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Degree Theory for VMO Maps on Metric Spaces and Applications to Hörmander Operators

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Abstract. We construct a degree theory for Vanishing Mean Oscillation functions in metric spaces, following some ideas of Brezis & Nirenberg. The underlying sets of our metric spaces are bounded open subsets of $\mathbb{R}^N$ and their boundaries. Then, we apply our results in order to analyze the surjectivity properties of the L-harmonic extensions of VMO vector-valued functions. The operators L we are dealing with are second order linear differential operators sum of squares of vector fields satisfying the hypoellipticity condition of Hörmander.

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1. – Introduction

Given a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, any continuous function $\varphi : \partial\Omega \to S^{N-1}$ has a continuous extension $\Phi : \bar{\Omega} \to \mathbb{R}^N$ such that $\Delta\Phi = 0$ in Ω. Moreover, if $\deg(\varphi, \partial\Omega, S^{N-1}) \neq 0$ then $\Phi(\Omega) \supseteq D(0, 1)$ (Δ is the Laplace operator and $D(0, r) = \{x \in \mathbb{R}^N \mid |x| < r\}$). The first assertion follows from classical results of potential theory while the second one is a consequence of the properties of topological degree for continuous maps (see e.g. [40]).

Motivated by some problems arising in nonlinear PDE and calculus of variations, Brezis and Nirenberg extended the previous result to functions $\varphi \in \mathrm{VMO}(\partial\Omega, S^{N-1})$, i.e. to functions with vanishing mean oscillation with respect to the surface measure and to the Euclidean balls centered on $\partial\Omega$. In doing that they were led to construct a degree theory for VMO functions generalizing the classical topological one. Brezis and Nirenberg extension of the above result can be stated as follows [9].

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(BN) *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$ *with smooth boundary and denote by* $P(x, y)$, $x \in \Omega$, $y \in \partial\Omega$, *its Poisson kernel. Then*

(i) *for any* $\varphi \in \mathrm{VMO}(\partial\Omega, S^{N-1})$,

$$\Phi(x) = \int_{\partial\Omega} P(x, y)\varphi(y)d\sigma(y), \qquad x \in \Omega,$$

is well defined and harmonic in Ω *(here* $d\sigma$ *denotes the surface measure).*

(ii) $\Phi \in \mathrm{VMO}_\varphi$, *i.e.* $\Phi \in \mathrm{VMO}(\bar{\Omega}, \mathbb{R}^N)$ *and its "trace" on* $\partial\Omega$ *is the function* φ.

(iii) *If* $\deg(\varphi, \partial\Omega, S^{N-1}) \neq 0$ *then* $\Phi(\Omega) \supseteq D(0, 1)$.

We refer to [9] for the precise meaning of VMO_φ; we also refer to Section 3.3 of this paper, where the notion of VMO_φ is extended to more general settings. The first assertion of this theorem is a mere consequence of the $\mathrm{VMO}(\partial\Omega, S^{N-1})$-function summability with respect to the harmonic measures $P(x, y)d\sigma(y)$, $x \in \Omega$. (ii) follows from the properties of VMO_φ and P. Assertion (iii) is the strongest one: its proof requires the degree theory for VMO functions developed in [8]-[9].

The aim of this note is to extend Theorem (BN) to the more general setting of the linear differential operators that are sums of squares of vector fields satisfying the Hörmander hypoellipticity condition. These operators appear in many theoretical fields besides being used in the mathematical modeling of non-isotropic and highly non-homogeneous materials (see e.g. [37], [13] and the bibliography therein). Hörmander operators have been widely studied in the past twenty years and there is plenty of references in literature. For our purposes, however, we may only quote the papers by Nagel-Stein-Wainger [38] and by Jerison-Sanchez Calle [31].

An operator

$$L = \sum_{j=1}^{p} X_j^2$$

satisfies the Hörmander condition in $\mathbb{R}^N$ if the X_j are C^∞ vector fields such that

$$\operatorname{rank} \operatorname{Lie}(X_1, \ldots, X_p)(x) = N \qquad \forall x \in \mathbb{R}^N. \tag{1}$$

We have denoted by $\operatorname{Lie}(X_1, \ldots, X_p)$ the Lie algebra generated by $X_1, \ldots, X_p$. If (1) is satisfied, then L is hypoelliptic and one of its main features is the structure of its fundamental solution Γ. Indeed, the following local estimates have been proved in [38] and [41]

$$\Gamma(x, y) \cong \frac{d(x, y)^2}{|B_d(x, d(x, y))|},$$

where d denotes the *control distance* related to $X_1, \ldots, X_p$ and $|\cdot|$ the Lebesgue measure on $\mathbb{R}^N$.

In extending Theorem (BN) to the Hörmander operators L, our main task is the construction of a degree theory for functions in VMO spaces modeled on control metrics.

In fact, our construction relies on distances d on $\mathbb{R}^N$ that satisfy very general and mild conditions. At this stage we shall work in abstract setting in order to emphasize the metric nature of our results.

We also stress that our degree theory seems to be applicable to settings other than Hörmander operators. It could be used for extending Theorem (BN) to X-elliptic operators, i.e. to linear second order partial differential operators with nonnegative characteristic form that are "elliptic" with respect to suitable control distances (see [32]). Moreover, our degree could be applied for studying lifting problems in Sobolev spaces related to general families of vector fields (for such Sobolev spaces we refer to [26], [24], [36], [29] and to the bibliography therein).

In recent years various extensions of the classical topological degree have been proposed by several authors (see [18], [7], [21], [3], [27], [28]). In our generalization we follow the lines of [8]-[9] since they appear the most easily adaptable to a general metric setting. The starting point in the construction consists in approximating a VMO function u by means of its integral averages $\bar{u}_\varepsilon$ over d-balls of radius ε. However, in doing so, one encounters a difficulty: the function $(x, \varepsilon) \mapsto \bar{u}_\varepsilon(x)$ need not be continuous even though the d-topology is the Euclidean one. We evade this obstacle by taking suitable weighted means $\hat{u}_\varepsilon$ instead of $\bar{u}_\varepsilon$. We then prove that the resulting degree is independent of the weight if the measure of the d-balls satisfies a doubling condition (see Section 2 for the details).

In generalizing the theory in [8]-[9] a major difficulty arises when dealing with the trace problem for VMO functions. For general control distances this problem is much more difficult than that related to the Euclidean metric (see [23], [15]-[17], [2], [34], [1]). The approach we propose is the most original contribution of this work: in the remaining of this introduction we shall give a brief description of the steps we have taken to address the problem. The process is split in two basic parts: (1) the development of a trace theory for VMO functions from a purely metric and measure theoretical point of view; (2) the analysis of the link between our definition of trace and the L-harmonic extensions of VMO functions. The first step is accomplished in the following way: we assume the existence of a doubling Borel measure σ on $(\partial\Omega, d)$ and we give a definition of *trace* modeled on the structure of the integral averages in the *doubling space* $(\partial\Omega, d, \sigma)$. We would like to quote the papers [14], [29] where the main properties of the doubling spaces are studied. We stress that our definition of the class VMO_φ differs from the one given in [9] which does not seem to work in the general setting of control distances. Nevertheless our definition allows to establish the same relation proved in [9] between the degree of a map $u \in \mathrm{VMO}_\varphi(\Omega)$ and the degree of its trace $\varphi \in \mathrm{VMO}(\partial\Omega)$ (see Theorem 39). Moreover our definition of trace seems to be somehow comparable with that given by Danielli-Garofalo-Nhieu [15]-[17] and by Monti-Morbidelli [34] for Sobolev functions (see Remark 54). Then, the abstract theory developed in this first step seems to be satisfactory. As far as the second step goes, the main difficulties are due to: (2-i) the possible presence of characteristic points on $\partial\Omega$, which makes harder to find a suitable doubling measure σ; (2-ii) the need of finding suitable estimates of the L-harmonic measures on $\partial\Omega$ in terms of σ. These two problems are quite delicate and intimately related to each other. In the special but significant case when L is a sub-Laplacian on a Carnot group $\mathbb{G}$, we are able to find the required estimates of the L-harmonic measures μ_x (see Theorem 50) for some particular regular domains of $\mathbb{G}$, with the same arguments used in our previous paper [42]. The natural choice of the measure σ which is suggested by our estimates of μ_x is the following:

$$d\sigma = \langle A\nu, \nu\rangle^{\frac{1}{2}} d\mathcal{H}_{N-1}, \tag{2}$$

where $\langle A\nu, \nu\rangle$ is the characteristic form of the operator L applied to the outer unit normal ν to $\partial\Omega$, and $\mathcal{H}_{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. We would like to remark that this measure plays a crucial role in the trace problem for generalized Sobolev functions, as first pointed out by Danielli-Garofalo-Nhieu [15]-[17] (see also [35] where σ is studied in a geometric measure theory context). In such papers (see also [11]) the authors also prove that the above measure σ turns out to be doubling on $(\partial\Omega, d)$ in some significant cases. As a consequence we finally get our generalization of Theorem (BN) to the sub-Laplacians on these domains (see Theorem 52 and Remark 53). In the case of general Hörmander operators, we give an abstract theorem (Theorem 49) which simply summarizes the hypotheses we need in order to apply our degree theory for obtaining Theorem (BN). We plan to look for more explicit conditions in a forthcoming paper.

We next give a rather detailed plan of the paper and also list the hypotheses we shall make during the construction of our degree. In Section 2.1, after introducing the main notation, we shall define the degree for maps on a bounded domain Ω of $\mathbb{R}^N$ which are VMO with respect to d, and we shall establish its basic properties. We shall also prove an extension to VMO maps of the Brouwer fixed point theorem by using the BMO-homotopy invariance of the degree. Throughout this Section 2.1 d will be any distance on $\mathbb{R}^N$ only satisfying the following hypothesis:

(H_1) the topology on $\mathbb{R}^N$ induced by d is the Euclidean one.

In Section 2.2, together with (H_1) we shall assume that

(H_2) d satisfies the doubling condition

and that

(H_3) the d-balls enjoy a chain-connectivity property.

We directly refer to Section 2.2 for the exact meaning of (H_2) and (H_3). We explicitly remark that (H_3) is satisfied by any control distance while (H_1) and

(H_2) hold for control distances related to vector fields satisfying the Hörmander condition. With (H_2) and (H_3) in hand we shall be able to prove that our definitions of VMO and degree do not depend on the choice of the weight used in defining the averages $\hat{u}_\varepsilon$.

Sections 3.1 and 3.2 are devoted to an analogous construction of the degree for maps defined on the boundary of Ω, with values in a compact manifold Y, which are VMO in the metric space $(\partial\Omega, d) := (\partial\Omega, d_{|\partial\Omega})$ endowed with a Borel measure σ satisfying

(H_4) $0 < \sigma(B) < +\infty$ for every metric ball B of $(\partial\Omega, d)$

and the crucial assumption:

(H_6) $(\partial\Omega, d, \sigma)$ is a doubling space.

In this construction we also assume that

(H_5) $\partial\Omega$ is smooth,

so that the classical degree for continuous maps from $\partial\Omega$ to Y is defined. Section 3.3 contains the core of the paper, i.e. the definition of $\mathrm{VMO}_\varphi(\Omega)$ that we have already discussed above. The domain Ω will be supposed to also satisfy the following boundary regularity condition with respect to the d-balls:

(H_7) there exist $c > 0$ and $\varepsilon_0 > 0$ such that $|B_d(y, \varepsilon) \cap \Omega^c| \geq c|B_d(y, \varepsilon)|$ for every $y \in \Omega^c$ and $0 < \varepsilon < \varepsilon_0$.

In Section 3.4 we shall examine the link between the degree of a map $u \in \mathrm{VMO}_\varphi(\Omega)$ and the degree of its trace $\varphi \in \mathrm{VMO}(\partial\Omega)$. In Section 3.5 we shall take up the study of a class of *extension integral operators*. In particular we shall show that the "L-harmonic extension" of a function $\varphi \in \mathrm{VMO}(\partial\Omega)$ belongs to $\mathrm{VMO}_\varphi(\Omega)$ and takes the value φ at the boundary, nontangentially pointwise σ-a.e., if the "L-Poisson kernel" K related to $d\sigma$ can be estimated as follows

$$|K(x, \xi)| \leq c \frac{d(x, \partial\Omega)}{d(x, \xi)\sigma(B_d(\xi, d(x, \xi)))}, \qquad x \in \Omega, \xi \in \partial\Omega$$

(see Theorem 42, Theorem 45 and Proposition 46). This condition also assures the surjectivity result (BN)-(iii) (see Remark 43). Finally, in Section 4 we apply the results of Section 3.5 to our Hörmander operators (Theorem 49). We then specialize our analysis to Carnot groups $\mathbb{G}$ and prove Theorem (BN) for the sub-Laplacians $\Delta_{\mathbb{G}}$ on some domains Ω satisfying a *uniform exterior ball condition* (see Theorem 52 and Remark 53).

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2. – Construction of the degree

2.1. Throughout the paper Ω will denote a bounded domain (non-empty connected open subset) of $\mathbb{R}^N$ and d a distance on $\mathbb{R}^N$ satisfying the following hypothesis:

(H_1) the topology on $\mathbb{R}^N$ induced by d is the Euclidean one.

We shall denote by $B_d(x,r)$ the balls in the metric d while the notation $D(x,r)$ will be used for Euclidean balls. The distance d we have in mind is the Carnot-Carathéodory control distance associated to some family of vector fields (in the sequel we shall refer to such a distance simply as a control distance); however we work in the abstract setting in order to emphasize the metric nature of the results.

We want to extend the classical degree for continuous maps to the larger class of the functions VMO with respect to d. Following [9], we will perform such extension by approximating a VMO function $u(x)$ with its means $\bar{u}_\varepsilon(x)$ over d-balls of radius ε. In doing so the first difficulty that one encounters is that $\bar{u}$ need not be continuous as a function of (x,ε). We evade this obstacle by taking suitable weighted means $\hat{u}_\varepsilon(x)$ instead of the $\bar{u}_\varepsilon(x)$. The drawback is that we have to modify our definition of VMO taking into account the weight; of course the resulting definition of degree shall depend on the choice of the weight too. As far as this construction of the degree, (H_1) is the only hypothesis we need; further assumptions (among which the doubling condition for the d-balls $B_d(x,r)$) will be required later. In particular we shall see (Theorem 17 and Remark 18) that for a doubling control distance our definitions of the class VMO and of the degree actually do not depend on the weight.

