair* $(u, \mu) \in X$, *satisfying the following three conditions*

(i) $\mu(\overline{\Omega}) < 1$;
(ii) $\mu = |\nabla u|^2 + \widetilde{\mu} + \sum_{i=0}^N \mu_i \delta_{x_i}$;
(iii) $\mathrm{dist}(\overline{\mathrm{supp}(|u| + \widetilde{\mu})}, \bigcup_{i=0}^N \{x_i\}) > 0$.

Then $\mathcal{F}^-(u, \mu) \geq \widetilde{\mathcal{F}}(u, \mu)$ *for every pair* $(u, \mu) \in X$.

PROOF. We shall prove the lemma in three steps. Let us first consider an arbitrary pair $(u,\mu)\in X$ satisfying (i) and (ii), i.e. $\mu=|\nabla u|^2+\widetilde{\mu}+\sum_{i=0}^N\mu_i\delta_{x_i}$ and $\mu(\overline{\Omega})<1$. We shall construct a sequence $\{(u_\varrho,\mu_\varrho)\}$, $\varrho>0$, with the following properties:

(1) $(u_\varrho,\mu_\varrho)\in X$ and $\mu_\varrho=|\nabla u_\varrho|^2+\widetilde{\mu}_\varrho+\sum_{i=0}^N\mu_i\delta_{x_i}$;
(2) $\mu_\varrho(\overline{\Omega})<1$ for ϱ small enough;
(3) $\mathrm{dist}(\overline{\mathrm{supp}(|u_\varrho|+\widetilde{\mu}_\rho)},\bigcup_{i=0}^N\{x_i\})>0$;
(4) $u_\varrho\to u$ s-$L^{2^*}(\Omega)$ and $(u_\varrho,\mu_\varrho)\xrightarrow{\tau}(u,\mu)$;
(5) $\lim_{\varrho\to0^+}\widetilde{\mathcal{F}}(u_\varrho,\mu_\varrho)=\widetilde{\mathcal{F}}(u,\mu)$.

For every $\varrho>0$ and every $i=0,\dots,N$, let us define $V_\varrho(x_i)=B_\varrho(x_i)\cap\Omega$ and let us choose a cut-off function $\phi_\varrho\in\mathcal{C}^\infty(\overline{\Omega})$, such that $0\le\phi_\varrho\le1$, $\phi_\varrho=0$ in $\cup_iV_\varrho(x_i)$, $\phi_\varrho=1$ in $\Omega\setminus\cup_iV_{2\varrho}(x_i)$, $|\nabla\phi_\varrho|\le1/\varrho$. Thus we set

$$(u_\varrho,\mu_\varrho)=\left(u\cdot\phi_\varrho,|\nabla(u\cdot\phi_\varrho)|^2+\widetilde{\mu}\cdot\phi_\varrho+\sum_{i=0}^N\mu_i\delta_{x_i}\right)=\left(u_\varrho,|\nabla u_\varrho|^2+\widetilde{\mu}_\varrho+\sum_{i=0}^N\mu_i\delta_{x_i}\right).$$

Clearly, $u_\varrho\in H_0^1(\Omega)$ and $\mu_\varrho\in\mathcal{M}(\overline{\Omega})$ and $\mathrm{dist}(\overline{\mathrm{supp}(|u_\varrho|+\widetilde{\mu}_\rho)},\cup_i\{x_i\})\ge\varrho>0$. Moreover

$$\int_\Omega|u\cdot\phi_\varrho-u|^{2^*}\,dx\le\sum_{i=0}^N\int_{V_{2\varrho}(x_i)}|u|^{2^*}\to0\qquad\text{for}\quad\varrho\to0^+,$$

and, for every $\psi\in\mathcal{C}^0(\overline{\Omega})$,

$$\begin{aligned}
&\left|\int_{\overline{\Omega}}\psi\,d\mu_\varrho-\int_{\overline{\Omega}}\psi\,d\mu\right|\le\int_{\overline{\Omega}}|\psi|\,d(|\mu_\varrho-\mu|)\\
&\le\sum_{i=0}^N\int_{V_{2\varrho}(x_i)}|\psi||(|\nabla(u\cdot\phi_\varrho)|^2-|\nabla u|^2)|\,dx+\sum_{i=0}^N\int_{\overline{V}_{2\varrho}(x_i)}|\psi|\,d\widetilde{\mu}\\
&\le2\sum_{i=0}^N\int_{V_{2\varrho}(x_i)}|\psi|\,|\nabla u|^2\,dx+2\sum_{i=0}^N\int_{V_{2\varrho}(x_i)\setminus V_\varrho(x_i)}|\psi|\,|u|^2\,|\nabla\phi_\varrho|^2\,dx\\
&\quad+\sum_{i=0}^N\int_{\overline{V}_{2\varrho}(x_i)}|\psi|\,d\widetilde{\mu}\\
&\le2\sum_{i=0}^N\int_{V_{2\varrho}(x_i)}|\psi|\,|\nabla u|^2\,dx+\frac{2}{\varrho^2}\sum_{i=0}^N\int_{V_{2\varrho}(x_i)}|\psi|\,|u|^2\,dx+\sum_{i=0}^N\int_{\overline{V}_{2\varrho}(x_i)}|\psi|\,d\widetilde{\mu}\\
&\le2\sum_{i=0}^N\int_{V_{2\varrho}(x_i)}|\psi|\,|\nabla u|^2\,dx+C\frac{\varrho^2}{\varrho^2}\sum_{i=0}^N\left(\int_{V_{2\varrho}(x_i)}|\psi|^{2^*/2}|u|^{2^*}\,dx\right)^{2/2^*}\\
&\quad+\sum_{i=0}^N\int_{\overline{V}_{2\varrho}(x_i)}|\psi|\,d\widetilde{\mu}\to0,
\end{aligned}$$

where we used the Hölder inequality, the absolute continuity of the integral and the fact that $\widetilde{\mu}$ is a non atomic measure. This proves that $u_\varrho \to u$ strongly in $L^{2^*}(\Omega)$ and that $\mu_\varrho \rightharpoonup \mu$ in w^*-$\mathcal{M}(\overline{\Omega})$, when $\varrho \to 0^+$; hence, $(u_\varrho, \mu_\varrho) \stackrel{\tau}{\to} (u, \mu)$ and, since $\mu(\overline{\Omega}) < 1$, it follows that μ_ϱ satisfies property (2). In particular $(u_\varrho, \mu_\varrho) \in X$. Finally, (5) follows directly by the definition of $\widetilde{\mathcal{F}}$ and the strong L^{2^*}-convergence of u_ϱ to u. Hence, by hypotheses, we have

$$\mathcal{F}^-(u_\varrho, \mu_\varrho) \geq \widetilde{\mathcal{F}}(u_\varrho, \mu_\varrho) \qquad \forall \varrho > 0\,.$$

