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H^∞ Functional Calculus for an Elliptic Operator on a Half-Space with General Boundary Conditions

GIOVANNI DORE – ALBERTO VENNI

Abstract. Let A be the L^p realization ($1 < p < \infty$) of a differential operator $P(D_x, D_t)$ on $\mathbb{R}^n \times \mathbb{R}^+$ with general boundary conditions $B_k(D_x, D_t)u(x, 0) = 0$ ($1 \leq k \leq m$). Here P is a homogeneous polynomial of order $2m$ in $n+1$ complex variables that satisfies a suitable ellipticity condition, and for $1 \leq k \leq m$ B_k is a homogeneous polynomial of order $m_k < 2m$; it is assumed that the usual complementing condition is satisfied. We prove that A is a sectorial operator with a bounded H^∞ functional calculus.

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1. – Introduction

When A is a sectorial operator in a complex Banach space X, with spectrum contained in some closed sector $\overline{S_\omega}$ of the complex plane, the H^∞ functional calculus for A is a function $f \mapsto f(A)$ from the algebra of the bounded holomorphic functions on some open sector $S_{\omega+\varepsilon}$ to the set of the closed operators acting in X, which has some reasonable algebraic properties; if $f(A)$ is a bounded operator for any bounded holomorphic function f on $S_{\omega+\varepsilon}$ then we say that on that sector the H^∞ functional calculus is bounded. As it is well-known, the boundedness (on some sector) of the H^∞ functional calculus for a sectorial operator is a stronger property than the boundedness of its imaginary powers; on its turn the boundedness of the imaginary powers of A has important consequences concerning the domain of A^r with $r \in]0, 1[$ (see [30], Theorem 1.15.3) and the maximal L^p-regularity for the Cauchy problem

$$(1.1) \qquad \begin{cases} u' + Au = f \\ u(0) = 0 \end{cases}$$

(see [7], [16], [22]).

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In some cases it seems more convenient to study the H^∞ functional calculus instead of the imaginary powers, also because the technical difficulties that one has to overcome to prove the weaker result are essentially the same that ensure the stronger one: in the L^p setting for a differential operator one has to apply in both cases some kind of Fourier multiplier theorem.

Several results on the boundedness of the imaginary powers and on the boundedness of the H^∞ functional calculus for elliptic operators in L^p can be found in the existing literature. Among the oldest papers on this subject there are [24], [25], [26] that go back to the late '60s; in these papers the boundedness of the imaginary powers is proved for elliptic systems with C^∞ coefficients on a compact manifold without boundary or on a bounded C^∞ domain. In the same framework the more recent paper [10] proves the boundedness of the H^∞ functional calculus.

A certain number of papers deals with the case of second order operators with boundary conditions of various types. Without aiming at completeness, we quote [4], [11], [12], [13], [21], [23], [27]. For the case of operators (or systems of operators) of arbitrary order on the whole space we quote the papers [3], [14], in which the boundedness of the H^∞ functional calculus is proved with minimal assumptions on the regularity of the coefficients.

The aim of this paper is to prove that if we call A the realization in L^p ($1 < p < \infty$) of an elliptic operator $P(D_x, D_t)$ of order $2m$ on a half-space ($x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$), with constant coefficients and top order terms only, under general boundary conditions of the type $B_k(D_x, D_t)u = 0$, then A is a sectorial operator and has a bounded H^∞ functional calculus. The ellipticity requirement on P is the following: if $(0, 0) \neq (x, t) \in \mathbb{R}^{n+1}$, then $P(ix, it) \notin \mathbb{R}^- \cup \{0\}$, and the boundary operators are expressed by m homogeneous polynomials of degree $< 2m$ that satisfy the usual complementing condition with respect to P.

The techniques that we use to obtain our result consist in studying the ordinary differential operators A_z on $\mathbb{R}^+$ that one gets by replacing (both in P and in B_k) the operators $D_{x_1}, \dots, D_{x_n}$ with complex parameters $z_1, \dots, z_n$ belonging to a suitable conical neighbourhood of $(i\,\mathbb{R})^n$, that is, to a conical neighbourhood of the cartesian product of the spectra of the operators $D_{x_1}, \dots, D_{x_n}$. The study is carried out by using standard tools that rely on integration on circuits embracing zeros of polynomials like $\mu - P(z, \cdot)$. This part of the paper requires some results on "elliptic" polynomials that we expose in detail, even if they are essentially known (see, e.g., [1], [2], [28]), in order to have a reference fitting precisely our needs. By means of suitable estimates, we obtain the boundedness in $\mathcal{L}(L^p(\mathbb{R}^+))$ of the H^∞ functional calculus for the operators A_z. Here we have to estimate two integral operators, one of convolution type, and the other containing boundary terms: we use a Mihlin type theorem for the former and an estimate concerning the Hilbert kernel for the latter. However we prove something more, i.e. that the function $z \mapsto h(A_z)$ is holomorphic and R-bounded for any function h bounded and holomorphic on some sector containing the spectrum of A_z. That allows us to use a recent result of N. Kalton and L. Weis [19] in order to replace z with $D_x = (D_{x_1}, \dots, D_{x_n})$ in

$h(A_z)$, obtaining a bounded operator. Formally that should imply the boundedness of $h(A)$; however we first have to prove that A is sectorial and to give a representation formula for the resolvents of A.

The paper is organized as follows. In Section 2 we fix some notations and we state our main result. In Section 3 we collect several auxiliary propositions concerning polynomials and ordinary differential equations. In Section 4 we give some information about the notion of R-boundedness for a set of bounded linear operators, and about H^∞ functional calculus. In Section 5 we obtain a number of preliminary results concerning elliptic polynomials and boundary operators. In Section 6 we study the operators A_z: we show that they are sectorial, and we prove properties of analyticity and R-boundedness with respect to z of the resolvent operators $(\mu - A_z)^{-1}$; we also prove that the operators A_z have a bounded H^∞ functional calculus, which moreover is R-bounded with respect to z. In Section 7 we prove that the derivative operators $D_{x_1}, \dots , D_{x_n}$ have a joint bounded H^∞ functional calculus. Finally in Section 8 we prove our main result, showing that A is a sectorial operator with a bounded H^∞ functional calculus. From our results it follows easily that the usual *a priori* estimate holds for the elliptic operator with vanishing boundary conditions.

2. – Notations. Statement of the main result

We establish some notational conventions, that we keep for the whole paper.

(I) The symbol of a norm, $\|\cdot\|$, may have different meanings: when convenient, we shall use suitable indices to distinguish between them. In particular, if $z = (z_1, \dots , z_r) \in \mathbb{C}^r$, $\|z\|$ will always denote the euclidean norm of z, i.e. $\left(\sum_{k=1}^r |z_k|^2\right)^{1/2}$. In every Banach space, $B(x, r)$ and $\overline{B}(x, r)$ denote, respectively, the open and closed ball centred at x, with radius r.

(II) When X and Y are Banach spaces, $\mathcal{L}(X, Y)$ denotes the Banach space of the bounded linear operators on X to Y, and $\mathcal{L}(X) := \mathcal{L}(X, X)$. I_X denotes the identity operator on X.

(III) When T is a linear operator, its domain and range are denoted by $\mathcal{D}(T)$ and $\mathcal{R}(T)$, respectively. When X is a complex Banach space and T is a linear operator acting in X (i.e. when $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are vector subspaces of X), $\sigma(T)$ and $\rho(T)$ denote, as usual, the spectrum and the resolvent set of T, respectively.

(IV) The function "principal argument" denoted by "arg" is meant to have $\mathbb{C}\setminus\,]-\infty, 0]$ as domain and $]-\pi, \pi[$ as range.

(V) When γ is a *circuit* (that is a finite family of piecewise C^1 oriented closed curves) in $\mathbb{C}$, and $a \in \mathbb{C} \setminus \gamma$, the *winding number* of γ with respect to a is defined by

$$w(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a}\, dz.$$

(VI) $\forall \theta \in]0, \pi[$ we set

$$S_\theta = \{\rho\, e^{i\alpha};\, \rho \in \mathbb{R}^+,\ \alpha \in]-\theta, \theta[\,\} = \{z \in \mathbb{C} \setminus]-\infty, 0];\, |\arg z| < \theta\}$$

and $\forall \beta \in]0, \frac{\pi}{2}[$ we set

$$\begin{aligned} \Sigma_\beta &= S_{\beta+\frac{\pi}{2}} \cap (-S_{\beta+\frac{\pi}{2}}) = (i\, S_\beta) \cup (-i\, S_\beta) \\ &= \left\{\rho\, e^{i\alpha};\, \rho \in \mathbb{R} \setminus \{0\},\ \alpha \in \left]\frac{\pi}{2} - \beta, \frac{\pi}{2} + \beta\right[\right\}. \end{aligned}$$

Thus S_θ is the (open) sector around $\mathbb{R}^+$, with opening angle equal to 2θ, and Σ_β is the (open) "double-sector" around $i\,\mathbb{R}$ with opening angle equal to 2β. When $\overline{S_0}$ is mentioned (as it happens e.g. in Definition 5.1) it is understood that it equals $\mathbb{R}^+ \cup \{0\}$.

(VII) Derivatives are always meant in the distribution sense. We use the symbol D for the derivative of a function of one variable. In most cases, however, we are dealing with functions of $n+1$ variables, and we denote these variables by means of (x, t), with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$; in this case $D_1, \dots, D_n$ denote the derivative operators with respect to $x_1, \dots, x_n$, and D_t the derivative operator with respect to t. In Section 7 and 8 we also use the notation D_x for $(D_1, \dots, D_n)$.
In many cases these derivative operators will be considered as unbounded operators in some function Banach space, but we will not introduce any special notation to emphasize this fact.

(VIII) When Ω is an open subset of $\mathbb{R}^N$, r is a positive integer and $q \in [1, \infty]$, we denote by $W^{r,q}(\Omega)$ the Banach space (with the natural norm) of the functions $u \in L^q(\Omega)$ whose distributional derivatives up to the order r belong to $L^q(\Omega)$.

(IX) The symbol $\mathcal{F}$ denotes the Fourier transformation, formally defined by $(\mathcal{F} f)(\xi) = \int_{\mathbb{R}^N} e^{-i\langle x, \xi\rangle} f(x)\, dx$.

After fixing these notations, we can state in a more precise form the result that we have obtained and the techniques that we used.

Let P be a homogeneous polynomial of degree $2m$ in $n+1$ variables, with complex coefficients. We shall emphasize the last variable by writing the values of P in the form $P(z, \lambda)$, with $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. We assume that the polynomial P satisfies the following ellipticity condition:

$$\text{if } x \in \mathbb{R}^n,\ t \in \mathbb{R} \text{ and } (x, t) \neq (0, 0), \text{ then } P(ix, it) \notin]-\infty, 0].$$

As we shall see in Section 5, this condition implies that there exists $\omega \in [0, \pi[$ such that if $\mu \in (\mathbb{C} \setminus \overline{S_\omega}) \cup \{0\}$, $x \in \mathbb{R}^n$ and $(x, \mu) \neq (0, 0)$, then $\forall \lambda \in \mathbb{C}$

$$\mu - P(ix, \lambda) = P^+_{ix,\mu}(\lambda)\, P^-_{ix,\mu}(\lambda)$$

where the polynomials $P^\pm_{ix,\mu}$ have degree m, and moreover all the roots of $P^+_{ix,\mu}$ have positive real part and all the roots of $P^-_{ix,\mu}$ have negative real part.

The boundary conditions for the elliptic operator will be given by means of a family $B_1, \dots, B_m$ of polynomials in $n+1$ variables. It is assumed that each B_k is a homogeneous polynomial of degree $m_k < 2m$, and moreover that if $x \in \mathbb{R}^n$, $\mu \in (\mathbb{C} \setminus \overline{S_\omega}) \cup \{0\}$ and $(x, \mu) \neq (0, 0)$, then the polynomials (in one variable) $B_1(ix, \cdot), \dots, B_m(ix, \cdot)$ are linearly independent modulo $P^-_{ix,\mu}$.

Finally, we fix $p \in]1, \infty[$ and call A the realization in $L^p(\mathbb{R}^n \times \mathbb{R}^+)$ of the differential operator $P(D_x, D_t)$, obtained by taking as domain of A the space of the functions $u \in W^{2m,p}(\mathbb{R}^n \times \mathbb{R}^+)$ satisfying the boundary conditions $B_k(D_x, D_t)u|_{t=0} = 0$. Our main result is

Theorem 2.1. *A is a sectorial operator with spectral angle ω and has a bounded H^∞ functional calculus on the sector S_θ, $\forall \theta \in]\omega, \pi[$.*

This means that: (i) A has dense domain and range, (ii) $\sigma(A) \subseteq \overline{S_\omega}$, (iii) $\forall \varepsilon \in]0, \pi - \omega[$ $\lambda(\lambda - A)^{-1}$ is bounded outside of $\overline{S_{\omega+\varepsilon}}$, (iv) for any complex valued bounded holomorphic function f on S_θ one can define $f(A)$ as a bounded linear operator on $L^p(\mathbb{R}^n \times \mathbb{R}^+)$ (see Subsection 4.2 for more details).

3. – Polynomials and ordinary differential equations

We denote by $\mathcal{P}$ the algebra of the polynomials in one argument, with complex coefficients.

Let $Q, Q_1, \dots, Q_r \in \mathcal{P}$. We say that $Q_1, \dots, Q_r$ are linearly dependent modulo Q or linearly independent modulo Q according that the equivalence classes of $Q_1, \dots, Q_r$ in the quotient algebra of $\mathcal{P}$ modulo the ideal generated by Q are linearly dependent or independent; this means that there does or does not exist a linear combination of $Q_1, \dots, Q_r$ with coefficients not all equal to 0 which belongs to the ideal generated by Q, i.e. to $\{QR;\, R \in \mathcal{P}\}$.

Definition 3.1. $\forall Q \in \mathcal{P} \setminus \{0\}$ and for $k \in \{0, 1\}$ we denote by $\Lambda_k(Q)$ the set of the circuits γ in $\mathbb{C}$ such that $Q(a) = 0 \Rightarrow \Big(a \notin \gamma,\ w(\gamma, a) = k\Big)$.

Remark 3.2. Let γ be a circuit, and f a meromorphic function on $\mathbb{C}$ such that for any singular point b of f we have $b \notin \gamma$ and $w(\gamma, b) = 0$. Let $Q \in \mathcal{P} \setminus \{0\}$. The following two statements are straightforward consequences of the residue theorem and will be used in the sequel without any further reference.

(a) If $\gamma \in \Lambda_1(Q)$, then $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{Q(z)}\,dz$ is the sum of the residues of f/Q at the roots of Q, and therefore does not depend on $\gamma \in \Lambda_1(Q)$.

(b) If $\gamma \in \Lambda_0(Q)$, then $\int_\gamma \frac{f(z)}{Q(z)}\,dz = 0$. □

LEMMA 3.3. *Let* $P_0,\ P_1 \in \mathcal{P} \setminus \{0\}$, *with* $\deg P_1 = q$, *and let* $\gamma \in \Lambda_0(P_0) \cap \Lambda_1(P_1)$. *Then* $\forall\, Q \in \mathcal{P}$ *the following statements are equivalent:*

(a) $\forall R \in \mathcal{P}$ *with* $\deg R \le q-1$, $\int_\gamma \frac{R\,Q}{P_0\,P_1} = 0$

(b) $\forall R \in \mathcal{P}$ $\int_\gamma \frac{R\,Q}{P_0\,P_1} = 0$

(c) *Q belongs to the ideal of $\mathcal{P}$ generated by P_1.*

PROOF. If $q = 0$, then P_1 is constant, so that (a) and (b) hold trivially; moreover in this case the ideal of $\mathcal{P}$ generated by P_1 is $\mathcal{P}$, so that also (c) holds. Hence we assume $q \ge 1$.

(a) $\Rightarrow$ (b) Let $R \in \mathcal{P}$. Then $R = SP_1 + T$, with $S,\ T \in \mathcal{P}$ and $\deg T \le q-1$. Therefore

$$\int_\gamma \frac{RQ}{P_0 P_1} = \int_\gamma \frac{SQ}{P_0} + \int_\gamma \frac{TQ}{P_0 P_1} = 0.$$

(b) $\Rightarrow$ (c) Let a be a root of P_1, with multiplicity r. If $h \in \{1, \dots, r\}$, then $P_1(\lambda) = (\lambda - a)^h\, R(\lambda)$ for a suitable polynomial R. Hence

$$0 = \int_\gamma \frac{R\,P_0\,Q}{P_0 P_1} = \int_\gamma \frac{Q(\lambda)}{(\lambda - a)^h}\,d\lambda = \frac{2\pi i\, w(\gamma, a)}{(h-1)!}\, Q^{(h-1)}(a),$$

so that $Q^{(h-1)}(a) = 0$. Therefore a is also a root of Q, with multiplicity $\ge r$. This proves the existence of a polynomial S such that $Q = S\,P_1$.

(c) $\Rightarrow$ (a) We have $Q = S\,P_1$, with $S \in \mathcal{P}$; hence $\forall R \in \mathcal{P}$ with $\deg R \le q-1$ we get $\int_\gamma \frac{R\,Q}{P_0 P_1} = \int_\gamma \frac{RS}{P_0} = 0$. □

LEMMA 3.4. *Let* $P_0,\ P_1 \in \mathcal{P} \setminus \{0\}$, $\gamma \in \Lambda_0(P_0) \cap \Lambda_1(P_1)$, $Q_1, \dots, Q_r \in \mathcal{P}$. *We call Σ the $r \times r$ matrix whose entries are*

$$\sigma_{k,j} = \frac{1}{2\pi i}\int_\gamma \frac{\lambda^{j-1}\, Q_k(\lambda)}{P_0(\lambda)\, P_1(\lambda)}\,d\lambda.$$

Then a sufficient condition for $Q_1, \dots, Q_r$ to be linearly independent modulo P_1 is that $\det \Sigma \ne 0$. *If moreover* $\deg P_1 \le r$, *then this condition is also necessary.*

PROOF. Let $c_1, \dots, c_r \in \mathbb{C}$ such that $\sum_{k=1}^r c_k Q_k = RP_1$ for some $R \in \mathcal{P}$. Then $\forall j$

$$\sum_{k=1}^r c_k \sigma_{k,j} = \sum_{k=1}^r \frac{c_k}{2\pi i}\int_\gamma \frac{\lambda^{j-1}\, Q_k(\lambda)}{P_0(\lambda)\, P_1(\lambda)}\,d\lambda = \frac{1}{2\pi i}\int_\gamma \frac{\lambda^{j-1}\, R(\lambda)}{P_0(\lambda)}\,d\lambda = 0$$

and this proves the first part of the lemma.

Suppose that $\det \Sigma = 0$. Then there exists a non-zero r-tuple $(c_1, \ldots, c_r) \in \mathbb{C}^r$ such that, with $Q := \sum_{k=1}^r c_k Q_k$, we have $\forall j \in \{1, \ldots, r\}$

$$0 = \sum_{k=1}^{r} c_k \sigma_{k,j} = \frac{1}{2\pi i} \int_\gamma \frac{\lambda^{j-1} Q(\lambda)}{P_0(\lambda) P_1(\lambda)} d\lambda,$$

whence $\int_\gamma \frac{R\,Q}{P_0\,P_1} = 0 \;\forall R \in \mathcal{P}$ with $\deg R \leq r-1$. If in particular $r \geq \deg P_1 =: q$, then this equality holds $\forall R \in \mathcal{P}$ with $\deg R \leq q-1$, and so Q satisfies condition (a) of Lemma 3.3. Therefore Q belongs to the ideal of $\mathcal{P}$ generated by P_1, so that $Q_1, \ldots, Q_r$ are linearly dependent modulo P_1. □

In the sequel, we will be concerned with polynomials (in one variable) with no roots on the imaginary axis. As we shall see in Theorem 3.7 the roots with negative real part will be interesting, and those with positive real part will not.

As usual, $Q \mapsto Q(D)$ denotes the natural isomorphism between the algebra $\mathcal{P}$ of the polynomials in one variable and the algebra of the linear differential operators on $\mathbb{R}$ (or on some open interval of $\mathbb{R}$) with constant coefficients.

THEOREM 3.5. *Let* $P_0,\ P_1 \in \mathcal{P} \setminus \{0\}$, $\deg P_1 = m$, $\gamma \in \Lambda_0(P_0) \cap \Lambda_1(P_1)$. *The following statements hold:*

(a) $\forall Q \in \mathcal{P}$, *the function* $\mathbb{R} \ni t \mapsto \int_\gamma \frac{Q(\lambda)\, e^{\lambda t}}{P_0(\lambda)\, P_1(\lambda)} d\lambda$ *is a solution of the differential equation* $P_1(D)u = 0$*;*

(b) *if we set* $u_j(t) = \frac{1}{2\pi i} \int_\gamma \frac{\lambda^{j-1} e^{\lambda t}}{P_0(\lambda) P_1(\lambda)} d\lambda$, *then* $\{u_1, \ldots, u_m\}$ *is a basis of* $\ker P_1(D)$*;*

(c) *if* $Q_1,\ldots,Q_m$ *are polynomials linearly independent modulo* P_1, *and* $(b_1,\ldots,b_m) \in \mathbb{C}^m$, *then there is a unique* $u \in \ker P_1(D)$ *such that* $(Q_k(D)u)(0) = b_k$ *for* $1 \leq k \leq m$.

PROOF.

(a) Let $Q \in \mathcal{P}$ and $u(t) = \int_\gamma \frac{Q(\lambda)\, e^{\lambda t}}{P_0(\lambda)\, P_1(\lambda)} d\lambda$. Since it is obvious that we can differentiate with respect to t within the integral, we get $(P_1(D)u)(t) = \int_\gamma \frac{Q(\lambda)\, e^{\lambda t}}{P_0(\lambda)} d\lambda = 0$.

(b) Since the dimension of $\ker P_1(D)$ equals m, we have only to show that $u_1, \ldots, u_m$ are linearly independent. Let $(c_1, \ldots, c_m) \in \mathbb{C}^m$, with $\sum_{k=1}^m c_k u_k = 0$. We set $Q(\lambda) = \sum_{k=1}^m \frac{c_k}{2\pi i} \lambda^{k-1}$. Then $\forall R \in \mathcal{P}$

$$0 = \left(R(D) \sum_{k=1}^{m} c_k u_k \right)(0) = \sum_{k=1}^{m} c_k \,(R(D) u_k)(0)$$

$$= \sum_{k=1}^{m} \frac{c_k}{2\pi i} \int_\gamma \frac{\lambda^{k-1} R(\lambda)}{P_0(\lambda)\, P_1(\lambda)} d\lambda = \int_\gamma \frac{QR}{P_0\, P_1}.$$

It follows from Lemma 3.3 that Q belongs to the ideal generated by P_1, which is possible only if $Q = 0$ (that is $c_k = 0\ \forall k$), since $\deg Q < m = \deg P_1$.

(c) Let $u = \sum_{j=1}^{m} c_j\, u_j$. Then for $1 \le k \le m$

$$
\begin{aligned}
(Q_k(D)u)(0) &= \sum_{j=1}^{m} c_j\, (Q_k(D)u_j)(0) = \sum_{j=1}^{m} \frac{c_j}{2\pi i} \int_\gamma \frac{\lambda^{j-1}\, Q_k(\lambda)}{P_0(\lambda)\, P_1(\lambda)}\, d\lambda \\
&= \sum_{j=1}^{m} c_j\, \sigma_{k,j},
\end{aligned}
$$

where $\sigma_{k,j}$ is the same as in Lemma 3.4. By the same lemma we have $\det \Sigma \ne 0$, so that the system $\sum_{j=1}^{m} c_j\, \sigma_{k,j} = b_k$ $(1 \le k \le m)$ has a unique solution. □

LEMMA 3.6. *Let $\lambda_1, \dots, \lambda_r$ be complex numbers such that $\lambda_h \ne \lambda_k$ for $h \ne k$ and $\mathrm{Re}\, \lambda_k \ge 0\ \forall k$. Let $Q_1, \dots, Q_r$ be polynomials, and suppose that the function $t \mapsto \sum_{k=1}^{r} Q_k(t)\, e^{\lambda_k t}$ belongs to $L^p(\mathbb{R}^+)$ for some $p \in [1, \infty[$. Then $Q_k = 0\ \forall k$.*

PROOF. We perform the proof in three steps.