As weight function we choose the following $\Lambda : \mathbb{R} \to \mathbb{R}$, $\Lambda(t) = 1$ for $t \le \frac{1}{2}$, $\Lambda(t) = 0$ for $t \ge 1$ and defined to be the linear join between 1 and 0 for $\frac{1}{2} < t < 1$. Then we set

$$\Lambda_{x,r} = \Lambda\left(\frac{d(x,\cdot)}{r}\right)$$

for every $x \in \mathbb{R}^N$, $r > 0$. We remark that $\Lambda_{x,r}$ is continuous (by (H_1)). We now set

$$\mathcal{B} = \{B_d(x,r) | 0 < r < d(x,\Omega^c)\}, \qquad \mathcal{C} = \left\{B_d(x,r) | 0 < r \le \frac{1}{2} d(x,\Omega^c)\right\}$$

(where Ω^c denotes the complementary set $\mathbb{R}^N \setminus \Omega$ and $d(x,A) = \inf_{y\in A} d(x,y)$ for any A non-empty subset of $\mathbb{R}^N$). We remark that $d(x,\Omega^c)$ may be different from $d(x,\partial\Omega)$ (however when d is a control distance they are actually the same). For $u \in L^1_{\mathrm{loc}}(\Omega)$ and $B_d(x,r) \in \mathcal{B}$, we define the weighted mean

$$\hat{u}(x,r) = \hat{u}_r(x) = \hat{\fint}_{B_d(x,r)} u = \frac{1}{\int_{\mathbb{R}^N} \Lambda_{x,r}(y)dy} \int_{\mathbb{R}^N} u(y)\Lambda_{x,r}(y)dy \tag{3}$$

and the weighted mean oscillation

$$\widehat{\mathrm{MO}}_{x,r}(u) = \widehat{\fint}_{B_d(x,r)} |u(y) - \hat{u}_r(x)|dy.$$

We remark that $0 < \int \Lambda_{x,r} < +\infty$ for every $B_d(x,r) \in \mathcal{B}$, since

$$\chi_{B_d(x,\frac{r}{2})} \leq \Lambda_{x,r} \leq \chi_{B_d(x,r)} \leq \chi_\Omega \tag{4}$$

and $|B| > 0$ for any $B \in \mathcal{B}$ (χ_A denotes the characteristic function of the set A and $|\cdot|$ the Lebesgue measure on $\mathbb{R}^N$). We also notice that, by (H_1),

$$\inf_{x\in K} |B_d(x,r)| > 0 \tag{5}$$

for every $K \subset\subset \mathbb{R}^N$ and $r > 0$. We can now define the weighted BMO space $\widehat{\mathrm{BMO}}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) \big| \ \|u\|_{\widehat{\mathrm{BMO}}(\Omega)} < +\infty\}$, where

$$\|u\|_{\widehat{\mathrm{BMO}}(\Omega)} = \sup_{B_d(x,r)\in\mathcal{C}} \widehat{\mathrm{MO}}_{x,r}(u). \tag{6}$$

Taking $\mathcal{B}$ instead of $\mathcal{C}$ in (6) one obtains a different definition of BMO. Under further assumptions on the metric d (in particular when d is a doubling control distance) these definitions turn out to be equivalent (see e.g. [10]).

For $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\varepsilon > 0$ we set

$$\hat{\eta}_\varepsilon(u) = \sup_{B_d(x,r)\in\mathcal{C}, r\leq\varepsilon} \widehat{\mathrm{MO}}_{x,r}(u),$$

$$\hat{\eta}_0(u) = \inf_{\varepsilon>0} \hat{\eta}_\varepsilon(u) = \lim_{\varepsilon\to 0^+} \hat{\eta}_\varepsilon(u)$$

and we finally define $\widehat{\mathrm{VMO}}(\Omega) = \{u \in \widehat{\mathrm{BMO}}(\Omega) | \hat{\eta}_0(u) = 0\}$. We also set

$$\widehat{\mathrm{MO}}^*_{x,r}(u) = \widehat{\fint}_{B_d(x,r)} \widehat{\fint}_{B_d(x,r)} |u(y) - u(z)|dydz.$$

and define $\hat{\eta}^*_\varepsilon$, $\hat{\eta}^*_0$, $\|\cdot\|^*_{\widehat{\mathrm{BMO}}}$ accordingly. Then $\widehat{\mathrm{MO}}_{x,r}$, $\widehat{\mathrm{MO}}^*_{x,r}$, $\hat{\eta}_\varepsilon$, $\hat{\eta}^*_\varepsilon$, $\hat{\eta}_0$, $\hat{\eta}^*_0$, $\|\cdot\|_{\widehat{\mathrm{BMO}}}$, $\|\cdot\|^*_{\widehat{\mathrm{BMO}}}$ are all seminorms on $\widehat{\mathrm{BMO}}(\Omega)$; moreover

$$\widehat{\mathrm{MO}}_{x,r}(u) \leq \widehat{\mathrm{MO}}^*_{x,r}(u) \leq 2\widehat{\mathrm{MO}}_{x,r}(u) \tag{7}$$

and analogous estimates hold for $\hat{\eta}^*_\varepsilon$, $\hat{\eta}^*_0$, $\|\cdot\|^*_{\widehat{\mathrm{BMO}}}$. In particular $\|\cdot\|_{\widehat{\mathrm{BMO}}}$ and $\|\cdot\|^*_{\widehat{\mathrm{BMO}}}$ are equivalent norms on $\widehat{\mathrm{BMO}}(\Omega)$ modulo constants, which is a Banach space under these norms. It is easy to verify that $\widehat{\mathrm{VMO}}(\Omega)$ is a closed subspace of $\widehat{\mathrm{BMO}}(\Omega)$; when d is doubling, $\widehat{\mathrm{VMO}}(\Omega)$ turns out to be the closure of $C(\bar{\Omega})$ in $\widehat{\mathrm{BMO}}(\Omega)$ (see Theorem 19). An example of function $u \in \widehat{\mathrm{BMO}}(\Omega)$ is given by $u(x) = \log d(x, \Omega^c)$ (this is used in the proof of Theorem 19); examples of functions $u \in \widehat{\mathrm{VMO}}(\Omega)$ are given by $u(x) = \log(\log \frac{M}{d(x,\Omega^c)})$ and $u(x) = |\log d(x,\Omega^c)|^\alpha$, for $0 < \alpha < 1$ and $M > 0$ suitable large.

All the above definitions depend on our choice of the weight Λ. Taking $\chi_{]-\infty,1[}$ instead of Λ we obtain the usual definitions of the BMO and VMO spaces with respect to d. We shall refer to this case by dropping the symbol $\hat{}$ in the above notation ($\fint$ instead of $\widehat{\fint}$, $\mathrm{MO}_{x,r}$ instead of $\widehat{\mathrm{MO}}_{x,r}$, $\mathrm{BMO}(\Omega)$ instead of $\widehat{\mathrm{BMO}}(\Omega)$, and so on). For our purpose, the main feature of the weight Λ is continuity, which allows us to prove the following lemma.

LEMMA 1. *For every* $u \in L^1_{loc}(\Omega)$, *the function* $\hat{u}$ *defined in* (3) *is continuous on its set of definition* $A = \{(x, r) \in \Omega\times]0, +\infty[\mid B_d(x, r) \in \mathcal{B}\}$.

PROOF. Let $(x_n, r_n) \to (x_0, r_0) \in A$. Setting $K_0 = \{x \in \Omega \mid d(x, \Omega^c) \geq \frac{1}{2}(d(x_0, \Omega^c) - r_0)\}$ we have $\overline{B_d(x_n, r_n) \cup B_d(x_0, r_0)} \subseteq K_0 \subset\subset \Omega$, for large n (we remark that $d(\cdot, \Omega^c)$ is continuous by (H$_1$)). Hence

$$\Big| \int u\Lambda_{x_n,r_n} - \int u\Lambda_{x_0,r_0}\Big| \leq \int_{K_0} \Big|\Lambda\left(\frac{d(x_n, y)}{r_n}\right) - \Lambda\left(\frac{d(x_0, y)}{r_0}\right)\Big||u(y)|dy$$

which vanishes as $n \to +\infty$ by dominated convergence, being Λ and d continuous, $u \in L^1(K_0)$ and $|\Lambda| \leq 1$. In the same way we see that $\int \Lambda_{x_n,r_n} \to \int \Lambda_{x_0,r_0}$. Thus $\hat{u}(x_n, r_n) \to \hat{u}(x_0, r_0)$. □

From the previous lemma it follows in particular that $\hat{u}_\varepsilon \in C(\overline{\Omega}_{2\varepsilon})$ for every $\varepsilon > 0$, where we have set

$$\Omega_\varepsilon = \{x \in \Omega \mid d(x, \Omega^c) > \varepsilon\}.$$

DEFINITION 2. Let $u \in \mathrm{VMO}\hat{}(\Omega, \mathbb{R}^N)$ and let $p \in \mathbb{R}^N$. We will say that u does not take the value p at the boundary of Ω in the VMO$\hat{}$ sense iff there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that

$$\inf_{0<r=\frac{1}{2}d(x,\Omega^c)<\varepsilon_0} \hat{\fint}_{B_d(x,r)} |u - p| \quad \geq \delta_0. \tag{8}$$

We denote by $\hat{\mathcal{M}} = \hat{\mathcal{M}}(\Omega)$ the set of couples (u, p) with this property.

PROPOSITION 3. *Let* $(u, p) \in \hat{\mathcal{M}}$ *(satisfying* (8)*). Then there exists* $\varepsilon_1 > 0$ *such that*

(i) $|\hat{u}_\varepsilon(x) - p| \geq \frac{\delta_0}{2}$ *whenever* $0 < d(x, \Omega^c) = 2\varepsilon < 2\varepsilon_1$;

(ii) *the classical degree* $\deg(\hat{u}_\varepsilon, \Omega_{2\varepsilon}, p)$ *is defined for every* $\varepsilon < \varepsilon_1$;

(iii) $\deg(\hat{u}_\varepsilon, \Omega_{2\varepsilon}, p) = \deg(\hat{u}_{\varepsilon'}, \Omega_{2\varepsilon'}, p)$ *for every* $\varepsilon, \varepsilon' \in]0, \varepsilon_1[$.

DEFINITION 4. We can then define $\deg\hat{} : \hat{\mathcal{M}} \to \mathbb{Z}$, $(u, p) \mapsto \deg\hat{}(u,\Omega,p) = \deg(\hat{u}_\varepsilon,\Omega_{2\varepsilon},p)$ for small $\varepsilon > 0$.

PROOF OF PROPOSITION 3. (i) is an easy consequence of (8) and of our definition of $u \in \mathrm{VMO}\hat{}$. (ii) follows from (i), since $\hat{u}_\varepsilon \in C(\overline{\Omega}_{2\varepsilon})$ by means of Lemma 1. (iii) is of course of crucial importance. However we will not give a detailed proof since it follows the lines of the one in [9]. We only say that the proof relies on the homotopy and excision properties of the classical degree for continuous functions, making use of (i) and of Lemma 1. We also would like to emphasize that (H_1) is used throughout. In particular we notice that $\partial\Omega_{2\varepsilon} \subseteq \{x \in \Omega \mid d(x, \Omega^c) = 2\varepsilon\}$ and $\max_{d(y,\Omega^c)=2t} d(y, \{x \in \Omega \mid d(x, \Omega^c) = 2\varepsilon\}) \to 0$, as $t \to \varepsilon$. □

REMARK 5. $\deg\hat{}$ is an extension of the usual degree for continuous maps. Indeed $C(\bar{\Omega}) \subseteq \mathrm{VMO}\hat{}(\Omega)$ (this follows from (H_1), the Heine-Cantor theorem in the compact metric space $(\bar{\Omega}, d)$ and (7); we also remark that $C(\bar{\Omega}) \subseteq L^\infty(\Omega) \hookrightarrow \mathrm{BMO}\hat{}(\Omega)$); moreover, if $u \in C(\bar{\Omega}, \mathbb{R}^N)$ and $p \in \mathbb{R}^N$ then the condition $(u, p) \in \hat{\mathcal{M}}$ is equivalent to $p \notin u(\partial\Omega)$ and (when this condition holds) $\deg\hat{}(u, \Omega, p) = \deg(u, \Omega, p)$. This follows again from the homotopy and excision properties of deg, observing that for $u \in C(\bar{\Omega})$, beside Lemma 1 one also has

$$\sup_{\Omega_\varepsilon} |\hat{u}_\varepsilon - u| \to 0, \qquad \text{as } \varepsilon \to 0^+. \tag{9}$$

We now establish some properties of the degree just defined. We first recall the definition of the essential range of a measurable map $f : A \to \mathbb{R}^m$ (here A is a measurable subset of $\mathbb{R}^N$):

$$\operatorname{ess} f(A) = \left\{ p \in \mathbb{R}^m \;\middle|\; \forall \delta > 0 \; |\{x \in A \;|\; |f(x) - p| < \delta\}| > 0 \right\}. \tag{10}$$

As pointed out in [8], $\operatorname{ess} f(A)$ can be characterized as the smallest closed set Σ in $\mathbb{R}^m$ such that $f(x) \in \Sigma$ a.e. Moreover, given $u \in L^1_{\mathrm{loc}}(\Omega)$, if we denote by L_u the set of its Lebesgue points $L_u = \{x \in \Omega \mid ⨍_{D(x,r)} |u - c| \to 0$, as $r \to 0^+$, for some constant $c\,(=: u^*(x))\}$, then we have

$$\operatorname{ess} u(\Omega) = \overline{u^*(L_u)} \tag{11}$$

(note that both L_u and u^* depend only on the class of the functions equal to u a.e.; moreover u^* defines a somehow privileged element of such class).

PROPOSITION 6. *Let* $(u, p) \in \hat{\mathcal{M}}$. *If* $\deg\hat{}(u, \Omega, p) \neq 0$ *then* $p \in \operatorname{ess} u(\Omega)$.

PROOF. It is a consequence of the analogous property of the degree of continuous functions. Indeed, for small $\varepsilon > 0$, $\deg(\hat{u}_\varepsilon, \Omega_{2\varepsilon}, p) = \deg\hat{}(u, \Omega, p) \neq 0$ and then there exists $x_\varepsilon \in \Omega_{2\varepsilon}$ such that $\hat{u}_\varepsilon(x_\varepsilon) = p$. Now, if we assume by contradiction that $p \notin \operatorname{ess} u(\Omega)$, then $|u - p| \geq \delta$ a.e. for some $\delta > 0$ and then

$$\delta \leq \hat{⨍}_{B_d(x_\varepsilon, \varepsilon)} |u - p| = \hat{⨍}_{B_d(x_\varepsilon, \varepsilon)} |u - \hat{u}_\varepsilon(x_\varepsilon)| \leq \hat{\eta}_\varepsilon(u),$$

contradicting $u \in \mathrm{VMO}\hat{}(\Omega)$. □

REMARK 7. Let $u \in \mathrm{VMO}\hat{}(\Omega, \mathbb{R}^N)$. Then the set $\hat{\mathcal{M}}_u = \{p \in \mathbb{R}^N \,|\, (u, p) \in \hat{\mathcal{M}}\}$ is open in $\mathbb{R}^N$; moreover $\deg\hat{}(u, \Omega, \cdot)$ is constant on the connected components of $\hat{\mathcal{M}}_u$. In particular, in the hypotheses of Proposition 6 we have $D(p, \delta) \subseteq \operatorname{ess} u(\Omega)$ for some $\delta > 0$.

REMARK 8. $(u, p) \in \hat{\mathcal{M}}$ iff $(u - p, 0) \in \hat{\mathcal{M}}$ and in this case we have $\deg\hat{}(u, \Omega, p) = \deg\hat{}(u - p, \Omega, 0)$. This is immediate from the definitions of $\hat{\mathcal{M}}$ and $\deg\hat{}$ and from the analogous property of the classical deg.