Taking into account the upper semicontinuity of the Γ^+-$\liminf$, letting $\varrho \to 0^+$ and using property (5), we obtain

$$\mathcal{F}^-(u, \mu) \geq \widetilde{\mathcal{F}}(u, \mu) \qquad \forall\ (u, \mu) \in X \text{ satisfying (i) and (ii)}\,. \tag{4.1}$$

As a second step let us consider a pair $(u, \mu) \in X$ satisfying (i). For any $N \geq 0$, define (u_N, μ_N) in the following way

$$u_N = u \quad \text{and} \quad \mu_N = |\nabla u|^2 + \widetilde{\mu} + \sum_{i=0}^{N} \mu_i \delta_{x_i}\,.$$

Clearly, $(u_N, \mu_N) \in X$ and $(u_N, \mu_N) \stackrel{\tau}{\to} (u, \mu)$, when $N \to +\infty$. Hence by (4.1) we have

$$\mathcal{F}^-(u_N, \mu_N) \geq \widetilde{\mathcal{F}}(u_N, \mu_N) = F_0 \int_\Omega |u|^{2^*}\, dx + S^F \sum_{i=0}^{N} \mu_i^{2^*/2} \qquad \forall N \geq 0\,.$$

Then taking into account the upper semicontinuity of the Γ^+-$\liminf$ and letting $N \to +\infty$, it follows

$$\mathcal{F}^-(u, \mu) \geq \limsup_{N \to +\infty} \mathcal{F}^-(u_N, \mu_N) \geq \lim_{N \to +\infty} \widetilde{\mathcal{F}}(u_N, \mu_N) = \widetilde{\mathcal{F}}(u, \mu)\,.$$

Finally we recover the general case $(u, \mu) \in X$ considering for any $\delta > 0$, the sequence $\{(u_\delta, \mu_\delta)\}$ defined as follows

$$u_\delta = \frac{u}{1+\delta} \quad \text{and} \quad \mu_\delta = \frac{\mu}{(1+\delta)^2} = |\nabla u_\delta|^2 + \frac{\widetilde{\mu}}{(1+\delta)^2} + \sum_{i=0}^{+\infty} \frac{\mu_i}{(1+\delta)^2} \delta_{x_i}\,.$$

Clearly, $(u_\delta, \mu_\delta) \in X$, $\mu_\delta(\Omega) < 1$ and $(u_\delta, \mu_\delta) \stackrel{\tau}{\to} (u, \mu)$, when $\delta \to 0^+$. Hence by the previous step

$$\mathcal{F}^-(u_\delta, \mu_\delta) \geq \widetilde{\mathcal{F}}(u_\delta, \mu_\delta) = \frac{F_0}{(1+\delta)^{2^*}} \int_\Omega |u|^{2^*}\, dx + \frac{S^F}{(1+\delta)^{2^*}} \sum_{i=0}^{+\infty} \mu_i^{2^*/2}\,.$$

Thus the conclusion follows taking into account the upper semicontinuity of the Γ^+-$\liminf$ and letting $\delta \to 0^+$. $\square$

5. – The non homogeneous case

In this section using the approach of the Γ-convergence we will consider a non homogeneous version of the sequence $\{\mathcal{F}_\varepsilon\}$, i.e.

$$\mathcal{F}_\varepsilon^A(u) = \varepsilon^{-2^*} \int_\Omega A(x) F(\varepsilon u_\varepsilon)\ dx\,.$$

As we saw in the previous section, the Γ^+-limit of the sequence $\{\mathcal{F}_\varepsilon\}$ is not a local functional. Thus, its non homogeneous version cannot be treated, as usual, simply by a localization argument, but it requires a slightly more careful analysis (see Theorem 5.1).

To this purpose, we assume that $A : \Omega \to \mathbb{R}$ is a nonnegative function belonging to $L^\infty(\Omega)$, which satisfies the following conditions

(H1) $A(x) = \lim_{\varrho\to 0^+}(\text{ess-}\sup_{B_\varrho(x)} A(y))$ for every $x \in \Omega$.

(H2) for every $x \in \Omega$, for every $\varrho > 0$ and for every $\eta > 0$ the set $B_\varrho(x) \cap \{y \in \Omega : A(y) > A(x) - \eta\}$ contains an open set.

We say that a function satisfying (H1) is essentially upper semicontinuous. Note that property (H1) implies also the usual upper semicontinuity. In particular, if A is continuous or piecewise continuos and essentially upper semicontinuous, it clearly satisfies (H1) and (H2). Moreover, we emphasize that the essential upper semicontinuity cannot be obtained from any measurable function satisfying (H2), simply modifying it on a set of zero Lebesgue measure.

As in Sections 2 and 3, let us define the functionals $\mathcal{F}_\varepsilon^A$ and $\overline{\mathcal{F}}^A : X \to [0, +\infty)$ as follows

$$\mathcal{F}_\varepsilon^A(u,\mu) = \begin{cases} \varepsilon^{-2^*} \displaystyle\int_\Omega A(x) F(\varepsilon u)\ dx & \text{if } (u,\mu) \in X \text{ and } \mu = |\nabla u|^2 \\ 0 & \text{otherwise in } X \end{cases} ;$$

$$\overline{\mathcal{F}}^A(u,\mu) = \Gamma^+\text{-}\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^A(u,\mu)\,.$$

In the sequel we shall prove the Γ^+-convergence and the concentration results in the non homogeneous case.