(I) Let $c_1, \dots, c_r$ be complex numbers, let $\alpha_1, \dots, \alpha_r$ be real numbers such that $\alpha_h \ne \alpha_k$ for $h \ne k$, and assume that $t \mapsto \sum_{k=1}^{r} c_k\, t^{-1} e^{i\alpha_k t}$ belongs to $L^1(1, \infty)$. Then $c_k = 0\ \forall k$. Indeed let us assume that, say, $c_r \ne 0$, and let us set $f(t) = e^{-i\alpha_r t} \sum_{k=1}^{r} c_k\, t^{-1} e^{i\alpha_k t}$. Then $f \in L^1(1, \infty)$, so that $\int_1^\infty e^{-\varepsilon t}\, f(t)\, dt \underset{\varepsilon \to 0}{\to} \int_1^\infty f(t)\, dt$. However

$$
\begin{aligned}
\int_1^\infty e^{-\varepsilon t}\, f(t)\, dt &= \sum_{k=1}^{r} c_k \int_1^\infty t^{-1}\, e^{-\varepsilon t + i(\alpha_k - \alpha_r)t}\, dt = c_r \int_1^\infty t^{-1}\, e^{-\varepsilon t}\, dt \\
&\quad + \sum_{k=1}^{r-1} c_k \left(\frac{e^{-\varepsilon + i(\alpha_k - \alpha_r)}}{\varepsilon - i(\alpha_k - \alpha_r)} - \frac{1}{\varepsilon - i(\alpha_k - \alpha_r)} \int_1^\infty t^{-2}\, e^{-\varepsilon t + i(\alpha_k - \alpha_r)t}\, dt \right)
\end{aligned}
$$

which does not converge as $\varepsilon \to 0$.

(II) Let $Q_1, \dots, Q_r$ be polynomials, and let $\alpha_1, \dots, \alpha_r$ be as in step (I). Assume that $t \mapsto \sum_{k=1}^{r} Q_k(t) e^{i\alpha_k t}$ belongs to $L^p(\mathbb{R}^+)$ for some $p \in [1, \infty[$. Then $Q_k = 0\ \forall k$. Indeed if this is not the case, setting $s := \max_{1 \le k \le r} \deg Q_k$, since $t \mapsto t^{-s-1}$ belongs to $L^{p'}(1, \infty)$ (where p' is the exponent conjugate to p), the function $t \mapsto t^{-s-1} \sum_{k=1}^{r} Q_k(t) e^{i\alpha_k t}$ belongs to $L^1(1, \infty)$, and so we obtain a contradiction with step (I).

(III) In the situation of the statement, assume that some Q_k is $\neq 0$. Then we can suppose that $Q_k \neq 0$ $\forall k$. We set $\rho = \max_{1\leq k\leq q} Re\,\lambda_k$. As the function $t \mapsto e^{-\rho t}$ belongs to $L^\infty(\mathbb{R}^+)$, also the function

$$t \mapsto \sum_{k=1}^{q} Q_k(t)\, e^{(\lambda_k-\rho)t} = \sum_{Re\,\lambda_k<\rho} Q_k(t)\, e^{(\lambda_k-\rho)t} + \sum_{Re\,\lambda_k=\rho} Q_k(t)\, e^{(\lambda_k-\rho)t}$$

belongs to $L^p(\mathbb{R}^+)$. Here $t \mapsto \sum_{Re\,\lambda_k<\rho} Q_k(t)\, e^{(\lambda_k-\rho)t}$ belongs to $L^p(\mathbb{R}^+)$, and so also the function $t \mapsto \sum_{Re\,\lambda_k=\rho} Q_k(t)\, e^{(\lambda_k-\rho)t}$ belongs to $L^p(\mathbb{R}^+)$. Remark that in the last sum there is at least one summand, that all the polynomials that appear in it are $\neq 0$ and that $\lambda_k - \rho = i r_k$, with $r_k \in \mathbb{R}$ and $r_h \neq r_k$ for $h \neq k$. Hence we have contradicted step (II). □

THEOREM 3.7. *Let $P_+,\ P_- \in \mathcal{P}\setminus\{0\}$, and assume that all the roots of P_+ have non-negative real part and all the roots of P_- have negative real part. Then* $\ker(P_+\,P_-)(D) = \ker P_+(D) \oplus \ker P_-(D)$, *and* $\forall p \in [1,\infty[$ *we have*

$$\ker P_-(D) \subseteq \bigcap_{n\in\mathbb{N}} W^{n,p}(\mathbb{R}^+), \qquad \ker P_+(D) \cap L^p(\mathbb{R}^+) = \{0\}.$$

PROOF. It is well known that $\forall Q \in \mathcal{P}\setminus\{0\}$ $\dim\ker Q(D) = \deg Q$. Since it is obvious that $\ker P_+(D)$ and $\ker P_-(D)$ are linear subspaces of $\ker(P_+\,P_-)(D)$, in order to prove that $\ker(P_+\,P_-)(D) = \ker P_+(D) \oplus \ker P_-(D)$ it is sufficient to show that $\ker P_+(D) \cap \ker P_-(D) = \{0\}$, and this follows from the two formulas of the last line of the statement.

By Theorem 3.5 (b) any $u \in \ker P_-(D)$ is of the type

$$u(t) = \frac{1}{2\pi i}\int_\gamma \frac{Q(\lambda)\, e^{\lambda t}}{P_+(\lambda)\, P_-(\lambda)}\, d\lambda,$$

with $Q \in \mathcal{P}$, $\deg Q < \deg P_-$ and $\gamma \in \Lambda_1(P_-) \cap \Lambda_0(P_+)$. We can take γ in such a way that $\max_{\lambda\in\gamma} Re\,\lambda = -M < 0$; then for $t \geq 0$ and $\forall n \in \mathbb{N}$

$$|u^{(n)}(t)| \leq \frac{1}{2\pi}\int_\gamma \left|\frac{\lambda^n\, Q(\lambda)}{P_+(\lambda)\, P_-(\lambda)}\right| d|\lambda|\, e^{-Mt},$$

and this proves that $u^{(n)} \in L^p(\mathbb{R}^+)$. Therefore $u \in \bigcap_{n\in\mathbb{N}} W^{n,p}(\mathbb{R}^+)$.

Likewise, any $u \in \ker P_+(D)$ is of the type $u(t) = \frac{1}{2\pi i}\int_\gamma \frac{Q(\lambda)\, e^{\lambda t}}{P_+(\lambda)\, P_-(\lambda)}\, d\lambda$ with $Q \in \mathcal{P}$, $\deg Q < \deg P_+$ and $\gamma \in \Lambda_1(P_+) \cap \Lambda_0(P_-)$. Let $a_1, \ldots, a_r$ be the roots of P_+ (with $a_j \neq a_k$ for $j \neq k$), and let m_k be the multiplicity of a_k. Then the residue theorem yields

$$\begin{aligned}
u(t) &= \sum_{k=1}^{r} \frac{1}{(m_k-1)!}\,\frac{d^{m_k-1}}{d\lambda^{m_k-1}}\left(\frac{Q(\lambda)\,(\lambda-a_k)^{m_k}}{P_+(\lambda)\, P_-(\lambda)}\, e^{\lambda t}\right)\Big|_{\lambda=a_k} \\
&= \sum_{k=1}^{r} \frac{1}{(m_k-1)!} \sum_{h=0}^{m_k-1} \binom{m_k-1}{h} \frac{d^{m_k-1-h}}{d\lambda^{m_k-1-h}}\left(\frac{Q(\lambda)\,(\lambda-a_k)^{m_k}}{P_+(\lambda)\, P_-(\lambda)}\right)\Big|_{\lambda=a_k} t^h\, e^{a_k t} \\
&= \sum_{k=1}^{r} Q_k(t)\, e^{a_k t}
\end{aligned}$$

for suitable polynomials $Q_1, \dots, Q_r$. Now, if $u \in L^p(\mathbb{R}^+)$, from Lemma 3.6 it follows that $u = 0$. □

Summing up, we have

THEOREM 3.8. *Let P_-, P_+ be as in Theorem 3.7, and let $m = \deg P_-$, $r = \deg P_+ + \deg P_-$. Let $Q_1, \dots, Q_m$ be polynomials linearly independent modulo P_-. Let $p \in [1, \infty[$ and $(b_1, \dots, b_m) \in \mathbb{C}^m$. Then the problem*

$$\begin{cases} u \in W^{r,p}(\mathbb{R}^+) \\ (P_+ P_-)(D)u = 0 \\ (Q_k(D)u)(0) = b_k \quad 1 \le k \le m \end{cases}$$

has a unique solution.

PROOF. By Theorem 3.7 we have $W^{r,p}(\mathbb{R}^+) \cap \ker(P_+ P_-)(D) = \ker P_-(D)$. Then the result follows from Theorem 3.5 (c). □

4. – R-boundedness and functional calculus

4.1. – R-boundedness

Let X, Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{L}(X, Y)$ is said to be R-bounded if $\exists\, C > 0$ such that for any positive integer N and for arbitrary choices of $T_1, \dots, T_N \in \mathcal{T}$ and $x_1, \dots, x_N \in X$ one has

$$\sum_{\varepsilon \in \{-1,1\}^N} \Big\| \sum_{k=1}^N \varepsilon_k T_k x_k \Big\|_Y \le C \sum_{\varepsilon \in \{-1,1\}^N} \Big\| \sum_{k=1}^N \varepsilon_k x_k \Big\|_X . \tag{4.1}$$

We call R_1-bound of $\mathcal{T}$ the best constant C that can be put in the right-hand side of (4.1).

Due to the Khintchine-Kahane inequality (see [6], 11.1), formula (4.1) can be written in an equivalent form as

$$\left(\sum_{\varepsilon \in \{-1,1\}^N} \Big\| \sum_{k=1}^N \varepsilon_k T_k x_k \Big\|_Y^q \right)^{1/q} \le C_q \left(\sum_{\varepsilon \in \{-1,1\}^N} \Big\| \sum_{k=1}^N \varepsilon_k x_k \Big\|_X^q \right)^{1/q}$$

where $q \in \mathbb{R}^+$. The best C_q will be called the R_q-bound of $\mathcal{T}$. By setting $N = 1$ in this inequality, one sees at once that if $\mathcal{T}$ is a R-bounded subset of $\mathcal{L}(X, Y)$, with R_q-bound C, then $\mathcal{T}$ is bounded, and $\sup\limits_{T \in \mathcal{T}} \|T\| \le C$. Moreover a simple application of the triangle inequality proves the following lemma.

LEMMA 4.2. *Let X, Y be Banach spaces, and let $\mathcal{T}'$, $\mathcal{T}''$ be R-bounded subsets of $\mathcal{L}(X,Y)$. Then $\mathcal{T}'+\mathcal{T}''$ is R-bounded, and $\forall p \in [1,\infty[$ the R_p-bound of $\mathcal{T}'+\mathcal{T}''$ is not greater than the sum of the R_p-bounds of $\mathcal{T}'$ and $\mathcal{T}''$.*

The simplest example of R-bounded set is given in the following well-known result (see [6], 12.2).

THEOREM 4.3 (Kahane's contraction principle). *Let X be a Banach space, $M \in \mathbb{R}^+$. Then the set $\{\lambda I_X;\, \lambda \in \mathbb{C},\ |\lambda| \le M\}$ is R-bounded, and $\forall p \in [1,\infty[$ its R_p-bound is $\le 2M$.*

The following theorem (which can be found in [31]) provides a nontrivial example of a R-bounded set of operators. Recall that $\mathcal{F}$ is the Fourier transformation.

THEOREM 4.4. *Let $(\psi_i)_{i\in I}$ be a family of elements of $L^1(\mathbb{R})$ and assume that $\forall i \in I$ $\mathcal{F}\psi_i \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}\setminus\{0\})$, while the function $m_1\,(\mathcal{F}\psi_i)'$ (where $m_1(t) = t\ \forall t \in \mathbb{R}$) belongs to $L^\infty(\mathbb{R})$. Assume moreover that*

$$\sup_{i\in I}\max\left\{\|\mathcal{F}\psi_i\|_{L^\infty(\mathbb{R})},\ \|m_1\,(\mathcal{F}\psi_i)'\|_{L^\infty(\mathbb{R})}\right\} =: \eta < +\infty.$$

Let $p \in]1,\infty[$ and let $T_i : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be the bounded linear operator defined by $T_i f = \psi_i * f$. Then $\{T_i : i \in I\}$ is a R-bounded set of operators, and its R_p-bound is $\le C(p)\,\eta$.

Another sufficient condition for R-boundedness is given in the following result.

THEOREM 4.5. *Let $M \in \mathbb{R}^+$ and let $\mathcal{K}_M$ be the set of the measurable functions on $\mathbb{R}^+\times\mathbb{R}^+$ to $\mathbb{C}$ such that*

$$\operatorname*{ess\,sup}_{t,s\in\mathbb{R}^+}(t+s)\,|K(t,s)| \le M.$$

Let $p \in]1,\infty[$ and let X be a Banach space. $\forall K \in \mathcal{K}_M$ let T_K be the operator formally defined on $L^p(\mathbb{R}^+, X)$ by

$$T_K f(t) = \int_0^\infty K(t,s)\,f(s)\,ds.$$

Then $\forall K \in \mathcal{K}_M$ T_K is a bounded linear operator on $L^p(\mathbb{R}^+,X)$ to itself, and the set $\{T_K;\, K \in \mathcal{K}_M\}$ is R-bounded, its R_p-bound being $\le \frac{2M\pi}{\sin(\pi/p)}$.

PROOF. The statement that $T_K \in \mathcal{L}(L^p(\mathbb{R}^+,X))$ and the inequality

$$\|T_K\|_{\mathcal{L}(L^p(\mathbb{R}^+,X))} \le \frac{M\pi}{\sin(\pi/p)}$$

are well-known: a proof can be found in ([17], Theorem 319). In order to prove the R-boundedness, we choose $f_j \in L^p(\mathbb{R}^+, X)$ and $K_j \in \mathcal{K}_M$ $(1 \le j \le N)$. Then

$$\left(\sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{j=1}^N \varepsilon_j T_{K_j} f_j \Big\|^p_{L^p(\mathbb{R}^+,X)}\right)^{1/p}$$

$$= \left(\int_0^\infty \sum_{\varepsilon\in\{-1,1\}^N} \Big\| \int_0^\infty \sum_{j=1}^N \varepsilon_j K_j(t,s) f_j(s)\, ds \Big\|^p_X dt\right)^{1/p} .$$

The last term can be interpreted in the following way. In the space $\prod_{\varepsilon\in\{-1,1\}^N} X_\varepsilon$, where each X_ε is a copy of X, we consider the norm $\|x\|_p := \Big(\sum\limits_{\varepsilon\in\{-1,1\}^N} \|x_\varepsilon\|^p_X\Big)^{1/p}$ (where $x = (x_\varepsilon)_{\varepsilon\in\{-1,1\}^N}$). If we set $x_\varepsilon(t,s) = \sum\limits_{j=1}^N \varepsilon_j K_j(t,s) f_j(s)$ and $x(t,s) = (x_\varepsilon(t,s))_{\varepsilon\in\{-1,1\}^N}$, then

$$\sum_{\varepsilon\in\{-1,1\}^N} \Big\| \int_0^\infty \sum_{j=1}^N \varepsilon_j K_j(t,s) f_j(s)\, ds \Big\|^p_X = \Big\| \int_0^\infty x(t,s)\, ds \Big\|^p_p$$

$$\le \left(\int_0^\infty \|x(t,s)\|_p\, ds\right)^p = \left(\int_0^\infty \left(\sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{j=1}^N \varepsilon_j K_j(t,s) f_j(s) \Big\|^p_X\right)^{1/p} ds\right)^p .$$

Therefore

$$\left(\sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{j=1}^N \varepsilon_j T_{K_j} f_j \Big\|^p_{L^p(\mathbb{R}^+,X)}\right)^{1/p}$$

$$\le \left(\int_0^\infty \left(\int_0^\infty \left(\sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{j=1}^N \varepsilon_j K_j(t,s) f_j(s) \Big\|^p_X\right)^{1/p} ds\right)^p dt\right)^{1/p}$$

(by the contraction principle, as $K_j \in \mathcal{K}_M$)

$$\le \left(\int_0^\infty \left(\int_0^\infty \frac{2M}{t+s} \left(\sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{j=1}^N \varepsilon_j f_j(s) \Big\|^p_X\right)^{1/p} ds\right)^p dt\right)^{1/p}$$

$$\le \frac{2M\pi}{\sin(\pi/p)} \Bigg\| \left(\sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{j=1}^N \varepsilon_j f_j(\cdot) \Big\|^p_X\right)^{1/p} \Bigg\|_{L^p(\mathbb{R}^+)}$$

$$= \frac{2M\pi}{\sin(\pi/p)} \left(\sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{j=1}^N \varepsilon_j f_j \Big\|^p_{L^p(\mathbb{R}^+,X)}\right)^{1/p} . \qquad \square$$

REMARK 4.6. Let X, Y be Banach spaces, and for $k = 1, 2$ let (Ω_k, μ_k) be a σ-finite measure space. We call (Ω, μ) the product measure space. Assume that $T \in \mathcal{L}(L^p(\mu_2, X), L^p(\mu_2, Y))$. Then we can define an operator $\hat{T}$ on $L^p(\mu, X)$ to $L^p(\mu, Y)$ by the formula

$$(\hat{T} f)(t_1, t_2) = \Big(T(f(t_1, \cdot))\Big)(t_2)$$

and we have $\|\hat{T} f\|_{L^p(\mu,Y)} \le \|T\| \, \|f\|_{L^p(\mu,X)}$. Since the linearity of T obviously implies the linearity of $\hat{T}$, we have that $\hat{T} \in \mathcal{L}(L^p(\mu, X), L^p(\mu, Y))$, with $\|\hat{T}\| \le \|T\|$. Remark that the transformation $T \mapsto \hat{T}$, from $\mathcal{L}(L^p(\mu_2, X), L^p(\mu_2, Y))$ to $\mathcal{L}(L^p(\mu, X), L^p(\mu, Y))$, is linear and continuous. □

LEMMA 4.7. *With the notations of Remark* 4.6*, assume that $\mathcal{T}$ is a R-bounded subset of $\mathcal{L}(L^p(\mu_2, X), L^p(\mu_2, Y))$. We set $\hat{\mathcal{T}} = \{\hat{T}; T \in \mathcal{T}\}$. Then $\hat{\mathcal{T}}$ is R-bounded, and the R_p-bound of $\hat{\mathcal{T}}$ is not greater than the R_p-bound of $\mathcal{T}$.*

PROOF. If M is the R_p-bound of $\mathcal{T}$, then

$$\begin{aligned}
\sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{k=1}^N \varepsilon_k \hat{T}_k f_k \Big\|^p_{L^p(\mu,Y)} &= \int_{\Omega_1} \sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{k=1}^N \varepsilon_k T_k(f_k(t_1, \cdot)) \Big\|^p_{L^p(\mu_2,Y)} d\mu_1(t_1) \\
&\le M^p \int_{\Omega_1} \sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{k=1}^N \varepsilon_k f_k(t_1, \cdot) \Big\|^p_{L^p(\mu_2,X)} d\mu_1(t_1) \\
&= M^p \sum_{\varepsilon\in\{-1,1\}^N} \Big\| \sum_{k=1}^N \varepsilon_k f_k \Big\|^p_{L^p(\mu,X)}.
\end{aligned}$$

□

In Section 8, we shall apply Remark 4.6 and Lemma 4.7 to the case of $\Omega_1 = \mathbb{R}^n$ and $\Omega_2 = \mathbb{R}^+$ (with the Lebesgue measure, in both cases).

4.2. – Joint functional calculus for sectorial operators

What follows is a short review of some part of the theory of H^∞ functional calculus for n-tuples of sectorial and bisectorial operators. Proofs and details can be found in [8]. We also refer the reader to the papers [5] and [20].

DEFINITION 4.8. Let T be a linear operator in the complex Banach space X, and let $\beta \in [0, \pi[$. We say that T is sectorial with spectral angle β if:

(i) $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are dense in X;
(ii) $\sigma(T) \subseteq \overline{S_\beta}$;
(iii) $\forall \varepsilon \in]0, \pi - \beta[\ \exists C_\varepsilon \in \mathbb{R}^+$ such that $\|\lambda(\lambda - T)^{-1}\| \le C_\varepsilon \ \forall \lambda \in \mathbb{C} \setminus \overline{S_{\beta+\varepsilon}}$.

From condition 4.8(iii) it follows that $\ker T \cap \overline{\mathcal{R}(T)} = \{0\}$; therefore from condition 4.8(i) one obtains that any sectorial operator is injective.

Conversely, if X is reflexive, then conditions (ii) and (iii) of Definition 4.8 imply that $\mathcal{D}(T)$ is dense in X, and that $X = \ker T \oplus \overline{\mathcal{R}(T)}$, so that T has dense range if and only if it is injective (see [5], Theorem 3.8).

DEFINITION 4.9. Let X be a complex Banach space, $\Omega = \prod_{j=1}^{N} S_{\beta_j}$, with $\beta_j \in]0, \pi[\ \forall j$. We call

$H(\Omega, X)$ the vector space of the X-valued holomorphic functions on Ω;
$H^\infty(\Omega, X)$ the Banach space of the X-valued bounded holomorphic functions on Ω, with the norm $\|f\|_\infty := \sup_{z\in\Omega} \|f(z)\|_X$;
$H_0^\infty(\Omega, X)$ the set of the holomorphic functions $f : \Omega \to X$ satisfying the following condition: $\exists\, C > 0$, $s > 0$ such that $\forall z = (z_1, \dots, z_N) \in \Omega$

$$\|f(z)\|_X \le C \prod_{j=1}^{N} \min\left\{|z_j|^s,\ |z_j|^{-s}\right\}.$$

In the notations $H(\Omega, X)$ etc., the mention of X will be omitted when $X = \mathbb{C}$. Remark, however, that when $X = \mathcal{L}(Y)$ (for some Banach space Y), then the scalar valued functions can be identified with the X-valued functions in a natural way, by replacing f with $f(\cdot)\, I_Y$.

It is obvious that $H_0^\infty(\Omega, X)$ is a vector subspace of $H^\infty(\Omega, X)$; moreover, if X is a Banach algebra, then also $H^\infty(\Omega, X)$ is a Banach algebra, and $H_0^\infty(\Omega, X)$ is a two-sided ideal of $H^\infty(\Omega, X)$.

Let $T_1, \dots, T_N$ be sectorial operators in the complex Banach space X (with spectral angles $\alpha_1, \dots, \alpha_N$). We assume that the resolvent operators of $T_1, \dots, T_N$ commute, and call $\mathcal{B}$ the set all bounded linear operators on X that commute with these resolvent operators. Then $\mathcal{B}$ is a closed subalgebra of $\mathcal{L}(X)$. Let us set $\Omega = \prod_{j=1}^{N} S_{\beta_j}$, with $\alpha_j < \beta_j < \pi$. If $f \in H_0^\infty(\Omega, \mathcal{B})$, then the operator $f(T_1, \dots, T_N) \in \mathcal{B}$ is defined as follows.

Let $\gamma_j \in]\alpha_j, \beta_j[$. We set $\Gamma = \prod_{j=1}^{N} \Gamma_j$, where Γ_j is the curve parametrized by $t \mapsto |t|\, e^{-i\gamma_j \operatorname{sgn} t}$ for $t \in \mathbb{R} \setminus \{0\}$, and oriented according to the increasing values of t (i.e. according to the decreasing imaginary parts). If $f \in H_0^\infty(\Omega, \mathcal{B})$, then the function $z \mapsto f(z) \prod_{j=1}^{N}(z_j - T_j)^{-1}$ is summable on Γ, and its integral (which belongs to $\mathcal{B}$) does not depend on the choice of the angular values $\gamma_j \in]\alpha_j, \beta_j[$. Therefore we set

$$f(T_1, \dots, T_N) = (2\pi i)^{-N} \int_\Gamma f(z) \prod_{j=1}^{N} (z_j - T_j)^{-1}\, dz$$

and we obtain that the map $f \mapsto f(T_1, \dots, T_N)$ is an algebra homomorphism of $H_0^\infty(\Omega, \mathcal{B})$ to $\mathcal{B}$.

Among the (scalar valued) functions that belong to $H_0^\infty(\Omega)$ there are the very useful functions introduced in the following definition.