PROPOSITION 9. *Let* $u \in \mathrm{VMO}\hat{}(\Omega, \mathbb{R}^N)$*, let* $p \in \mathbb{R}^N$ *and let* u_n *be a sequence in* $\mathrm{VMO}\hat{}(\Omega, \mathbb{R}^N)$ *such that*

(i) $u_n \to u$ *in* $\mathrm{BMO}\hat{}(\Omega)$ *and in* $L^1_{\mathrm{loc}}(\Omega)$*;*

(ii) *there exist* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that* (u_n, p) *verifies* (8) *for every* $n \in \mathbb{N}$ (*uniformly*).

Then $(u, p) \in \hat{\mathcal{M}}$ *and* $\deg\hat{}(u, \Omega, p) = \deg\hat{}(u_n, \Omega, p)$ *for large* n.

PROOF. This result can be achieved arguing exactly as in [9]. We only say that the L^1_{loc} convergence of the u_n implies the uniform convergence of the weighted means $\widehat{u_n}_\varepsilon \to \hat{u}_\varepsilon$ in $\overline{\Omega}_{2\varepsilon}$, as $n \to +\infty$, thanks to (4) and (5). □

COROLLARY 10 (homotopy invariance). *Let* $p \in \mathbb{R}^N$ *and let* $[0, 1] \ni t \mapsto F_t \in \mathrm{VMO}\hat{}(\Omega, \mathbb{R}^N)$ *be a continuous map in the* $\mathrm{BMO}\hat{}$*-topology and in the* L^1_{loc}*-topology. If there exist* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that* (F_t, p) *verifies* (8) *for every* $t \in [0, 1]$ (*uniformly*)*, then the map* $[0, 1] \ni t \mapsto \deg\hat{}(F_t, \Omega, p) \in \mathbb{Z}$ *is constant.*

COROLLARY 11 (Rouché-type theorem). *Let* $u, v \in \mathrm{VMO}\hat{}(\Omega, \mathbb{R}^N)$*,* $p \in \mathbb{R}^N$*. If there exist* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that* (u, p) *verifies* (8) *and*

$$\sup_{0<r=\frac{1}{2}d(x,\Omega^c)<\varepsilon_0} \quad \hat{\fint}_{B_d(x,r)} |u - v| \quad < \delta_0,$$

then $(v, p) \in \hat{\mathcal{M}}$ *and* $\deg\hat{}(v, \Omega, p) = \deg\hat{}(u, \Omega, p)$.

PROPOSITION 12 (Brouwer fixed point-type theorem). *Let* $u \in \mathrm{VMO}\hat{}(D(0, 1), \overline{D(0, 1)})$*. Then* $0 \in \mathrm{ess}\,(\mathrm{id} - u)(D(0,1))$ *where* id *denotes the identity function.*

PROOF. It follows from Proposition 6 and Corollary 10, using the homotopy $t \mapsto F_t = \mathrm{id} - tu$. In order to see that $(F_t, 0)$ verifies (8) uniformly, one assumes by contradiction that $0 \notin \mathrm{ess}\,(\mathrm{id} - u)(D(0, 1))$ and uses the fact that $|\mathrm{id} - u| \geq \delta$ a.e. for some $\delta > 0$, to prove that $|F_t(x)| \geq \frac{\delta}{4}$ if $1 - \frac{\delta}{2} \leq |x| < 1$, for every $t \in [0, 1]$. This allows to apply Corollary 10. □

2.2. From now on we shall assume that d satisfies the following hypothesis (doubling condition):

(H$_2$) for every $K \subset\subset \mathbb{R}^N$ and for every $r_0 > 0$ there exists $A > 0$ such that $|B_d(x, r)| \leq A|B_d(x, \frac{r}{2})|$ for every $x \in K$ and $0 < r \leq r_0$.

Actually we only need that (H$_2$) holds near Ω, for example when K is a fixed compact set containing a neighborhood of $\bar{\Omega}$ and $r_0 = \max\limits_{x,y\in K} d(x, y)$. Moreover we shall always deal with balls contained in such compact set K and thus with finite measure. In the sequel A shall denote the (smallest) constant in (H$_2$) related to this choice of K and r_0. Moreover we set $Q = \log_2 A$. It is standard to derive the following

$$(12) \qquad |B_d(x, tr)| \leq At^Q|B_d(x, r)| \qquad \text{whenever } x \in K, t \geq 1, \text{ and } tr \leq r_0.$$

Another important consequence of the doubling condition is that it allows us to compare the weighted means with the usual ones. Indeed, for $u \in L^1_{\mathrm{loc}}(\Omega)$ and $B_d(x,r) \in \mathcal{B}$, from (4) and (H_2) we get

$$\frac{1}{A} \fint_{B_d(x,\frac{r}{2})} |u| \leq \hat{\fint}_{B_d(x,r)} |u| \leq A \fint_{B_d(x,r)} |u| \tag{13}$$

and then

$$\frac{1}{A^2} \mathrm{MO}^*_{x,\frac{r}{2}}(u) \leq \widehat{\mathrm{MO}}^*_{x,r}(u) \leq A^2 \mathrm{MO}^*_{x,r}(u). \tag{14}$$

Moreover (13) also yields

$$\begin{aligned} |\hat{u}_r(x) - \bar{u}_r(x)| &\leq \hat{\fint}_{B_d(x,r)} |u(y) - \bar{u}_r(x)| dy \\ &\leq A \fint_{B_d(x,r)} |u(y) - \bar{u}_r(x)| dy = A\, \mathrm{MO}_{x,r}(u) \end{aligned} \tag{15}$$

(we have used the notation $\bar{u}_r(x) = \fint_{B_d(x,r)} u$). From (14) it follows that $\mathrm{BMO}(\Omega) \subseteq \widehat{\mathrm{BMO}}(\Omega)$ and $\mathrm{VMO}(\Omega) \subseteq \widehat{\mathrm{VMO}}(\Omega)$ but not the reverse inclusions. Indeed we can only say that $\widehat{\mathrm{BMO}}(\Omega) \subseteq \mathrm{BMO}(\Omega, \mathcal{C}_{\frac{1}{4}})$ and $\widehat{\mathrm{VMO}}(\Omega) \subseteq \mathrm{VMO}(\Omega, \mathcal{C}_{\frac{1}{4}})$, where the spaces $\mathrm{BMO}(\Omega, \mathcal{C}_\mu)$ and $\mathrm{VMO}(\Omega, \mathcal{C}_\mu)$, for $0 < \mu < 1$, are obtained replacing $\mathcal{C} = \mathcal{C}_{\frac{1}{2}}$ with

$$\mathcal{C}_\mu = \{B_d(x,r) \mid 0 < r \leq \mu d(x, \Omega^c)\}$$

in the relevant definitions. On the other hand (see Proposition 16 below) these spaces turn out to be not depending on the parameter $\mu \in]0,1[$ if the metric d satisfies, for example, the further assumption

(H_3) there exists $c > 0$ such that, if $y, z \in B_d(x,r) \in \mathcal{B}$ and $n \in \mathbb{N}$, then there exists a chain of points $x_0, \dots, x_n \in B_d(x,r)$ such that $x_0 = y$, $x_n = z$ and $d(x_{i-1}, x_i) \leq c\frac{r}{n}$ for $i = 1, \dots, n$.

REMARK 13. The condition (H_3) holds true for any control distance d. This is easily seen by arguing with admissible paths connecting y with x and x with z and taking the x_i along these paths.

From now on, we then assume also (H_3) (beside (H_1) and (H_2)). Indeed we use (H_3) only in the proof of Lemma 15 below.

LEMMA 14. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and let $B_i = B_d(x_i, r_i) \in \mathcal{B}$ for $i = 1, 2, 3$. If $B_3 \subseteq B_1 \cap B_2$, then*

$$|\bar{u}_{r_1}(x_1) - \bar{u}_{r_2}(x_2)| \leq \frac{|B_1|}{|B_3|}\mathrm{MO}_{x_1,r_1}(u) + \frac{|B_2|}{|B_3|}\mathrm{MO}_{x_2,r_2}(u). \tag{16}$$

If moreover $r_2 \leq r_1 \leq \frac{r_0}{2}$, then

$$|\bar{u}_{r_1}(x_1) - \bar{u}_{r_2}(x_2)| \leq A\left(\frac{2r_1}{r_3}\right)^Q \big(\mathrm{MO}_{x_1,r_1}(u) + \mathrm{MO}_{x_2,r_2}(u)\big). \tag{17}$$

PROOF. To prove (16), we observe that

$$|\bar{u}_{r_1}(x_1)-\bar{u}_{r_2}(x_2)| \leq \sum_{i=1}^{2} \fint_{B_3} |u(y)-\bar{u}_{r_i}(x_i)|dy \leq \sum_{i=1}^{2} \frac{1}{|B_3|}\int_{B_i} |u(y)-\bar{u}_{r_i}(x_i)|dy.$$

(16) and (12) immediately give (17), since $B_1 \cup B_2 \subseteq B_d(x_3, 2r_1)$. □

LEMMA 15. *There exists $c > 0$ such that, if $y, z \in B_d(x, r)$, $0 < \mu < 1$, $0 < \varrho \leq \min\{\frac{r_0}{2}, \mu(d(x, \Omega^c) - r)\}$, then*

$$|\bar{u}_\varrho(y) - \bar{u}_\varrho(z)| \leq c\left(1 + \frac{r}{\varrho}\right)\eta_\varrho(u, \mathcal{C}_\mu)$$

for every $u \in L^1_{\mathrm{loc}}(\Omega)$.

PROOF. By (H_3), if we choose n to be the smallest integer greater than $\frac{2cr}{\varrho}$ (where c is the constant in (H_3)), we can find a chain of points $x_0, \ldots, x_n \in B_d(x, r)$ such that $x_0 = y$, $x_n = z$ and $B_d(x_i, \frac{\varrho}{2}) \subseteq B_d(x_{i-1}, \varrho) \cap B_d(x_i, \varrho)$, for $i = 1, \ldots, n$. Since moreover $B_d(x_i, \varrho) \in \mathcal{C}_\mu$, from (17) we get

$$|\bar{u}_\varrho(y) - \bar{u}_\varrho(z)| \leq \sum_{i=1}^{n} |\bar{u}_\varrho(y_i) - \bar{u}_\varrho(y_{i-1})| \leq nA4^Q 2\eta_\varrho(u, \mathcal{C}_\mu).$$

Recalling the choice of n, this proves the lemma. □

PROPOSITION 16. *For every $\mu \in]0, 1[$ $\mathrm{BMO}(\Omega) = \mathrm{BMO}(\Omega, \mathcal{C}_\mu)$ and $\mathrm{VMO}(\Omega) = \mathrm{VMO}(\Omega,\mathcal{C}_\mu)$. Moreover the relevant norms are equivalent.*

PROOF. The result follows from the lemmas above, by means of the same covering argument used in Lemma A1.1 of [9]. □

THEOREM 17. *We have*

$$\mathrm{BMO}\hat{}(\Omega) = \mathrm{BMO}(\Omega), \qquad \mathrm{VMO}\hat{}(\Omega) = \mathrm{VMO}(\Omega) \tag{18}$$

and the relevant norms are equivalent. Moreover

$$\hat{\mathcal{M}}(\Omega) = \mathcal{M}(\Omega) \tag{19}$$

where $\mathcal{M}(\Omega)$ is defined replacing the weighted means $\hat{\fint}$ by the usual means $\fint$ in Definition 2.

REMARK 18. The definition of the degree $\deg\hat{}$ does not depend on our choice of the weight Λ. Indeed that choice was rather arbitrary. We might have made the same construction of the degree as above choosing any nonnegative continuous weight $\tilde{\Lambda}$ such that $\tilde{\Lambda}(0) > 0$ and $\tilde{\Lambda}(t) = 0$ for $t \geq 1$. The resulting degree would have been the same. This follows from Theorem 17, (15) and the Rouché theorem (for the classical degree of continuous maps) applied to the weighted means $\hat{u}_\varepsilon$ and $\tilde{\hat{u}}_\varepsilon$ on $\Omega_{2\varepsilon}$. In light of these considerations (and of Remark 5) in the sequel we shall always denote $\deg\hat{}$ simply by deg.

PROOF OF THEOREM 17. (18) is an easy consequence of Proposition 16 and (14). (19) follows, observing that for $u \in \mathrm{VMO}(\Omega, \mathbb{R}^N)$, $p \in \mathbb{R}^N$ and $0 < r = \frac{1}{2}d(x, \Omega^c) < \varepsilon$, by (15) and (7) we have

$$\begin{aligned}\left|\hat{\fint}_{B_d(x,r)} |u-p| - \fint_{B_d(x,r)} |u-p|\right| &\leq A\,\mathrm{MO}_{x,r}(|u-p|)\\ &\leq A\,\mathrm{MO}^*_{x,r}(|u-p|) \leq A\,\mathrm{MO}^*_{x,r}(u-p)\\ &= A\,\mathrm{MO}^*_{x,r}(u) \leq A\,\eta^*_\varepsilon(u) \to 0, \quad \text{as } \varepsilon \to 0^+. \quad \square\end{aligned}$$

We end this section stating the following density result (which we will never use in this paper). We omit the proof, since it is an adaptation of the proof of Theorem 1 of [9].

THEOREM 19. $\mathrm{VMO}\hat{}(\Omega)$ *is the closure of* $C(\bar{\Omega})$ *in* $\mathrm{BMO}\hat{}(\Omega)$. *Furthermore, for every* $u \in \mathrm{VMO}\hat{}(\Omega)$ *there exists a sequence* $u_j \in C_0^\infty(\Omega)$ *converging to* u *in* $\mathrm{BMO}\hat{}(\Omega)$ *and in* $L^1_{\mathrm{loc}}(\Omega)$.

3. – The trace problem

3.1. Let σ be a (fixed) positive Borel measure on $\partial\Omega$ satisfying the following condition:

(H$_4$) $\sigma(\partial\Omega) < +\infty$ and $\sigma(B_d(x, r)) > 0$ for every $x \in \partial\Omega$ and $r > 0$.