Theorem 5.1. *Let* $A \in L^\infty(\Omega)$ *satisfying* (H1) *and* (H2) *and* $F : \mathbb{R} \to [0, +\infty)$ *as in* (2.3) *and* (2.4). *Then there exists the* Γ^+*-limit of the sequence* $\{\mathcal{F}_\varepsilon^A\}$, *denoted by* $\overline{\mathcal{F}}^A$, *and*

$$\overline{\mathcal{F}}^A(u,\mu) = F_0 \int_\Omega A(x)|u|^{2^*}\ dx + S^F \sum_{i=0}^{+\infty} A(x_i)\mu_i^{2^*/2} \qquad \forall (u,\mu) \in X\,.$$

Proof. Let us denote

$$\widetilde{\mathcal{F}}^A(u,\mu) = F_0 \int_\Omega A(x)|u|^{2^*}\ dx + S^F \sum_{i=0}^{+\infty} A(x_i)\mu_i^{2^*/2}\,.$$

Let us first prove the Γ^+-limsup inequality, i.e.

$$\Gamma^+\text{-}\limsup_{\varepsilon\to 0^+} \mathcal{F}^A_\varepsilon(u,\mu) \le \widetilde{\mathcal{F}}^A(u,\mu) \qquad \forall (u,\mu)\in X\,. \tag{5.1}$$

For every $\varepsilon > 0$, let $(u_\varepsilon, \mu_\varepsilon) \in X$ with $\mu_\varepsilon = |\nabla u_\varepsilon|^2$ be a sequence such that $(u_\varepsilon,\mu_\varepsilon) \stackrel{\tau}{\to} (u,\mu) \in X$. By (3.1) and (3.2), it follows that

$$\frac{F(\varepsilon u_\varepsilon)}{\varepsilon^{2^*}} \rightharpoonup \nu := g + \sum_{i=0}^{+\infty} \nu_i \delta_{x_i}$$

with $g \in L^1(\Omega)$ and $0 \le g \le F_0|u|^{2^*}$ a.e. in Ω and $\nu_i \le S^F \mu_i^{2^*/2}$. Hence, by (H1)

$$\limsup_{\varepsilon\to 0} \int_\Omega A(x)\frac{F(\varepsilon u_\varepsilon)}{\varepsilon^{2^*}}\,dx \le \int_\Omega A(x)\,d\nu \le \widetilde{\mathcal{F}}^A(u,\mu)$$

and this gives (5.1).

In order to prove the inequality

$$\Gamma^+\text{-}\liminf_{\varepsilon\to 0^+} \mathcal{F}^A_\varepsilon(u,\mu) \ge \widetilde{\mathcal{F}}^A(u,\mu) \qquad \forall (u,\mu)\in X$$

we proceed as follows. We fix $(u,\mu)\in X$. Without loss of generality, we may assume that the atomic part of μ is given by a single Dirac mass $\mu_0\delta_{x_0}$, with $x_0 \in \overline{\Omega}$ and $\mu_0 \in [0,1]$. The general case follows trivially. By (H2), we have that for every $x \in \overline{\Omega}$, every $\rho > 0$ and every $\eta > 0$, we can find an open set Ω^ρ_η, which is contained in the interior of the set $\{y \in \Omega \ :\ A(y) > A(x) - \eta\}$ and such that there exists $x^\rho_\eta \in \Omega^\rho_\eta$ with $\operatorname{dist}(x, x^\rho_\eta) < \rho$, i.e. $x^\rho_\eta \to x$ as $\rho \to 0^+$ for every choice of $\eta > 0$.

Fix $x = x_0$, for every $n \in \mathbb{N}$, set $\rho_n = \eta_n = 1/n$, and consider the corresponding set $\Omega_n := \Omega^{\rho_n}_{\eta_n}$ and the point $x_n := x^{\rho_n}_{\eta_n} \in \Omega_n$ such that $d(x_n, x_0) < 1/n$. Set $\mu_n = |\nabla u|^2 + \widetilde{\mu} + \mu_0\delta_{x_n}$. Since $x_n \to x_0$, we have that $\mu_n \rightharpoonup \mu$ in $w^*\text{-}\mathcal{M}(\overline{\Omega})$ as $n \to +\infty$ and hence $(u,\mu_n) \stackrel{\tau}{\to} (u,\mu)$. By Theorem 3.1 and Remark 3.2, given $n\in\mathbb{N}$, there exists a sequence $\{(u^n_\varepsilon, \mu^n_\varepsilon)\} \subseteq X$, with $\mu^n_\varepsilon = |\nabla u^n_\varepsilon|^2$, such that $(u^n_\varepsilon, |\nabla u^n_\varepsilon|^2) \stackrel{\tau}{\to} (u,\mu^n)$, when $\varepsilon \to 0^+$, and

$$\text{(5.2)}\qquad \begin{aligned} &\text{(i)} && \frac{F_\varepsilon(\varepsilon u^n_\varepsilon)}{\varepsilon^{2^*}} \rightharpoonup F_0|u|^{2^*} + S^F\mu_0^{2^*/2}\delta_{x_n} && w^*\text{-}\mathcal{M}(\overline{\Omega})\,;\\ &\text{(ii)} && \frac{F_\varepsilon(\varepsilon u^n_\varepsilon)}{\varepsilon^{2^*}} \to F_0|u|^{2^*} && \text{strongly in } L^{2^*}_{\mathrm{loc}}(\Omega\setminus\{x_n\})\,. \end{aligned}$$

This implies

$$\begin{aligned} \mathcal{F}^A_\varepsilon(u^n_\varepsilon,\mu^n_\varepsilon) &= \varepsilon^{-2^*}\int_\Omega A(x)F(\varepsilon u^n_\varepsilon)\,dx = \varepsilon^{-2^*}\int_{\Omega\setminus\Omega_n} A(x)F(\varepsilon u^n_\varepsilon)\,dx \\ &\quad + \varepsilon^{-2^*}\int_{\Omega_n} A(x)F(\varepsilon u^n_\varepsilon)\,dx \\ &\ge \varepsilon^{-2^*}\int_{\Omega\setminus\Omega_n} A(x)F(\varepsilon u^n_\varepsilon)\,dx + \left[A(x_0) - \frac{1}{n}\right]\varepsilon^{-2^*}\int_{\Omega_n} F(\varepsilon u^n_\varepsilon)\,dx\,. \end{aligned}$$