DEFINITION 4.10. For any positive integer k, we define the function $\Psi_{k,N} : (\mathbb{C} \setminus \{-k,\ -k^{-1}\})^N \to \mathbb{C}$ by

$$\Psi_{k,N}(z) = \prod_{j=1}^{N} \frac{k^2 z_j}{(k + z_j)\,(1 + k z_j)}.$$

We usually will not mention the dimensional parameter N, and we set

$$\Psi(z) = \Psi_1(z) = \prod_{j=1}^{N} \frac{z_j}{(1 + z_j)^2}.$$

One has

LEMMA 4.11.

$$\Psi(T_1, \ldots, T_N) = \prod_{j=1}^{N} T_j\,(1 + T_j)^{-2}.$$

Moreover $\Psi(T_1, \ldots, T_N)$ is injective and has dense range.

Then one can extend the definition of $f(T_1, \ldots, T_N)$ to the case of $f \in H^\infty(\Omega, \mathcal{B})$ (and even to a larger space, actually) with the following device. If $f \in H^\infty(\Omega, \mathcal{B})$, then $\Psi\, f \in H_0^\infty(\Omega, \mathcal{B})$. Then we can set

$$f(T_1, \ldots, T_N) := \Psi(T_1, \ldots, T_N)^{-1}\,(\Psi f)(T_1, \ldots, T_N).$$

This definition extends the one given for $f \in H_0^\infty(\Omega, \mathcal{B})$, but in general $f(T_1, \ldots, T_N)$ is a closed, densely defined, and not necessarily bounded, operator. One can prove that

LEMMA 4.12. *If $f,\ g \in H^\infty(\Omega, \mathcal{B})$, then*

$$f(T_1, \ldots, T_N) + g(T_1, \ldots, T_N) \subseteq (f + g)(T_1, \ldots, T_N)$$

and

$$f(T_1, \ldots, T_N)\, g(T_1, \ldots, T_N) \subseteq (fg)(T_1, \ldots, T_N).$$

LEMMA 4.13. *If $S \in \mathcal{B}$ and $f(z) = S\ \forall z \in \Omega$, then $f(T_1, \ldots, T_N) = S$.*

LEMMA 4.14. $\forall x \in X\ \lim_{k\to+\infty} \Psi_k(T_1, \ldots, T_N)x = x$.

REMARK 4.15. If $f \in H^\infty(\Omega, \mathcal{B})$ and $f(T_1, \ldots, T_N) \in \mathcal{L}(X)$, then from Lemma 4.12 we get

$$(\Psi_k f)(T_1, \ldots, T_N) = \Psi_k(T_1, \ldots, T_N)\, f(T_1, \ldots, T_N);$$

therefore $\forall x \in X$ we have, by Lemma 4.14,

$$f(T_1, \ldots, T_N)x = \lim_{k\to+\infty} (\Psi_k f)(T_1, \ldots, T_N)x. \qquad \square$$

DEFINITION 4.16. Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{B}$. We say that $(T_1, \ldots, T_N)$ has a bounded joint $H^\infty(\Omega, \mathcal{A})$ functional calculus if $f(T_1, \ldots, T_N) \in \mathcal{L}(X)$ $\forall f \in H^\infty(\Omega, \mathcal{A})$.

Remark that this definition includes the case of scalar valued functions f, when $\mathcal{A} = \{\lambda\, I_X;\ \lambda \in \mathbb{C}\}$.

Another useful result is the following.

LEMMA 4.17. *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{B}$. Then a necessary and sufficient condition for $(T_1, \dots, T_N)$ to have a bounded joint $H^\infty(\Omega, \mathcal{A})$ functional calculus is that $\exists\, C \in \mathbb{R}^+$ such that $\forall f \in H_0^\infty(\Omega, \mathcal{A})$ $\|f(T_1, \dots, T_N)\| \le C\,\|f\|_\infty$. In this case, $\forall f \in H^\infty(\Omega, \mathcal{A})$ $\|f(T_1, \dots, T_N)\| \le C \prod_{j=1}^N \cos^{-2}(\beta_j/2)\,\|f\|_\infty$.*

In some concrete cases, e.g. when $X = L^p(\mathbb{R}^n)$, in order to prove the estimate $\|f(T_1, \dots, T_N)\| \le C\,\|f\|_\infty$ for $f \in H_0^\infty(\Omega, \mathcal{A})$ it may be useful to use multiplier theorems. Therefore the "vector-valued case" (that is, $f \in H_0^\infty(\Omega, \mathcal{A})$) is expected to be much more difficult than the "scalar-valued case" (i.e. $f \in H_0^\infty(\Omega, \mathbb{C})$), for in the former case one should work with operator valued multipliers, instead of scalar valued multipliers. Other difficulties can be found if we do not know anything about the existence of a bounded joint $H^\infty(\Omega, \mathcal{A})$ functional calculus, and we want to prove that $f(T_1, \dots, T_N)$ is bounded for a given $f \in H^\infty(\Omega, \mathcal{A})$. In this connection we have the following theorem, that we shall use later.

THEOREM 4.18 (see [19], Theorem 4.4; [8] Theorem 6.7). *Assume that $(T_1, \dots, T_N)$ have a bounded joint $H^\infty(\Omega')$ functional calculus, where Ω' is a set of the same type as Ω, defined with respect to smaller angles. Let $f : \Omega \to \mathcal{B}$ be a holomorphic function whose range is R-bounded. Then $f(T_1, \dots, T_N) \in \mathcal{L}(X)$. Moreover $\exists\, C(T_1, \dots, T_N, p, X) \in \mathbb{R}^+$ such that $\|f(T_1, \dots, T_N)\|$ is not greater than $C(T_1, \dots, T_N, p, X)$ times the R_p-bound of the range of f.*

In the sequel, we shall deal also with a different situation. To explain what is the matter, we first give a definition.

DEFINITION 4.19. Let T be a linear operator acting in the complex Banach space X, $\delta \in [0, \pi/2[$. We say that T is bisectorial with spectral angle δ if:

(i) $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are dense in X;
(ii) $\overline{\Sigma_\delta} \supseteq \sigma(T)$;
(iii) $\forall \delta' \in]\delta, \frac{\pi}{2}[\ \exists\, C(\delta') \in \mathbb{R}^+$ such that $\forall \lambda \in \mathbb{C} \setminus \overline{\Sigma_{\delta'}}$ $\|\lambda(\lambda - T)^{-1}\| \le C(\delta')$.

Since $\Sigma_\delta \subset S_{\delta+\frac{\pi}{2}}$, it is obvious that a bisectorial operator with spectral angle $\delta \in]0, \frac{\pi}{2}[$ is also a sectorial operator with spectral angle $\delta + \frac{\pi}{2}$. However in this case we wish to define $f(T)$ also when f is holomorphic on $\Sigma_{\delta'}$, with $\delta < \delta' < \pi/2$, and not necessarily on the whole sector $S_{\frac{\pi}{2}+\delta'}$. In a similar way, if $T_1, \dots, T_N$ are bisectorial operators with commuting resolvents, we wish to define $f(T_1, \dots, T_N)$ when f is holomorphic on the set $\Omega = \prod_{k=1}^n \Sigma_{\beta_k}$, where $\forall k$ β_k is greater than the spectral angle α_k of the bisectorial operator T_k.

In order to do this, let us agree that in the present situation the meaning of such symbols as $H(\Omega, X)$, $H^\infty(\Omega, X)$, $H_0^\infty(\Omega, X)$ is analogous to the one of the sectorial case, given in Definition 4.9. Then we define at first $f(T_1, \dots, T_N)$ when $f \in H_0^\infty(\Omega, \mathcal{B})$. If $\gamma_k \in]\alpha_k + \frac{\pi}{2}, \beta_k + \frac{\pi}{2}[$, we call Γ_k the curve parametrized by $\mathbb{R} \setminus \{0\} \ni t \mapsto |t|\, e^{-i\gamma_k \operatorname{sgn} t}$, and set $\widetilde{\Gamma}_k = \Gamma_k \cup (-\Gamma_k)$, where Γ_k is oriented, as above, according to the decreasing imaginary parts, while $-\Gamma_k$ is oriented according to the increasing imaginary parts. With $\widetilde{\Gamma} = \prod_{k=1}^N \widetilde{\Gamma}_k$, we can set

$$f(T_1, \dots, T_N) := (2\pi i)^{-N} \int_{\widetilde{\Gamma}} f(z) \prod_{k=1}^N (z_k - T_k)^{-1}\, dz.$$

One can prove again that this integral exists, and does not depend on the choice of the $\gamma_k \in]\alpha_k + \frac{\pi}{2}, \beta_k + \frac{\pi}{2}[$. Moreover, when $f \in H_0^\infty\Big(\prod_{k=1}^N S_{\beta_k}, \mathcal{B}\Big)$, if we set $\Gamma = \prod_{k=1}^N \Gamma_k$, we have

$$\int_{\Gamma} f(z) \prod_{k=1}^N (z_k - T_k)^{-1}\, dz = \int_{\widetilde{\Gamma}} f(z) \prod_{k=1}^N (z_k - T_k)^{-1}\, dz$$

and this means that, for such functions f, the definition of $f(T_1, \dots, T_N)$ for the bisectorial case coincides with its definition for the sectorial case. This happens, for instance, for the scalar function Ψ; hence also in the bisectorial case we can extend the definition of $f(T_1, \dots, T_N)$ to the functions $f \in H^\infty(\Omega, \mathcal{B})$ by the formula

$$f(T_1, \dots, T_N) := \Psi(T_1, \dots, T_N)^{-1} (\Psi f)(T_1, \dots, T_N).$$

Then it is possible to prove that also in the bisectorial case Lemmas 4.12, 4.13, 4.14, 4.17, and Theorem 4.18 hold true. Details can be found in the paper [8].

5. – Some preliminary results on elliptic polynomials

In the sequel, m and n are positive integers, and $P : \mathbb{C}^{n+1} \to \mathbb{C}$ is a homogeneous polynomial of degree $2m$, with complex coefficients. We'll emphasize the last argument of P by writing $P(z, \lambda)$ with $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$.

Definition 5.1. Let $\omega \in [0, \pi[$, $L \in \mathbb{R}^+$. We say that P is (L, ω)-elliptic if the following conditions hold:

(i) $|P(ix, it)| \geq L^{-1} \|(x, t)\|^{2m}$ $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}$;
(ii) $P(ix, it) \in \overline{S_\omega}$ $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}$;
(iii) the maximum of the moduli of the coefficients of P is $\leq L$.

Remark 5.2.

(a) It is quite obvious that a sufficient condition for a homogeneous polynomial P to be (L, ω)-elliptic for some L and ω is that $P(ix, it) \notin \mathbb{R}^- \cup \{0\}$ $\forall (x, t) \in (\mathbb{R}^n \times \mathbb{R}) \setminus \{(0, 0)\}$.
(b) Assume that P is (L, ω)-elliptic. If we write

$$P(z, \lambda) = \sum_{k=0}^{2m} \Big(\sum_{|\alpha| = 2m-k} c_{k,\alpha}\, z^\alpha \Big) \lambda^k,$$

then by condition (i) of Definition 5.1 $c_{2m,0} = (-1)^m P(0,i) \neq 0$; in particular $\forall z \in \mathbb{C}^n$ the polynomial $\mathbb{C} \ni \lambda \mapsto P(z,\lambda)$ has degree $2m$.

(c) It follows from condition (iii) that $L \geq |c_{2m,0}| = |P(0,i)| \geq L^{-1}$; therefore $L \geq 1$. □

Henceforth it is understood that P is (L,ω)-elliptic, for some fixed $L \geq 1$ and $\omega \in [0,\pi[$. In all the inequalities that will follow, the constants will depend on P, but only through L and ω.

We now study the behaviour of $P(z,\lambda)$ when $(z,\lambda) \in (\overline{\Sigma_\varphi})^n \times \overline{\Sigma_\varphi}$. We need some lemmas.

LEMMA 5.3. *If $w = |w|\, e^{i\alpha}$, $z = |z|\, e^{i\beta}$, with $\alpha,\ \beta \in \mathbb{R}$, then*

$$|w - z| \geq (\,|w| + |z|\,)\,|\sin((\alpha-\beta)/2)|.$$

PROOF.

$$\begin{aligned}|w-z|^2 &= \Big(|w|\cos\alpha - |z|\cos\beta\Big)^2 + \Big(|w|\sin\alpha - |z|\sin\beta\Big)^2\\ &= |w|^2 + |z|^2 - 2\,|w|\,|z|\cos(\alpha-\beta)\\ &= \Big(|w| - |z|\Big)^2 \cos^2((\alpha-\beta)/2) + \Big(|w| + |z|\Big)^2 \sin^2((\alpha-\beta)/2).\end{aligned}$$ □

LEMMA 5.4. *Let $z, w \in \mathbb{C} \setminus (\mathbb{R}^- \cup \{0\})$ such that $|w-z| < |z|$. Then*

$$|\arg w| \leq |\arg z| + \arcsin \frac{|w-z|}{|z|}.$$

PROOF. Put $\lambda = w/z$. Then $1 - Re\,\lambda \leq |\lambda - 1| = \frac{|w-z|}{|z|} < 1$ so that $Re\,\lambda > 0$ and hence $|\arg\lambda| < \frac{\pi}{2}$. From $w = \lambda z$ it follows that $|\arg w| \leq |\arg\lambda + \arg z| \leq |\arg z| + |\arg\lambda|$; hence in order to prove the lemma it is enough to show that $|\arg\lambda| \leq \arcsin|\lambda - 1|$, and since $|\arg\lambda| < \frac{\pi}{2}$, this is equivalent to $\sin^2\arg\lambda \leq |\lambda-1|^2$, i.e. to $\sin^2\arg\lambda \leq |\lambda|^2 - 2\,|\lambda|\,\cos\arg\lambda + 1$, that is obviously true. □

When $N \geq 1$ and $q \geq 0$ are integers, we set $\ell_{N,q} = \binom{q+N-1}{q}$, i.e. the number of indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $|\alpha| = q$.

LEMMA 5.5. *Let N and q be positive integers and let $Q : \mathbb{C}^N \to \mathbb{C}$ be a homogeneous polynomial with* $\deg Q = q$. *We call M the maximum of the moduli of the coefficients of Q. Then for arbitrary $\xi,\ \eta \in \mathbb{C}^N$*

$$|Q(\xi+\eta) - Q(\xi)| \leq 2^q\, \ell_{N,q}\, M\, \|\eta\|\ \max\Big\{\|\xi\|^{q-1},\ \|\eta\|^{q-1}\Big\}.$$

PROOF. We have obviously

$$(\xi+\eta)^\alpha - \xi^\alpha = \sum_{\beta\leq\alpha,\beta\neq\alpha} \binom{\alpha}{\beta}\, \eta^{\alpha-\beta}\xi^\beta.$$

If $|\alpha| = q$, $\beta \leq \alpha$ and $\beta \neq \alpha$, then $|\beta| \leq q-1$; in this case we have

$$|\eta^{\alpha-\beta}\xi^\beta| \leq \|\eta\|\,\|\eta\|^{q-1-|\beta|}\,\|\xi\|^{|\beta|} \leq \|\eta\|\,\Big(\max\Big\{\|\eta\|,\ \|\xi\|\Big\}\Big)^{q-1}.$$

Since $\sum_{\beta\leq\alpha}\binom{\alpha}{\beta} = 2^{|\alpha|} = 2^q$, the assertion is proved. □

THEOREM 5.6. *Let* $\theta \in]\omega, \pi[$. *We set* $\varphi = \left(2^{2m+1}\,\ell_{n+1,2m}\,L^2\right)^{-1}\,\sin(\theta-\omega)$. *If* $(0,0) \neq (z,\lambda) \in (\overline{\Sigma_\varphi})^n \times \overline{\Sigma_\varphi}$, *then* $P(z,\lambda) \in S_\theta$ *and* $|P(z,\lambda)| \geq \frac{1}{2L}\,\|(z,\lambda)\|^{2m}$.

PROOF. Let $(0,0) \neq (z,\lambda) \in (\overline{\Sigma_\varphi})^n \times \overline{\Sigma_\varphi}$. For suitable $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $\alpha_1,\dots,\alpha_n, \beta \in [-\varphi,\varphi]$ we have $(z,\lambda) = \left(e^{i\alpha_1} i x_1, \dots, e^{i\alpha_n} i x_n,\ e^{i\beta} i t\right)$. Obviously $\|(x,t)\| = \|(z,\lambda)\|$.

From Lemma 5.5 we get

$$|P(z,\lambda) - P(ix,it)|$$
$$\leq 2^{2m}\,\ell_{n+1,2m} L \|(z,\lambda)-(ix,it)\| \max\left\{\|(ix,it)\|^{2m-1},\ \|(z,\lambda)-(ix,it)\|^{2m-1}\right\}$$

and

$$\begin{aligned}\|(z,\lambda)-(ix,it)\|^2 &= \sum_{k=1}^n |e^{i\alpha_k}-1|^2\,x_k^2 + |e^{i\beta}-1|^2\,t^2\\ &= 4\left(\sum_{k=1}^n \sin^2(\alpha_k/2)\,x_k^2 + \sin^2(\beta/2)\,t^2\right)\\ &\leq 4\,\sin^2(\varphi/2)\,\|(x,t)\|^2 < \varphi^2\,\|(x,t)\|^2;\end{aligned}$$

so that (as it is obvious that $\varphi < 1$)

$$|P(z,\lambda) - P(ix,it)| < 2^{2m}\,\ell_{n+1,2m}\,L\,\varphi\,\|(x,t)\|^{2m} = \frac{1}{2L}\,\sin(\theta-\omega)\,\|(x,t)\|^{2m}.$$

On the other hand we have $|P(ix,it)| \geq L^{-1}\,\|(x,t)\|^{2m}$, hence

$$\frac{|P(z,\lambda) - P(ix,it)|}{|P(ix,it)|} < \frac{1}{2}\,\sin(\theta-\omega) \leq \frac{1}{2}.$$

Therefore

$$|P(z,\lambda)| > \frac{1}{2}\,|P(ix,it)| \geq \frac{1}{2L}\,\|(x,t)\|^{2m} = \frac{1}{2L}\,\|(z,\lambda)\|^{2m}.$$

Moreover from Lemma 5.4 it follows that

$$\begin{aligned}|\arg P(z,\lambda)| &\leq |\arg P(ix,it)| + \arcsin\frac{|P(z,\lambda)-P(ix,it)|}{|P(ix,it)|}\\ &\leq \omega + \arcsin\left(\tfrac{1}{2}\,\sin(\theta-\omega)\right) \leq \omega + \frac{\pi}{4}\,\sin(\theta-\omega) < \theta. \qquad \square\end{aligned}$$

In the sequel we shall denote with φ_0 the function on $]\omega,\pi[$ to $]0,\frac{\pi}{2}[$ defined by

$$\varphi_0(\theta) := \left(2^{2m+1}\,\ell_{n+1,2m}\,L^2\right)^{-1}\,\sin\frac{\theta-\omega}{2}.$$

COROLLARY 5.7. *If* $\theta \in]\omega,\pi[$, $(0,0) \neq (z,\lambda) \in (\overline{\Sigma_{\varphi_0(\theta)}})^n \times \overline{\Sigma_{\varphi_0(\theta)}}$, *then* $P(z,\lambda) \in S_{(\theta+\omega)/2}$ *and* $|P(z,\lambda)| \geq \frac{1}{2L}\,\|(z,\lambda)\|^{2m}$.

Proof. Immediate consequence of Theorem 5.6. $\square$

Theorem 5.8. *Let $\theta \in]\omega, \pi[$. If $\mu \in \mathbb{C} \setminus S_\theta$ and $(z, \lambda) \in (\overline{\Sigma_{\varphi_0(\theta)}})^n \times \overline{\Sigma_{\varphi_0(\theta)}}$, then*

$$|\mu - P(z,\lambda)| \geq \frac{1}{2L} \sin \frac{\theta - \omega}{4} \left(\|(z,\lambda)\|^{2m} + |\mu| \right).$$

Proof. Suppose that $\mu \neq 0$ and $(z, \lambda) \neq (0, 0)$, otherwise the inequality is trivial. We set $\mu = |\mu| e^{i\alpha}$, with $\theta \leq |\alpha| \leq \pi$. Then by Lemma 5.3 and Corollary 5.7 we have

$$\begin{aligned} |\mu - P(z,\lambda)| &\geq \sin \frac{|\alpha - \arg P(z,\lambda)|}{2} \Big(|\mu| + |P(z,\lambda)| \Big) \\ &\geq \sin \frac{\theta - (\theta + \omega)/2}{2} \Big(|\mu| + (2L)^{-1} \|(z,\lambda)\|^{2m} \Big) \\ &\geq (2L)^{-1} \sin \frac{\theta - \omega}{4} \Big(|\mu| + \|(z,\lambda)\|^{2m} \Big). \end{aligned}$$ $\square$

Remark 5.9. Let us fix $\theta \in]\omega, \pi[$, $\alpha \in [\theta - \pi, \pi - \theta]$, $\beta = (\beta_1, \dots, \beta_n) \in [-\varphi_0(\theta), \varphi_0(\theta)]^n$. For $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ and $\nu \in \mathbb{C}$ we set

$$Q(\zeta, \lambda, \nu) = (-1)^m e^{i\alpha} \nu^{2m} + P(e^{i\beta_1} \zeta_1, \dots, e^{i\beta_n} \zeta_n, \lambda).$$

Then Q is a homogeneous polynomial of degree $2m$ in $n+2$ variables; moreover Theorem 5.8 implies that Q does not vanish on $(i\,\mathbb{R})^{n+2} \setminus \{0\}$. Since $n+2 \geq 3$, by means of well-known arguments (see e.g. [2], Proposition 2.2), one obtains that for $(x, \xi) \in \mathbb{R}^{n+1} \setminus \{(0,0)\}$ the equation (in the unknown λ) $Q(ix, \lambda, i\xi) = 0$ has m roots with positive real part and m with negative real part. This means that for $(0,0) \neq (z, \mu) \in (\overline{\Sigma_{\varphi_0(\theta)}})^n \times (\mathbb{C} \setminus S_\theta)$ the equation (in the unknown λ) $\mu = P(z, \lambda)$ has m roots with positive real part and m with negative real part. Then we shall write

$$\mu - P(z, \lambda) = P^+_{z,\mu}(\lambda)\, P^-_{z,\mu}(\lambda)$$

where the polynomial $P^+_{z,\mu}$ ($P^-_{z,\mu}$) collects the roots with positive (negative) real part of $\mu - P(z, \cdot)$. $\square$

Definition 5.10. $\forall (z, \mu) \in \mathbb{C}^{n+1}$ we set $\beta(z, \mu) = \|z\| + |\mu|^{\frac{1}{2m}}$.

Theorem 5.11. *$\forall \theta \in]\omega, \pi[$ there exist positive constants $r(L, \omega, \theta)$ and $M_1(L, \omega, \theta)$ such that if $\mu \in \mathbb{C} \setminus S_\theta$, $z \in (\overline{\Sigma_{\varphi_0(\theta)}})^n$, $\lambda \in \mathbb{C}$ and*

$$|\mu - P(z, \lambda)| < M_1(L, \omega, \theta)\, \beta(z, \mu)^{2m}$$

then $\lambda \notin \overline{\Sigma_{\varphi_0(\theta)}}$ and $r(L, \omega, \theta)^{-1} \beta(z, \mu) < |\lambda| < r(L, \omega, \theta)\, \beta(z, \mu)$. In particular any root λ of the polynomial $\mu - P(z, \cdot)$ satisfies these conditions.

PROOF. Let $\theta \in]\omega, \pi[$, $\mu \in \mathbb{C} \setminus S_\theta$, $z \in (\overline{\Sigma_{\varphi_0(\theta)}})^n$ and $\lambda \in \mathbb{C}$. We set $M_0 := \frac{1}{2^{2m} L} \sin \frac{\theta-\omega}{4}$.