Here $B_d(x, r)$ denotes the metric ball in $\partial\Omega$, which is understood to be equipped with the restriction of the distance d. In the sequel we shall always use the same notation for the balls in $(\mathbb{R}^N, d)$ and $(\partial\Omega, d)$. We first want to introduce the weighted BMO and VMO spaces on $(\partial\Omega, d, \sigma)$. Taking the same weight Λ as in Section 2, we define the weighted means of a map $u \in L^1_\sigma(\partial\Omega)$ as

$$\hat{u}(x, r) = \hat{u}_r(x) = \hat{\fint}_{B_d(x,r)} u\,d\sigma = \frac{1}{\int_{\partial\Omega} \Lambda_{x,r}(y)\,d\sigma(y)} \int_{\partial\Omega} u(y)\Lambda_{x,r}(y)\,d\sigma(y)$$

for every $x \in \partial\Omega$ and $r > 0$. Denoting $\mathrm{MO\hat{}}_{x,r}(u) = \hat{\fint}_{B_d(x,r)} |u(y) - \hat{u}_r(x)| d\sigma(y)$, we then set

$$\|u\|_{\mathrm{BMO\hat{}}(\partial\Omega)} = \sup_{x\in\partial\Omega,\, r>0} \mathrm{MO\hat{}}_{x,r}(u)$$

and $\mathrm{BMO\hat{}}(\partial\Omega) = \{u \in L^1_\sigma(\partial\Omega) \big| \ \|u\|_{\mathrm{BMO\hat{}}(\partial\Omega)} < +\infty\}$. We then define $\hat{\eta}_\varepsilon$, $\hat{\eta}_0$ and $\mathrm{VMO\hat{}}(\partial\Omega)$ accordingly, as in Section 2 (in the same way we also introduce the seminorms $\mathrm{MO\hat{}}^{\,*}_{x,r}$, $\hat{\eta}^*_\varepsilon$, $\|\cdot\|^*_{\mathrm{BMO\hat{}}(\partial\Omega)}$). Taking the usual means $\bar{u}_r(x) = \frac{1}{\sigma(B_d(x,r))}\int_{B_d(x,r)} u\, d\sigma$ instead of $\hat{u}_r(x)$ we define the spaces $\mathrm{BMO}(\partial\Omega)$ and $\mathrm{VMO}(\partial\Omega)$ analogously (as in Section 2, we agree to use the same notation, but dropping the symbol $\hat{}$, for the relevant seminorms). Since Λ is continuous, we can easily prove the following lemma.

LEMMA 20. *Let $u \in L^1_\sigma(\partial\Omega)$. Then $\hat{u} \in C(\partial\Omega\times]0, +\infty[)$. In particular $\hat{u}_\varepsilon \in C(\partial\Omega)$ for every $\varepsilon > 0$. If moreover $u \in C(\partial\Omega)$ and we set $\hat{u}(x, 0) = u(x)$, then $\hat{u} \in C(\partial\Omega \times [0, +\infty[)$ and $\hat{u}_r \to u$ uniformly on $\partial\Omega$, as $r \to 0^+$.*

Let now Y be a connected compact smooth $(N-1)$-dimensional oriented manifold (without boundary) embedded in some $\mathbb{R}^m$ space. From now on we also suppose that

(H$_5$) $\partial\Omega$ is a smooth $(N-1)$-dimensional oriented manifold,

so that $\deg(u, \partial\Omega, Y)$ is defined for continuous maps $u : \partial\Omega \to Y$. Following [8] we will extend this degree to the maps $u \in \mathrm{VMO\hat{}}(\partial\Omega, Y) = \{u \in \mathrm{VMO\hat{}}(\partial\Omega, \mathbb{R}^m) |\, u(x) \in Y\ \sigma\text{-a.e.}\}$. We first observe that, for $u \in \mathrm{VMO\hat{}}(\partial\Omega, Y)$, we have

$$\sup_{x\in\partial\Omega} \mathrm{dist}(\hat{u}_\varepsilon(x), Y) \le \sup_{x\in\partial\Omega} \mathrm{MO\hat{}}_{x,\varepsilon}(u) \le \hat{\eta}_\varepsilon(u) \to 0, \qquad \text{as } \varepsilon \to 0^+ \tag{20}$$

(we have denoted $\mathrm{dist}(z, Y) = \inf_{y\in Y} |y - z|$). Denoting by P_Y the projection operator in $\mathbb{R}^m$ to the nearest point on Y (this is well defined and continuous in a tubular neighborhood of Y), we can then consider $P\hat{u}_\varepsilon = P_Y \circ \hat{u}_\varepsilon : \partial\Omega \to Y$, for small $\varepsilon > 0$.

PROPOSITION 21. *Let $u \in \mathrm{VMO\hat{}}(\partial\Omega, Y)$. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon, \varepsilon' \in]0, \varepsilon_0[$ we have*

(i) *$P\hat{u}_\varepsilon \in C(\partial\Omega, Y)$, so that $\deg(P\hat{u}_\varepsilon, \partial\Omega, Y)$ is defined,*

(ii) $\deg(P\hat{u}_\varepsilon, \partial\Omega, Y) = \deg(P\hat{u}_{\varepsilon'}, \partial\Omega, Y)$.

PROOF. (i) follows from Lemma 20 and the continuity of P_Y; (ii) follows from the homotopy invariance of the degree, using the continuous (by Lemma 20) homotopy $[0, 1] \times \partial\Omega \ni (t, x) \mapsto P_Y(\hat{u}(x, t\varepsilon + (1-t)\varepsilon')) \in Y$. □

DEFINITION 22. We can then define $\deg\hat{} : \mathrm{VMO\hat{}}(\partial\Omega, Y) \to \mathbb{Z}$, $u \mapsto \deg\hat{}(u, \partial\Omega, Y) = \deg(P\hat{u}_\varepsilon, \partial\Omega, Y)$ for small $\varepsilon > 0$.

REMARK 23. It is easy to recognize that $\deg\hat{}$ is an extension of the usual degree for continuous maps. Indeed, if $u \in C(\partial\Omega, Y) \subseteq \mathrm{VMO}\hat{}(\partial\Omega, Y)$, then (for small $\varepsilon > 0$) $[0,1] \times \partial\Omega \ni (t,x) \mapsto P_Y(\hat{u}(x, t\varepsilon)) \in Y$ is a continuous homotopy by Lemma 20.

The degree just defined has the following properties, which can be proved using arguments similar to those in [8].

PROPOSITION 24. *Let* $u \in \mathrm{VMO}\hat{}(\partial\Omega, Y)$.

(i) *If* $\deg\hat{}(u, \partial\Omega, Y) \neq 0$, *then* $\operatorname{ess} u(\partial\Omega) = Y$. *Here* $\operatorname{ess} u(\partial\Omega)$ *is the essential range of u with respect to the measure* σ.

(ii) *There exists* $\delta > 0$ *(depending on u) such that* $\deg\hat{}(u, \partial\Omega, Y) = \deg\hat{}(v, \partial\Omega, Y)$ *for every* $v \in \mathrm{VMO}\hat{}(\partial\Omega, Y)$ *such that* $\|u - v\|_{\mathrm{BMO}\hat{}(\partial\Omega)} < \delta$. *Hence* $\deg\hat{}$ *is constant on the connected components of* $\mathrm{VMO}\hat{}(\partial\Omega, Y)$.

(iii) *(homotopy invariance) Let* $[0,1] \ni t \mapsto F_t \in \mathrm{VMO}\hat{}(\partial\Omega, Y)$ *be a continuous map (in the* $\mathrm{BMO}\hat{}(\partial\Omega)$*-topology). Then* $[0,1] \ni t \mapsto \deg\hat{}(F_t, \partial\Omega, Y) \in \mathbb{Z}$ *is constant.*

3.2. We now assume that the measure σ is doubling on $(\partial\Omega, d)$:

$$(\mathrm{H}_6) \qquad \sup_{x\in\partial\Omega,\, r>0} \frac{\sigma(B_d(x, 2r))}{\sigma(B_d(x, r))} < +\infty.$$

Of our hypotheses, this is the strongest one. We would like to remark that, in the setting of control distances d related to Hörmander operators, (H_6) is satisfied by the measure σ defined in (2) if the boundary of Ω has no characteristic points (see e.g. [34]); less restrictive sufficient conditions for having (H_6) are given in [11] and [17] in the setting of Carnot groups.

We denote by α the sup in (H_6) and we set $q = \log_2 \alpha$. From (H_6) it follows in a standard way that

$$(21) \qquad \sigma(B_d(x, tr)) \le \alpha t^q \sigma(B_d(x, r)) \qquad \forall t \ge 1,\ x \in \partial\Omega,\ r > 0.$$

Moreover, for every $u \in L^1_\sigma(\partial\Omega)$, $x \in \partial\Omega$ and $r > 0$, from (4) and (H_6) we easily obtain

$$(22) \qquad \frac{1}{\alpha} \fint_{B_d(x,\frac{r}{2})} |u| d\sigma \ \le \widehat{\fint}_{B_d(x,r)} |u| d\sigma \ \le \alpha \fint_{B_d(x,r)} |u| d\sigma,$$

$$(23) \qquad \frac{1}{\alpha^2} \mathrm{MO}^*_{x,\frac{r}{2}}(u) \ \le \mathrm{MO}\hat{}\,^*_{x,r}(u) \ \le \alpha^2 \mathrm{MO}^*_{x,r}(u),$$

$$(24) \qquad \left| \widehat{\fint}_{B_d(x,r)} u d\sigma \ - \fint_{B_d(x,r)} u d\sigma \right| \le \alpha \mathrm{MO}_{x,r}(u).$$

THEOREM 25. *We have* $\mathrm{BMO}\hat{}(\partial\Omega) = \mathrm{BMO}(\partial\Omega)$, $\mathrm{VMO}\hat{}(\partial\Omega) = \mathrm{VMO}(\partial\Omega)$ *and the relevant norms are equivalent. Moreover* $\mathrm{VMO}(\partial\Omega)$ *is the closure of* $C(\partial\Omega)$

in BMO$(\partial\Omega)$. *Furthermore* $\widehat{\deg}(\cdot,\partial\Omega,Y)$ (*defined in Section* 3.1) *turns out to be the unique possible extension to* VMO$(\partial\Omega,Y)$ *of the classical degree* $\deg(\cdot,\partial\Omega,Y)$ *for continuous maps, satisfying the* BMO*-homotopy invariance property in Proposition* 24-(iii).

In light of the above theorem, in the sequel we shall always denote $\widehat{\deg}$ simply by deg. The following lemmas will be used in the proof of Theorem 25.

LEMMA 26. *Fixed any $M>0$ there exists $c_M>0$ such that, if $x_1,x_2\in\partial\Omega$, $d(x_1,x_2)\le M\varepsilon$ and $r_1,r_2\in[\varepsilon,M\varepsilon]$ for some $\varepsilon>0$, then*

$$|\hat{u}_{r_1}(x_1)-\hat{u}_{r_2}(x_2)|\le c_M\hat{\eta}_{c_M\varepsilon}(u)$$

for every $u\in L^1_\sigma(\partial\Omega)$.

LEMMA 27. *Let $u\in\widehat{\text{VMO}}(\partial\Omega)$. Then $\hat{u}_\varepsilon\to u$ in $\widehat{\text{BMO}}(\partial\Omega)$, as $\varepsilon\to0^+$. If moreover $u\in\widehat{\text{VMO}}(\partial\Omega,Y)$, then also $P\hat{u}_\varepsilon\to u$ in $\widehat{\text{BMO}}(\partial\Omega)$, as $\varepsilon\to0^+$.*

PROOF OF LEMMA 26. Choosing $R=\max\{r_1,2d(x_1,x_2)+r_2\}$ we get $\Lambda_{x_i,r_i}\le\Lambda_{x_1,R}$ (for $i=1,2$) so that

$$\begin{aligned}|\hat{u}_{r_i}(x_i)-\hat{u}_R(x_1)|&\le\frac{1}{\int\Lambda_{x_i,r_i}d\sigma}\int\Lambda_{x_i,r_i}|u-\hat{u}_R(x_1)|d\sigma\\&\le\frac{1}{\sigma(B_d(x_i,\frac{r_i}{2}))}\int\Lambda_{x_1,R}|u-\hat{u}_R(x_1)|d\sigma\\&\le\alpha\frac{\sigma(B_d(x_1,R))}{\sigma(B_d(x_i,r_i))}\widehat{\text{MO}}_{x_1,R}(u),\end{aligned}$$

by (4) and (H_6). Moreover by (21) we have $\frac{\sigma(B_d(x_1,R))}{\sigma(B_d(x_1,r_1))}\le\alpha(\frac{R}{r_1})^q\le c_M$ and $\frac{\sigma(B_d(x_1,R))}{\sigma(B_d(x_2,r_2))}\le\frac{\sigma(B_d(x_2,R+d(x_1,x_2)))}{\sigma(B_d(x_2,r_2))}\le\alpha(\frac{R+d(x_1,x_2)}{r_2})^q\le c_M$. Therefore

$$|\hat{u}_{r_1}(x_1)-\hat{u}_{r_2}(x_2)|\le|\hat{u}_{r_1}(x_1)-\hat{u}_R(x_1)|+|\hat{u}_R(x_1)-\hat{u}_{r_2}(x_2)|\le c\hat{\eta}_R(u)\le c\hat{\eta}_{c\varepsilon}(u)$$

with c depending on M. □

PROOF OF LEMMA 27. The proof of the first statement is an easy adaptation of the proof of (A.6) of [8], by means of the doubling condition (H_6) (see also (22)-(23)) and of Lemma 26. The second statement follows from the first one and from (20). □

PROOF OF THEOREM 25. The first assertion follows immediately from (23). The second assertion follows from Lemma 27 and Lemma 20. The last assertion of the theorem follows from the second statement of Lemma 27, by considering (once fixed $u\in\text{VMO}(\partial\Omega,Y)$) the homotopy $F_t=P\hat{u}_{t\varepsilon}$ for $0<t\le1$, $F_0=u$. □

3.3. Following [9] we denote by $\mathrm{VMO}_0(\Omega)$ the set of functions $u \in \mathrm{VMO}(\Omega)$ which, extended as identically zero outside Ω, belong to $\mathrm{VMO}(\tilde{\Omega})$ for some bounded domain $\tilde{\Omega}$ containing $\bar{\Omega}$. Assuming the boundary regularity condition below:

(H_7) there exist $c > 0$ and $\varepsilon_0 > 0$ such that $|B_d(y, \varepsilon) \cap \Omega^c| \geq c|B_d(y, \varepsilon)|$ for every $y \in \Omega^c$ and $0 < \varepsilon < \varepsilon_0$,

and adapting the proof of [9] one can see that the following characterization of $\mathrm{VMO}_0(\Omega)$ holds also in our case.

PROPOSITION 28. *Let* $u \in \mathrm{VMO}(\Omega)$. *Then* $u \in \mathrm{VMO}_0(\Omega)$ *iff*

$$\sup_{0<\varepsilon=\frac{1}{2}d(x,\Omega^c)} \fint_{B_d(x,\varepsilon)} |u| \quad \to 0, \qquad \text{as } \varepsilon \to 0^+ \tag{25}$$

(*or equivalently replacing* $\fint$ *with* $\hat{\fint}$ *in* (25)).