Passing to the lim inf as $\varepsilon \to 0^+$, taking into account (5.2) and the lower semicontinuity of measures on open sets, it follows

$$\begin{aligned}(\Gamma^+\text{-}\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon^A)(u,\mu^n) \geq F_0\int_{\Omega\setminus\Omega_n} A(x)|u|^{2^*}\,dx + \left[A(x_0)-\frac{1}{n}\right]\int_{\Omega_n}[F_0|u|^{2^*}dx \\ + S^F\mu_0^{2^*/2}\delta_{x_n}] \\ = F_0\left(\int_{\Omega\setminus\Omega_n} A(x)|u|^{2^*}\,dx + \left[A(x_0)-\frac{1}{n}\right]\int_{\Omega_n}|u|^{2^*}dx\right) \\ + S^F\left[A(x_0)-\frac{1}{n}\right]\mu_0^{2^*/2}.\end{aligned}$$

Finally, letting $n \to +\infty$, taking into account the upper semicontinuity of the Γ^+-lim inf and using the fact that $|\Omega_n| \to 0$, we obtain

$$(\Gamma^+\text{-}\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon^A)(u,\mu) \geq F_0\int_\Omega A(x)|u|^{2^*}\;dx + S^F A(x_0)\mu_0^{2^*/2}.$$

The theorem is then proved. □

REMARK 5.2. Clearly $\overline{\mathcal{F}}^A(u,\mu) \leq (\max A)\; S^F$ and the equality is achieved at least when $(u,\mu) = (0,\delta_{x_0})$, with x_0 a maximizer of A in $\overline{\Omega}$, i.e.

$$\max_{(u,\mu)\in X}\overline{\mathcal{F}}^A(u,\mu) = (\max A)\; S^F.$$

Hence, if we define

$$S_\varepsilon^{F,A} := \sup\{\mathcal{F}_\varepsilon^A(u)\;:\; u\in H_0^1(\Omega),\;\; \|\nabla u\|_{L^2(\Omega)}\leq 1\},$$

then, by Theorem 5.1 and the properties of Γ^+-convergence, it follows that $S_\varepsilon^{F,A} \to (\max A)\; S^F$.

Finally, we have the following concentration result.

THEOREM 5.3. *Let $(u_\varepsilon,\mu_\varepsilon)\in X$, with $\mu_\varepsilon = |\nabla u_\varepsilon|^2$, be a maximizing sequence for $\mathcal{F}_\varepsilon^A$. Then, for $\varepsilon \to 0^+$, it concentrates at a single point, which is a maximum of A in $\overline{\Omega}$; i.e. $(u_\varepsilon,\mu_\varepsilon) \xrightarrow{\tau} (0,\delta_{x_0})$, with x_0 among the maximum points of A in $\overline{\Omega}$. Moreover, every maximum point of A is the concentration point of some maximizing sequence.*

PROOF. Note that, for every $(u,\mu)\in X$, we have

$$\overline{\mathcal{F}}^A(u,\mu) \leq (\max A)\;\overline{\mathcal{F}}(u,\mu) \leq (\max A)\; S^F \tag{5.3}$$

and $\overline{\mathcal{F}}^A(0,\delta_{x_0}) = (\max A)\; S^F$, when x_0 is a maximizer of A.

Assume, by contradiction, that there exists $(u,\mu)\in X$ such that $(u,\mu)\neq(0,\delta_{x_0})$, with x_0 a maximizer for A, and $\overline{\mathcal{F}}^A(u,\mu) = (\max A)\; S^F$. Then, by (5.3), it follows that $\overline{\mathcal{F}}(u,\mu) = S^F$, and this implies, by Theorem 3.1 and Lemma 3.6, that $(u,\mu) = (0,\delta_{x_0})$. By the hypothesis, x_0 is not a maximizer for A, hence

$$\overline{\mathcal{F}}^A(0,\delta_{x_0}) = A(x_0)\; S^F < (\max A)\; S^F$$

which is a contradiction. The thesis is then accomplished. □

6. – Asymptotic expansion

In [12], a second order expansion of S_ε^F with respect to ε is given. Namely, the authors prove that

$$S_\varepsilon^F = S^F \left(1 - \frac{n}{n-2} w_\infty^2 \min_\Omega \mathcal{T}_\Omega \ \varepsilon^2 + o(\varepsilon^2)\right) \tag{6.1}$$

with the constant w_∞ defined by

$$w_\infty^2 := \frac{2(n-1)}{nS^F} \inf \left\{ \liminf_{k\to+\infty} \int_{\mathbb{R}^n} \frac{F(w_k)}{K(|x|)}\, dx \ : \ \{w_k\} \in \mathcal{B}^F \right\} , \tag{6.2}$$

where $K(r)$ is the fundamental solution of $-\Delta$, $\mathcal{B}^F$ is the class of maximizing sequences for S^F consisting of radial functions, and $\mathcal{T}_\Omega$ is the Robin function of Ω, i.e. the leading term of the regular part of the Green's function of Ω, evaluated in every $x \in \overline{\Omega}$.

This expansion together with a sharp estimate of $\int_\Omega F(\varepsilon u_\varepsilon)dx$ for any concentrating sequence $\{u_\varepsilon\}$ permits to identify the concentration point as the minimum of the Robin function.

The main idea, as usual in this framework, is to construct maximizing sequences by modifying the ground states for S^F, i.e. the maximizers of the corresponding problem in $\mathbb{R}^n$. Thus, a crucial point is the accurate study of the behaviour of these extremals, given in the following theorem.