(I) If $\lambda \in \overline{\Sigma_{\varphi_0(\theta)}}$ then from Theorem 5.8 it follows that

$$|\mu - P(z, \lambda)| \geq M_0\, \beta(z, \mu)^{2m}.$$

(II) Taking into account step (I) and Lemma 5.5 we have

$$\begin{aligned}|\mu - P(z, \lambda)| &\geq |\mu - P(z, 0)| - |P(z, 0) - P(z, \lambda)|\\ &\geq M_0\, \beta(z, \mu)^{2m} - 2^{2m}\, \ell_{n+1,2m}\, L\, |\lambda| \max\left\{\|z\|^{2m-1},\ |\lambda|^{2m-1}\right\}\\ &\geq M_0\, \beta(z, \mu)^{2m} - 2^{2m}\, \ell_{n+1,2m}\, L\, |\lambda| \max\left\{\beta(z, \mu)^{2m-1},\ |\lambda|^{2m-1}\right\}.\end{aligned}$$

Therefore if $C \in]0, 1]$ and $|\lambda| \leq C\beta(z, \mu)$ then we have

$$|\mu - P(z, \lambda)| \geq \left(M_0 - 2^{2m}\, \ell_{n+1,2m}\, L\, C\right)\beta(z, \mu)^{2m},$$

so that in particular for $C = (2^{2m+1}\, \ell_{n+1,2m}\, L)^{-1}\, M_0$ we get

$$|\mu - P(z, \lambda)| \geq \frac{1}{2}\, M_0\, \beta(z, \mu)^{2m}.$$

(III) Since $|c_{2m,0}| \geq L^{-1}$ (see Remark 5.2 (c)) we have

$$\begin{aligned}|\mu - P(z, \lambda)| &\geq |c_{2m,0}|\, |\lambda|^{2m} - |\mu| - \sum_{j=0}^{2m-1} \Bigl(\sum_{|\alpha|=2m-j} |c_{j,\alpha}|\, |z^\alpha|\Bigr)\, |\lambda|^j\\ &\geq L^{-1}\, |\lambda|^{2m} - |\mu| - \sum_{j=0}^{2m-1} \ell_{n,2m-j}\, L\, \|z\|^{2m-j}\, |\lambda|^j\\ &\geq L^{-1}\, |\lambda|^{2m} - |\mu| - \ell_{n+1,2m}\, L\, \|z\| \max\left\{\|z\|^{2m-1},\ |\lambda|^{2m-1}\right\}.\end{aligned}$$

Therefore if $|\lambda| \geq \beta(z, \mu)$, then we have

$$\begin{aligned}|\mu - P(z, \lambda)| &\geq L^{-1}\, |\lambda|^{2m} - \beta(z, \mu)^{2m} - \ell_{n+1,2m}\, L\, \beta(z, \mu)\, |\lambda|^{2m-1}\\ &= |\lambda|^{2m-1} \left(L^{-1}\, |\lambda| - \ell_{n+1,2m}\, L\, \beta(z, \mu)\right) - \beta(z, \mu)^{2m};\end{aligned}$$

so that in particular for $|\lambda| \geq 2\, \ell_{n+1,2m}\, L^2\, \beta(z, \mu)$ we get

$$|\mu - P(z, \lambda)| \geq (2^{2m-1}\, \ell_{n+1,2m}^{2m}\, L^{4m-1} - 1)\, \beta(z, \mu)^{2m}.$$

Hence the assertion is proved, with

$$M_1 := \min\left\{M_0/2,\ 2^{2m-1}\,\ell_{n+1,2m}^{2m}\,L^{4m-1}-1\right\} = M_0/2 = \frac{1}{2^{2m+1}\,L}\,\sin\frac{\theta-\omega}{4}$$

and

$$r := \max\left\{2^{2m+1}\,\ell_{n+1,2m}\,L\,M_0^{-1},\ 2\,\ell_{n+1,2m}\,L^2\right\} = 2^{2m+1}\,\ell_{n+1,2m}\,L\,M_0^{-1}$$
$$= \frac{2^{4m+1}\,\ell_{n+1,2m}\,L^2}{\sin((\theta-\omega)/4)}. \qquad \square$$

If γ is a circuit parametrized by the function ψ, and $c \in \mathbb{C}$, we denote by $c\gamma$ the circuit parametrized by $c\psi$; it is obvious that the length of $c\gamma$ equals $|c|$ times the length of γ; moreover if $c \neq 0$ and $a \notin \gamma$, then $ca \notin c\gamma$ and $w(c\gamma, ca) = w(\gamma, a)$.

Definition 5.12. Let $\theta \in]\omega, \pi[$, and let $r(L,\omega,\theta)$ have the same meaning as in Theorem 5.11. We call γ_θ^+ the closed curve (oriented counterclockwise) composed by:

(i) the arc of the circle centred at 0 with radius $r(L,\omega,\theta)$, from $r(L,\omega,\theta)\,e^{-i(\frac{\pi}{2}-\varphi_0(\theta))}$ to $r(L,\omega,\theta)\,e^{i(\frac{\pi}{2}-\varphi_0(\theta))}$;
(ii) the segment from $r(L,\omega,\theta)\,e^{i(\frac{\pi}{2}-\varphi_0(\theta))}$ to $r(L,\omega,\theta)^{-1}\,e^{i(\frac{\pi}{2}-\varphi_0(\theta))}$;
(iii) the arc of the circle centred at 0 with radius $r(L,\omega,\theta)^{-1}$, from $r(L,\omega,\theta)^{-1}\,e^{i(\frac{\pi}{2}-\varphi_0(\theta))}$ to $r(L,\omega,\theta)^{-1}\,e^{-i(\frac{\pi}{2}-\varphi_0(\theta))}$;
(iv) the segment from $r(L,\omega,\theta)^{-1}\,e^{-i(\frac{\pi}{2}-\varphi_0(\theta))}$ to $r(L,\omega,\theta)\,e^{-i(\frac{\pi}{2}-\varphi_0(\theta))}$.

Moreover, we set $\gamma_\theta^- = -\gamma_\theta^+$, and for $(0,0) \neq (z,\mu) \in \mathbb{C}^n \times \mathbb{C}$ $\gamma_\theta^\pm(z,\mu) = \beta(z,\mu)\gamma_\theta^\pm$.

It follows from Definition 5.12 that if $\lambda \in \gamma_\theta^+ \cup \gamma_\theta^-$, then either $\lambda \in \overline{\Sigma_{\varphi_0(\theta)}}$ or $|\lambda| \in \{r(L,\omega,\theta)^{-1}, r(L,\omega,\theta)\}$. Hence the following theorem is a straightforward consequence of Theorem 5.11 and Definition 5.12.

Theorem 5.13. *Let* $\theta \in]\omega,\pi[$, $(0,0) \neq (z,\mu) \in \mathbb{C}^n \times \mathbb{C}$. *Then we have:*

(a) *if* $\lambda \in \gamma_\theta^\pm(z,\mu)$, *then* $\pm Re\,\lambda \geq M_2(L,\omega,\theta)\,\beta(z,\mu)$, *where*

$$M_2(L,\omega,\theta) = r(L,\omega,\theta)^{-1}\,\sin\varphi_0(\theta);$$

(b) $\gamma_\theta^+(z,\mu)$ *and* $\gamma_\theta^-(z,\mu)$ *are disjoint from* $\Sigma_{\varphi_0(\theta)}$;
(c) *if* $\mu \in \mathbb{C}\setminus S_\theta$ *and* $z \in (\overline{\Sigma_{\varphi_0(\theta)}})^n$, *then any solution* λ *with positive (negative) real part of the equation* $P(z,\lambda) = \mu$ *does not belong to* $\gamma_\theta^+(z,\mu)$ *(to* $\gamma_\theta^-(z,\mu)$*) and has winding number equal to* 1 *with respect to* $\gamma_\theta^+(z,\mu)$ *(to* $\gamma_\theta^-(z,\mu)$*);*
(d) *if* $\mu \in \mathbb{C}\setminus S_\theta$, $z \in (\overline{\Sigma_{\varphi_0(\theta)}})^n$ *and* $\lambda \in \gamma_\theta^+(z,\mu) \cup \gamma_\theta^-(z,\mu)$, *then*

$$|\mu - P(z,\lambda)| \geq M_1(L,\omega,\theta)\,\beta(z,\mu)^{2m}.$$

The boundary operators relative to the differential operator $P(D_x, D_t)$ will be expressed through m homogeneous polynomials $B_1, \dots, B_m$ in $n+1$ variables, with $\deg B_k = m_k < 2m$. We assume that these polynomials satisfy an "ω-complementing condition", that is:

$$\text{(5.14)}\qquad \left[\begin{array}{l} \text{if } (0,0) \neq (x,\mu) \in \mathbb{R}^n \times ((\mathbb{C}\setminus \overline{S_\omega}) \cup \{0\}) \text{ then the polynomials} \\ \\ B_1(ix,\cdot), \dots, B_m(ix,\cdot) \text{ are linearly independent modulo } P^-_{ix,\mu}. \end{array}\right.$$

Here one could make a remark similar to 5.2 (a) concerning the fact that (5.14) is satisfied for some $\omega \in \,]0,\pi[$ provided that it holds for $\mu \in \mathbb{R}^- \cup \{0\}$. What we actually need, however, is a similar condition with ix replaced by $z \in (\overline{\Sigma_\alpha})^n$ for some $\alpha > 0$, and so in the last part of this section we are going to deduce it from (5.14).

For the sequel of this section we fix $\theta \in \,]\omega,\pi[$.

Assume that $(0,0) \neq (z,\mu) \in (\overline{\Sigma_{\varphi_0(\theta)}})^n \times (\mathbb{C}\setminus S_\theta)$. For $j, k \in \{1,\dots,m\}$ we set

$$g_{k,j}(z,\mu) = \frac{1}{2\pi i}\int_{\gamma_\theta^-(z,\mu)} \frac{\lambda^{j-1} B_k(z,\lambda)}{\mu - P(z,\lambda)}\, d\lambda \tag{5.15}$$

and we call $G(z,\mu)$ the $m \times m$ matrix with entries $g_{k,j}(z,\mu)$.

LEMMA 5.16. *Let* $(0,0) \neq (z,\mu) \in (\overline{\Sigma_{\varphi_0(\theta)}})^n \times (\mathbb{C}\setminus S_\theta)$, $\tau \in \mathbb{R}^+$. *Then*

$$\det G(\tau z, \tau^{2m}\mu) = \tau^{((m-3m^2)/2)+\sum_{k=1}^m m_k} \det G(z,\mu).$$

PROOF. $\det G(z,\mu)$ is a sum of $m!$ addenda, each one of which is $\pm\prod_{k=1}^m g_{k,\sigma(k)}(z,\mu)$, where σ is a permutation of $\{1,\dots,m\}$. Since, for $\tau \in \mathbb{R}^+$

$$g_{k,j}(\tau z, \tau^{2m}\mu) = \frac{1}{2\pi i}\int_{\gamma_\theta^-(\tau z,\tau^{2m}\mu)} \frac{\lambda^{j-1} B_k(\tau z,\lambda)}{\tau^{2m}\mu - P(\tau z,\lambda)}\, d\lambda = \tau^{j+m_k-2m} g_{k,j}(z,\mu)$$

an easy computation concludes the proof. □

LEMMA 5.17. $\exists\, L_0(\theta) \in \mathbb{R}^+$ *such that for* $(0,0) \neq (x,\mu) \in \mathbb{R}^n \times (\mathbb{C}\setminus S_\theta)$

$$|\det G(ix,\mu)| \geq L_0(\theta)\, \beta(ix,\mu)^{((m-3m^2)/2)+\sum_{k=1}^m m_k}.$$

PROOF. The complementing condition (5.14) and Lemma 3.4 imply that $\det G(ix,\mu) \neq 0$ when $(0,0) \neq (x,\mu) \in \mathbb{R}^n \times (\mathbb{C}\setminus S_\theta)$. Since the set

$$V_\theta := \{(ix,\mu) \in (i\,\mathbb{R})^n \times (\mathbb{C}\setminus S_\theta);\ \beta(ix,\mu) = 1\}$$

is compact, the result follows from Lemma 5.16 with $L_0(\theta) = \min_{V_\theta} |\det G|$. □

In the sequel L_0 will denote the function on $]\omega,\pi[$ introduced in Lemma 5.17. We denote with L_1 the maximum of the moduli of the coefficients of the polynomials $B_1, \dots, B_m$.

THEOREM 5.18. *There exists* $\varphi \in]0, \varphi_0(\theta)]$ (*depending on* L, L_1, ω, θ, $L_0(\theta)$) *such that if* $(0, 0) \neq (z, \mu) \in (\overline{\Sigma_\varphi})^n \times (\mathbb{C} \setminus S_\theta)$ *then*

$$|\det G(z, \mu)| \geq \frac{L_0(\theta)}{2}\, \beta(z, \mu)^{((m-3m^2)/2)+\sum_{k=1}^m m_k}.$$

PROOF. Let $\varphi \in]0, \varphi_0(\theta)]$ and $(0, 0) \neq (z, \mu) \in (\overline{\Sigma_\varphi})^n \times (\mathbb{C} \setminus S_\theta)$. For suitable $x \in \mathbb{R}^n$ and $\alpha_1, \ldots, \alpha_n \in [-\varphi, \varphi]$ we have $z = (e^{i\alpha_1} i x_1, \ldots, e^{i\alpha_n} i x_n)$. Because of Lemma 5.17, the theorem will be proved if we show that when φ is suitably small

$$|\det G(z, \mu) - \det G(ix, \mu)| \leq \frac{L_0(\theta)}{2}\, \beta(z, \mu)^{((m-3m^2)/2)+\sum_{k=1}^m m_k}.$$

To this end we remark that $\det G(z, \mu) - \det G(ix, \mu)$ can be expressed as the sum of $m!$ addenda, each one of the type

$$\begin{aligned}
&\prod_{k=1}^m g_{k,\sigma(k)}(z, \mu) - \prod_{k=1}^m g_{k,\sigma(k)}(ix, \mu)\\
&= \sum_{h=1}^m \left(\prod_{k=1}^{h-1} g_{k,\sigma(k)}(ix, \mu) \left(g_{h,\sigma(h)}(z, \mu) - g_{h,\sigma(h)}(ix, \mu) \right) \prod_{k=h+1}^m g_{k,\sigma(k)}(z, \mu) \right)
\end{aligned}$$

where σ describes the set of the permutations of $\{1, \ldots, m\}$. Therefore we are going to prove the following estimates:

$$|g_{k,j}(z, \mu)| \leq C(L, L_1, \omega, \theta)\, \beta(z, \mu)^{m_k+j-2m} \tag{5.19}$$

(and likewise for $g_{k,j}(ix, \mu)$, noticing that $\beta(ix, \mu) = \beta(z, \mu)$) and

$$|g_{k,j}(z, \mu) - g_{k,j}(ix, \mu)| \leq C(L, L_1, \omega, \theta)\, \varphi\, \beta(z, \mu)^{m_k+j-2m}. \tag{5.20}$$

Once we have obtained these estimates, it follows that

$$\begin{aligned}
\left| \prod_{k=1}^m g_{k,\sigma(k)}(z, \mu) - \prod_{k=1}^m g_{k,\sigma(k)}(ix, \mu) \right| &\leq C\, \varphi\, \beta(z, \mu)^{\sum_{k=1}^m (m_k+\sigma(k)-2m)}\\
&= C\, \varphi\, \beta(z, \mu)^{((m-3m^2)/2)+\sum_{k=1}^m m_k},
\end{aligned}$$

so that for a sufficiently small φ we get the result.

Let us prove (5.19). We have

$$|B_k(z, \lambda)| \leq L_1\, \ell_{n+1,m_k}\, \|(z, \lambda)\|^{m_k}; \tag{5.21}$$

moreover the length of $\gamma_\theta^-(z,\mu)$ is $\leq 2(1+\pi)\,r\,\beta(z,\mu)$ and on $\gamma_\theta^-(z,\mu)$ $|\mu - P(z,\lambda)| \geq M_1\,\beta(z,\mu)^{2m}$ (see Theorem 5.13 (d)) and $|\lambda| \leq r\,\beta(z,\mu)$. Therefore

$$\begin{aligned} |g_{k,j}(z,\mu)| &\leq \frac{1}{2\pi}\int_{\gamma_\theta^-(z,\mu)} \frac{|\lambda|^{j-1}\,|B_k(z,\lambda)|}{|\mu - P(z,\lambda)|}\,d|\lambda| \\ &\leq \frac{1}{2\pi}\int_{\gamma_\theta^-(z,\mu)} \frac{|\lambda|^{j-1}\,L_1\,\ell_{n+1,m_k}\,\|(z,\lambda)\|^{m_k}}{M_1\,\beta(z,\mu)^{2m}}\,d|\lambda| \leq C\,\beta(z,\mu)^{m_k+j-2m}. \end{aligned}$$

Let us prove (5.20). Since $\beta(ix,\mu) = \beta(z,\mu)$, we have $\gamma_\theta^-(ix,\mu) = \gamma_\theta^-(z,\mu)$; hence

$$\begin{aligned} |g_{k,j}(z,\mu) - g_{k,j}(ix,\mu)| &= \frac{1}{2\pi}\left| \int_{\gamma_\theta^-(ix,\mu)} \frac{\lambda^{j-1}\,(B_k(z,\lambda) - B_k(ix,\lambda))}{\mu - P(z,\lambda)}\,d\lambda \right. \\ &\qquad \left. + \int_{\gamma_\theta^-(ix,\mu)} \left(\frac{\lambda^{j-1}\,B_k(ix,\lambda)}{\mu - P(z,\lambda)} - \frac{\lambda^{j-1}\,B_k(ix,\lambda)}{\mu - P(ix,\lambda)} \right) d\lambda \right| \\ &\leq \frac{1}{2\pi}\left(\int_{\gamma_\theta^-(ix,\mu)} \frac{|\lambda|^{j-1}\;|B_k(z,\lambda) - B_k(ix,\lambda)|}{|\mu - P(z,\lambda)|}\,d|\lambda| \right. \\ &\qquad \left. + \int_{\gamma_\theta^-(ix,\mu)} \frac{|\lambda|^{j-1}|B_k(ix,\lambda)||P(z,\lambda) - P(ix,\lambda)|}{|\mu - P(ix,\lambda)|\,|\mu - P(z,\lambda)|}\,d|\lambda| \right). \end{aligned}$$

As in the proof of Theorem 5.6, we have $\|(z,\lambda) - (ix,\lambda)\| \leq \varphi\,\|z\|$; therefore from Lemma 5.5 we get

$$\begin{aligned} &|B_k(z,\lambda) - B_k(ix,\lambda)| \\ &\leq 2^{m_k}\ell_{n+1,m_k}L_1\,\|(z,\lambda) - (ix,\lambda)\|(\max\{\|(ix,\lambda)\|, \|(z,\lambda) - (ix,\lambda)\|\})^{m_k-1} \\ &\leq 2^{m_k}\ell_{n+1,m_k}L_1\,\varphi\,\|z\|\,\|(ix,\lambda)\|^{m_k-1} \leq 2^{m_k}\ell_{n+1,m_k}L_1\,\varphi\,\|(z,\lambda)\|^{m_k}. \end{aligned}$$

Analogously

$$\begin{aligned} &|P(z,\lambda) - P(ix,\lambda)| \\ &\leq 2^{2m}\ell_{n+1,2m}L\,\|(z,\lambda) - (ix,\lambda)\|(\max\{\|(ix,\lambda)\|, \|(z,\lambda) - (ix,\lambda)\|\})^{2m-1} \\ &\leq 2^{2m}\ell_{n+1,2m}L\,\varphi\,\|z\|\,\|(ix,\lambda)\|^{2m-1} \leq 2^{2m}\ell_{n+1,2m}L\,\varphi\,\|(z,\lambda)\|^{2m}. \end{aligned}$$

Therefore

$$
\begin{aligned}
&|g_{k,j}(z,\mu)-g_{k,j}(ix,\mu)| \\
&\leq \frac{1}{2\pi}\left(\int_{\gamma_\theta^-(ix,\mu)} \frac{|\lambda|^{j-1}\, 2^{m_k}\ell_{n+1,m_k} L_1\,\varphi\,\|(z,\lambda)\|^{m_k}}{M_1\,\beta(z,\mu)^{2m}}\, d|\lambda|\right. \\
&\quad \left. + \int_{\gamma_\theta^-(ix,\mu)} \frac{|\lambda|^{j-1}\, L_1\ell_{n+1,m_k}\|(z,\lambda)\|^{m_k}\, 2^{2m}\ell_{n+1,2m} L\,\varphi\,\|(z,\lambda)\|^{2m}}{M_1^2\,\beta(z,\mu)^{4m}}\, d|\lambda|\right) \\
&\leq C\,\varphi\,\beta(z,\mu)^{m_k+j-2m}. \qquad \square
\end{aligned}
$$

In the sequel we shall use the function $\theta \mapsto \varphi(\theta)$, from $]\omega,\pi[$ to $]0,\frac{\pi}{2}[$ implicitly defined in the statement of Theorem 5.18.

6. – The ordinary differential operators A_z

In this section, as well as in the next ones, we fix $p \in]1,\infty[$, and denote by p' the exponent conjugate to p. The polynomials P and $B_1,\ldots,B_m$ are the same as in Section 5.

$\forall z \in \mathbb{C}^n$ we consider the operator A_z defined by

$$
\begin{aligned}
&\mathcal{D}(A_z) = \{u \in W^{2m,p}(\mathbb{R}^+);\, (B_k(z,D)u)(0) = 0 \text{ for } 1 \leq k \leq m\} \\
&A_z u = P(z,D)u \quad \forall u \in \mathcal{D}(A_z).
\end{aligned}
$$

Concerning the definition of A_z, we recall that a function $v \in W^{1,p}(\mathbb{R}^+)$ is almost everywhere equal to a continuous function on $[0,+\infty[$, so that when $u \in W^{2m,p}(\mathbb{R}^+)$ the value at $t=0$ of $B_k(z,D)u$ is well defined. Since it is obvious that both the domain and the range of A_z are subspaces of $L^p(\mathbb{R}^+)$, we look at A_z as an unbounded operator in the Banach space $L^p(\mathbb{R}^+)$.

6.1. – The operators A_z are sectorial

The aim of this subsection is to prove the following result.

THEOREM 6.1. *Let $\theta \in]\omega,\pi[$ and $z \in (\overline{\Sigma_{\varphi(\theta)}})^n$. Then A_z is a sectorial operator with spectral angle θ.*

Actually, what we shall prove is

LEMMA 6.2. *Let $\theta \in]\omega,\pi[$ and assume that $(0,0) \neq (z,\mu) \in (\overline{\Sigma_{\varphi(\theta)}})^n \times (\mathbb{C}\setminus S_\theta)$. Then $\mu \in \rho(A_z)$ and*

$$
\|(\mu - A_z)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}^+))} \leq \frac{C(L,L_1,\omega,\theta,L_0(\theta))}{\|z\|^{2m}+|\mu|}. \tag{6.3}
$$

Once proved Lemma 6.2, going back to Theorem 6.1 we remark that

(i) the domain of A_z is dense in $L^p(\mathbb{R}^+)$ as it contains $C_0^\infty(\mathbb{R}^+)$;
(ii) when $z \neq 0$ Lemma 6.2 implies that $0 \in \rho(A_z)$, so that A_z is boundedly invertible, and in particular $\mathcal{R}(A_z) = L^p(\mathbb{R}^+)$;
(iii) A_0 is the restriction to $\mathcal{D}(A_0)$ of $c_{2m,0}\, D^{2m}$ (where D is the derivative operator in $L^p(\mathbb{R}^+)$), and hence it is injective, because such is D: this implies that $\mathcal{R}(A_0)$ is dense in $L^p(\mathbb{R}^+)$ since this is a reflexive space;
(iv) for $\mu \in \mathbb{C} \setminus \overline{S_\theta}$ the inequality (6.3) yields

$$\|\mu(\mu - A_z)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}^+))} \leq C(L, L_1, \omega, \theta, L_0(\theta)).$$

Thus Theorem 6.1 will be proved.

However in Section 8 we'll need something more, and so instead of (6.3) we are going to prove that

$$\|D^\ell(\mu - A_z)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}^+))} \leq C(L, L_1, \omega, \theta, L_0(\theta))\, \beta(z, \mu)^{\ell - 2m} \tag{6.4}$$

(for $\ell \leq 2m$) of which (6.3) is a particular case, since $\beta(z, \mu)^{2m} \geq \|z\|^{2m} + |\mu|$.

* * *

Solving in $\mathcal{D}(A_z)$ the equation

$$\mu\, u - A_z u = f$$

with $f \in L^p(\mathbb{R}^+)$, is the same as solving the problem

$$\begin{cases} u \in W^{2m,p}(\mathbb{R}^+) \\ \mu\, u(t) - P(z, D)u(t) = f(t) & t \in \mathbb{R}^+ \\ (B_k(z, D)u)(0) = 0 & 1 \leq k \leq m. \end{cases} \tag{6.5}$$

From the ω-complementing condition and Theorems 3.8 and 5.18 we get that the homogeneous equation with arbitrary initial data has a unique solution: this proves that problem (6.5) has at most one solution; in order to prove the existence of the solution of (6.5) we only need prove the existence in $W^{2m,p}(\mathbb{R}^+)$ of a solution of the equation $\mu\, u - P(z, D)u = f$: after that it is enough to sum this solution v with the solution w of the homogeneous equation with such initial conditions that annihilate the initial data of v.