We now want to introduce the class $\mathrm{VMO}_\varphi(\Omega)$ of the VMO functions which have trace $\varphi \in \mathrm{VMO}(\partial\Omega)$ in a suitable sense. Since the definition of [9] does not work in the general setting, we shall make a different construction (which however, in the Euclidean case, turns out to give the same class VMO_φ). We start with defining a projection on $\partial\Omega$. Since $\partial\Omega$ is compact and d is continuous we can find a map $\pi : \Omega \to \partial\Omega$ such that

$$d(x, \pi(x)) = d(x, \partial\Omega)$$

for every $x \in \Omega$. We claim that we can choose π to be measurable. Indeed, $\Gamma(x) = \{y \in \partial\Omega \mid d(y, x) = d(x, \partial\Omega)\}$ defines a measurable multifunction from Ω to nonempty compact subsets of $\partial\Omega$. Thus Γ admits a measurable selection π (see e.g. [12], Theorem III.6). Of course the choice of our measurable projection π is not unique. However we shall see that this choice will not affect the resulting definition of $\mathrm{VMO}_\varphi(\Omega)$. We now want to define an extension operator E from $\mathrm{VMO}(\partial\Omega)$ to $\mathrm{VMO}(\Omega)$. Let $\varphi \in L^1_\sigma(\partial\Omega)$. We set

$$F\varphi(x) = \hat{\fint}_{B_d(\pi(x), d(x,\partial\Omega))\cap\partial\Omega} \varphi \, d\sigma \qquad \Big(= \hat{\varphi}(\pi(x), d(x, \partial\Omega))\Big), \quad x \in \Omega.$$

Since π is measurable, $F\varphi \in L^\infty_{\mathrm{loc}}(\Omega)$ (notice that $\inf_{y\in\partial\Omega} \sigma(B_d(y, r)) > 0$ for every $r > 0$, by (H_4)). If π were continuous then for every $\varphi \in C(\partial\Omega)$ we would have $F\varphi \in C(\bar{\Omega})$, $(F\varphi)_{|\partial\Omega} = \varphi$, by means of Lemma 20. Since π may fail to be continuous (even in a small neighborhood of $\partial\Omega$) F is not a good extension operator for our purpose. So we introduce E instead, by taking further means:

$$E\varphi(x) = \hat{\fint}_{B_d(x, \frac{1}{2}d(x,\Omega^c))} F\varphi, \quad x \in \Omega.$$

We remark that in the construction of the extension operator E no regularity assumption on Ω is needed. By Lemma 1, E is a linear operator from $L^1_\sigma(\partial\Omega)$ to $C(\Omega)$. Moreover E has the following properties.

PROPOSITION 29.

(i) $E \in \mathcal{L}(\mathrm{BMO}(\partial\Omega), \mathrm{BMO}(\Omega))$*;*

(ii) $E \in \mathcal{L}(C(\partial\Omega), C(\bar{\Omega}))$*,* $(E\varphi)_{|\partial\Omega} = \varphi$ *for every* $\varphi \in C(\partial\Omega)$*;*

(iii) $E(\mathrm{VMO}(\partial\Omega)) \subseteq \mathrm{VMO}(\Omega)$.

We have denoted by $\mathcal{L}(X, Y)$ *the set of the linear continuous functionals from* X *to* Y*.*

We first prove the following lemma.

LEMMA 30. *There exists* $c > 0$ *such that for every* $\varphi \in L^1_\sigma(\partial\Omega)$ *we have*

$$|E\varphi(x) - F\varphi(x)| \le c\hat{\eta}_{cd(x,\partial\Omega)}(\varphi) \qquad \text{for every } x \in \Omega; \tag{26}$$

$$|F\varphi(y) - F\varphi(z)| \le c\hat{\eta}_{cd(x,\partial\Omega)}(\varphi) \qquad \text{if } y, z \in B_d(x, r) \in \mathcal{C}; \tag{27}$$

$$|E\varphi(y) - E\varphi(z)| \le c\hat{\eta}_{cd(x,\partial\Omega)}(\varphi) \qquad \text{if } y, z \in B_d(x, r) \in \mathcal{C}. \tag{28}$$

PROOF. Let us prove (27) first. Since $B_d(x, r) \in \mathcal{C}$, we have $r \le \frac{1}{2}d(x, \Omega^c) \le \frac{1}{2}d(x, \partial\Omega)$ and thus, setting $\varepsilon = \frac{1}{2}d(x, \partial\Omega)$, $r_y = d(y, \partial\Omega)$, $r_z = d(z, \partial\Omega)$, we have $\varepsilon \le d(x, \partial\Omega) - r \le d(x, \partial\Omega) - d(y, x) \le r_y \le d(y, x) + d(x, \partial\Omega) \le r + d(x, \partial\Omega) \le 3\varepsilon$ and analogously $\varepsilon \le r_z \le 3\varepsilon$. Moreover $d(\pi(y), \pi(z)) \le d(\pi(y), y) + d(y, z) + d(z, \pi(z)) \le r_y + 2r + r_z \le 8\varepsilon$. We can then apply Lemma 26 and obtain

$$|\hat{\varphi}_{r_y}(\pi(y)) - \hat{\varphi}_{r_z}(\pi(z))| \le c\hat{\eta}_{c\varepsilon}(\varphi).$$

This proves (27). We now turn to the proof of (26). For every $x \in \Omega$ we have

$$|E\varphi(x) - F\varphi(x)| \le \hat{\fint}_{B_d(x, \frac{1}{2}d(x,\Omega^c))} |F\varphi(y) - F\varphi(x)| dy \le c\hat{\eta}_{cd(x,\partial\Omega)}(\varphi)$$

by (27), since $B_d(x, \frac{1}{2}d(x, \Omega^c)) \in \mathcal{C}$. Finally, (28) follows from (27) and (26), recalling that $d(y, \partial\Omega) \le \frac{3}{2}d(x, \partial\Omega)$ for every $y \in B_d(x, r) \in \mathcal{C}$. $\square$

PROOF OF PROPOSITION 29. From (28) it follows that $\hat{\mathrm{MO}}^*_{x,r}(E\varphi) \le c\hat{\eta}_{cd(x,\partial\Omega)}(\varphi) \le c\|\varphi\|_{\mathrm{BMO}(\partial\Omega)}$ for every $B_d(x, r) \in \mathcal{C}$. This yields

$$\|E\varphi\|_{\mathrm{BMO}(\Omega)} \le c\|\varphi\|_{\mathrm{BMO}(\partial\Omega)}$$

and proves (i). (In the same way, using (27), one can also see that $F \in \mathcal{L}(\mathrm{BMO}(\partial\Omega), \mathrm{BMO}(\Omega))$). To prove (ii), we only need to show that, for every $\varphi \in C(\partial\Omega)$ and $\xi \in \partial\Omega$, we have $E\varphi(x) \to \varphi(\xi)$, as $x \to \xi$, $x \in \Omega$. Fix $\varepsilon > 0$. By the Heine-Cantor theorem on the compact metric space $(\partial\Omega, d)$, there exists $\delta > 0$ such that $|\varphi(y) - \varphi(\xi)| \le \varepsilon$ whenever $d(y, \xi) \le 3\delta$. Thus, if $d(x, \xi) \le \delta$ we get

$$|F\varphi(x) - \varphi(\xi)| \le \hat{\fint}_{B_d(\pi(x), d(x,\partial\Omega))} |\varphi(y) - \varphi(\xi)| d\sigma(y) \le \varepsilon,$$

since $d(y,\xi) \le d(y,\pi(x)) + d(\pi(x),x) + d(x,\xi) \le 3d(x,\xi)$ for every $y \in B_d(\pi(x), d(x,\partial\Omega))$. On the other hand, (26) gives $|E\varphi(x) - F\varphi(x)| \le c\hat{\eta}_{cd(x,\xi)}(\varphi)$ which vanishes as x goes to ξ, since $\varphi \in C(\partial\Omega) \subseteq \mathrm{VMO}(\partial\Omega)$. By the triangle inequality $|E\varphi(x) - \varphi(\xi)| \le |E\varphi(x) - F\varphi(x)| + |F\varphi(x) - \varphi(\xi)|$, the proof of (ii) is complete. Finally (iii) follows from (i) and (ii) by a standard density argument (since $\mathrm{VMO}(\partial\Omega)$ is the closure of $C(\partial\Omega)$ in $\mathrm{BMO}(\partial\Omega)$ (see Theorem 25)). $\square$

We are now in position to define the class $\mathrm{VMO}_\varphi(\Omega)$. Given $\varphi \in \mathrm{VMO}(\partial\Omega)$, we set

$$\mathrm{VMO}_\varphi(\Omega) = \{u \in \mathrm{VMO}(\Omega) \mid u - E\varphi \in \mathrm{VMO}_0(\Omega)\}. \tag{29}$$

Clearly this class is not empty, since $E\varphi \in \mathrm{VMO}_\varphi(\Omega)$.

REMARK 31. If $\Phi \in C(\bar{\Omega})$ and we set $\varphi = \Phi_{|\partial\Omega}$, then $\Phi \in \mathrm{VMO}_\varphi(\Omega)$. This follows immediately from Proposition 29-(ii) and the definition of $\mathrm{VMO}_0(\Omega)$.

REMARK 32. The definition of $\mathrm{VMO}_\varphi(\Omega)$ does not depend on the choice of π. Indeed, if π' is an other measurable projection and E' denotes the associated extension operator, then we have $E\varphi - E'\varphi \in \mathrm{VMO}_0(\Omega)$. To show this fact we use Proposition 28. By (26), we only need to prove that

$$\hat{\fint}_{B_d(x,\frac{1}{2}d(x,\Omega^c))} |F\varphi - F'\varphi| \le c\hat{\eta}_{cd(x,\partial\Omega)}(\varphi) \tag{30}$$

(with c independent of $x \in \Omega$) and observe that

$$d(x,\partial\Omega) \to 0, \qquad \text{as } d(x,\Omega^c) \to 0, \quad \text{uniformly in } x \in \Omega. \tag{31}$$

On the other hand (30) follows from Lemma 26, since

$$|F\varphi(y) - F'\varphi(y)| = |\hat{\varphi}_{d(y,\partial\Omega)}(\pi(y)) - \hat{\varphi}_{d(y,\partial\Omega)}(\pi'(y))|,$$

with $d(\pi(y),\pi'(y)) \le d(\pi(y),y) + d(y,\pi'(y)) = 2d(y,\partial\Omega)$ and $d(y,\partial\Omega) \le \frac{3}{2}d(x,\partial\Omega)$ for $y \in B_d(x,\frac{1}{2}d(x,\Omega^c))$.

REMARK 33. If there exists a continuous projection π, defined even only in a neighborhood of $\partial\Omega$ in Ω, then one might define $\mathrm{VMO}_\varphi(\Omega)$ by taking $F\varphi$ (extended to all of Ω by a suitable cut-off function) instead of $E\varphi$ in (29). Such definition would be equivalent to ours.

REMARK 34. Assume that Ω satisfies the following regularity condition: for σ-a.e. $x \in \partial\Omega$ there exists a sequence x_k in Ω converging to x such that

$$\sup_{k\in\mathbb{N}} \frac{d(x_k,x)}{d(x_k,\partial\Omega)} < +\infty. \tag{32}$$

If $\varphi \in \mathrm{VMO}(\partial\Omega)$ is such that $E\varphi \in \mathrm{VMO}_0(\Omega)$, then $\varphi = 0$ σ-a.e. (In particular $E : \mathrm{VMO}(\partial\Omega) \to \mathrm{VMO}(\Omega)$ is injective.) As a consequence, if (32) holds and

$\varphi \neq \psi$ ($\in \mathrm{VMO}(\partial\Omega)$) then the corresponding classes $\mathrm{VMO}_\varphi(\Omega)$ and $\mathrm{VMO}_\psi(\Omega)$ are disjoint. In other words the same VMO function cannot have two different traces.

PROOF. We first recall that, since σ is regular and doubling (see (H_6)), for every $\varphi \in L^1_\sigma(\partial\Omega)$ and σ-a.e. $x \in \partial\Omega$ (the Lebesgue points of φ) we have

$$(33) \qquad \fint_{B_d(x,\varepsilon)} |\varphi(y) - \varphi(x)| d\sigma(y) \to 0, \qquad \text{as } \varepsilon \to 0^+.$$

From (33) and (22) we obtain also

$$(34) \quad \hat{\fint}_{B_d(x,\varepsilon)} |\varphi(y) - \varphi(x)| d\sigma(y) \to 0, \qquad \text{as } \varepsilon \to 0^+, \text{ for } \sigma\text{-a.e. } x \in \partial\Omega.$$

Let now $\varphi \in \mathrm{VMO}(\partial\Omega)$ be such that $E\varphi \in \mathrm{VMO}_0(\Omega)$ and let us fix x (and a sequence (x_k)) verifying both (32) and (34). We have

$$\begin{aligned} |\varphi(x)| \le{}& |\varphi(x) - \hat{\varphi}_{d(x_k,\partial\Omega)}(x)| + |\hat{\varphi}_{d(x_k,\partial\Omega)}(x) - \hat{\varphi}_{d(x_k,\partial\Omega)}(\pi(x_k))| \\ &+ |F\varphi(x_k) - E\varphi(x_k)| + \hat{\fint}_{B_d(x_k,\frac{1}{2}d(x_k,\Omega^c))} |F\varphi - E\varphi| + \hat{\fint}_{B_d(x_k,\frac{1}{2}d(x_k,\Omega^c))} |E\varphi|. \end{aligned}$$

Letting $k \to +\infty$, the first and the last term in the right hand side go to zero by (34) and by Proposition 28, respectively. We claim that all the other terms are smaller than $c\hat{\eta}_{cd(x_k,\partial\Omega)}(\varphi)$, for a positive constant c which may depend on x. Thus they vanish as well, and $\varphi(x)$ has to be zero. Let us prove the claim. Since $d(x,\pi(x_k)) \le d(\pi(x_k),x_k) + d(x_k,x) \le d(x_k,\partial\Omega)\big(1 + \sup_{j\in\mathbb{N}} \frac{d(x_j,x)}{d(x_j,\partial\Omega)}\big)$, the desired estimate of $|\hat{\varphi}_{d(x_k,\partial\Omega)}(x) - \hat{\varphi}_{d(x_k,\partial\Omega)}(\pi(x_k))|$ can be derived from Lemma 26. On the other hand, the estimate of the remaining terms follows from (26). □

3.4. In this section we want to compare the degree of a map $u \in \mathrm{VMO}_\varphi(\Omega)$ (defined in Section 2.1) with the degree of its trace φ (defined in Section 3.1). Our aim is to show that our construction of the class VMO_φ allows to establish the same relation proved in [9]. This is done in Theorem 39 below. Note that, while the degree of u of course does not depend on the measure σ on $\partial\Omega$, both the definition of the degree of φ and the one of the class VMO_φ depend on σ.

LEMMA 35. *There exists a positive constant c such that, given any $\varphi \in \mathrm{VMO}(\partial\Omega)$ with $|\varphi| \ge \delta$ σ-a.e. for some $\delta > 0$, we have*

$$|E\varphi(x)| \ge \delta - c\hat{\eta}_{cd(x,\partial\Omega)}(\varphi)$$

for every $x \in \Omega$.