THEOREM 6.1 (see, [13, Lemma 2]). *Let w be an extremal for S^F, then:*

(i) *either $w > 0$ or $w < 0$;*
(ii) *there exists a ball $B_{r_0}(x_0)$ (we may assume $x_0 = 0$) such that w is a radial function outside this ball;*
(iii) *if we assume that $w > 0$, then the function $r \mapsto w(r)$ is strictly decreasing on $(r_0, +\infty)$ and*

$$w(r) = W_\infty(r)(1 + O(r^{-2})) , \qquad w'(r) = W_\infty K'(r)(1 + O(r^{-2})) \tag{6.3}$$

for $r \to +\infty$, where

$$W_\infty^2 = \frac{2(n-1)}{nS^F} \int_{\mathbb{R}^n} \frac{F(w)}{K(|x|)}\, dx ;$$

(iv) *$w(r) \le Cr^{2-n}$, $F(w(r)) \le Cr^{-2n}$ and*

$$\int_{\mathbb{R}^n \setminus B_R(0)} |\nabla w|^2\, dx \le CR^{2-n} , \qquad \int_{\mathbb{R}^n \setminus B_R(0)} F(w)\, dx \le CR^{-n} ,$$

for every $R > 0$;
(v) *if F is non-decreasing on R^+ and non-increasing on R^-, then $\overline{B_{r_0}(0)} = \{w = \max w\}$.*

Clearly, if w is an extremal for S^F, then $w_\infty^2 \leq W_\infty^2$. In order to obtain (6.1), it is necessary to require a further assumption on the integrand F. Namely, the following condition

$$0 < w_\infty < +\infty \tag{6.4}$$

is assumed. In particular, it is verified by the volume functional ($F(t) = \chi_{\{t\geq 1\}}(t)$), while it is not in the critical case $F(t) = |t|^{2^*}$. Nevertheless, we can easily construct a function F with critical growth satisfying (6.4), for instance

$$F(t) = \begin{cases} |t|^{2^*} & \text{if } 0 \leq t \leq 1\,; \\ 2|t|^{2^*} & \text{if } 1 < t < 2\,; \\ |t|^{2^*} & \text{if } t \geq 2\,. \end{cases}$$

Assumption (6.4) has an influence on the behaviour of the sequences in $\mathcal{B}^F$; in particular, in [12, Theorem 3], it is proved that it is equivalent to the fact that no sequences of $\mathcal{B}^F$ concentrates at 0 and there exists a radial extremal for S^F.

REMARK 6.2. Using the generalized version of the Concentration Compactness Alternative of P. L. Lions (see Theorem 2.2), one can deduce that, among all extremals of S^F, there exists one function, say w, with optimal decay at $+\infty$ (i.e. (6.3) holds with W_∞ replaced by w_∞). In fact, w can be obtained as the limit of a compact sequence in $\mathcal{B}^F$ on which the infimum in (6.2) is attained, and this implies that

$$w_\infty^2 = W_\infty^2\,.$$

In this section, we shall interpret this result of asymptotic expansion in terms of Γ^+-convergence, in order to extend it also to the non homogeneous case, taking advantage of the properties of Γ^+-convergence. This will be done in Theorem 6.8, under some additional assumptions on the coefficient A. Those assumptions may be not optimal. However, in the general case, the result is not always true, as shown in Example 6.6.

In order to avoid further difficulties, in the sequel we always assume that Ω is a regular bounded open subset of $\mathbb{R}^n$, even if in [12], the authors consider the general case, with a suitable definition of the Green and the Robin functions in the whole of $\overline{\Omega}$.

It is well-known that the Green function with pole at x_0 of a regular bounded open set Ω is defined by

$$G_{x_0}^\Omega(x) = K(|x - x_0|) - H(x, x_0)$$

where H is the solution of

$$\begin{cases} \Delta H(\cdot, x_0) = 0 & \text{in } \Omega \\ H(\cdot, x_0) = K(|\cdot\ - x_0|) & \text{on } \partial\Omega \end{cases}$$

and the Robin function is defined by

$$\mathcal{T}_\Omega(x) = H(x, x)\,.$$

By its definition, the Robin function turns out to be continuous in Ω. Moreover, since Ω is regular, $\mathcal{T}_\Omega$ diverges to $+\infty$ approaching the boundary; hence, in particular, $\mathcal{T}_\Omega$ always admits a minimum point in Ω. We recall that a minimum point of the Robin function on Ω is called a harmonic center of Ω and the harmonic radius r_Ω is defined by the relation $K(r_\Omega) = \min_\Omega \mathcal{T}_\Omega$. This, in particular, implies $r_\Omega > 0$.

Let us now define the following functionals:

$$\mathcal{H}_\varepsilon(u,\mu) = \begin{cases} \dfrac{\varepsilon^{-2^*}\displaystyle\int_\Omega F(\varepsilon u)\ dx - S^F}{\varepsilon^2} & \text{if } (u,\mu)\in X,\ \ \mu = |\nabla u|^2\,, \\ -\dfrac{S^F}{\varepsilon^2} & \text{otherwise in } X\,, \end{cases}$$

$$\overline{\mathcal{H}}(u,\mu) = \begin{cases} -\dfrac{n}{n-2} w_\infty^2 \mathcal{T}_\Omega(x_0) S^F & \text{if } (u,\mu) = (0,\delta_{x_0}) \text{ and } x_0 \in \Omega\,, \\ -\infty & \text{otherwise in } X\,. \end{cases}$$

Using this notation, the result proven in [12] reads as follows.

THEOREM 6.3. *Assume* $0 < w_\infty < +\infty$. *There exists the* Γ^+*-limit of the sequence of functionals* $\{\mathcal{H}_\varepsilon\}$ *and*

$$\Gamma^+\text{-}\lim_{\varepsilon\to 0^+} \mathcal{H}_\varepsilon(u,\mu) = \overline{\mathcal{H}}(u,\mu) \qquad \forall (u,\mu)\in X\,.$$

The proof is essentially contained in [12]. Indeed, the first condition

$$\text{(i)} \qquad \forall\ (u_\varepsilon,\mu_\varepsilon) \xrightarrow{\tau} (u,\mu) \qquad \overline{\mathcal{H}}(u,\mu) \ge \limsup_{\varepsilon\to 0} \mathcal{H}_\varepsilon(u_\varepsilon,\mu_\varepsilon)$$

follows by [12, Part 1 of Theorem 17], while the second one

$$\text{(ii)} \qquad \exists\ (\widetilde{u}_\varepsilon,\widetilde{\mu}_\varepsilon) \xrightarrow{\tau} (u,\mu), \quad \text{s.t.} \qquad \overline{\mathcal{H}}(u,\mu) \le \liminf_{\varepsilon\to 0} \mathcal{H}_\varepsilon(\widetilde{u}_\varepsilon,\widetilde{\mu}_\varepsilon)$$

can be obtained taking into account [12, Remark 21]. Nevertheless, in order to underline the structure of the optimal sequences, we will sketch the proof of (ii), following the lines of Step 1 in the proof of [12, Theorem 17].