As usual, we need some preliminary results.

Lemma 6.6. *Let $Q \in \mathcal{P}$ with* $\deg Q = r \geq 1$, $Q(\lambda) = \sum_{k=0}^r a_k\, \lambda^k$. *Let $\gamma \in \Lambda_1(Q)$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{\lambda^j}{Q(\lambda)}\, d\lambda = \begin{cases} 0 & \text{if } 0 \leq j < r - 1 \\ a_r^{-1} & \text{if } j = r - 1. \end{cases}$$

Proof. Since the integral does not depend on $\gamma \in \Lambda_1(Q)$, it is not restrictive to assume that γ be a circle centred at 0, with large radius ρ, oriented counterclockwise. Since the integral does not depend on ρ, the result can be easily obtained by letting $\rho \to +\infty$. □

DEFINITION 6.7. Let $\theta \in]\omega, \pi[$, and $(0,0) \neq (z,\mu) \in (\overline{\Sigma_{\varphi(\theta)}})^n \times (\mathbb{C} \setminus S_\theta)$. We set

$$H_{z,\mu}(x) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_\theta^-(z,\mu)} \frac{e^{\lambda x}}{\mu - P(z,\lambda)}\, d\lambda & \text{if } x \geq 0 \\ -\dfrac{1}{2\pi i} \displaystyle\int_{\gamma_\theta^+(z,\mu)} \frac{e^{\lambda x}}{\mu - P(z,\lambda)}\, d\lambda & \text{if } x \leq 0. \end{cases}$$

REMARK 6.8. It follows from Theorem 5.13 (a), (c) that

$$\gamma_\theta^+(z,\mu) \cup \gamma_\theta^-(z,\mu) \in \Lambda_1(\mu - P(z,\cdot)).$$

Since $\deg(\mu - P(z,\cdot)) = 2m \geq 2$, we get from Lemma 6.6 that

$$\int_{\gamma_\theta^+(z,\mu)} \frac{1}{\mu - P(z,\lambda)}\, d\lambda + \int_{\gamma_\theta^-(z,\mu)} \frac{1}{\mu - P(z,\lambda)}\, d\lambda = 0$$

and that ensures that $H_{z,\mu}$ is correctly defined at $x = 0$. The independence of $H_{z,\mu}$ from θ, and more generally the independence from θ of any integral of the type

$$\int_{\gamma_\theta^\pm(z,\mu)} \frac{Q(z,\lambda)\, e^{\lambda x}}{\mu - P(z,\lambda)}\, d\lambda$$

where Q is a polynomial, is an obvious consequence of Cauchy's theorem. $\square$

REMARK 6.9. Let us deduce some properties of the functions $H_{z,\mu}$. It is understood that $\theta \in]\omega, \pi[$ and $(0,0) \neq (z,\mu) \in (\overline{\Sigma_{\varphi(\theta)}})^n \times (\mathbb{C} \setminus S_\theta)$.

(a) It is obvious that $H_{z,\mu} \in C^\infty(\mathbb{R} \setminus \{0\})$, with

$$H_{z,\mu}^{(j)}(x) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_\theta^-(z,\mu)} \frac{\lambda^j e^{\lambda x}}{\mu - P(z,\lambda)}\, d\lambda & \text{if } x > 0 \\ -\dfrac{1}{2\pi i} \displaystyle\int_{\gamma_\theta^+(z,\mu)} \frac{\lambda^j e^{\lambda x}}{\mu - P(z,\lambda)}\, d\lambda & \text{if } x < 0. \end{cases}$$

In particular the limits $H_{z,\mu}^{(j)}(0^+)$ and $H_{z,\mu}^{(j)}(0^-)$ exist; by Lemma 6.6 we get

$$H_{z,\mu}^{(j)}(0^+) - H_{z,\mu}^{(j)}(0^-) = 0 \text{ if } 0 \leq j \leq 2m-2$$

$$H_{z,\mu}^{(2m-1)}(0^+) - H_{z,\mu}^{(2m-1)}(0^-) = -c_{2m,0}^{-1}$$

(see the representation of P given in Remark 5.2). In particular $H_{z,\mu} \in C^{2m-2}(\mathbb{R})$.
Here, as always in the following, we have denoted by $H_{z,\mu}^{(j)}$ the j-th derivative of $H_{z,\mu}$ on $\mathbb{R} \setminus \{0\}$; the symbol $D^j H_{z,\mu}$ will always denote its j-th distributional derivative on $\mathbb{R}$.

(b) Taking into account the estimates of Theorem 5.13 (a), (d), for $x \in \mathbb{R}^\pm$ we obtain from (a)

$$|H^{(j)}_{z,\mu}(x)| \leq \frac{1}{2\pi} \int_{\gamma_\theta^\mp(z,\mu)} \frac{|\lambda|^j}{M_1\,\beta(z,\mu)^{2m}}\, d|\lambda|\, e^{-M_2\beta(z,\mu)|x|} = C\,\beta(z,\mu)^{j+1-2m}\, e^{-M_2\beta(z,\mu)|x|}.$$

Then we get that $H^{(j)}_{z,\mu} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\forall q \in [1,\infty]$

$$\|H^{(j)}_{z,\mu}\|_{L^q(\mathbb{R})} \leq C(q)\,\beta(z,\mu)^{j-2m+1-(1/q)}.$$

(c) Let D_0 denote the distributional derivative on $\mathbb{R}\setminus\{0\}$. By (a), we get $D^j H_{z,\mu} = D_0^j H_{z,\mu}$ for $0 \leq j \leq 2m-1$, while $D^{2m} H_{z,\mu} = D_0^{2m} H_{z,\mu} - c_{2m,0}^{-1}\,\delta_0$ (where δ_0 is Dirac's measure at 0).
Let $f \in L^p(\mathbb{R}^+)$, and let us extend f to $\mathbb{R}$ by setting $f(x) = 0$ for $x < 0$. Then $H_{z,\mu} * f \in L^p(\mathbb{R})$, and

$$D^\ell(H_{z,\mu} * f) = \begin{cases} D_0^\ell H_{z,\mu} * f & \text{if } \ell < 2m \\ D_0^{2m} H_{z,\mu} * f - c_{2m,0}^{-1} f & \text{if } \ell = 2m \end{cases}$$

so that

$$(\mu - P(z,D))(H_{z,\mu} * f) = \Big((\mu - P(z,D_0))\,H_{z,\mu}\Big) * f + f.$$

Since for $x \in \mathbb{R}^\pm$

$$\Big((\mu - P(z,D_0))\,H_{z,\mu}\Big)(x) = \frac{\pm 1}{2\pi i} \int_{\gamma_\theta^\mp(z,\mu)} e^{\lambda x}\, d\lambda = 0$$

we have $(\mu - P(z,D))(H_{z,\mu} * f) = f$. $\square$

As we have seen in Remark 6.9 (c), $\forall f \in L^p(\mathbb{R}^+)$ $H_{z,\mu} * f$ (or more precisely the restriction to $\mathbb{R}^+$ of the convolution between $H_{z,\mu}$ and the natural extension of f to $\mathbb{R}$) is a solution of the inhomogeneous equation. As we noticed above, as a consequence of Theorem 3.8, this fact implies that problem (6.5) has a unique solution, so that $\mu - A_z$ is proved to be a bijective operator from $\mathcal{D}(A_z)$ onto $L^p(\mathbb{R}^+)$. We shall write this solution in the form

$$(\mu - A_z)^{-1} f = H_{z,\mu} * f + w_{z,\mu}(f).$$

where $w_{z,\mu}(f)$ is the solution of the homogeneous equation that annihilates the initial values of $H_{z,\mu} * f$. For $0 \leq \ell \leq 2m$, we have, by Remark 6.9 (b), $\|H^{(\ell)}_{z,\mu}\|_{L^1(\mathbb{R})} \leq C\beta(z,\mu)^{\ell-2m}$; therefore Young's inequality and Remark 6.9 (c) yield

$$\|D^\ell(H_{z,\mu} * f)\|_{L^p(\mathbb{R}^+)} \leq C\,\beta(z,\mu)^{\ell-2m}\,\|f\|_{L^p(\mathbb{R}^+)} \tag{6.10}$$

whenever $\theta \in]\omega,\pi[$, and $(0,0) \neq (z,\mu) \in (\overline{\Sigma_{\varphi(\theta)}})^n \times (\mathbb{C}\setminus S_\theta)$.

Now we turn to study $w_{z,\mu}(f)$. As $\deg B_k < 2m$, Remark 6.9 (c) yields also

$$
\begin{aligned}
\Big(B_k(z,D)(H_{z,\mu} * f)\Big)(0) &= \Big((B_k(z,D)H_{z,\mu}) * f\Big)(0) \\
&= \int_0^\infty (B_k(z,D)H_{z,\mu})(-s)\, f(s)\, ds \\
&= -\frac{1}{2\pi i}\int_0^\infty \int_{\gamma_\theta^+(z,\mu)} \frac{B_k(z,\lambda)\, e^{-\lambda s}}{\mu - P(z,\lambda)}\, d\lambda\, f(s)\, ds \\
&= -\int_0^\infty h_{k,z,\mu}(s)\, f(s)\, ds
\end{aligned}
$$

where we have set

$$
h_{k,z,\mu}(s) = \frac{1}{2\pi i}\int_{\gamma_\theta^+(z,\mu)} \frac{B_k(z,\lambda)\, e^{-\lambda s}}{\mu - P(z,\lambda)}\, d\lambda. \tag{6.11}
$$

Then $w_{z,\mu}(f)$ is the solution of the problem

$$
\begin{cases}
w \in W^{2m,p}(\mathbb{R}^+) & \\
\mu\, w(t) - P(z,D)w(t) = 0 & t \in \mathbb{R}^+ \\
(B_k(z,D)w)(0) = \int_0^\infty h_{k,z,\mu}(s)\, f(s)\, ds & 1 \le k \le m.
\end{cases} \tag{6.12}
$$

Lemma 6.13. *Let* $(b_1, \ldots, b_m) \in \mathbb{C}^m$, *and let* $(0,0) \neq (z,\mu) \in (\overline{\Sigma_{\varphi(\theta)}})^n \times (\mathbb{C} \setminus S_\theta)$ *for some* $\theta \in]\omega, \pi[$. *The problem*

$$
\begin{cases}
w \in W^{2m,p}(\mathbb{R}^+) & \\
\mu\, w(t) - P(z,D)w(t) = 0 & t \in \mathbb{R}^+ \\
(B_k(z,D)w)(0) = b_k & 1 \le k \le m.
\end{cases} \tag{6.14}
$$

has a unique solution $w_{z,\mu}$ *of the form* $\sum_{j,k=1}^m \delta_{j,k}(z,\mu)\, b_k\, u_{j,z,\mu}$ *where* $\delta_{j,k}(z,\mu) \in \mathbb{C}$, $u_{j,z,\mu} \in L^p(\mathbb{R}^+)$ *and they satisfy the estimates*

$$
|\delta_{j,k}(z,\mu)| \le C(L, L_1, \omega, \theta, L_0(\theta))\, \beta(z,\mu)^{2m-m_k-j}
$$

$$
\|u_{j,z,\mu}\|_{L^p(\mathbb{R}^+)} \le C(L, \omega, \theta)\, \beta(z,\mu)^{j-2m-(1/p)}.
$$

Moreover the inequality

$$
\|D^\ell w_{z,\mu}\|_{L^p(\mathbb{R}^+)} \le C(L, L_1, \omega, \theta, L_0(\theta)) \sum_{k=1}^m \beta(z,\mu)^{\ell - m_k - (1/p)} |b_k|
$$

holds for $\ell \le 2m$.

PROOF. The existence and uniqueness of the solution of problem (6.14) are direct consequences of Theorem 3.8, since $\det G(z,\mu) \neq 0$ by Theorem 5.18. We set (for $1 \le j \le m$ and $t > 0$)

$$u_{j,z,\mu}(t) = H_{z,\mu}^{(j-1)}(t) = \frac{1}{2\pi i} \int_{\gamma_\theta^-(z,\mu)} \frac{\lambda^{j-1} e^{\lambda t}}{\mu - P(z,\lambda)}\, d\lambda. \tag{6.15}$$

Then, according to Theorem 3.5 (b), $\{u_{1,z,\mu}, \ldots, u_{m,z,\mu}\}$ is a basis of $\ker P^-_{z,\mu}(D)$, so that by Theorem 3.7, $w_{z,\mu}$ is a linear combination of $u_{1,z,\mu}, \ldots, u_{m,z,\mu}$:

$$w_{z,\mu} = \sum_{j=1}^m c_j(z,\mu)\, u_{j,z,\mu}$$

for some coefficients $c_j(z,\mu) \in \mathbb{C}$. Therefore

$$D^\ell w_{z,\mu} = \sum_{j=1}^m c_j(z,\mu)\, u_{j,z,\mu}^{(\ell)}. \tag{6.16}$$

We know from Remark 6.9 (b) that

$$\|u_{j,z,\mu}^{(\ell)}\|_{L^p(\mathbb{R}^+)} = \|H_{z,\mu}^{(\ell+j-1)}\|_{L^p(\mathbb{R}^+)} \le C\,\beta(z,\mu)^{\ell+j-2m-(1/p)}. \tag{6.17}$$

As for the coefficients $c_j(z,\mu)$, they have to satisfy the equalities

$$\begin{aligned} b_k = (B_k(z,D) w_{z,\mu})(0) &= \sum_{j=1}^m c_j(z,\mu)\,(B_k(z,D) u_{j,z,\mu})(0) \\ &= \sum_{j=1}^m \frac{c_j(z,\mu)}{2\pi i} \int_{\gamma_\theta^-(z,\mu)} \frac{\lambda^{j-1} B_k(z,\lambda)}{\mu - P(z,\lambda)}\, d\lambda = \sum_{j=1}^m g_{k,j}(z,\mu) c_j(z,\mu) \end{aligned}$$

(see (5.15)). Let us call $\delta_{j,k}(z,\mu)$ the coefficients of $G^{-1}(z,\mu)$; then

$$\delta_{j,k}(z,\mu) = (-1)^{j+k} \frac{\det G^{kj}(z,\mu)}{\det G(z,\mu)} \tag{6.18}$$

where $G^{kj}(z,\mu)$ is the matrix obtained from $G(z,\mu)$ by deleting the k-th row and the j-th column. Each addendum of $\det G^{kj}(z,\mu)$ is of the type $\pm \prod_{h\neq k} g_{h,\sigma(h)}(z,\mu)$, where σ is a bijective function of $\{1, \ldots, k-1, k+1, \ldots, m\}$ onto $\{1, \ldots, j-1, j+1, \ldots, m\}$. Therefore, by applying the estimate of Theorem 5.18 for $\det G(z,\mu)$ and formula (5.19) for $g_{k,j}(z,\mu)$, we obtain

$$|\delta_{j,k}(z,\mu)| \le C\,\beta(z,\mu)^{2m-m_k-j}, \tag{6.19}$$

where $C = C(L, L_1, \omega, \theta, L_0(\theta))$. Finally, from

$$c_j(z,\mu) = \sum_{k=1}^{m} \delta_{j,k}(z,\mu)\, b_k \tag{6.20}$$

we get the desired expression of the solution $w_{z,\mu}$. Moreover we have

$$|c_j(z,\mu)| \le C \sum_{k=1}^{m} \beta(z,\mu)^{2m-m_k-j}\, |b_k|$$

and (6.16), (6.17) yield

$$\begin{aligned}\|D^\ell w_{z,\mu}\|_{L^p(\mathbb{R}^+)} &\le \sum_{j=1}^{m} |c_j(z,\mu)|\, \|u^{(\ell)}_{j,z,\mu}\|_{L^p(\mathbb{R}^+)} \\ &\le C \sum_{j,k=1}^{m} \beta(z,\mu)^{2m-m_k-j}\, |b_k|\, \beta(z,\mu)^{\ell+j-2m-(1/p)} \\ &= m\, C \sum_{k=1}^{m} \beta(z,\mu)^{\ell-m_k-(1/p)}\, |b_k|. \qquad \square\end{aligned}$$

The following lemma, combined with (6.10), concludes the proof of Lemma 6.2 and of the inequality (6.4).

LEMMA 6.21. *Let* $(0,0) \neq (z,\mu) \in (\overline{\Sigma_{\varphi(\theta)}})^n \times (\mathbb{C} \setminus S_\theta)$ *for some* $\theta \in]\omega, \pi[$. $\forall f \in L^p(\mathbb{R}^+)$ *the solution of problem* (6.12) *is given by*

$$w_{z,\mu}(f) = \sum_{j,k=1}^{m} \delta_{j,k}(z,\mu) \int_0^\infty h_{k,z,\mu}(s)\, f(s)\, ds\, u_{j,z,\mu}$$

and

$$\|D^\ell(w_{z,\mu}(f))\|_{L^p(\mathbb{R}^+)} \le C\, \beta(z,\mu)^{\ell-2m}\, \|f\|_{L^p(\mathbb{R}^+)}.$$

PROOF. The expression of $w_{z,\mu}(f)$ is given by Lemma 6.13. By the same lemma we have

$$\|D^\ell(w_{z,\mu}(f))\|_{L^p(\mathbb{R}^+)} \le C \sum_{k=1}^{m} \beta(z,\mu)^{\ell-m_k-(1/p)} \left| \int_0^\infty h_{k,z,\mu}(s)\, f(s)\, ds \right|. \tag{6.22}$$

By the definition of $h_{k,z,\mu}$ (see (6.11)) and Theorem 5.13, we have

$$\begin{aligned}|h_{k,z,\mu}(t)| &\le \frac{1}{2\pi} \int_{\gamma_\theta^+(z,\mu)} \frac{|B_k(z,\lambda)|\, e^{-tRe\lambda}}{|\mu - P(z,\lambda)|}\, d|\lambda| \\ &\le \frac{1}{2\pi} \int_{\gamma_\theta^+(z,\mu)} |B_k(z,\lambda)|\, d|\lambda|\, \frac{e^{-tM_2\beta(z,\mu)}}{M_1\, \beta(z,\mu)^{2m}}\end{aligned}$$

and from (5.21), we get

$$|h_{k,z,\mu}(t)| \leq C\,\beta(z,\mu)^{m_k+1-2m}\, e^{-tM_2\beta(z,\mu)} \tag{6.23}$$

where $C = C(L, L_1, \omega, \theta)$, and so

$$\|h_{k,z,\mu}\|_{L^{p'}(\mathbb{R}^+)} \leq C\,\beta(z,\mu)^{m_k-2m+(1/p)}.$$

Now Hölder's inequality gives

$$\Big|\int_0^\infty h_{k,z,\mu}(s)\, f(s)\, ds\Big| \leq C\,\beta(z,\mu)^{m_k-2m+(1/p)}\, \|f\|_{L^p(\mathbb{R}^+)}. \tag{6.24}$$

and by inserting this inequality in (6.22) we get the result. $\square$

Summing up we have proved that for $\ell \leq 2m$ $D^\ell(\mu - A_z)^{-1}$ is a bounded operator on $L^p(\mathbb{R}^+)$, and that

$$D^\ell(\mu - A_z)^{-1} f = D^\ell(H_{z,\mu} * f) + \int_0^\infty D_t^\ell K_{z,\mu}(\cdot, s)\, f(s)\, ds \tag{6.25}$$

where we have set

$$K_{z,\mu}(t,s) = \sum_{j,k=1}^{m} \delta_{j,k}(z,\mu)\, h_{k,z,\mu}(s)\, u_{j,z,\mu}(t).$$

6.2. – Analyticity with respect to z

In this subsection we prove that $\forall \ell \in \mathbb{N}$ with $\ell \leq 2m$ the function $z \mapsto D^\ell\,(\mu - A_z)^{-1}$ is analytic.

LEMMA 6.26. *Let (M, ν) be a σ-finite measure space, let $q \in [1, \infty[$, and let Ω be an open subset of $\mathbb{C}^r$. We are given a function $F : M \times \Omega \to \mathbb{C}$, and we assume that:*

(a) *$\forall z \in \Omega$ the function $t \mapsto F(t, z)$ is measurable;*
(b) *for every compact subset W of Ω there exists a non-negative function $F_W \in L^q(\nu)$ such that $|F(t,z)| \leq F_W(t)\ \forall (t,z) \in M \times W$;*
(c) *$\forall t \in M$ the function $z \mapsto F(t, z)$ is holomorphic on Ω.*

Then $F(\cdot, z) \in L^q(\nu)\ \forall z \in \Omega$ and the function $z \mapsto F(\cdot, z)$ is holomorphic from Ω to $L^q(\nu)$.

Proof. We have obviously

$$\int_M |F(t,z)|^q \, d\nu(t) \le \int_M F_{\{z\}}(t)^q \, d\nu(t) < +\infty$$

and so $F(\cdot,z) \in L^q(\nu)$ $\forall z \in \Omega$. Next, if $z_0 \in \Omega$, and z belongs to a compact neighbourhood $W \subset \Omega$ of z_0, $\forall t \in M$ we have $F(t,z) - F(t,z_0) \underset{z\to z_0}{\longrightarrow} 0$ and $|F(t,z) - F(t,z_0)|^q \le 2^q F_W(t)^q$, so that the dominated convergence theorem yields the continuity of $z \mapsto F(\cdot,z)$ as a function from Ω to $L^q(\nu)$.

In order to prove that the same function is holomorphic with respect to $z = (z_1, \dots, z_n)$ it is sufficient to show that it is holomorphic with respect to each variable z_k. That amounts to show that $\int_\gamma F(\cdot,z)\, dz_k = 0$ whenever γ is a small circle which embraces a disk contained in the z_k-section of the open set Ω, and the integral on γ is understood in the sense of $L^q(\nu)$. Now, by ([9], Theorem III.11.17), this integral, as an element of $L^q(\nu)$, is the function $t \mapsto \int_\gamma F(t,z)\, dz_k$, and this integral is constantly 0 by assumption (c). □

Lemma 6.27. *Let $\theta \in]\omega,\pi[$, and $\mu \in \mathbb{C} \setminus S_\theta$. Then the following functions are holomorphic on $(\Sigma_{\varphi(\theta)})^n$:*

$z \mapsto \delta_{j,k}(z,\mu)$
$z \mapsto H^{(\ell)}_{z,\mu}$ *($\ell \le 2m$) with respect to the norm of $L^1(\mathbb{R})$*
$z \mapsto u^{(\ell)}_{j,z,\mu}$ *($\ell \le 2m$) with respect to the norm of $L^p(\mathbb{R}^+)$*
$z \mapsto h_{k,z,\mu}$ *with respect to the norm of $L^{p'}(\mathbb{R}^+)$*

where $H_{z,\mu}$ was defined in 6.7*, $h_{k,z,\mu}$ was defined in* (6.11)*, $u_{j,z,\mu}$ was defined in* (6.15) *and $\delta_{j,k}(z,\mu)$ was introduced in the proof of Lemma* 6.13.

Proof. The coefficients $\delta_{j,k}(z,\mu)$ are the entries of the inverse of the matrix $G(z,\mu)$, whose entries $g_{k,j}(z,\mu)$ were defined in (5.15); therefore the analyticity of $\delta_{j,k}(z,\mu)$ follows from the analyticity of $g_{k,j}(z,\mu)$. Now we remark that $g_{k,j}(z,\mu)$, $h_{k,z,\mu}(s)$ (when $s \in \mathbb{R}^+$), $u^{(\ell)}_{j,z,\mu}(t)$ (when $t \in \mathbb{R}^+$) and $H^{(\ell)}_{z,\mu}(t)$ (when $t \in \mathbb{R} \setminus \{0\}$) are defined as integrals of the type

$$\int_{\gamma_\theta^\pm(z,\mu)} \frac{f(z,\lambda)}{\mu - P(z,\lambda)} \, d\lambda$$

where f is a holomorphic function on $\mathbb{C}^{n+1}$. However it is obvious that if $z_0 \in (\Sigma_{\varphi(\theta)})^n$ and z belongs to a suitable neighbourhood of z_0, then

$$\int_{\gamma_\theta^\pm(z,\mu)} \frac{f(z,\lambda)}{\mu - P(z,\lambda)} \, d\lambda = \int_{\gamma_\theta^\pm(z_0,\mu)} \frac{f(z,\lambda)}{\mu - P(z,\lambda)} \, d\lambda,$$

and this proves the analyticity of all these complex valued functions of z.