PROOF. Recalling (26), we only need to prove that

$$|F\varphi(x)| \geq \delta - \hat{\eta}_{d(x,\partial\Omega)}(\varphi),$$

which is easily seen in the following way:

$$\begin{aligned}|F\varphi(x)| \geq -\hat{⨍}_{B_d(\pi(x),d(x,\partial\Omega))} |F\varphi(x) - \varphi(y)| d\sigma(y) \\ + \hat{⨍}_{B_d(\pi(x),d(x,\partial\Omega))} |\varphi(y)| d\sigma(y) \geq -\hat{\mathrm{MO}}_{\pi(x),d(x,\partial\Omega)}(\varphi) + \delta.\end{aligned} \qquad \square$$

LEMMA 36. *Let $\varphi \in \mathrm{VMO}(\partial\Omega, S^{N-1})$, where $(Y =) S^{N-1}$ is the unit sphere of $\mathbb{R}^N$. Then we have*

(i) *$\hat{\varphi}_\varepsilon$ and $P\hat{\varphi}_\varepsilon$ converge to φ in $\mathrm{BMO}(\partial\Omega)$ and σ-a.e., as $\varepsilon \to 0^+$.*
(ii) *$F(P\hat{\varphi}_\varepsilon) \to F\varphi$ pointwise, in $L^1(\Omega)$ and in $\mathrm{BMO}(\Omega)$.*
(iii) *$E(P\hat{\varphi}_\varepsilon) \to E\varphi$ pointwise, in $L^1(\Omega)$ and in $\mathrm{BMO}(\Omega)$.*
(iv) *$(E\varphi, 0) \in \mathcal{M}(\Omega)$ and*

$$\deg(E\varphi, \Omega, 0) = \deg(\varphi, \partial\Omega, S^{N-1}). \tag{35}$$

PROOF. (i) follows from Lemma 27, (34) and (20). The pointwise and L^1 convergence in (ii) follow from (i) by dominated convergence. The BMO convergence in (ii) follows from (i) since $F \in \mathcal{L}(\mathrm{BMO}(\partial\Omega), \mathrm{BMO}(\Omega))$ (see the proof of Proposition 29). (iii) follows from (ii), (i) and Proposition 29-(i) in the same way. We now prove (iv). Setting $\varphi_k = P\hat{\varphi}_{\frac{1}{k}}$, we have $\mathrm{VMO}(\partial\Omega, S^{N-1}) \supseteq C(\partial\Omega, S^{N-1}) \ni \varphi_k \to \varphi$ in BMO. Thus, it is not difficult to show that

$$\sup_{k\in\mathbb{N}} \hat{\eta}_\varepsilon(\varphi_k) \to 0, \qquad \text{as } \varepsilon \to 0^+.$$

We now apply Lemma 35 to φ_k. Recalling (31) we obtain that $(E\varphi_k, 0)$ verify (8) uniformly in $k \in \mathbb{N}$. Since also (iii) holds, we can use Proposition 9 and get that $(E\varphi, 0) \in \mathcal{M}$ and $\deg(E\varphi, \Omega, 0) = \deg(E\varphi_k, \Omega, 0)$ (for large k). On the other hand $E\varphi_k \in C(\bar{\Omega})$, $(E\varphi_k)_{|\partial\Omega} = \varphi_k$ by Proposition 29-(ii) and then $\deg(E\varphi_k, \Omega, 0) = \deg(\varphi_k, \partial\Omega, S^{N-1})$ by the classical degree theory. Therefore (35) follows, recalling Definition 22. □

LEMMA 37. *Let $u, v \in \mathrm{VMO}(\Omega, \mathbb{R}^N)$ and suppose that $u - v \in \mathrm{VMO}_0(\Omega)$. Then $(u, 0) \in \mathcal{M}(\Omega)$ iff $(v, 0) \in \mathcal{M}(\Omega)$ and in such case $\deg(u,\Omega,0) = \deg(v,\Omega,0)$.*

PROOF. It is a straightforward consequence of Proposition 28 and Corollary 11. □

REMARK 38. If $\varphi \in \mathrm{VMO}(\partial\Omega, \mathbb{R}^N)$ and $|\varphi| \geq \delta$ σ-a.e. for some positive constant δ, then $\frac{\varphi}{|\varphi|} \in \mathrm{VMO}(\partial\Omega, S^{N-1})$. Indeed the composition of a VMO map with any Lipschitz map is VMO as well. This is immediately seen by considering the mean oscillations $\mathrm{MO}^*_{x,r}$.

THEOREM 39. *Let $\varphi \in \mathrm{VMO}(\partial\Omega, \mathbb{R}^N)$ and let $p \in \mathbb{R}^N$. Suppose that $|\varphi - p| \geq \delta$ σ-a.e. for some positive constant δ. Then for every $u \in \mathrm{VMO}_\varphi(\Omega)$ we have $(u, p) \in \mathcal{M}(\Omega)$ and*

$$\deg(u, \Omega, p) = \deg\left(\frac{\varphi - p}{|\varphi - p|}, \partial\Omega, S^{N-1}\right).$$

PROOF. It is not restrictive to assume $p = 0$. Observing that

$$\hat{\fint}_{B_d(x,\frac{1}{2}d(x,\Omega^c))} |u| \geq \hat{\fint}_{B_d(x,\frac{1}{2}d(x,\Omega^c))} |E\varphi| - \hat{\fint}_{B_d(x,\frac{1}{2}d(x,\Omega^c))} |u - E\varphi|$$

and using Lemma 35, (31) and Proposition 28, we obtain that $(u,0) \in \mathcal{M}(\Omega)$. We now set $\psi = \frac{\varphi}{|\varphi|}$ ($\in \mathrm{VMO}(\partial\Omega, S^{N-1})$ by Remark 38). We have $\deg(E\psi, \Omega, 0) = \deg(\psi, \partial\Omega, S^{N-1})$ by Lemma 36-(iv) and $\deg(E\varphi, \Omega, 0) = \deg(u, \Omega, 0)$ by Lemma 37. Thus, to complete the proof, we only need to show that $\deg(E\psi, \Omega, 0) = \deg(E\varphi, \Omega, 0)$. This follows from Corollary 10, by using the homotopy $H_t = tE\psi + (1-t)E\varphi = E(t\psi + (1-t)\varphi)$, if we prove that $(H_t, 0)$ verifies (8) uniformly in $t \in [0, 1]$. We have $|t\psi + (1-t)\varphi| = t + (1-t)|\varphi| \geq t + (1-t)\delta \geq \min\{1, \delta\} = \delta' > 0$ σ-a.e. and then, by Lemma 35,

$$\begin{aligned}
|H_t(x)| &\geq \delta' - c\hat{\eta}_{cd(x,\partial\Omega)}(t\psi + (1-t)\varphi) \\
&\geq \delta' - c\big(t\hat{\eta}_{cd(x,\partial\Omega)}(\psi) + (1-t)\hat{\eta}_{cd(x,\partial\Omega)}(\varphi)\big) \\
&\geq \delta' - c\max\big\{\hat{\eta}_{cd(x,\partial\Omega)}(\psi), \hat{\eta}_{cd(x,\partial\Omega)}(\varphi)\big\}.
\end{aligned}$$

Recalling (31), this completes the proof. □

COROLLARY 40. *Let $\varphi \in \mathrm{VMO}(\partial\Omega, S^{N-1})$ and let $p \in \mathbb{R}^N$, $|p| < 1$. Then* $\deg(\varphi, \partial\Omega, S^{N-1}) = \deg(\frac{\varphi - p}{|\varphi - p|}, \partial\Omega, S^{N-1})$.

PROOF. It follows from Theorem 39 and Remark 7, taking $u = E\varphi$. □

COROLLARY 41. *Let $\varphi \in \mathrm{VMO}(\partial\Omega, S^{N-1})$. If $\deg(\varphi, \partial\Omega, S^{N-1}) \neq 0$ and $u \in \mathrm{VMO}_\varphi(\Omega)$, then $\overline{D(0,1)} \subseteq \operatorname{ess} u(\Omega)$ (we recall that $\operatorname{ess} u(\Omega)$ is the essential range of u, defined in* (10)).

PROOF. It follows from Theorem 39, Corollary 40 and Proposition 6, recalling also that $\operatorname{ess} u(\Omega)$ is closed. □

3.5. In this section we introduce a class of integral operators and show that they are extension operators from $\mathrm{VMO}(\partial\Omega)$ to $\mathrm{VMO}(\Omega)$ in the sense that they map a $\varphi \in \mathrm{VMO}(\partial\Omega)$ into $\mathrm{VMO}_\varphi(\Omega)$. In other words these extensions of φ have trace φ on $\partial\Omega$ in the VMO sense; moreover we shall see that such extensions also take the value φ at the boundary, nontangentially pointwise σ-a.e. Along the lines of all the paper we give an abstract presentation, but the extensions we have in mind are the L-harmonic extensions with respect to some differential operator L, whose associated control distance is d. Thus, the kernels K below have to be thought as generalized Poisson kernels.

Let $K : \Omega \times \partial\Omega \to \mathbb{R}$ be the kernel of a well-defined integral operator H,

$$H\varphi(x) = \int_{\partial\Omega} K(x,\xi)\varphi(\xi)d\sigma(\xi), \qquad x \in \Omega, \tag{36}$$

which maps $L^1_\sigma(\partial\Omega)$ into the measurable functions on Ω. We assume that H has the following properties:

$$H(C(\partial\Omega)) \subseteq C(\bar{\Omega}), \qquad (H\varphi)_{|\partial\Omega} = \varphi \quad \forall\varphi \in C(\partial\Omega), \tag{37}$$

$$H(1) \equiv 1. \tag{38}$$

We also assume that $K(x,\cdot) \in L^\infty_\sigma(\partial\Omega)$ for every $x \in \Omega$ and we set

$$B^x_j = B_d(\pi(x), 2^j d(x,\partial\Omega)) \cap \partial\Omega,$$

$$a_0(x) = \|K(x,\cdot)\|_{L^\infty_\sigma(B^x_0)}, \quad a_j(x) = \|K(x,\cdot)\|_{L^\infty_\sigma(B^x_j \smallsetminus B^x_{j-1})} \quad (j \in \mathbb{N}).$$

Finally, we assume that the following condition holds:

$$\operatorname*{ess\,sup}_{x\in\Omega} \left(a_0(x)\sigma(B^x_0) + \sum_{j=1}^{+\infty} j\, a_j(x)\sigma(B^x_j) \right) < +\infty. \tag{39}$$

Under these hypotheses we shall prove the following theorems.

THEOREM 42.
(i) $H \in \mathcal{L}(\mathrm{BMO}(\partial\Omega), \mathrm{BMO}(\Omega))$;
(ii) $H \in \mathcal{L}(C(\partial\Omega), C(\bar{\Omega}))$;
(iii) $H(\mathrm{VMO}(\partial\Omega)) \subseteq \mathrm{VMO}(\Omega)$;
(iv) $H\varphi \in \mathrm{VMO}_\varphi(\Omega)$ *for every* $\varphi \in \mathrm{VMO}(\partial\Omega)$.

REMARK 43. In particular collecting (iv) and Corollary 41 we obtain a generalization of Theorem (BN) of the Introduction.

DEFINITION 44. Let $u : \Omega \to \mathbb{R}^m$, $\ell \in \mathbb{R}^m$ and $y_0 \in \partial\Omega$. We shall say that

$$u(x) \to \ell, \quad \text{nontangentially as } x \to y_0,$$

iff for every $M > 0$ we have

$$\operatorname*{ess\,sup}_{x\in B_d(y_0,\varepsilon)\cap\Omega,\, d(x,y_0)\le Md(x,\partial\Omega)} |u(x) - \ell| \to 0, \qquad \text{as } \varepsilon \to 0^+.$$

(We agree to let $\operatorname{ess\,sup} \varnothing = 0$.)

THEOREM 45. *Let* $\varphi \in \mathrm{VMO}(\partial\Omega)$. *Then for* σ*-a.e.* $y \in \partial\Omega$ *we have*

$$H\varphi(x) \to \varphi(y), \quad \textit{nontangentially as } x \to y.$$

(*The same holds true also for the "extensions"* $F\varphi(x)$ *and* $E\varphi(x)$ *defined in Section* 3.3.)

Before proving the above results, we point out the following sufficient condition for having (39).

PROPOSITION 46. *If the kernel K can be estimated as*

$$|K(x,\xi)| \leq c\frac{d(x,\partial\Omega)}{d(x,\xi)\sigma(B_d(\xi,d(x,\xi)))}, \tag{40}$$

then (39) *holds. We explicitly remark that the estimate* (40) *can be deduced in particular from the estimates*

$$|K(x,\xi)| \leq c\frac{d(x,\partial\Omega)}{|B_d(\xi,d(x,\xi))|}, \tag{41}$$

and

$$\sigma(B_d(\xi,r)) \leq c_R\frac{|B_d(\xi,r)|}{r} \qquad (\xi \in \partial\Omega,\ 0 < r \leq R). \tag{42}$$

PROOF. We first notice that for every $x \in \Omega$ and $\xi \in \partial\Omega$ we have

$$\frac{1}{3}(d(\xi,\pi(x)) + d(x,\partial\Omega)) \leq d(x,\xi) \leq d(\xi,\pi(x)) + d(x,\partial\Omega). \tag{43}$$

Let us fix $\xi \in B_0^x = B_d(\pi(x), d(x,\partial\Omega))$. It is easy to recognize that $B_0^x \subseteq B_d(\xi, 3d(x,\xi))$. By the doubling condition (H_6), we get $\sigma(B_0^x) \leq \alpha^2\sigma(B_d(\xi,d(x,\xi)))$. We now use (40) and obtain $|K(x,\xi)| \leq \frac{c}{\sigma(B_0^x)}$. Therefore we have proved that $a_0(x)\sigma(B_0^x) \leq c$ with c not depending on $x \in \Omega$. We now fix $\xi \in B_j^x \setminus B_{j-1}^x$ ($j \in \mathbb{N}$). One can show that $B_j^x \subseteq B_d(\xi, 6d(x,\xi))$ and then get $\sigma(B_j^x) \leq \alpha^3\sigma(B_d(\xi,d(x,\xi)))$, by ($H_6$). Using (40), this yields $|K(x,\xi)| \leq \frac{c}{2^j\sigma(B_j^x)}$. Therefore we have proved that $a_j(x)\sigma(B_j^x) \leq \frac{c}{2^j}$ with c not depending on $x \in \Omega$. (39) follows straightforwardly. □

We now take up the proof of Theorem 42.