It is clearly enough to consider $(u,\mu) = (0,\delta_{x_0})$, for any $x_0 \in \Omega$. Let $r(x_0)$ (simply r) be defined by the relation $K(r) = \mathcal{T}_\Omega(x_0)$. Since $\mathcal{T}_\Omega(x_0)$ is finite, this implies $r > 0$. For every $\varepsilon > 0$, define

$$R_\varepsilon = \varepsilon^{-\frac{2}{n-2}} r \qquad \text{and} \qquad r_\varepsilon = \varepsilon^{-\frac{2}{n-1}} r\,. \tag{6.5}$$

Let w be a radial extremal for S^F, with optimal decay, as in Remark 6.2. We define the comparison function $W_\varepsilon \in H_0^1(B_{R_\varepsilon}(0))$ by $W_\varepsilon = w$ in $B_{r_\varepsilon}(0)$ and $\Delta W_\varepsilon = 0$ in $B_{R_\varepsilon}(0)\setminus B_{r_\varepsilon}(0)$. It follows that

$$\int_{B_{R_\varepsilon}(0)} F(W_\varepsilon)\ dx \ge S^F - o(\varepsilon^2) \ \text{ and } \ \|\nabla W_\varepsilon\|^2_{L^2(\Omega)} \le 1 + w_\infty^2 \mathcal{T}_\Omega(x_0)\varepsilon^2 + o(\varepsilon^2) \tag{6.6}$$

(see [12, Section 6]). Hence, set $\lambda_\varepsilon = 1 + w_\infty^2 \mathcal{T}_\Omega(x_0)\varepsilon^2 + o(\varepsilon^2)$, $s_\varepsilon = \lambda_\varepsilon^{-\frac{1}{n-2}}$ and $\widetilde{W_\varepsilon}(|x|) = W_\varepsilon(\frac{|x|}{s_\varepsilon})$. Then $\widetilde{W_\varepsilon} \in H_0^1(B_{s_\varepsilon R_\varepsilon}(0)) \subset H_0^1(B_{R_\varepsilon}(0))$, $\|\nabla \widetilde{W_\varepsilon}\|_{L^2(B_{R_\varepsilon}(0))} \leq 1$ and, by (6.6) and a change of variables, we get

$$(6.7)\quad \int_{B_{R_\varepsilon}(0)} F(\widetilde{W_\varepsilon})\,dx = \int_{B_{s_\varepsilon R_\varepsilon}(0)} F(\widetilde{W_\varepsilon})\,dx \geq S^F\left(1 - \frac{n}{n-2} w_\infty^2 \mathcal{T}_\Omega(x_0)\varepsilon^2 + o(\varepsilon^2)\right).$$

Let $U_\varepsilon(x) = \widetilde{W_\varepsilon}(\varepsilon^{-\frac{2^*}{n}}|x|)$. This is a radial function and hence there exists $\phi_\varepsilon : [0, +\infty) \to \mathbb{R}$ such that $U_\varepsilon = \phi_\varepsilon \circ G_0$, where G_0 denotes the Green's function of $B_r(0)$ with pole at zero. Thus we can define the harmonic transplantation of U_ε with respect to Ω and x_0 as $u_\varepsilon = \phi_\varepsilon \circ G_{x_0}^\Omega$, where $G_{x_0}^\Omega$ is the Green's function of Ω with pole at x_0 (see [2]). This transformation preserves the Dirichlet integral and

$$(6.8)\qquad \int_\Omega F(u_\varepsilon)\;dx \geq \int_{B_r(0)} F(U_\varepsilon)\;dx\,.$$

By construction, the sequence of functions $\widetilde{u}_\varepsilon$, defined by $\widetilde{u}_\varepsilon = u_\varepsilon/\varepsilon$, is such that $(\widetilde{u}_\varepsilon, |\nabla \widetilde{u}_\varepsilon|^2) \in X$ and $(\widetilde{u}_\varepsilon, |\nabla \widetilde{u}_\varepsilon|^2) \xrightarrow{\tau} (0, \delta_{x_0})$. Finally, by a change of variables and taking into account (6.7) and (6.8), we deduce (ii).

Remark 6.4. This procedure gives an almost explicit construction of an optimal sequence. Indeed $\widetilde{u}_\varepsilon = \varepsilon^{-1}\left(\phi_\varepsilon \circ G_{x_0}^\Omega\right)$ and ϕ_ε can be deduced by the definition of U_ε as follows. Since

$$\begin{aligned} U_\varepsilon(x) &= W_\varepsilon\left(\varepsilon^{-\frac{2^*}{n}}\frac{|x|}{s_\varepsilon}\right) \\ &= \begin{cases} w\left(\varepsilon^{-\frac{2^*}{n}}\dfrac{|x|}{s_\varepsilon}\right) & \text{if } |x| \leq s_\varepsilon \varepsilon^\gamma r\,, \\ w(r_\varepsilon)\dfrac{s_\varepsilon^{n-2}K(|x|) - K(r)}{\varepsilon^{-\frac{2}{n-1}}K(r) - K(r)} & \text{if } s_\varepsilon\varepsilon^\gamma r < |x| < s_\varepsilon r\,, \\ 0 & \text{if } s_\varepsilon r \leq |x| \leq r\,, \end{cases} \end{aligned}$$

with $\gamma = \frac{2}{(n-2)(n-1)}$, at least in the interval $(0, K(s_\varepsilon\varepsilon^\gamma r) - K(r))$, the function ϕ_ε is piecewise affine. More explicitly

$$(6.9)\qquad \phi_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 \leq t \\ & \quad \leq K(s_\varepsilon r) - K(r), \\ w(r_\varepsilon)\dfrac{s_\varepsilon^{n-2}t + (s_\varepsilon^{n-2} - 1)K(r)}{\varepsilon^{-\frac{2}{n-1}}K(r) - K(r)} & \text{if } K(s_\varepsilon r) - K(r) < t \\ & \quad < K(s_\varepsilon\varepsilon^\gamma r) - K(r), \end{cases}$$

where, for $\varepsilon \to 0$, $K(s_\varepsilon r) - K(r) \to 0$ and $K(s_\varepsilon\varepsilon^\gamma r) - K(r) \to +\infty$.