Next, with the aim of applying Lemma 6.26 (in order to obtain the analyticity in the sense of L^q) we remark that the measurable dependence on the variables s and t is obvious, and so we only have to prove that $H^{(\ell)}_{z,\mu}$, $h_{k,z,\mu}$ and $u^{(\ell)}_{j,z,\mu}$ satisfy condition (b) of that lemma, with $q = 1$, $q = p'$, $q = p$, respectively. This follows at once from formula (6.23) (for $h_{k,z,\mu}$) and the pointwise estimates of $H_{z,\mu}$ and its derivatives (which include $u_{j,z,\mu}$ and their derivatives) given in Remark 6.9 (b). $\square$

THEOREM 6.28. *Let ℓ be a non-negative integer, $\ell \le 2m$. Assume that $\theta \in]\omega, \pi[$, and $\mu \in \mathbb{C} \setminus S_\theta$. Then the function $z \mapsto D^\ell\,(\mu - A_z)^{-1}$ is holomorphic from $(\Sigma_{\varphi(\theta)})^n$ to $\mathcal{L}(L^p(\mathbb{R}^+))$.*

PROOF. We recall that the operator valued function $z \mapsto D^\ell\,(\mu - A_z)^{-1}$ is holomorphic if and only if $\forall f \in L^p(\mathbb{R}^+)$ the function $z \mapsto D^\ell(\mu - A_z)^{-1} f$ is holomorphic. Therefore we fix $f \in L^p(\mathbb{R}^+)$. From (6.25) we know that

$$D^\ell(\mu - A_z)^{-1} f = D^\ell(H_{z,\mu} * f) + \sum_{j,k=1}^{m} \delta_{j,k}(z, \mu) \int_0^\infty h_{k,z,\mu}(s)\, f(s)\, ds\, u^{(\ell)}_{j,z,\mu}.$$

Now we recall that $D^\ell(H_{z,\mu} * f) = H^{(\ell)}_{z,\mu} * f$ if $\ell < 2m$ and $D^{2m}(H_{z,\mu} * f) = H^{(2m)}_{z,\mu} * f - c^{-1}_{2m,0} f$. Since the convolution by f is a bounded linear operator from L^1 to L^p, and since we already know (Lemma 6.27) that $z \mapsto H^{(\ell)}_{z,\mu}$ is holomorphic as a function from $(\Sigma_{\varphi(\theta)})^n$ to $L^1(\mathbb{R})$, we get immediately that $z \mapsto H^{(\ell)}_{z,\mu} * f$ is holomorphic with respect to the norm of $L^p(\mathbb{R}^+)$.

The same argument works for the second summand, since the functional

$$g \mapsto \int_0^\infty g(s)\, f(s)\, ds$$

is bounded on $L^{p'}(\mathbb{R}^+)$. $\square$

6.3. – R-boundedness

Now we prove the R-boundedness of the function $z \mapsto z^\alpha\, D^\ell\,(\mu - A_z)^{-1}$ when $|\alpha| + \ell \le 2m$. In order to apply a suitable version of the Mihlin multiplier theorem, we need the following lemma.

LEMMA 6.29. *Let $\beta_1, \dots, \beta_N \in]0, \pi[$, and let $g \in H^\infty\Big(\prod_{k=1}^N S_{\beta_k}\Big)$. Then $\forall \alpha \in \mathbb{N}^N$ and $\forall \tau \in (\mathbb{R}^+)^N$ we have*

$$|\tau^\alpha\, D^\alpha g(\tau)| \le \alpha! \prod_{k=1}^N (\sin \beta_k)^{-\alpha_k}\, \|g\|_\infty.$$

PROOF. We take $\delta_k \in]0, \sin \beta_k[$, so that $\forall \tau \in (\mathbb{R}^+)^N$ it is $\prod_{k=1}^N \overline{B}(\tau_k, \delta_k\, \tau_k) \subseteq \prod_{k=1}^N S_{\beta_k}$. If we fix $\tau \in (\mathbb{R}^+)^N$ and call γ_k the boundary of $\overline{B}(\tau_k, \delta_k\, \tau_k)$ oriented counterclockwise, then

$$D^\alpha g(\tau) = \frac{\alpha!}{(2\pi i)^N} \int_{\prod_{k=1}^N \gamma_k} \frac{g(z)}{\prod_{k=1}^n (z_k - \tau_k)^{\alpha_k + 1}}\, dz.$$

Hence

$$|\tau^\alpha D^\alpha g(\tau)| \le \frac{\alpha!\,\tau^\alpha}{(2\pi)^N}\prod_{k=1}^N (2\pi\,\delta_k\,\tau_k)\prod_{k=1}^N(\delta_k\,\tau_k)^{-\alpha_k-1}\,\|g\|_\infty = \alpha!\prod_{k=1}^n \delta_k^{-\alpha_k}\,\|g\|_\infty .$$

Now we let δ_k tend to $\sin\beta_k$ $\forall k$, and the proof is ended. □

LEMMA 6.30. *Let $\theta \in]\omega,\pi[$ and $(0,0)\neq(z,\mu)\in(\overline{\Sigma_{\varphi(\theta)}})^n\times(\mathbb{C}\setminus S_\theta)$. Then $\forall\xi\in\mathbb{R}$*

$$\mathcal{F}H_{z,\mu}(\xi) = \frac{1}{\mu - P(z,i\xi)}.$$

PROOF. In the definition of $H_{z,\mu}(x)$ when, say, $x\ge 0$, the circuit $\gamma_\theta^-(z,\mu)$ can be replaced by the segment $[-iR,iR]$ oriented upwards and followed by the semicircle $[\pi/2,3\pi/2]\ni\alpha\mapsto R\,e^{i\alpha}$ if R is large enough. For the integral on the semicircle we have

$$\left|\int_{\pi/2}^{3\pi/2}\frac{e^{R\,x\,e^{i\alpha}}}{\mu-P(z,Re^{i\alpha})}\,iRe^{i\alpha}\,d\alpha\right| \le \pi\sup_{|\lambda|=R}\frac{R}{|\mu-P(z,\lambda)|}\underset{R\to+\infty}{\longrightarrow}0$$

so that

$$H_{z,\mu}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{\lambda x}}{\mu-P(z,\lambda)}\,d\lambda = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{ix\xi}}{\mu-P(z,i\xi)}\,d\xi .$$

When $x\le 0$ a similar argument gives the same equality for $H_{z,\mu}(x)$. This proves that $x\mapsto 2\pi\,H_{z,\mu}(-x)$ is the Fourier transform of the function $\xi\mapsto\frac{1}{\mu-P(z,i\xi)}$, whence the assertion of the lemma follows immediately. □

THEOREM 6.31. *Let ℓ be a non-negative integer, α an n-tuple of non negative integers, and suppose that $|\alpha|+\ell\le 2m$. Assume that $\theta\in]\omega,\pi[$. Then for $0\neq\mu\in\mathbb{C}\setminus S_\theta$ the function*

$$(\overline{\Sigma_{\varphi(\theta)}})^n\ni z\mapsto z^\alpha D^\ell(\mu-A_z)^{-1}\in\mathcal{L}(L^p(\mathbb{R}^+))$$

has R-bounded range, and its R_p-bound is $\le C(L,L_1,\omega,\theta,L_0(\theta))\,|\mu|^{\frac{\ell+|\alpha|}{2m}-1}$.

PROOF. Because of Lemma 4.2 it is sufficient to prove the R-boundedness of the sets $\{S_{z,\mu};\, z\in(\overline{\Sigma_{\varphi(\theta)}})^n\}$ and $\{T_{z,\mu};\, z\in(\overline{\Sigma_{\varphi(\theta)}})^n\}$, where

$$S_{z,\mu}f = z^\alpha D^\ell(H_{z,\mu}*f)$$

and

$$\begin{aligned}T_{z,\mu}f &= z^\alpha D^\ell w_{z,\mu}(f) = z^\alpha\sum_{j,k=1}^m \delta_{j,k}(z,\mu)\int_0^\infty h_{k,z,\mu}(s)\,f(s)\,ds\,u_{j,z,\mu}^{(\ell)}(t)\\ &= z^\alpha\int_0^\infty D_t^\ell K_{z,\mu}(t,s)\,f(s)\,ds.\end{aligned}$$

Here we have set

$$K_{z,\mu}(t,s) = \sum_{j,k=1}^{m} \delta_{j,k}(z,\mu)\, h_{k,z,\mu}(s)\, u_{j,z,\mu}(t).$$

We have $z^\alpha D^\ell(H_{z,\mu} * f) = (D^\ell(z^\alpha H_{z,\mu})) * f$, and from Lemma 6.30 we get

$$\mathcal{F}(D^\ell(z^\alpha H_{z,\mu}))(\xi) = \frac{z^\alpha\,(i\xi)^\ell}{\mu - P(z, i\xi)}.$$

Therefore Theorem 5.8 yields

$$|\mathcal{F}(D^\ell(z^\alpha H_{z,\mu}))(\xi)| \le C(L,\omega,\theta)\, \frac{\|z\|^{|\alpha|}\,|\xi|^\ell}{\|z\|^{2m} + |\xi|^{2m} + |\mu|} \le C(L,\omega,\theta)\,|\mu|^{\frac{\ell+|\alpha|}{2m}-1}.$$

Again by Theorem 5.8, if $\xi \in i\,\Sigma_{\varphi(\theta)}$, then $P(z,i\xi) \ne \mu$, so that $\mathcal{F}(D^\ell(z^\alpha H_{z,\mu}))$ can be extended holomorphically to $i\,\Sigma_{\varphi(\theta)}$ (which is a double-sector containing $\mathbb{R}\setminus\{0\}$) and satisfies there the same estimate. Hence Lemma 6.29 implies that we have also

$$\sup_{z\in(\overline{\Sigma_{\varphi(\theta)}})^n,\ \xi\in\mathbb{R}\setminus\{0\}} \left|\xi\,\frac{d}{d\xi}\mathcal{F}(D^\ell(z^\alpha H_{z,\mu}))(\xi)\right| \le C(L,\omega,\theta)\,|\mu|^{\frac{\ell+|\alpha|}{2m}-1}.$$

Now we can apply Theorem 4.4, and get the R-boundedness of the set of operators $\{S_{z,\mu};\, z \in (\overline{\Sigma_{\varphi(\theta)}})^n\}$.

We turn to the operators $T_{z,\mu}$. From the inequality (6.23), Lemma 6.13, and Remark 6.9(b) we know that

$$|h_{k,z,\mu}(s)| \le C\,\beta(z,\mu)^{m_k+1-2m}\, e^{-sM_2\beta(z,\mu)}$$

$$|\delta_{j,k}(z,\mu)| \le C\,\beta(z,\mu)^{2m-m_k-j}$$

$$|u^{(\ell)}_{j,z,\mu}(t)| = |H^{(\ell+j-1)}_{z,\mu}(t)| \le C\,\beta(z,\mu)^{j+\ell-2m}\, e^{-tM_2\beta(z,\mu)}$$

so that, taking into account the elementary equality $\sup_{r\in\mathbb{R}^+} r\,e^{-r} = e^{-1}$, we get

$$\begin{aligned} |z^\alpha D^\ell_t K_{z,\mu}(t,s)| &\le C\,\|z\|^{|\alpha|}\,\beta(z,\mu)^{\ell-2m+1}\, e^{-(t+s)M_2\beta(z,\mu)} \\ &\le \frac{C}{e\,M_2}\,\frac{\beta(z,\mu)^{\ell+|\alpha|-2m}}{t+s} \le \frac{C}{e\,M_2}\,\frac{|\mu|^{\frac{\ell+|\alpha|}{2m}-1}}{t+s}. \end{aligned} \tag{6.32}$$

Now an application of Theorem 4.5 concludes the proof. □

6.4. – H^∞ functional calculus

In this subsection we prove that the operators A_z have a bounded H^∞ functional calculus, and that for any bounded holomorphic function h the operator valued function $z \mapsto h(A_z)$ is R-bounded and holomorphic.

LEMMA 6.33. *Let $\theta \in]\omega, \pi[$, $\delta \in]0, \pi - \theta[$. Then $\forall h \in H_0^\infty(S_{\theta+\delta})$ the set $\{h(A_z);\, z \in (\overline{\Sigma_{\varphi(\theta)}})^n\}$ is R-bounded, and its R_p-bound is $\leq C(L, L_1, \omega, \theta, L_0(\theta))\|h\|_\infty$.*

PROOF. Let $h \in H_0^\infty(S_{\theta+\delta})$. We have

$$h(A_z) = \frac{1}{2\pi i} \int_\gamma h(\mu)\,(\mu - A_z)^{-1}\,d\mu$$

where γ is the curve parametrized by $\mathbb{R} \setminus \{0\} \ni t \mapsto |t|\,e^{-i\theta\,\mathrm{sgn}\,t}$, oriented according to the increasing values of t. Now we take $f \in L^p(\mathbb{R}^+)$, and recall that

$$(\mu - A_z)^{-1} f = H_{z,\mu} * f + \int_0^\infty K_{z,\mu}(\cdot, s) f(s)\,ds$$

(see (6.25)). Since $|h(\mu)| \leq C \min\left\{|\mu|^s,\ |\mu|^{-s}\right\}$ for some $s \in \mathbb{R}^+$ and

$$\|H_{z,\mu} * f\|_{L^p(\mathbb{R}^+)} \leq \frac{C}{\|z\|^{2m} + |\mu|}\,\|f\|_{L^p(\mathbb{R}^+)}$$

(see (6.10)) the integral $\int_\gamma h(\mu)\,(H_{z,\mu} * f)\,d\mu$ exists in the norm of $L^p(\mathbb{R}^+)$ and $f \mapsto \int_\gamma h(\mu)\,(H_{z,\mu} * f)\,d\mu$ is a bounded linear operator on $L^p(\mathbb{R}^+)$ to itself. Moreover from

$$\begin{aligned}
&\int_\gamma \int_{\mathbb{R}} |h(\mu)|\,|H_{z,\mu}(x)|\,dx\,d\,|\mu| \\
&\qquad = \int_\gamma |h(\mu)|\,\|H_{z,\mu}\|_{L^1(\mathbb{R})}\,d\,|\mu| \\
&\qquad \leq C \int_\gamma \min\{|\mu|^s,\ |\mu|^{-s}\}\,(|\mu| + \|z\|^{2m})^{-1}\,d\,|\mu| < \infty
\end{aligned} \tag{6.34}$$

it follows that the function

$$x \mapsto W_{z,h}(x) := \frac{1}{2\pi i} \int_\gamma h(\mu)\,H_{z,\mu}(x)\,d\mu$$

is well defined and summable on $\mathbb{R}$. We want to show that $\forall f \in L^p(\mathbb{R}^+)$

$$W_{z,h} * f = \frac{1}{2\pi i} \int_\gamma h(\mu)\,(H_{z,\mu} * f)\,d\mu \tag{6.35}$$

and it is not restrictive to assume that $f \in L^p(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, since we know that both sides of (6.35) define bounded operators on $L^p(\mathbb{R}^+)$. Then (6.35) follows immediately from an exchange of order of integration, which is allowed by inequality (6.34), since f is bounded.

Now we compute the Fourier transform of $W_{z,h}$. By means of the usual exchange of order of integration we obtain, by Lemma 6.30,

$$\begin{aligned}(\mathcal{F}W_{z,h})(\xi) &= \frac{1}{2\pi i}\int_\gamma h(\mu)\,(\mathcal{F}H_{z,\mu})(\xi)\,d\mu \\ &= \frac{1}{2\pi i}\int_\gamma \frac{h(\mu)}{\mu - P(z,i\xi)}\,d\mu = h(P(z,i\xi))\end{aligned}$$

as it follows from the residue theorem, since $P(z,i\xi) \in S_\theta$ (at least when $\xi \neq 0$, see Theorem 5.6). More generally 5.6 implies that the function $\lambda \mapsto h(P(z,i\lambda))$ is defined and hence holomorphic on $i\,\Sigma_{\varphi(\theta)}$; therefore from Lemma 6.29 we get

$$\sup_{z\in(\overline{\Sigma_{\varphi(\theta)}})^n} \max\left\{ \sup_{0\neq\xi\in\mathbb{R}} |(\mathcal{F}W_{z,h})(\xi)|,\ \sup_{0\neq\xi\in\mathbb{R}} \left|\xi\,\frac{d}{d\xi}\,(\mathcal{F}W_{z,h})(\xi)\right|\right\} \le C\,\|h\|_\infty.$$

By Theorem 4.4, it follows that $\left\{ f \mapsto W_{z,h} * f;\, z \in (\overline{\Sigma_{\varphi(\theta)}})^n \right\}$ is a R-bounded subset of $\mathcal{L}(L^p(\mathbb{R}^+))$, with R_p-bound $\le C\,\|h\|_\infty$.

Concerning the kernel $K_{z,\mu}$, the inequality (6.32) yields

$$|K_{z,\mu}(t,s)| \le C\,|\mu|^{\frac{1}{2m}-1}\,e^{-(t+s)\,M_2\,|\mu|^{\frac{1}{2m}}}$$

so that

$$\begin{aligned}\int_\gamma |h(\mu)|\,|K_{z,\mu}(t,s)|\,d\mu &\le C\,\|h\|_\infty \int_\gamma |\mu|^{\frac{1}{2m}-1}\,e^{-(t+s)\,M_2\,|\mu|^{\frac{1}{2m}}}\,d\,|\mu| \\ &= 4m\,C\,\|h\|_\infty \int_0^\infty e^{-(t+s)\,M_2\,r}\,dr \\ &= \frac{4m\,C}{M_2\,(t+s)}\,\|h\|_\infty.\end{aligned} \tag{6.36}$$

Therefore $\forall f \in L^p(\mathbb{R}^+)$ and $\forall t \in \mathbb{R}^+$

$$\begin{aligned}\int_\gamma\int_0^\infty |h(\mu)|\,|K_{z,\mu}(t,s)|\,|f(s)|\,ds\,d\,|\mu| &\le C\,\|h\|_\infty \int_0^\infty \frac{|f(s)|}{t+s}\,ds \\ &\le C\,t^{-1/p}\,\|h\|_\infty\,\|f\|_{L^p(\mathbb{R}^+)}.\end{aligned}$$

This allows us to perform the following exchange of order of integration

$$\int_\gamma h(\mu)\int_0^\infty K_{z,\mu}(t,s)\,f(s)\,ds\,d\mu = \int_0^\infty\int_\gamma h(\mu)\,K_{z,\mu}(t,s)\,d\mu\,f(s)\,ds$$

and since, by (6.36), the kernels $(t,s) \mapsto \int_\gamma h(\mu)\, K_{z,\mu}(t,s)\, d\mu$ satisfy the assumptions of Theorem 4.5, we obtain the R-boundedness of the set of operators

$$\left\{ f \mapsto \frac{1}{2\pi i} \int_\gamma h(\mu) \int_0^\infty K_{z,\mu}(\cdot, s)\, f(s)\, ds\, d\mu;\ z \in (\overline{\Sigma_{\varphi(\theta)}})^n \right\}$$

with R_p-bound $\le C\, \|h\|_\infty$. Now we put together the results concerning the two addenda, using Lemma 4.2, and we obtain that the set $\left\{ h(A_z);\ z \in (\overline{\Sigma_{\varphi(\theta)}})^n \right\}$ is R-bounded, with R_p-bound $\le C\, \|h\|_\infty$. □

Theorem 6.37. *Let $\theta \in]\omega, \pi[$, $\delta \in]0, \pi-\theta[$. Then $\forall z \in (\overline{\Sigma_{\varphi(\theta)}})^n$ the sectorial operator A_z has a bounded $H^\infty(S_{\theta+\delta})$ functional calculus. Moreover $\forall h \in H^\infty(S_{\theta+\delta})$ the set $\left\{ h(A_z);\ z \in (\overline{\Sigma_{\varphi(\theta)}})^n \right\}$ is R-bounded, and its R_p-bound is $\le C(L, L_1, \omega, \theta, L_0(\theta), \delta)\, \|h\|_\infty$.*

Proof. Let $z \in (\overline{\Sigma_{\varphi(\theta)}})^n$ be fixed. The result of Lemma 6.33 implies that $\forall h \in H_0^\infty(S_{\theta+\delta})$ we have $\|h(A_z)\| \le C\, \|h\|_\infty$. By Lemma 4.17 this proves that A_z has a bounded $H^\infty(S_{\theta+\delta})$ functional calculus. In order to prove the R-boundedness of $\left\{ h(A_z);\ z \in (\overline{\Sigma_{\varphi(\theta)}})^n \right\}$ and to estimate its R_p-bound we use the sequence $(\Psi_k)_{k\in\mathbb{N}}$ of functions introduced in Definition 4.10 (with $N=1$). It is easy to prove (see [8]) that $\sup\limits_{\mu \in S_{\theta+\delta}} |\Psi_k(\mu)| \le \cos^{-2}((\theta+\delta)/2)$. Now, by Remark 4.15 and Lemma 6.33, if $z^{(1)}, \dots, z^{(N)} \in (\overline{\Sigma_{\varphi(\theta)}})^n$ and $f_1, \dots, f_N \in L^p(\mathbb{R}^+)$, we have

$$\begin{aligned}
&\left(\sum_{\varepsilon \in \{-1,1\}^N} \left\| \sum_{r=1}^N \varepsilon_r\, h(A_{z^{(r)}})\, f_r \right\|_{L^p(\mathbb{R}^+)}^p \right)^{1/p} \\
&\quad = \lim_{k\to\infty} \Big(\sum_{\varepsilon \in \{-1,1\}^N} \left\| \sum_{r=1}^N \varepsilon_r\, (\Psi_k\, h)(A_{z^{(r)}})\, f_r \right\|_{L^p(\mathbb{R}^+)}^p \Big)^{1/p} \\
&\quad \le C\, \cos^{-2}((\theta+\delta)/2)\, \|h\|_\infty \left(\sum_{\varepsilon \in \{-1,1\}^N} \left\| \sum_{r=1}^N \varepsilon_r\, f_r \right\|_{L^p(\mathbb{R}^+)}^p \right)^{1/p}.
\end{aligned}$$

This proves that $\left\{ h(A_z);\ z \in (\overline{\Sigma_{\varphi(\theta)}})^n \right\}$ is R-bounded, and gives the required estimate of its R_p-bound. □

Theorem 6.38. *Let $\theta \in]\omega, \pi[$, $\delta \in]0, \pi-\theta[$, $h \in H^\infty(S_{\theta+\delta})$. Then the function $z \mapsto h(A_z)$ is holomorphic on $(\Sigma_{\varphi(\theta)})^n$.*

Proof. We first assume that $h \in H_0^\infty(S_{\theta+\delta})$. Then

$$h(A_z) = \frac{1}{2\pi i} \int_\gamma h(\mu)\, (\mu - A_z)^{-1}\, d\mu$$

where for $\mu \in \gamma$ and a suitable $s > 0$

$$\|h(\mu)\,(\mu - A_z)^{-1}\| \leq C \,\max\left\{|\mu|^s,\ |\mu|^{-s}\right\} |\mu|^{-1}$$

which is a summable function on γ. Since $z \mapsto (\mu - A_z)^{-1}$ is continuous on $(\Sigma_{\varphi(\theta)})^n$ (Theorem 6.28), the dominated convergence theorem yields the continuity of $z \mapsto h(A_z)$. Moreover for $k \in \{1, \dots, n\}$, if σ is a small circle that embraces a disk contained in the z_k-section of $(\Sigma_{\varphi(\theta)})^n$, we have, by Theorem 6.28,

$$\int_\sigma h(A_z)\,dz_k = \int_\gamma h(\mu) \int_\sigma (\mu - A_z)^{-1}\,dz_k\,d\mu = 0$$

and this proves that $z \mapsto h(A_z)$ is holomorphic.