LEMMA 47. *There exists a positive constant c such that $\|H\varphi - F\varphi\|_{L^\infty(\Omega)} \leq c\|\varphi\|_{\mathrm{BMO}(\partial\Omega)}$ for every $\varphi \in L^1_\sigma(\partial\Omega)$.*

PROOF. Using (38) and recalling that $F\varphi(x) = \hat\varphi_{d(x,\partial\Omega)}(\pi(x))$, we have

$$\begin{aligned}
|H\varphi(x) - F\varphi(x)| &\leq \int_{\partial\Omega} |K(x,\xi)|\,|\varphi(\xi) - F\varphi(x)|d\sigma(\xi) \\
&\leq a_0(x)\int_{B_0^x} |\varphi(\xi) - \hat\varphi_{d(x,\partial\Omega)}(\pi(x))|d\sigma(\xi) \\
&\quad + \sum_{j=1}^{+\infty} a_j(x)\int_{B_j^x \setminus B_{j-1}^x} \Big(|\varphi(\xi) - \hat\varphi_{2^j d(x,\partial\Omega)}(\pi(x))| \\
&\quad + \sum_{i=1}^{j} |\hat\varphi_{2^i d(x,\partial\Omega)}(\pi(x)) - \hat\varphi_{2^{i-1}d(x,\partial\Omega)}(\pi(x))|\Big)d\sigma(\xi).
\end{aligned}$$

Recalling (24), for every $j \in \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned}\int_{B_j^x} |\varphi(\xi) - \hat{\varphi}_{2^j d(x,\partial\Omega)}(\pi(x))| d\sigma(\xi) &\leq c\sigma(B_j^x)\mathrm{MO}_{\pi(x),2^j d(x,\partial\Omega)}(\varphi) \\ &\leq c\sigma(B_j^x)\|\varphi\|_{\mathrm{BMO}(\partial\Omega)}.\end{aligned}$$

On the other hand,

$$|\hat{\varphi}_{2^i d(x,\partial\Omega)}(\pi(x)) - \hat{\varphi}_{2^{i-1} d(x,\partial\Omega)}(\pi(x))| \leq c\|\varphi\|_{\mathrm{BMO}(\partial\Omega)},$$

by means of Lemma 26. Therefore

$$|H\varphi(x) - F\varphi(x)| \leq c\|\varphi\|_{\mathrm{BMO}(\partial\Omega)} \left(a_0(x)\sigma(B_0^x) + \sum_{j=1}^{+\infty} j\, a_j(x)\sigma(B_j^x) \right)$$

and from (39) we get the desired estimate. □

PROOF OF THEOREM 42. From Lemma 47 and (26), it follows that

$$\|H\varphi - E\varphi\|_{L^\infty(\Omega)} \leq c\|\varphi\|_{\mathrm{BMO}(\partial\Omega)} \qquad \forall\varphi \in L^1_\sigma(\partial\Omega) \tag{44}$$

and then $(H - E) \in \mathcal{L}(\mathrm{BMO}(\partial\Omega), L^\infty(\Omega)) \subseteq \mathcal{L}(\mathrm{BMO}(\partial\Omega), \mathrm{BMO}(\Omega))$ (since $L^\infty(\Omega) \hookrightarrow \mathrm{BMO}(\Omega)$). Recalling Proposition 29-(i), we get (i). Collecting (37), Proposition 29-(ii) and (44) and observing that $L^\infty_\sigma(\partial\Omega) \hookrightarrow \mathrm{BMO}(\partial\Omega)$, we get also (ii). Moreover, (iii) follows from (i) and (37) by a standard density argument (since $\mathrm{VMO}(\partial\Omega)$ is the closure of $C(\partial\Omega)$ in $\mathrm{BMO}(\partial\Omega)$, by Theorem 25). We now prove (iv). Recalling Proposition 28, it is sufficient to show that

$$\operatorname*{ess\,sup}_{0<d(x,\Omega^c)<\varepsilon} |H\varphi(x) - E\varphi(x)| \to 0, \qquad \text{as } \varepsilon \to 0^+. \tag{45}$$

In order to prove (45) we choose $\psi \in C(\partial\Omega)$ close to φ in the BMO norm (this can be done in light of Theorem 25); then we apply (44) to the function $\varphi - \psi$ and observe that (45) holds replacing φ with ψ, by means of (37) and Proposition 29-(ii). In this way we prove (45) for φ. □

REMARK 48. Let $\varphi \in \mathrm{VMO}(\partial\Omega)$. Then

$$\operatorname*{ess\,sup}_{0<d(x,\Omega^c)<\varepsilon} \mathrm{dist}(H\varphi(x), \operatorname{ess}\varphi(\partial\Omega)) \to 0, \qquad \text{as } \varepsilon \to 0^+ \tag{46}$$

(we recall that $\operatorname{ess}\varphi(\partial\Omega)$ denotes the essential range of φ, with respect to the measure σ, and $\mathrm{dist}(p, A) = \inf_{q\in A} |p-q|$). Indeed, from (26) and (31), it follows that

$$\sup_{0<d(x,\Omega^c)<\varepsilon} |E\varphi(x) - F\varphi(x)| \to 0, \qquad \text{as } \varepsilon \to 0^+. \tag{47}$$

Moreover, arguing as in (20), it is not difficult to show that

$$\sup_{0<d(x,\Omega^c)<\varepsilon} \operatorname{dist}(F\varphi(x), \operatorname{ess}\varphi(\partial\Omega)) \to 0, \qquad \text{as } \varepsilon \to 0^+. \tag{48}$$

Thus (46) follows from (45), (47) and (48).

Proof of Theorem 45. Since (45) and (47) hold, we only need to prove that $F\varphi(x) \to \varphi(y)$, nontangentially as $x \to y$, for every Lebesgue point y of φ (according to (33)-(34)). Let us fix such a Lebesgue point $y_0 \in \partial\Omega$. We also fix $M > 0$ and take $x \in \Omega$ such that $d(x, y_0) \leq Md(x, \partial\Omega)$. We have

$$|\varphi(y_0) - F\varphi(x)| \leq |\varphi(y_0) - \hat{\varphi}_{d(x,\partial\Omega)}(y_0)| + |\hat{\varphi}_{d(x,\partial\Omega)}(y_0) - \hat{\varphi}_{d(x,\partial\Omega)}(\pi(x))|,$$

where the first term in the right hand side vanishes as $x \to y_0$ by (34), and the other term is less then $c_M\hat{\eta}_{c_M d(x,\partial\Omega)}(\varphi)$ (and then it vanishes as $x \to y_0$, as well) by means of Lemma 26, since $d(y_0, \pi(x)) \leq d(y_0, x) + d(x, \partial\Omega) \leq (M + 1)d(x, \partial\Omega)$. □

4. – Hörmander operators and sub-Laplacians

We now want to give some examples of applications to PDE of the results established in the previous sections. We begin with the following Theorem 49 which summarizes the results of Section 3.5 in the framework of a Carnot-Carathéodory space $(\mathbb{R}^N, d)$, where d is the control distance associated to a family $X = (X_1, \dots, X_p)$ of C^∞ vector fields satisfying Hörmander condition (1) on $\mathbb{R}^N$. For the convenience of the reader we recall the definition of the control distance d related to X. Given $x, y \in \mathbb{R}^N$ one defines

$$d(x, y) = \inf\{T\}$$

where $\{T\}$ denotes the set of the real numbers $T > 0$ for which there exists a X-subunit path $\gamma : [0, T] \to \mathbb{R}^N$ connecting x and y. The path γ is called X-subunit if it is absolutely continuous and

$$\gamma'(t) = \sum_{j=1}^{p} \lambda_j(t)X_j(\gamma(t))$$

with $|\lambda_j(t)| \leq 1$ a.e. in $[0, T]$. The Hörmander condition (1) assures both the existence of a X-subunit path connecting any couple of points in $\mathbb{R}^N$, and the hypoellipticity of the operator

$$L = \sum_{j=1}^{p} X_j^2.$$

We assume, together with (1), that there exist two C^2-functions θ, θ^* on a connected open set $\Sigma \subseteq \mathbb{R}^N$ such that $\theta, \theta^* > 0$ and $L\theta < 0, L^*\theta^* < 0$ in Σ, where L^* is the formal adjoint operator to L. These conditions assure the solvability of the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u_{|\partial\Omega} = \varphi, & \varphi \in C(\partial\Omega) \end{cases} \tag{49}$$

for any open set $\Omega \subset\subset \Sigma$ satisfying the regularity assumption (H_7) (see [6] and [39]). If u is the (unique) solution of (49), then

$$u(x) = \int_{\partial\Omega} \varphi(\xi) d\mu_x(\xi), \qquad x \in \Omega,$$

where μ_x is the L-harmonic measure for Ω at x.

THEOREM 49. *Let $\Omega \subset\subset \Sigma$ be a smooth bounded domain of $\mathbb{R}^N$ satisfying* (H_7) *and let σ be a positive Borel measure given on $\partial\Omega$, satisfying* (H_4) *and* (H_6). *Suppose moreover that the L-harmonic measures μ_x on $\partial\Omega$, $x \in \Omega$, are absolutely continuous with respect to σ, with densities $\frac{d\mu_x(\xi)}{d\sigma(\xi)} =: K(x, \xi)$ ($x \in \Omega$, $\xi \in \partial\Omega$) satisfying the estimate* (40). *Finally let $\varphi \in \mathrm{VMO}(\partial\Omega)$ ($= \mathrm{VMO}(\partial\Omega, d, \sigma)$) and consider the L-harmonic extension $H\varphi$ of φ, $H\varphi(x) = \int_{\partial\Omega} \varphi d\mu_x$. Then we have:*

(i) $H\varphi \in \mathrm{VMO}_\varphi(\Omega)$ (*i.e. $H\varphi$ has trace φ in the VMO sense*).

(ii) *$H\varphi(x) \to \varphi(y)$, nontangentially as $x \to y$, for σ-a.e. $y \in \partial\Omega$.*

(iii) *If $\varphi \in \mathrm{VMO}(\partial\Omega, S^{N-1})$ and $\deg(\varphi, \partial\Omega, S^{N-1}) \neq 0$, then*

$$D(0, 1) \subseteq H\varphi(\Omega).$$

PROOF. We first observe that the hypotheses of the abstract theory that we have made along the paper, namely (H_1)-(H_7), are all verified. Indeed (H_1) follows from the Hölder continuity of the control distance d, see e.g. [22]. The doubling condition (H_2) has been proved in the deep paper [38], while (H_3) is true for any control distance (see Remark 13); (H_4)-(H_7) are assumed in the hypotheses of the theorem. Moreover K is a nonnegative kernel with the properties required in Section 3.5. Indeed from the estimate (40) it follows that the definition of H given in (36) is well posed and (39) holds (see Proposition 46). Moreover (38) is an immediate consequence of the properties of the harmonic measures μ_x. Finally (37) follows from (H_7), see [39]. We can also get (37) directly as a consequence of (40): it is enough to observe that, fixed $\varphi \in C(\partial\Omega)$ and $y \in \partial\Omega$, if $x \in \Omega$ and $d(x, y) < \frac{\delta}{2}$ we have

$$\begin{aligned} |H\varphi(x) - \varphi(y)| &\leq \int_{B_d(y,\delta)} K(x, \xi)|\varphi(\xi) - \varphi(y)| d\sigma(\xi) \\ &\quad + 2 \max_{\partial\Omega} |\varphi| \int_{\partial\Omega \setminus B_d(y,\delta)} c \frac{d(x, \partial\Omega)}{\frac{\delta}{2} \inf_{\eta \in \partial\Omega} \sigma(B_d(\eta, \frac{\delta}{2}))} d\sigma(\xi) \\ &\leq H(1) \sup_{d(\xi,y)<\delta,\, \xi \in \partial\Omega} |\varphi(\xi) - \varphi(y)| + c_\delta d(x, \partial\Omega). \end{aligned}$$

We can then apply Theorem 42 and Theorem 45 and obtain (i)-(ii). In order to obtain (iii), we use Corollary 41 and get

$$D(0,1) \subseteq \operatorname{ess}(H\varphi)(\Omega) = \overline{H\varphi(\Omega)}$$

by (11), since $H\varphi$ is L-harmonic and thus in particular continuous on Ω. Moreover we observe that $|H\varphi| \in C(\overline{\Omega})$, $|H\varphi|_{|\partial\Omega} \equiv 1$ by means of Remark 48. As a consequence we finally get $D(0,1) \subseteq H\varphi(\Omega)$. □

From now on we shall work on Carnot groups and consider suitable regular domains Ω; we plan to make a deeper study of the general case in a forthcoming paper.

Let us give the definition of a Carnot group. Let $\circ$ be an assigned Lie group law on $\mathbb{R}^N$. We suppose $\mathbb{R}^N$ is endowed with a homogeneous structure by a given family of Lie group automorphisms $\{\delta_\lambda\}_{\lambda>0}$ (called *dilations*) of the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}).$$

Here $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \dots, r$ and $N_1 + \cdots + N_r = N$. We denote by $\mathfrak{g}$ the Lie algebra of $(\mathbb{R}^N, \circ)$ i.e. the Lie algebra of left-invariant vector fields on $\mathbb{R}^N$. For $i = 1, \dots, N_1$, let X_i be the (unique) vector field in $\mathfrak{g}$ that agrees at the origin with $\partial/\partial x_i$. We make the following assumption: the Lie algebra generated by $X_1, \dots, X_{N_1}$ is the whole $\mathfrak{g}$. With the above hypotheses, we call $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a *Carnot group*. We also say that $\mathbb{G}$ is of *step* r and has $p := N_1$ *generators*. We denote by $Q = \sum_{j=1}^r j\, N_j$ the *homogeneous dimension* of $\mathbb{G}$. The *sub-Laplacian* on $\mathbb{G}$ is the second order differential operator

$$\Delta_{\mathbb{G}} = \textstyle\sum_{i=1}^p X_i^2.$$

We explicitly remark that $\Delta_{\mathbb{G}}$ is hypoelliptic since $X_1, \dots, X_p$ Lie-generate $\mathfrak{g}$ and hence they satisfy Hörmander's condition

$$\operatorname{rank}\big(\operatorname{Lie}\{X_1, \dots, X_p\}(x)\big) = N, \qquad \forall\, x \in \mathbb{R}^N.$$

The simplest example of Carnot group is $(\mathbb{R}^N, +)$ (in this case the sub-Laplacian is the classical Laplace operator). The most simple non-abelian example is the Heisenberg group $\mathbb{H}^n$ (with the Kohn-Laplace operator). We refer (e.g.) to [5]-[4] for other examples and for a more detailed presentation of Carnot groups.

We remark that in literature a Carnot group (or *stratified group*) $\mathbb{H}$ is usually defined as a connected and simply connected Lie group whose Lie algebra $\mathfrak{h}$ admits a *stratification* $\mathfrak{h} = V_1 \oplus \cdots \oplus V_r$ with $[V_1, V_i] = V_{i+1}$, $[V_1, V_r] = \{0\}$. The two definitions are actually equivalent up to isomorphism (see [5]).

A noteworthy property of the operator $\Delta_{\mathbb{G}}$ is the structure of its fundamental solution Γ. Indeed $\Gamma(x, y) = [y^{-1} \circ x]^{-Q+2}$, where $[\cdot]$ is a homogeneous norm on $\mathbb{G}$, i.e. a continuous function from $\mathbb{R}^N$ to $[0, \infty[$, smooth away from the origin, such that $[\delta_\lambda(x)] = \lambda\,[x]$, $[x^{-1}] = [x]$, and $[x] = 0$ iff $x = 0$. Then

$$d_{\mathbb{G}}(x, y) = [y^{-1} \circ x], \quad x, y \in \mathbb{R}^N, \tag{50}$$

is a quasi-distance that endows $\mathbb{R}^N$ with a metric structure which is the natural one for $\Delta_{\mathbb{G}}$. Denoting by d the Carnot-Carathéodory control distance associated to the family of vector fields $\{X_1, \dots, X_p\}$, d and $d_{\mathbb{G}}$ are equivalent in the sense that

$$c^{-1}d(x, y) \le d_{\mathbb{G}}(x, y) \le cd(x, y), \qquad x, y \in \mathbb{G}, \tag{51}$$

holds for some positive constant c.