This fact will be the crucial point in the proof of Theorem 6.8. Moreover, it permits to conclude that the level sets of $\widetilde{u}_\varepsilon$, namely the level sets of the Green function, are "almost" circles.

REMARK 6.5. Since the maximum of $\overline{\mathcal{H}}$ is reached when x_0 is a minimum point of the Robin function $\mathcal{T}_\Omega$, as a consequence of Theorem 6.3 we obtain, in particular, that the maximizer sequences of $\mathcal{F}_\varepsilon$ concentrates at a harmonic center of Ω.

As we saw in the previous section, the inhomogeneity of the functional $\overline{\mathcal{F}}^A$ plays a strong role in the choice of the concentrating points. Already at the first order, they have to be a maximum point of A. In this section, we investigate if the geometry of the set Ω has an influence in the choice of the concentration points among all the maximizers of A. In some cases, this fact will be highlighted by means of a second order expansion, which will be deduced by Theorem 6.3.

The following example shows that, in general, the behaviour of the coefficient $A(x)$ near its maximum points could affect the order of the convergence of the optimal sequences and hence the order of the second term in the asymptotic expansion in Γ^+-convergence of $\mathcal{F}^A_\varepsilon$.

EXAMPLE 6.6. Let $A(x) = 1 - |x|^\alpha$, with $\alpha \in (0, +\infty)$, and $\Omega = B_1(0) \subset \mathbb{R}^n$. Assume that

$$F(t) = \begin{cases} 0 & \text{if } \; t < 1\,; \\ 1 & \text{if } \; t \geq 1\,. \end{cases}$$

Then

$$\sup\{\mathcal{F}^A_\varepsilon(u,\mu) : (u,\mu) \in X\} = \sup\{\varepsilon^{-2^*} \int_B A(x)\, dx \;:\; B \subset \Omega\,,\ \operatorname{cap}_\Omega(B) \leq \varepsilon^2\}\,.$$

Since the functional is radially symmetric, by the properties of the capacity, it easily follows that

$$\max\{\mathcal{F}^A_\varepsilon(u,\mu) \;:\; (u,\mu) \in X\} = \varepsilon^{-2^*} \int_{B_{\rho_\varepsilon}(0)} A(x)\, dx$$

where $B_{\rho_\varepsilon}(0)$ is the optimal sequence for the volume functional and $\rho_\varepsilon \sim \varepsilon^{\frac{2^*}{n}}$. Hence,

$$\frac{1}{\varepsilon^2}\left(\varepsilon^{-2^*}\int_{B_{\rho_\varepsilon}(0)} A(x)\,dx - S^V\right) = \left(\frac{\varepsilon^{-2^*}|B_{\rho_\varepsilon}(0)| - S^V}{\varepsilon^2}\right) - \frac{1}{\varepsilon^{2+2^*}}\int_{B_{\rho_\varepsilon}(0)} |x|^\alpha\, dx$$

$$= \left(\frac{\varepsilon^{-2^*}|B_{\rho_\varepsilon}(0)| - S^V}{\varepsilon^2}\right) - \frac{C}{\varepsilon^{2+2^*}}\int_0^{\rho_\varepsilon} \rho^{n-1+\alpha}\, d\rho$$

$$= \left(\frac{\varepsilon^{-2^*}|B_{\rho_\varepsilon}(0)| - S^V}{\varepsilon^2}\right) - C\varepsilon^{\frac{-2n+4+2\alpha}{n-2}}$$

which does not diverge if and only if $\alpha \geq n - 2$.

In particular, if $\alpha > n-2$, the inhomogeneity does not change the optimal sequences. Then, the second order expansion in Γ^+-convergence can be computed using the result in the homogeneous case.

In the case $\alpha = n-2$, even though the order remains the same, the Γ^+-limit does not.

Finally, for $\alpha < n-2$, the exact order of the second term in the asymptotic expansion will be strictly less than two.

REMARK 6.7. Let us note that in the example above the threshold for the exponent α is strictly related with the behaviour of the integrand F at zero. Indeed, in a similar radial case ($A(x) = 1 - |x|^\alpha$), one can see that, if $F(t) \sim |t|^p$, $p > 2^*$, as $t \to 0$, the dependence on x does not affect the optimal sequences if $\alpha > \frac{2p(n-2)}{2p-2-2^*}$.

Having in mind this example, we can prove a result similar to the one in Theorem 6.3, under some additional hypothesis on the coefficient A. To this purpose, let us define the following functionals

$$\mathcal{H}_\varepsilon^A(u,\mu) = \begin{cases} \dfrac{\varepsilon^{-2^*}\displaystyle\int_\Omega A(x)F(\varepsilon u)\ dx - (\max A)S^F}{\varepsilon^2} & \text{if } (u,\mu)\in X,\, \mu = |\nabla u|^2\,, \\ -\dfrac{(\max A)S^F}{\varepsilon^2} & \text{otherwise in } X\,, \end{cases}$$

$$\overline{\mathcal{H}}^A(u,\mu) = \begin{cases} -\dfrac{n}{n-2} w_\infty^2 (\max A)\mathcal{T}_\Omega(x_0)S^F & \text{if } (u,\mu) = (0,\delta_{x_0}) \text{ and } x_0 \\ & \text{is a maximizer of } A\,, \\ -\infty & \text{otherwise in } X\,. \end{cases}$$