Next, we take $h \in H^\infty(S_{\theta+\delta})$. Then $(\Psi_k\,h)_{k\in\mathbb{N}}$ is a sequence in $H_0^\infty(S_{\theta+\delta})$ such that $(\Psi_k\,h)(A_z)f \underset{k\to\infty}{\to} h(A_z)f \ \forall z \in (\Sigma_{\varphi(\theta)})^n$ (see Remark 4.15), and in particular $\forall z$ the set $\{(\Psi_k h)(A_z)f ; k \in \mathbb{N}\}$ is relatively compact in $L^p(\mathbb{R}^+)$. Since we also have

$$\|(\Psi_k h)(A_z)\| \leq C\,\cos^{-2}((\theta+\delta)/2)\,\|h\|_\infty,$$

by Lemma 6.29 the functions $z \mapsto (\Psi_k h)(A_z)f$ are locally equicontinuous on $(\Sigma_{\varphi(\theta)})^n$; therefore by Ascoli's theorem we can extract a subsequence that converges uniformly on compacta, and hence $z \mapsto h(A_z)f$ is holomorphic $\forall f \in L^p(\mathbb{R}^+)$. As it is well known, this yields that $z \mapsto h(A_z)$ is holomorphic with values in $\mathcal{L}(L^p(\mathbb{R}^+))$. □

7. – The operators $D_1, \dots, D_n$ in $L^p(\mathbb{R}^n \times \mathbb{R}^+)$

In this section we work in the Banach space $L^p(\mathbb{R}^n \times \mathbb{R}^+)$ (with $1 < p < \infty$, as usual), and we are concerned with the derivative operators $D_1, \dots, D_n$ with respect to the "tangential" variables $x_1, \dots, x_n$. D_j is considered as an unbounded operator in $L^p(\mathbb{R}^n \times \mathbb{R}^+)$, with domain $\{u \in L^p(\mathbb{R}^n \times \mathbb{R}^+); D_j u \in L^p(\mathbb{R}^n \times \mathbb{R}^+)\}$. We will denote by D_x the n-tuple of operators $(D_1, \dots, D_n)$.

We quote, without proofs, some folk results on the spectral properties of the operators D_j.

Theorem 7.1. $\forall j \in \{1, \dots, n\}$ *the operator* D_j *is bisectorial with spectral angle* 0. *More precisely:*

(i) *if* $Re\,\lambda \neq 0$ *and* $g \in L^p(\mathbb{R}^n \times \mathbb{R}^+)$, *then*

$$(\lambda - D_j)^{-1} g(x,t) = e^{\lambda x_j} \int_{x_j}^\infty e^{-\lambda r_j} g(x', r_j, t)\,dr_j$$

for $Re\,\lambda > 0$ *and*

$$(\lambda - D_j)^{-1} g(x,t) = -e^{\lambda x_j} \int_{-\infty}^{x_j} e^{-\lambda r_j} g(x', r_j, t)\, dr_j$$

for $Re\,\lambda < 0$*;*

(ii) *if* $\alpha \in]0, \frac{\pi}{2}[$*, and* $\lambda \in \mathbb{C} \setminus \overline{\Sigma_\alpha}$*, then*

$$\|(\lambda - D_j)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+))} \le \frac{1}{\sin\alpha} \frac{1}{|\lambda|}.$$

THEOREM 7.2. *If* $n \ge 2$*, and* $j, k \in \{1, \dots, n\}$*, then* $\forall(\lambda, \mu) \in (\mathbb{C} \setminus (i\,\mathbb{R}))^2$ $(\lambda - D_j)^{-1}$ *commutes with* $(\mu - D_k)^{-1}$*.*

It is known from [18], Corollary 2 that each operator D_j has bounded H^∞ functional calculus on Σ_δ $\forall\delta > 0$. If one applies this result and [15], Theorem 4.3 one obtains the following theorem, of which we give a direct proof.

THEOREM 7.3. *Let* $\beta_1, \dots, \beta_n \in]0, \frac{\pi}{2}[$*, and let us set* $\Omega = \prod_{k=1}^n \Sigma_{\beta_k}$*. Then the* n*-tuple of operators* $D_x = (D_1, \dots, D_n)$ *has a bounded joint* $H^\infty(\Omega)$ *functional calculus.*

PROOF. We have to prove that $f(D_x) \in \mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+))$ $\forall f \in H^\infty(\Omega)$. To this end, by (the bisectorial analogous of) Lemma 4.17, it is sufficient to show that $\exists\, C \in \mathbb{R}^+$ such that $\forall f \in H_0^\infty(\Omega)$ $\|f(D_x)\| \le C\,\|f\|_\infty$. In order to prove this statement we have to show that

$$\|f(D_x)u\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \le C\|f\|_\infty\, \|u\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)}$$

when u belongs to a dense subspace of $L^p(\mathbb{R}^n \times \mathbb{R}^+)$, e.g. $L^p(\mathbb{R}^n \times \mathbb{R}^+) \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^+)$.

Let us take $f \in H_0^\infty(\Omega)$. The first step consists in writing $f(D_x)$ as a convolution operator (in the variables $x_1, \dots, x_n$).

$\forall k \in \{1, \dots, n\}$ we choose $\gamma_k \in]\frac{\pi}{2}, \frac{\pi}{2} + \beta_k[$ and set $\widetilde{\Gamma}_k := \Gamma_k \cup (-\Gamma_k)$ where Γ_k is the curve parametrized by $\mathbb{R} \setminus \{0\} \ni \tau \mapsto |\tau|\, e^{-i\gamma_k \operatorname{sgn}\tau}$. We set $\widetilde{\Gamma} := \prod_{k=1}^n \widetilde{\Gamma}_k$ and (when $\varepsilon \in \{-1, 1\}$) $h_\varepsilon : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$

$$h_\varepsilon(\lambda, r) = \begin{cases} e^{\lambda r}\, \chi_{\mathbb{R}^-}(r) & \text{if } \varepsilon = -1 \\ -e^{\lambda r}\, \chi_{\mathbb{R}^+}(r) & \text{if } \varepsilon = 1. \end{cases}$$

It follows immediately from Theorem 7.1 that

$$(\lambda - D_k)^{-1} u(x,t) = \int_{\mathbb{R}} h_{\operatorname{sgn} Re\lambda}(\lambda, r)\, u(x', x_k - r, t)\, dr.$$

Therefore

$$\begin{aligned}\Big(f(D_x)u\Big)(x,t) &= (2\pi i)^{-n}\int_{\widetilde{\Gamma}} f(z)\Big(\prod_{k=1}^{n}(z_k-D_k)^{-1}\,u\Big)(x,t)\,dz\\ &= \sum_{\varepsilon\in\{-1,1\}^n}(2\pi i)^{-n}\int_{\prod_k \varepsilon_k\Gamma_k} f(z)\int_{\mathbb{R}^n} H_\varepsilon(z,r)\,u(x-r,t)\,dr\,dz\end{aligned}$$

where we have set $H_\varepsilon(z,r)=\prod_{k=1}^n h_{\varepsilon_k}(z_k,r_k)$. Now we notice that for $z\in\widetilde{\Gamma}$

$$\text{(7.4)}\quad \int_{\mathbb{R}^n}|H_\varepsilon(z,r)|\,dr=\prod_{k=1}^{n}\int_{\mathbb{R}}|h_{\varepsilon_k}(z_k,\tau)|\,d\tau=\prod_{k=1}^{n}|Re\,z_k|^{-1}=\prod_{k=1}^{n}\Big(|z_k|\,|\cos\gamma_k|\Big)^{-1}$$

and so

$$\int_{\mathbb{R}^n}|H_\varepsilon(z,r)\,u(x-r,t)|\,dr\le\|u\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^+)}\prod_{k=1}^{n}\Big(|z_k|\,|\cos\gamma_k|\Big)^{-1}.$$

Since $f\in H_0^\infty(\Omega)$, this proves that

$$\int_{\prod_k \varepsilon_k\Gamma_k}|f(z)|\int_{\mathbb{R}^n}|H_\varepsilon(z,x-r)\,u(r,t)|\,dr\,d|z|<+\infty$$

and hence we can exchange the order of integration, and get

$$\begin{aligned}\Big(f(D_x)u\Big)(x,t) &= \sum_{\varepsilon\in\{-1,1\}^n}\int_{\mathbb{R}^n}(2\pi i)^{-n}\int_{\prod_k \varepsilon_k\Gamma_k} f(z)\,H_\varepsilon(z,r)\,dz\,u(x-r,t)\,dr\\ &= (F*u(\cdot,t))(x)\end{aligned}$$

with

$$F(x)=(2\pi i)^{-n}\sum_{\varepsilon\in\{-1,1\}^n}\int_{\prod_k\varepsilon_k\Gamma_k} f(z)\,H_\varepsilon(z,x)\,dz\,.$$

Since $f\in H_0^\infty(\Omega)$, from (7.4) and Fubini's theorem it follows that $F\in L^1(\mathbb{R}^n)$.

The next step is to show that $(\mathcal{F}F)(\tau)=f(i\tau_1,\dots,i\tau_n)$. We have

$$(\mathcal{F}F)(\tau)=(2\pi i)^{-n}\sum_{\varepsilon\in\{-1,1\}^n}\int_{\mathbb{R}^n}e^{-i\langle x,\tau\rangle}\int_{\prod_k \varepsilon_k\Gamma_k} f(z)\,H_\varepsilon(z,x)\,dz\,dx.$$

Since $|e^{-i\langle x,\tau\rangle}|=1$, by (7.4) we can exchange the order of integration, and we obtain

$$(\mathcal{F}F)(\tau)=(2\pi i)^{-n}\sum_{\varepsilon\in\{-1,1\}^n}\int_{\prod_k \varepsilon_k\Gamma_k} f(z)\,(\mathcal{F}H_\varepsilon(z,\cdot))(\tau)\,dz$$

where

$$(\mathcal{F}H_\varepsilon(z,\cdot))(\tau)=\prod_{k=1}^{n}(\mathcal{F}h_{\varepsilon_k}(z_k,\cdot))(\tau_k)=\prod_{k=1}^{n}\int_{\varepsilon_k\infty}^{0}e^{-is\tau_k+z_ks}\,ds=\prod_{k=1}^{n}(z_k-i\tau_k)^{-1}.$$

Hence

$$\begin{aligned}\mathcal{F}F(\tau)&=(2\pi i)^{-n}\sum_{\varepsilon\in\{-1,1\}^n}\int_{\prod_k\varepsilon_k\Gamma_k}f(z)\prod_{k=1}^{n}(z_k-i\tau_k)^{-1}\,dz\\&=(2\pi i)^{-n}\int_{\widetilde{\Gamma}}f(z)\prod_{k=1}^{n}(z_k-i\tau_k)^{-1}\,dz.\end{aligned}$$

Now we break every curve $\widetilde{\Gamma}_k=\Gamma_k\cup(-\Gamma_k)$ in a different way, setting $\widetilde{\Gamma}_k=\widetilde{\Gamma}_k^+\cup\widetilde{\Gamma}_k^-$, where $\widetilde{\Gamma}_k^\pm=\widetilde{\Gamma}_k\cap\{z\in\mathbb{C};\,Im\,z\in\mathbb{R}^\pm\}$. We give $\widetilde{\Gamma}_k^\pm$ the orientation induced by the orientation of $\widetilde{\Gamma}_k$: therefore $\widetilde{\Gamma}_k^+$ is positively oriented with respect to $i\,\mathbb{R}^+$ and $\widetilde{\Gamma}_k^-$ is positively oriented with respect to $i\,\mathbb{R}^-$. Hence

$$\mathcal{F}F(\tau)=(2\pi i)^{-n}\sum_{\varepsilon\in\{-1,1\}^n}\int_{\widetilde{\Gamma}_1^{\varepsilon_1}}\cdots\int_{\widetilde{\Gamma}_n^{\varepsilon_n}}f(z)\prod_{k=1}^{n}(z_k-i\tau_k)^{-1}\,dz_n\cdots dz_1.$$

Here we have the sum of 2^n integrals, and it is obvious that 2^n-1 of these integrals are equal to 0, namely those for which $(\varepsilon_1,\dots,\varepsilon_n)\neq(\operatorname{sgn}\tau_1,\dots,\operatorname{sgn}\tau_n)$; while the value of the remaining integral, i.e. of the one for which $(\varepsilon_1,\dots,\varepsilon_n)=(\operatorname{sgn}\tau_1,\dots,\operatorname{sgn}\tau_n)$, is equal to $f(i\tau_1,\dots,i\tau_n)$. This proves that $\mathcal{F}F(\tau)=f(i\tau_1,\dots,i\tau_n)$.

Now (third step) we estimate $\mathcal{F}F$ and its derivatives. We have obviously $\sup_{\tau\in\mathbb{R}^n}|\mathcal{F}F(\tau)|\le\|f\|_\infty$; moreover $\mathcal{F}F$ can be extended holomorphically to $\prod_{k=1}^n(i\,\Sigma_{\beta_k})$, and also for this extension we have $\sup_{\zeta\in\prod_{k=1}^n(i\,\Sigma_{\beta_k})}|\mathcal{F}F(\zeta)|\le\|f\|_\infty$. Then it follows from Lemma 6.29 that $\forall\alpha\in\mathbb{N}^n$ (and in particular for $|\alpha|=1$)

$$\sup_{\tau\in\mathbb{R}^n}|\tau^\alpha\,(D^\alpha\mathcal{F}F)(\tau)|\le C\,\|f\|_\infty.$$

As a final step, we apply the Mihlin multiplier theorem to the already proved equality $(f(D_x)u)(\cdot,t)=F*u(\cdot,t)$ and we get

$$\|(f(D_x)u)(\cdot,t)\|_{L^p(\mathbb{R}^n)}\le C\,\|f\|_\infty\,\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)}$$

whence

$$\begin{aligned}\|f(D_x)u\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}&=\Big(\int_{\mathbb{R}^+}\|(f(D_x)u)(\cdot,t)\|^p_{L^p(\mathbb{R}^n)}\,dt\Big)^{1/p}\\&\le C\,\|f\|_\infty\,\|u\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}.\end{aligned}$$

□

REMARK 7.5. In Theorem 7.3 the variable t did not play any rôle: we have preserved it only for the sake of a more direct application of that theorem in the next section. However we could have settled this result in the space $L^p(\mathbb{R}^n, L^p(\mathbb{R}^+))$, which is naturally isomorphic to $L^p(\mathbb{R}^n \times \mathbb{R}^+)$, or, more generally, in any space of the type $L^p(\mathbb{R}^n, X)$, provided that the Banach space X have such properties that allow to apply the Mihlin multiplier theorem to X-valued functions of several variables, with scalar-valued multipliers: such properties are the UMD property plus property (α) (see [29]). $\square$

8. – The elliptic operator: resolvent and functional calculus

In this section we have to deal with the trace operator at $t = 0$ in the space $L^p(\mathbb{R}^n \times \mathbb{R}^+)$, that we call T_0. Among the several equivalent ways to define it, the most useful for our purposes is the following. Let us consider the operator D_t in $L^p(\mathbb{R}^n \times \mathbb{R}^+)$: this is a closed operator whose domain is $\mathcal{D}(D_t) := \{u \in L^p(\mathbb{R}^n \times \mathbb{R}^+); D_t u \in L^p(\mathbb{R}^n \times \mathbb{R}^+)\}$. Remark that if $\mathcal{D}(D_t)$ is endowed with the graph norm, then $W^{1,p}(\mathbb{R}^n \times \mathbb{R}^+) \hookrightarrow \mathcal{D}(D_t) \hookrightarrow L^p(\mathbb{R}^n \times \mathbb{R}^+)$ (inclusions with continuous embeddings). While it is obvious that if $f \in L^p(\mathbb{R}^n \times \mathbb{R}^+)$, then $f(\cdot, t) \in L^p(\mathbb{R}^n)$ for almost every $t \in \mathbb{R}^+$, it is also easy to see that if $f \in \mathcal{D}(D_t)$, then as $t \to 0^+$ $f(\cdot, t)$ converges in $L^p(\mathbb{R}^n)$: the limit function is, by definition, its trace $T_0 f \in L^p(\mathbb{R}^n)$; T_0 is a bounded linear operator on $\mathcal{D}(D_t)$ to $L^p(\mathbb{R}^n)$. We also have, for almost every $x \in \mathbb{R}^n$, $(T_0 f)(x) = \lim_{t\to 0^+} f(x, t)$ in the pointwise sense. Moreover if $f \in W^{r,p}(\mathbb{R}^n \times \mathbb{R}^+)$ then $T_0 f \in W^{r-1,p}(\mathbb{R}^n)$, and for $|\alpha| \le r - 1$ one has $T_0 D_x^\alpha f = D^\alpha T_0 f$.

We shall also write $f(x, 0)$ instead of $(T_0 f)(x)$.

Let $P, B_1, \dots, B_m$ be the polynomials introduced in Section 5, and let L, L_1, ω, L_0 have same meaning as before. We recall that P is (L, ω)-elliptic and that the ω-complementing condition is satisfied. Since $\deg B_k = m_k < 2m$, the trace at $t = 0$ of $B_k u(x, t)$ is well defined $\forall u \in W^{2m,p}(\mathbb{R}^n \times \mathbb{R}^+)$.

Let A be the operator in $L^p(\mathbb{R}^n \times \mathbb{R}^+)$ (always with $1 < p < \infty$) defined as follows:

$$\mathcal{D}(A) = \left\{u \in W^{2m,p}(\mathbb{R}^n \times \mathbb{R}^+); (B_k(D_x, D_t)u)(x, 0) = 0,\ 1 \le k \le m\right\}$$

$$Au = P(D_x, D_t)u.$$

We want to prove the following:

THEOREM 8.1. *A is a sectorial operator with spectral angle ω, and $\forall\theta \in]\omega, \pi[$ A has a bounded $H^\infty(S_\theta)$ functional calculus. In particular $\forall\theta \in]\omega, \pi[$ we have:*

(i) *for $\mu \in \mathbb{C} \setminus \overline{S_\theta}$, $\|(\mu - A)^{-1}\| \le C(L, L_1, \omega, \theta, L_0(\theta))|\mu|^{-1}$;*
(ii) *if $h \in H^\infty(S_\theta)$, then $\|h(A)\| \le C(L, L_1, \omega, \theta, L_0(\theta))\, \|h\|_\infty$.*

The proof of Theorem 8.1 will be obtained as the conclusion of a long series of preliminary results. The underlying idea, however, is rather simple: in Section 6 we defined the operators A_z by means of a formal replacement of the derivative operators $D_1, \dots, D_n$ with complex parameters $z_1, \dots, z_n$; now we are going to make the inverse replacement: we shall construct $h(A)$ by substituting D_x to z in $h(A_z)$. This statement actually means that $h(A)$ will be proved to be the operator in $\mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+))$ that corresponds to the holomorphic and R-bounded function $z \mapsto h(A_z)$ (or, more precisely, $z \mapsto \widehat{(h(A_z))}$, see Remark 4.6) in the homomorphism $g \mapsto g(D_x)$. Here Theorem 4.18 plays a crucial rôle, because it allows to avoid any type of multiplier theorem with operator-valued multipliers.

The first goal is to show that $\rho(A) \supseteq \mathbb{C} \setminus \overline{S_\omega}$ and to estimate $\|(\mu - A)^{-1}\|$ on $\mathbb{C} \setminus \overline{S_\omega}$. Remark that if $\mu \notin \overline{S_\omega}$, then $\exists\, \theta \in]\omega, \pi[$ such that $\mu \notin \overline{S_\theta}$.

We fix $\theta \in]\omega, \pi[$ and $\mu \in \mathbb{C} \setminus \overline{S_\theta}$.

For the derivative operator D_t the following result holds, analogous to Theorems 7.1 and 7.2.

THEOREM 8.2. *The resolvent set of the operator D_t in $L^p(\mathbb{R}^n \times \mathbb{R}^+)$ contains the half-plane $\{\lambda \in \mathbb{C}; Re\,\lambda > 0\}$, and for $Re\,\lambda > 0$ one has*

$$(\lambda - D_t)^{-1} u(x,t) = e^{\lambda t} \int_t^\infty e^{-\lambda s}\, u(x,s)\, ds.$$

Moreover the resolvent operators of D_t commute with the resolvents of $D_1, \dots, D_n$.

In Remark 4.6 we constructed a bounded linear transformation $T \mapsto \hat{T}$ of norm ≤ 1, which in the present situation goes from $\mathcal{L}(L^p(\mathbb{R}^+))$ to $\mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+))$; and we saw (Lemma 4.7) that this transformation preserves the R-boundedness and the R_p-bound.

In the following lemma we state some more properties of this transformation. The proofs are straightforward, and we omit them.

LEMMA 8.3. *Let $T \in \mathcal{L}(L^p(\mathbb{R}^+))$. Then:*

(a) *$\hat{T}$ commutes with the resolvent operators of $D_1, \dots, D_n$;*
(b) *if $T \in \mathcal{L}(L^p(\mathbb{R}^+), W^{r,p}(\mathbb{R}^+))$, then $\hat{T} \in \mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+), \mathcal{D}(D_t^r))$, and $D_t^r(\hat{T} f) = \widehat{(D^r T)} f\ \forall f \in L^p(\mathbb{R}^n \times \mathbb{R}^+)$.*

DEFINITION 8.4. $\forall z \in (\overline{\Sigma_{\varphi(\theta)}})^n$ we set $R_\mu(z) = \widehat{\Big((\mu - A_z)^{-1}\Big)}$.

LEMMA 8.5. *The following statements hold:*

(a) *$\forall z \in (\overline{\Sigma_{\varphi(\theta)}})^n$ and $\forall \ell \in \{0, \dots, 2m\}$, $D_t^\ell R_\mu(z)$ is a bounded operator on $L^p(\mathbb{R}^n \times \mathbb{R}^+)$ to itself, and precisely $D_t^\ell R_\mu(z) = \widehat{\Big(D^\ell (\mu - A_z)^{-1}\Big)}$; moreover*

$$\|D_t^\ell R_\mu(z)\|_{\mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+))} \leq C(L, L_1, \omega, \theta, L_0(\theta))\, \beta(z,\mu)^{\ell - 2m};$$

(b) *if* $\alpha \in \mathbb{N}^n$, $\ell \in \mathbb{N}$ *and* $|\alpha|+\ell \le 2m$, *then the function* $z \mapsto z^\alpha D_t^\ell R_\mu(z)$ *is holomorphic and R-bounded on* $(\Sigma_{\varphi(\theta)})^n$ *(with respect to the norm of* $L^p(\mathbb{R}^n \times \mathbb{R}^+)$*) with* R_p*-bound* $\le C(L, L_1, \omega, \theta, L_0(\theta))\,|\mu|^{\frac{\ell+|\alpha|}{2m}-1}$.

Proof. The first part of (a) is an immediate consequence of Lemma 8.3(b); the other statements follow readily from the inequality (6.4) and Theorems 6.28 and 6.31, taking into account Lemma 4.7. □

Definition 8.6. For $\alpha \in \mathbb{N}^n$, $\ell \in \mathbb{N}$ and $|\alpha| + \ell \le 2m$ the function

$$G_{\mu,\alpha,\ell} : (\Sigma_{\varphi(\theta)})^n \to \mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+))$$

is defined by $G_{\mu,\alpha,\ell}(z) = z^\alpha\, D_t^\ell\, R_\mu(z)$.

Remark that the function $G_{\mu,\alpha,\ell}$ is holomorphic and bounded, so that $G_{\mu,\alpha,\ell}(D_x)$ is defined.

Lemma 8.7. *For* $\alpha \in \mathbb{N}^n$, $\ell \in \mathbb{N}$ *and* $|\alpha| + \ell \le 2m$ *we have*

$$G_{\mu,\alpha,\ell}(D_x) \in \mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+)),$$

and in particular $\|R_\mu(D_x)\|_{\mathcal{L}(L^p(\mathbb{R}^n\times\mathbb{R}^+))} \le C(L, L_1, \omega, \theta, L_0(\theta))|\mu|^{-1}$.

Proof. $\forall z \in (\Sigma_{\varphi(\theta)})^n$ $G_{\mu,\alpha,\ell}(z) = \widehat{\left(z^\alpha\, D^\ell(\mu - A_z)^{-1}\right)}$ commutes with the resolvents of $D_1, \dots, D_n$ by Lemma 8.3. Moreover $(D_1, \dots, D_n)$ has a bounded H^∞ functional calculus (Theorem 7.3). Since by Lemma 8.5 (b) the function $G_{\mu,\alpha,\ell}$ is holomorphic and R-bounded, with R$_p$-bound $\le C\,|\mu|^{\frac{\ell+|\alpha|}{2m}-1}$, both assertions follow from the bisectorial analogous of Theorem 4.18. □

Our next task is to show that $R_\mu(D_x) = (\mu - A)^{-1}$.

In the sequel it is understood that $\widetilde{\Gamma}$ is a system of curves, contained in $(\Sigma_{\varphi(\theta)})^n$, of the type introduced in the proof of Theorem 7.3.