Let Ω be a smooth bounded domain of $\mathbb{G}$ and let us endow $\partial\Omega$ with the measure

$$d\sigma = \langle A\nu, \nu\rangle^{\frac{1}{2}} d\mathcal{H}_{N-1}, \tag{52}$$

where $\mathcal{H}_{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure, ν is the outer unit normal to $\partial\Omega$ and A is the matrix which allows to write $\Delta_{\mathbb{G}}$ in the divergence form $\Delta_{\mathbb{G}} = \operatorname{div}(A\nabla)$. Following [42] we shall say that Ω satisfies the uniform exterior $d_{\mathbb{G}}$-ball condition iff there exists $r_0 > 0$ such that for every $\xi \in \partial\Omega$ and for every $r \in]0, r_0]$ there exists $x \in \mathbb{G}$ such that $B_{d_{\mathbb{G}}}(x, r) \cap \Omega = \varnothing$ and $\xi \in \partial B_{d_{\mathbb{G}}}(x, r)$. The following result can be obtained using the same arguments as in [42].

THEOREM 50. *Let $\mathbb{G}$ be a Carnot group. Let Ω be a smooth bounded domain of $\mathbb{G}$, satisfying the uniform exterior $d_{\mathbb{G}}$-ball condition. Then the $\Delta_{\mathbb{G}}$-harmonic measures μ_x on $\partial\Omega$, $x \in \Omega$, are absolutely continuous with respect to σ, with densities $\frac{d\mu_x(\xi)}{d\sigma(\xi)} =: K(x, \xi)$ ($x \in \Omega$, $\xi \in \partial\Omega$) satisfying the estimate*

$$0 \le K(x, \xi) \le c\frac{d_{\mathbb{G}}(x, \partial\Omega)}{d_{\mathbb{G}}(x, \xi)^Q}, \qquad x \in \Omega, \xi \in \partial\Omega. \tag{53}$$

We emphasize that here K denotes the Poisson kernel of $\Delta_{\mathbb{G}}$ with respect to the measure σ *defined in* (52).

PROOF. We only give a sketch of the proof, which is essentially contained in [42]. Actually in that paper we were concerned with the special case where $\mathbb{G}$ is the Heisenberg group. However the arguments of [42] can be easily adapted to general Carnot groups. Moreover, in [42] we proved an estimate weaker than (53): precisely we showed that the $\Delta_{\mathbb{G}}$-Poisson kernel P *relative to* $\mathcal{H}_{N-1}$ is well defined and can be estimated as

$$0 \le P(x, \xi) \le c\langle A(\xi)\nu(\xi), \nu(\xi)\rangle^{\frac{1}{2}} d_{\mathbb{G}}(x, \xi)^{1-Q}, \qquad x \in \Omega, \xi \in \partial\Omega. \tag{54}$$

(We would like to take this opportunity for observing that in [42] we made an unnecessary assumption, namely we asked the set of the characteristic points of $\partial\Omega$ to have surface measure zero. However this is always true for general Hörmander type vector fields, see [19]-[20]). We now want to briefly show how we can improve the estimate (54) and get

$$0 \le P(x, \xi) \le c\langle A(\xi)\nu(\xi), \nu(\xi)\rangle^{\frac{1}{2}} \frac{d_{\mathbb{G}}(x, \partial\Omega)}{d_{\mathbb{G}}(x, \xi)^Q}, \qquad x \in \Omega, \xi \in \partial\Omega. \tag{55}$$

We refer to the proof of Theorem 3.6 of [42], where we showed that

$$0 < G(x, y) \le c d_{\mathbb{G}}(y, \partial\Omega) d_{\mathbb{G}}(x, y)^{1-Q}, \qquad x, y \in \Omega \tag{56}$$

(here G is the Green function of Ω). (56) was obtained using a comparison argument based on the maximum principle for $\Delta_{\mathbb{G}}$, starting from the estimate

$$0 < G(x, y) \le c d_{\mathbb{G}}(x, y)^{2-Q}. \tag{57}$$

Now, observing that $G(x, y) = G(y, x)$ and using the same comparison argument, but starting from (56) instead of (57), we can obtain

$$0 < G(x, y) \le c d_{\mathbb{G}}(x, \partial\Omega) d_{\mathbb{G}}(y, \partial\Omega) d_{\mathbb{G}}(x, y)^{-Q}.$$

From this last estimate, (55) follows in the same way as (54) followed from (56) in [42]. Recalling the definition (52) of σ this concludes the proof, since $P(x, \xi) = K(x, \xi)\langle A(\xi)\nu(\xi), \nu(\xi)\rangle^{\frac{1}{2}}$. □

REMARK 51. An estimate of the $\Delta_{\mathbb{G}}$-Poisson kernel K weaker then (53) has been announced by Capogna-Garofalo-Nhieu in [11], in the case of groups of Heisenberg type $\mathbb{G}$.

We are finally able to prove our generalization of Theorem (BN) to Carnot groups.

THEOREM 52. *Let $\mathbb{G}$ be a Carnot group. Let Ω be a smooth bounded domain of $\mathbb{G}$, satisfying the uniform exterior $d_{\mathbb{G}}$-ball condition and such that the doubling condition* (H$_6$) *holds for the measure σ on $\partial\Omega$, defined in* (52). *Then, for every $\varphi \in$* VMO($\partial\Omega$)*, the conclusions* (i)-(ii)-(iii) *of Theorem* 49 *hold.*

PROOF. From Theorem 50 and (51) it follows that K satisfies the estimate (41) (we recall that $|B_d(x, r)| = cr^Q$ in a Carnot group). Moreover, (42) holds also (see [11], Theorem 1.3; see also [16]). As a consequence we get (40) and we can apply Theorem 49. We explicitly remark that also (H$_7$) holds since Ω satisfies the uniform exterior $d_{\mathbb{G}}$-ball condition. □

REMARK 53. The doubling condition (H$_6$) for σ holds if $\partial\Omega$ has no characteristic points. Less restrictive sufficient conditions for (H$_6$) to hold are given in the recent papers [11] and [17]. In [11] it is announced that (H$_6$) holds if Ω, beside satisfying the uniform exterior $d_{\mathbb{G}}$-ball condition, is an *X-nontangentially accessible* domain. In [17] it is proved that (H$_6$) holds, for any smooth Ω, when $\mathbb{G}$ is a Carnot group of Step 2.

In order to better exemplify our result, we now want to write more explicitly the sub-Laplacian $\Delta_{\mathbb{G}}$ when $\mathbb{G}$ is a Carnot group of step 2. We shall assume that $x^{-1} = -x$ for every $x \in \mathbb{G}$ (this is not restrictive, up to isomorphism). Then the group law takes the form

$$\begin{aligned} x \circ y = \Big(& x_1^{(1)} + y_1^{(1)}, \dots, x_{N_1}^{(1)} + y_{N_1}^{(1)}, x_1^{(2)} + y_1^{(2)} + \langle B_1 x^{(1)}, y^{(1)}\rangle, \dots \\ & \dots, x_{N_2}^{(2)} + y_{N_2}^{(2)} + \langle B_{N_2} x^{(1)}, y^{(1)}\rangle\Big), \end{aligned}$$

where $B_1, \dots, B_{N_2}$ are skew-symmetric $N_1 \times N_1$ constant matrices. As a consequence the vector fields X_i are

$$X_i = \partial_{x_i^{(1)}} + \sum_{h=1}^{N_2} (B_h x^{(1)})_i \partial_{x_h^{(2)}}, \qquad i = 1, \dots, N_1,$$

and the sub-Laplacian $\Delta_{\mathbb{G}}$ is given by

$$\Delta_{\mathbb{G}} u = \Delta_{x^{(1)}} u + 2 \sum_{h=1}^{N_2} \langle B_h x^{(1)}, \nabla_{x^{(1)}} \partial_{x_h^{(2)}} u \rangle + \sum_{h,k=1}^{N_2} \langle B_h x^{(1)}, B_k x^{(1)} \rangle \partial^2_{x_h^{(2)}, x_k^{(2)}} u.$$

In the special case when $\mathbb{G} = \mathbb{H}^n$ is the Heisenberg group, we have $N_1 = 2n$, $N_2 = 1$ and the matrix B takes the form $B = 2 \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$.

REMARK 54. We would like to end the paper by pointing out some connections between VMO and Sobolev spaces. In the setting of Hörmander operators L, the Poincaré inequality proved by Jerison [30] (see also [33]) allows us to get in a standard way the embedding $W_L^{1,Q}(\Omega) \hookrightarrow \mathrm{VMO}(\Omega)$. Here Q denotes the *homogeneous dimension* related to L and $W_L^{1,Q}$ stands for the first order Sobolev space related to the $X = (X_1, \dots, X_p)$-gradient. The comparison between the notion of trace in VMO sense and the usual one for $W_L^{1,Q}(\Omega)$-functions is more delicate. However, if we assume the estimate

$$\sigma(B_d(x,r)) \gtrsim \frac{|B_d(x,r)|}{r}, \tag{58}$$

it is quite easy to show that the trace Besov space $B^Q_{1-\frac{1}{Q}}(\partial\Omega)$ introduced and studied in [16], [34] is embedded in our $\mathrm{VMO}(\partial\Omega)$. We would like to refer to the same papers [16], [34] for some comments on the condition (58).

REFERENCES

[1] H. BAHOURI – J. Y. CHEMIN – C. J. XU, *Trace and trace lifting theorems in weight Sobolev spaces*, preprint.

[2] S. BERHANU – I. PESENSON, *The trace problem for vector fields satisfying Hörmander's condition*, Math. Z. **231** (1999), 103-122.

[3] F. BETHUEL – H. BREZIS – F. HÉLEIN, *Ginzburg-Landau vortices*, In: "Progress in Nonlinear Differential Equations and their Applications" **13** Birkhäuser, Boston, 1994.

[4] A. BONFIGLIOLI – E. LANCONELLI – F. UGUZZONI, *Uniform Gaussian estimates of the fundamental solutions for heat operators on Carnot groups*, to appear in Adv. Differential Equations.

[5] A. Bonfiglioli – F. Uguzzoni, *Families of diffeomorphic sub-Laplacians and free Carnot groups*, preprint.

[6] J. M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier Grenoble **19** (1969), 277-304.

[7] H. Brezis – J. M. Coron, *Large solutions for harmonic maps in two dimensions*, Comm. Math. Phys. **92** (1983), 203-215.

[8] H. Brezis – L. Nirenberg, *Degree theory and BMO; Part I: compact manifolds without boundaries*, Selecta Math. **1** (1995), 197-263.

[9] H. Brezis – L. Nirenberg, *Degree theory and BMO; Part II: compact manifolds with boundaries*, Selecta Math. **2** (1996), 309-368.

[10] S. M. Buckley, *Inequalities of John-Nirenberg type in doubling spaces*, J. Anal. Math. **79** (1999), 215-240.

[11] L. Capogna – N. Garofalo – D. M. Nhieu, *A version of a theorem of Dahlberg for the subelliptic Dirichlet problem*, Math. Res. Lett. **5** (1998), 541-549.

[12] C. Castaing – M. Valadier, "Convex analysis and measurable multifunctions", Lecture Notes in Math. **580**, Springer-Verlag, New York, 1977.

[13] D. Christodoulou, *On the geometry and dynamics of crystalline continua*, Ann. Inst. H. Poincaré, Phys. Théor. **69** (1998), 335-358.

[14] R. Coifman – G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. **242**, Springer-Verlag, Berlin-New York, 1971.

[15] D. Danielli – N. Garofalo – D. M. Nhieu, *Trace inequalities for Carnot-Carathéodory spaces and applications*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **27** (1998), 195-252.

[16] D. Danielli – N. Garofalo – D. M. Nhieu, *Sub-elliptic Besov spaces and the characterization of traces on lower dimensional manifolds*, preprint.

[17] D. Danielli – N. Garofalo – D. M. Nhieu, *Non-doubling Ahlfors measures, perimeter measures, and the characterization of the trace spaces of Sobolev functions in Carnot-Carathéodory spaces*, preprint.

[18] H. A. De Kleine – J. E. Girolo, *A degree theory for almost continuous functions*, Fund. Math. **101** (1978), 39-52.

[19] M. Derridj, *Un problème aux limites pour une classe d'opérateurs du second ordre hypoelliptiques*, Ann. Inst. Fourier Grenoble **21** (1971), 99-148.

[20] M. Derridj, *Sur un théorème de traces*, Ann. Inst. Fourier Grenoble **22** (1972), 73-83.

[21] M. J. Esteban – S. Müller, *Sobolev maps with integer degree and applications to Skyrme's problem*, Proc. Roy. Soc. London Ser. A **436** (1992), 197-201.

[22] C. Fefferman – D. H. Phong, *Subelliptic eigenvalue problems*, In: "Conference on harmonic analysis in honor of Antoni Zygmund", Vol. I, II, Wadsworth, 1983, 590-606.

[23] B. Franchi, *Trace theorems for anisotropic weighted Sobolev spaces in a corner*, Math. Nachr. **127** (1986), 25-50.

[24] B. Franchi – R. Serapioni – F. Serra Cassano, *Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields*, Boll. Un. Mat. Ital. **11**-B (1997), 83-117.

[25] L. Gallardo, "Capacités, mouvement Brownien et problème de l'épine de Lebesgue sur les groupes de Lie nilpotents", Lecture Notes in Math. **928**, Springer, Berlin-New York, 1982, 96-120.

[26] N. Garofalo – D. M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Commun. Pure Appl. Math. **49** (1996), 1081-1144.

[27] M. Giaquinta – G. Modica, – J. Soucek, *Remarks on the degree theory*, J. Funct. Anal. **125** (1994), 172-200.

[28] L. Greco – T. Iwaniec – C. Sbordone – B. Stroffolini, *Degree formulas for maps with nonintegrable Jacobian*, Topol. Methods Nonlinear Anal. **6** (1995), 81-95.

[29] P. Hajlasz – P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688.

[30] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), 503-523.

[31] D. Jerison – A. Sánchez-Calle, *Subelliptic, second order differential operators*, In: "Complex analysis III", Lecture Notes in Math. **1277** Springer, Berlin, 1987, 46-77.

[32] E. Lanconelli – A. E. Kogoj, *X-elliptic operators and X-control distances*, Ricerche Mat. **49** (2000), 223-243.

[33] E. Lanconelli – D. Morbidelli, *On the Poincaré inequality for vector fields*, Ark. Mat. **38** (2000), 327-342.

[34] R. Monti – D. Morbidelli, *Trace theorems for vector fields*, to appear in Math. Z.

[35] R. Monti – F. Serra Cassano, *Surface measures in Carnot-Carathéodory spaces*, preprint.

[36] D. Morbidelli, *Fractional Sobolev norms and structure of Carnot-Carathéodory balls for Hörmander vector fields*, Studia Math. **139** (2000), 213-244.

[37] U. Mosco, *Alcuni aspetti variazionali dei mezzi discontinui*, Boll. Un. Mat. Ital. (7) **7**-A (1993), 149-198.

[38] A. Nagel – E. M. Stein – S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math. **155** (1985), 103-147.

[39] P. Negrini – V. Scornazzani, *Wiener criterion for a class of degenerate elliptic operators*, J. Differential Equations **66** (1987), 151-164.

[40] L. Nirenberg, "Topics in nonlinear functional analysis", Courant Institute Lecture Notes, New York, 1974.

[41] A. Sánchez-Calle, *Fundamental solutions and geometry of the sum of squares of vector fields*, Invent. Math. **78** (1984), 143-160.

[42] F. Uguzzoni – E. Lanconelli, *On the Poisson kernel for the Kohn Laplacian*, Rend. Mat. Appl. **17** (1997), 659-677.

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