THEOREM 6.8. *Assume that $A \in L^\infty(\Omega)$ satisfies conditions* (H1) *and* (H2) *of previous section. Assume also that the set $B = \{x \in \overline{\Omega} \ : \ A(x) = \max A\}$ is the closure of an open set. Then*

$$\Gamma^+\text{-}\lim_{\varepsilon\to 0^+} \mathcal{H}_\varepsilon^A(u,\mu) = \overline{\mathcal{H}}^A(u,\mu) \qquad \forall (u,\mu)\in X\,.$$

PROOF. We have to prove that, for every $(u,\mu) \in X$, it follows

(i) for all sequences $(u_\varepsilon, \mu_\varepsilon) \xrightarrow{\tau} (u,\mu)$, we have

$$\overline{\mathcal{H}}^A(u,\mu) \geq \limsup_{\varepsilon\to 0} \mathcal{H}_\varepsilon^A(u_\varepsilon,\mu_\varepsilon)\,;$$

(ii) there exists a sequence $(\widetilde{u}_\varepsilon, \widetilde{\mu}_\varepsilon) \xrightarrow{\tau} (u,\mu)$, such that

$$\overline{\mathcal{H}}^A(u,\mu) \leq \liminf_{\varepsilon\to 0} \mathcal{H}_\varepsilon^A(\widetilde{u}_\varepsilon,\widetilde{\mu}_\varepsilon)\,.$$

The inequality (i) is trivially satisfied (see Theorem 6.3).

In order to prove (ii), we proceed as follows. Let x_0 be a point in the interior of B (i.e. $A(x_0) = \max A$) and $\{(\widetilde{u}_\varepsilon, \widetilde{\mu}_\varepsilon)\}$ be a sequence such that $(\widetilde{u}_\varepsilon, \widetilde{\mu}_\varepsilon) \xrightarrow{\tau} (0, \delta_{x_0})$ and

$$\liminf_{\varepsilon \to 0} \mathcal{H}_\varepsilon(\widetilde{u}_\varepsilon, \widetilde{\mu}_\varepsilon) \geq \overline{\mathcal{H}}(0, \delta_{x_0}) .$$

By Remark 6.4, it is possible to choose

$$\widetilde{u}_\varepsilon = \varepsilon^{-1}(\phi_\varepsilon \circ G^\Omega_{x_0}) \tag{6.10}$$

with ϕ_ε the piecewise affine function defined in (6.9). Hence,

$$\begin{aligned} \mathcal{H}^A_\varepsilon(\widetilde{u}_\varepsilon, \widetilde{\mu}_\varepsilon) &= \frac{\varepsilon^{-2^*} \displaystyle\int_{\Omega \setminus B} [A(x) - A(x_0)] F(\varepsilon \widetilde{u}_\varepsilon)\, dx}{\varepsilon^2} \\ &\quad + \frac{\varepsilon^{-2^*} \displaystyle\int_{\Omega} A(x_0)\; F(\varepsilon \widetilde{u}_\varepsilon)\, dx - (\max A)\; S^F}{\varepsilon^2} \\ &\geq -(\max A)\; \varepsilon^{-2-2^*} \int_{\Omega \setminus B} F(\varepsilon \widetilde{u}_\varepsilon)\, dx + (\max A) \mathcal{H}_\varepsilon(\widetilde{u}_\varepsilon, \widetilde{\mu}_\varepsilon) . \end{aligned}$$

The thesis will be accomplished, if we prove that

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-2-2^*} \int_{\Omega \setminus B} F(\varepsilon \widetilde{u}_\varepsilon)\, dx = 0 .$$

To this purpose, let us fix λ sufficiently large in such a way that $T_\lambda := \{G^\Omega_{x_0} > \lambda\} \subset B$ (which is possible since $\{G^\Omega_{x_0} > \lambda\} \subseteq \{K > \lambda\}$). When ε is small enough, we have $K(s_\varepsilon r) - K(r) < \lambda < K(s_\varepsilon \varepsilon^\gamma r) - K(r)$ and by (6.10) we have

$$\begin{aligned} \int_{\Omega \setminus B} F(\varepsilon \widetilde{u}_\varepsilon)\, dx &\leq \int_{\Omega \setminus T_\lambda} F(\varepsilon \widetilde{u}_\varepsilon)\, dx \\ &\leq C \varepsilon^{2^*} \int_{\Omega \setminus T_\lambda} |\widetilde{u}_\varepsilon|^{2^*}\, dx = C \int_{\Omega \setminus T_\lambda} |\phi_\varepsilon(G^\Omega_{x_0})|^{2^*}\, dx \\ &\leq C \int_{\Omega \setminus T_\lambda} |\phi_\varepsilon(\lambda)|^{2^*}\, dx \\ &\leq C \left(\frac{W_\infty}{1 - \varepsilon^{2/(n-1)}} \right)^{2^*} [s_\varepsilon^{n-2} \lambda + (s_\varepsilon^{n-2} - 1) K(r)]^{2^*} |\Omega \setminus T_\lambda| \varepsilon^{2 \cdot 2^*} , \end{aligned}$$

where we used (iii) of Theorem 6.1. This implies

$$\varepsilon^{-2-2^*} \int_{\Omega \setminus B} F(\varepsilon \widetilde{u}_\varepsilon)\, dx \leq C \varepsilon^{-2-2^*} \varepsilon^{2 \cdot 2^*} \to 0 ,$$

and concludes the proof of (ii), when x_0 belongs to the interior of B.

If $x_0 \in \partial B \setminus \partial \Omega$, the thesis follows by the upper semicontinuity of the Γ^+-limit and the continuity of the Robin function in the interior of Ω. Finally, if $x_0 \in \partial B \cap \partial \Omega$, the Robin function diverges to $+\infty$ and hence the inequality is trivially satisfied. This concludes the proof. □

The previous theorem gives, in particular, that the maximizers of $\mathcal{F}^A$ concentrate in the minimum points of the Robin function among the maximizers of A in Ω.

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