Lemma 8.8. $\forall f \in L^p(\mathbb{R}^n \times \mathbb{R}^+)\, R_\mu(D_x)f \in W^{2m,p}(\mathbb{R}^n \times \mathbb{R}^+)$ *and when* $|\alpha| + \ell \le 2m$, $D_x^\alpha D_t^\ell\, R_\mu(D_x)f = G_{\mu,\alpha,\ell}(D_x)f$.

Proof. In order to prove the lemma, it is enough to show that if $|\alpha|+\ell < 2m$ then $\forall f \in L^p(\mathbb{R}^n \times \mathbb{R}^+)$

(a) $G_{\mu,\alpha,\ell+1}(D_x)f = D_t\, G_{\mu,\alpha,\ell}(D_x)f$
(b) $G_{\mu,\alpha+e_j,\ell}(D_x)f = D_j G_{\mu,\alpha,\ell}(D_x)f$.

Indeed (a) and (b) imply that $\forall f \in L^p(\mathbb{R}^n \times \mathbb{R}^+)$ $G_{\mu,\alpha,\ell}(D_x)f$ belongs to the domain of D_t and of $D_1, \dots, D_n$, and hence it belongs to $W^{1,p}(\mathbb{R}^n \times \mathbb{R}^+)$. Moreover from (a) and (b) one obtains immediately that $R_\mu(D_x)f \in W^{2m,p}(\mathbb{R}^n \times \mathbb{R}^+)$ and $G_{\mu,\alpha,\ell}(D_x)f = D_x^\alpha D_t^\ell\, R_\mu(D_x)f$.

Let us prove (a) and (b). In both equalities, the left-hand side is a bounded operator on $L^p(\mathbb{R}^n \times \mathbb{R}^+)$, while the right-hand side is closed (because D_j and

D_t are closed operators in $L^p(\mathbb{R}^n \times \mathbb{R}^+)$); therefore it is enough to show that they coincide on some dense subspace of $L^p(\mathbb{R}^n \times \mathbb{R}^+)$: e.g. on the range of $\Psi(D_x)$, where $\Psi(z) = \prod_{j=1}^n \frac{z_j}{(1+z_j)^2}$ (see Lemma 4.11).

We take $g \in \mathcal{R}(\Psi(D_x))$, $g = \Psi(D_x) f$; then

$$\begin{aligned} G_{\mu,\alpha,\ell}(D_x) g &= G_{\mu,\alpha,\ell}(D_x)\, \Psi(D_x) f = (\Psi\, G_{\mu,\alpha,\ell})(D_x) f \\ &= (2\pi i)^{-n} \int_{\widetilde{\Gamma}} \Psi(z)\, G_{\mu,\alpha,\ell}(z) \prod_{r=1}^n (z_r - D_r)^{-1} f\, dz. \end{aligned}$$

As it is $\ell < 2m$, the range of $G_{\mu,\alpha,\ell}(z) = z^\alpha D_t^\ell R_\mu(z)$ is contained in $\mathcal{D}(D_t)$ (Lemma 8.5 (a)) and by definition $D_t G_{\mu,\alpha,\ell}(z) = G_{\mu,\alpha,\ell+1}(z)$ which is a bounded function of z (with values in $L^p(\mathbb{R}^n \times \mathbb{R}^+)$) by Lemma 8.5 (b). Therefore $z \mapsto \Psi(z)\, D_t\, G_{\mu,\alpha,\ell}(z) \prod_{r=1}^n (z_r - D_r)^{-1} f$ is summable on $\widetilde{\Gamma}$, and since D_t is closed we get $G_{\mu,\alpha,\ell}(D_x) g \in \mathcal{D}(D_t)$ and

$$\begin{aligned} D_t\, G_{\mu,\alpha,\ell}(D_x) g &= (2\pi i)^{-n} \int_{\widetilde{\Gamma}} \Psi(z)\, G_{\mu,\alpha,\ell+1}(z) \prod_{r=1}^n (z_r - D_r)^{-1} f\, dz \\ &= (\Psi\, G_{\mu,\alpha,\ell+1})(D_x) f = G_{\mu,\alpha,\ell+1}(D_x)\, \Psi(D_x) f \\ &= G_{\mu,\alpha,\ell+1}(D_x) g. \end{aligned}$$

This proves (a). To prove (b) we note that

$$\Psi(z) \prod_{r=1}^n (z_r - D_r)^{-1}\, G_{\mu,\alpha,\ell}(z) f \in \mathcal{D}(D_j)$$

and

$$\begin{aligned} &\Psi(z)\, D_j \prod_{r=1}^n (z_r - D_r)^{-1}\, G_{\mu,\alpha,\ell}(z) f \\ &= \Psi(z) \prod_{r=1}^n (z_r - D_r)^{-1}\, G_{\mu,\alpha+e_j,\ell}(z) f - \Psi(z)\, z_j^{-1} \prod_{r \neq j} (z_r - D_r)^{-1} G_{\mu,\alpha+e_j,\ell}(z) f. \end{aligned}$$

Here both the summands are summable on $\widetilde{\Gamma}$ and

$$\int_{\widetilde{\Gamma}} \Psi(z)\, z_j^{-1} \prod_{r \neq j} (z_r - D_r)^{-1}\, G_{\mu,\alpha+e_j,\ell}(z) f\, dz = 0$$

as one sees by integrating with respect to z_j. Since D_j is a closed operator, it follows that $G_{\mu,\alpha,\ell}(D_x) g \in \mathcal{D}(D_j)$ and

$$\begin{aligned} D_j\, G_{\mu,\alpha,\ell}(D_x) g &= (2\pi i)^{-n} \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^n (z_r - D_r)^{-1}\, G_{\mu,\alpha+e_j,\ell}(z) f\, dz \\ &= (\Psi\, G_{\mu,\alpha+e_j,\ell})(D_x) f = G_{\mu,\alpha+e_j,\ell}(D_x)\, \Psi(D_x) f \\ &= G_{\mu,\alpha+e_j,\ell}(D_x) g. \end{aligned}$$ □

LEMMA 8.9. *Let $(\Psi_j)_{j \in \mathbb{N}}$ be the sequence of functions introduced in Definition* 4.10, *and let $v \in \mathcal{D}(D_t)$. Then $\Psi_j(D_x) v \underset{j \to \infty}{\longrightarrow} v$ in the norm of $\mathcal{D}(D_t)$.*

PROOF. We know from Lemma 4.14 that as $j \to \infty$ the sequence $(\Psi_j(D_x))_{j\in\mathbb{N}}$ converges to the identity in the strong topology of $\mathcal{L}(L^p(\mathbb{R}^n\times\mathbb{R}^+))$. Moreover it follows from Theorem 8.2 that $\Psi_j(D_x)$ commutes with the resolvents of D_t; therefore if $v \in \mathcal{D}(D_t)$, we have $\Psi_j(D_x)v \in \mathcal{D}(D_t)$ and $D_t\Psi_j(D_x)v = \Psi_j(D_x)D_t v$. Hence

$$\|\Psi_j(D_x)v - v\|_{\mathcal{D}(D_t)} = \|\Psi_j(D_x)v - v\|_{L^p} + \|\Psi_j(D_x)D_t v - D_t v\|_{L^p} \underset{j\to+\infty}{\to} 0. \qquad \square$$

We are now ready to prove that $R_\mu(D_x)$ is a right inverse of $\mu - A$.

LEMMA 8.10. *$\forall\, f \in L^p(\mathbb{R}^n \times \mathbb{R}^+)$ we have $R_\mu(D_x)f \in \mathcal{D}(A)$ and $f = (\mu - A)\,R_\mu(D_x)f$.*

PROOF. Let $f \in L^p(\mathbb{R}^n\times\mathbb{R}^+)$ be fixed. We first prove that $R_\mu(D_x)f \in \mathcal{D}(A)$. By Lemma 8.8 we have $R_\mu(D_x)f \in W^{2m,p}(\mathbb{R}^n\times\mathbb{R}^+)$; hence it remains to show that the trace at $t=0$ of $B_k(D_x, D_t)\,R_\mu(D_x)f$ vanishes $\forall k \in \{1,\dots,m\}$.

Setting

$$G_k(z) = B_k(z, D_t)\,R_\mu(z) = \widehat{\Big(B_k(z, D)\,(\mu - A_z)^{-1}\Big)}$$

(see Lemma 8.5 (a)) we have, by Lemma 8.8, $B_k(D_x, D_t)\,R_\mu(D_x) = G_k(D_x)$, and so we have to show that $G_k(D_x)f \in \mathcal{D}(D_t)$ and $T_0\,G_k(D_x)f = 0$. The former assertion is a consequence of Lemma 8.3 (b), since $B_k(z, D)\,(\mu - A_z)^{-1} \in \mathcal{L}(L^p(\mathbb{R}^+), W^{1,p}(\mathbb{R}^+))$. In order to prove that $T_0\,G_k(D_x)f = 0$, we construct a sequence in $\ker T_0$ that converges to $G_k(D_x)f$ in the norm of $\mathcal{D}(D_t)$. This sequence is $(\Psi_j(D_x)G_k(D_x)f)_{j\in\mathbb{N}}$. Since $G_k(D_x)f \in \mathcal{D}(D_t)$, Lemma 8.9 implies that $\Psi_j(D_x)G_k(D_x)f \underset{j\to\infty}{\to} G_k(D_x)f$ in the norm of $\mathcal{D}(D_t)$; moreover, writing

$$\Psi_j(D_x)\,G_k(D_x)f = (\Psi_j\,G_k)(D_x)f = (2\pi i)^{-n}\int_{\widetilde{\Gamma}} \Psi_j(z)\,G_k(z)\prod_{r=1}^{n}(z_r - D_r)^{-1} f\,dz$$

the integral converges in the norm of $\mathcal{D}(D_t)$; therefore

$$T_0\Psi_j(D_x)\,G_k(D_x)f = (2\pi i)^{-n}\int_{\widetilde{\Gamma}} \Psi_j(z)\,T_0\,G_k(z)\prod_{r=1}^{n}(z_r - D_r)^{-1} f\,dz.$$

Finally we have

$$(T_0G_k(z)f)(x) = \lim_{t\to0} G_k(z)f(x,t) = \lim_{t\to0}\Big(B_k(z, D)\,(\mu - A_z)^{-1}f(x,\cdot)\Big)(t) = 0$$

because the function $t \mapsto (\mu - A_z)^{-1}f(x,t)$ belongs to $D(A_z)$, and hence it satisfies the initial conditions of problem (6.5).

This proves that $T_0\,B_k(D_x, D_t)\,R_\mu(D_x)f = 0$, and hence $R_\mu(D_x)f \in \mathcal{D}(A)$.

Now we prove that $(\mu - A)R_\mu(D_x) = I_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}$. $\forall z \in (\Sigma_{\varphi(\theta)})^n$ we have

$$(\mu - P(z, D))(\mu - A_z)^{-1} = I_{L^p(\mathbb{R}^+)}$$

so that, by Lemma 8.5 (a)

$$(\mu - P(z, D_t))R_\mu(z) = \widehat{(I_{L^p(\mathbb{R}^+)})} = I_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}.$$

On the other hand, by Lemma 8.8, $(\mu - A)R_\mu(D_x) = (\mu - P(D_x, D_t))R_\mu(D_x)$ is the operator that corresponds to the function $z \mapsto (\mu - P(z, D_t))R_\mu(z)$ in the homomorphism $g \mapsto g(D_x)$, and since that function is constantly equal to $I_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}$, it is proved that $(\mu - A)R_\mu(D_x) = I_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}$ (see Lemma 4.13). $\square$

Before proving that $R_\mu(D_x)$ is a left inverse of $\mu - A$, we need another preliminary result (similar to Lemma 8.8). $\forall r \in \mathbb{N}$ we denote by $W_x^{r,p}(\mathbb{R}^n\times\mathbb{R}^+)$ the Banach space (with the natural norm) of the functions $u \in L^p(\mathbb{R}^n\times\mathbb{R}^+)$ whose derivatives with respect to $x_1, \dots, x_n$, up to the order r, belong to $L^p(\mathbb{R}^n\times\mathbb{R}^+)$.

LEMMA 8.11. *Let $r \in \mathbb{N}$, and let $g : (\Sigma_{\varphi(\theta)})^n \to W_x^{r,p}(\mathbb{R}^n\times\mathbb{R}^+)$ be a holomorphic function (with respect to the norm of $W_x^{r,p}(\mathbb{R}^n\times\mathbb{R}^+)$). We assume that for any $\alpha,\ \beta \in \mathbb{N}^n$ such that $|\alpha|+|\beta| \le r$ the function $z \mapsto z^\alpha D_x^\beta(g(z))$ be bounded on $(\Sigma_{\varphi(\theta)})^n$ with respect to the norm of $L^p(\mathbb{R}^n\times\mathbb{R}^+)$. Then for $|\alpha| \le r$ we have*

$$\int_{\widetilde{\Gamma}} \Psi(z) \prod_{k=1}^n (z_k - D_k)^{-1} D_x^\alpha(g(z))\, dz = \int_{\widetilde{\Gamma}} z^\alpha \Psi(z) \prod_{k=1}^n (z_k - D_k)^{-1} g(z)\, dz.$$

PROOF. It is sufficient to prove the result when $r = 1$. By assumption, the functions $z \mapsto z_j g(z)$ and $z \mapsto D_j(g(z))$ are holomorphic and bounded in the norm of $L^p(\mathbb{R}^n\times\mathbb{R}^+)$. Then

$$\begin{aligned}&\int_{\widetilde{\Gamma}} \Psi(z) \prod_{k=1}^n (z_k - D_k)^{-1}\, D_j(g(z))\, dz\\ &\quad = \int_{\widetilde{\Gamma}} \Psi(z) \Big(z_j \prod_{k=1}^n (z_k - D_k)^{-1} - \prod_{k\neq j} (z_k - D_k)^{-1} \Big)\, g(z)\, dz.\end{aligned}$$

Now it is easy to see that

$$\int_{\widetilde{\Gamma}_j} \Psi(z) \prod_{k\neq j} (z_k - D_k)^{-1}\, g(z)\, dz_j = 0$$

and this concludes the proof. $\square$

LEMMA 8.12. *$\forall u \in \mathcal{D}(A)$ we have $R_\mu(D_x)\,(\mu - A)u = u$.*

PROOF. Let $u \in \mathcal{D}(A)$. If $\ell \in \{0, \dots, 2m\}$, then $D_t^\ell u \in W_x^{2m-\ell,p}(\mathbb{R}^n \times \mathbb{R}^+)$. We know that $R_\mu(z)$ commutes with the resolvent operators of $D_1, \dots, D_n$ (Lemma 8.3 (a)), therefore for $|\beta| \le 2m - \ell$ we have that $D_x^\beta R_\mu(z)\, D_t^\ell u = R_\mu(z)\, D_x^\beta\, D_t^\ell u$. Hence for $|\alpha| + |\beta| \le 2m - \ell$ we get (by Lemma 6.2)

$$\|z^\alpha\, D_x^\beta R_\mu(z)\, D_t^\ell u\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} \le C\, \frac{\|z\|^{|\alpha|}}{\|z\|^{2m} + |\mu|}\, \|D_x^\beta\, D_t^\ell u\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}$$

and this allows us to apply Lemma 8.11 obtaining

$$\int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^n (z_r - D_r)^{-1} R_\mu(z)\, D_x^\alpha\, D_t^\ell u\, dz = \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^n (z_r - D_r)^{-1} z^\alpha\, R_\mu(z)\, D_t^\ell u\, dz$$

whenever $|\alpha| + \ell \le 2m$. Hence

$$\begin{aligned}
&\int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^n (z_r - D_r)^{-1}\, R_\mu(z)\, (\mu - A)u\, dz \\
&\quad = \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^n (z_r - D_r)^{-1}\, R_\mu(z)\, (\mu - P(z, D_t))u\, dz.
\end{aligned}$$

The function $(\mu - A_z)^{-1}\, (\mu - P(z, D))u(x, \cdot)$ is (for almost every x) the unique solution of the problem

$$\begin{cases} v \in W^{2m,p}(\mathbb{R}^+) & \\ \mu\, v(t) - P(z, D)v(t) = (\mu - P(z, D_t))u(x, t) & t \in \mathbb{R}^+ \\ (B_k(z, D)v)(0) = 0 & 1 \le k \le m \end{cases}$$

so that $(\mu - A_z)^{-1}\, (\mu - P(z, D))u(x, \cdot) - u(x, \cdot)$ is the solution of

$$\begin{cases} v \in W^{2m,p}(\mathbb{R}^+) & \\ \mu\, v(t) - P(z, D)v(t) = 0 & t \in \mathbb{R}^+ \\ (B_k(z, D)v)(0) = -(B_k(z, D_t)u)(x, 0) & 1 \le k \le m. \end{cases}$$

By Lemma 6.13 this solution is the function

$$-\sum_{j,k=1}^m \delta_{j,k}(z, \mu)\, (B_k(z, D_t)u)(x, 0)\, u_{j,z,\mu}$$

with $\delta_{j,k}(z, \mu) \in \mathbb{C}$ that depends holomorphically on z and $u_{j,z,\mu} \in L^p(\mathbb{R}^+)$ that depends holomorphically (in L^p norm) on z (Lemma 6.27). Therefore

$$R_\mu(D_x)\, (\mu - A)u = \Psi(D_x)^{-1}\, (2\pi i)^{-n} \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^n (z_r - D_r)^{-1}\, R_\mu(z)\, (\mu - A)u\, dz$$

(as we have seen above)

$$\begin{aligned}
&= \Psi(D_x)^{-1}\,(2\pi i)^{-n} \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^{n} (z_r - D_r)^{-1}\, R_\mu(z)\,(\mu - P(z, D_t))u\,dz \\
&= \Psi(D_x)^{-1}\,(2\pi i)^{-n} \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^{n} (z_r - D_r)^{-1}\, u\,dz \\
&\quad - \Psi(D_x)^{-1}\,(2\pi i)^{-n} \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^{n} (z_r - D_r)^{-1} \\
&\quad \times \sum_{j,k=1}^{m} \delta_{j,k}(z,\mu)\,(B_k(z, D_t)u)(\cdot, 0) \otimes u_{j,z,\mu}\,dz.
\end{aligned}$$

Here

$$\Psi(D_x)^{-1}\,(2\pi i)^{-n} \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^{n} (z_r - D_r)^{-1}\, u\,dz = \Psi(D_x)^{-1}\,\Psi(D_x)u = u,$$

so that the lemma will be proved if we show that $\forall j, k$ we have

$$\int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^{n} (z_r - D_r)^{-1}\, \delta_{j,k}(z,\mu)\,(B_k(z, D_t)u)(\cdot, 0) \otimes u_{j,z,\mu}\,dz = 0.$$

Let us fix $j, k \in \{1, \dots, m\}$ and $\ell \in \{0, \dots, m_k\}$. Then we set

$$g(z) = \delta_{j,k}(z,\mu)\,(D_t^\ell u)(\cdot, 0) \otimes u_{j,z,\mu}.$$

From $(D_t^\ell u)(\cdot, 0) \in W^{2m-\ell-1,p}(\mathbb{R}^n)$, $u_{j,z,\mu} \in L^p(\mathbb{R}^+)$ and the holomorphy of $\delta_{j,k}(z,\mu)$ and of $u_{j,z,\mu}$ we obtain that g is holomorphic with values in $W_x^{2m-\ell-1,p}(\mathbb{R}^n \times \mathbb{R}^+)$. Moreover it follows from Lemma 6.13 that if $|\alpha| + |\beta| \le 2m - \ell - 1$ then

$$\begin{aligned}
\|z^\alpha D_x^\beta (g(z))\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} &= |z^\alpha|\,|\delta_{j,k}(z,\mu)|\,\|u_{j,z,\mu}\|_{L^p(\mathbb{R}^+)}\,\|D_x^\beta D_t^\ell u(\cdot, 0)\|_{L^p(\mathbb{R}^n)} \\
&\le C\,|z^\alpha|\,\beta(z,\mu)^{-m_k-(1/p)}\,\|D_x^\beta D_t^\ell u(\cdot, 0)\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

and this is a bounded function of z when $|\alpha| \le m_k$. Then Lemma 8.11 implies that for $|\alpha| \le m_k - \ell$ we have

$$\begin{aligned}
&\int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^{n} (z_r - D_r)^{-1}\, \delta_{j,k}(z,\mu)\, z^\alpha\, D_t^\ell u(\cdot, 0) \otimes u_{j,z,\mu}\,dz \\
&= \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^{n} (z_r - D_r)^{-1}\, \delta_{j,k}(z,\mu)\, D_x^\alpha D_t^\ell u(\cdot, 0) \otimes u_{j,z,\mu}\,dz
\end{aligned}$$

and hence

$$\int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^{n} (z_r - D_r)^{-1} \delta_{j,k}(z,\mu)(B_k(z, D_t)u)(\cdot, 0) \otimes u_{j,z,\mu}\, dz$$
$$= \int_{\widetilde{\Gamma}} \Psi(z) \prod_{r=1}^{n} (z_r - D_r)^{-1} \delta_{j,k}(z,\mu)(B_k(D_x, D_t)u)(\cdot, 0) \otimes u_{j,z,\mu}\, dz = 0$$

because $(B_k(D_x, D_t)u)(\cdot, 0) = 0$, as $u \in \mathcal{D}(A)$. □

Putting together Lemmas 8.7, 8.10, 8.12, we have proved that $\rho(A) \supseteq \mathbb{C}\setminus\overline{S_\omega}$, and that $\forall\theta \in]\omega,\pi[\ \exists C > 0$ such that $\forall\mu \in \mathbb{C} \setminus \overline{S_\theta}$ one has $\|(\mu - A)^{-1}\| \leq \frac{C}{|\mu|}$.

In order to prove that A is sectorial with spectral angle ω, we still have to show that $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are dense. For $\mathcal{D}(A)$, it is enough to remark that $\mathcal{D}(A) \supset C_0^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. Concerning the range of A, it is known that, as $L^p(\mathbb{R}^n \times \mathbb{R}^+)$ is a reflexive Banach space, from the inequality $\sup_{\mu\in\mathbb{R}^-} \|\mu\,(\mu - A)^{-1}\| < +\infty$ (that we have just proved) it follows that $L^p(\mathbb{R}^n \times \mathbb{R}^+) = \ker A \oplus \overline{\mathcal{R}(A)}$, as we remarked in Subsection 4.2; therefore proving that $\mathcal{R}(A)$ is dense becomes equivalent to proving that A is injective, and this we do.

LEMMA 8.13. *A is injective (and hence $\mathcal{R}(A)$ is dense in $L^p(\mathbb{R}^n \times \mathbb{R}^+)$).*

PROOF. Assume that $u \in \ker A$. Then $\forall\mu \in \mathbb{C}\setminus\overline{S_\omega}$ we have $u = \mu(\mu - A)^{-1}u = \mu\, R_\mu(D_x)u$. Let us take $\alpha \in \mathbb{N}^n$ such that $|\alpha| = 2m$, and an integer $q \geq \max_{1\leq j\leq n}(1 + \alpha_j)$. Then $|z^\alpha| \leq \|z\|^{2m}$ and hence, for $z \in \widetilde{\Gamma}$,

$$\left\|\Psi(z)^q R_\mu(z) \prod_{r=1}^{n} (z_r - D_r)^{-1}\right\| \leq C \prod_{r=1}^{n} \frac{|z_r|^{q-1}}{|1 + z_r|^{2q}} (|\mu| + \|z\|^{2m})^{-1}$$
$$\leq C \prod_{r=1}^{n} \frac{|z_r|^{q-1}}{|1 + z_r|^{2q}} \|z\|^{-2m} \leq C \prod_{r=1}^{n} \frac{|z_r|^{q-1-\alpha_r}}{|1 + z_r|^{2q}}.$$

Therefore, for $\mu \in \mathbb{R}^-$,

$$\|\Psi(D_x)^q u\| = \|\mu\, \Psi(D_x)^q R_\mu(D_x)u\|$$
$$= \left\|(2\pi i)^{-n} \mu \int_{\widetilde{\Gamma}} \Psi(z)^q R_\mu(z) \prod_{r=1}^{n} (z_r - D_r)^{-1} u\, dz\right\|$$
$$\leq (2\pi)^{-n} C\, |\mu| \prod_{r=1}^{n} \int_{\widetilde{\Gamma}_r} \frac{|z_r|^{q-1-\alpha_r}}{|1 + z_r|^{2q}}\, d\,|z_r|\, \|u\|$$

and as $\mu \to 0$ we get $\Psi(D_x)^q u = 0$. Since $\Psi(D_x)$ is injective (Lemma 4.11), it follows that $u = 0$. □

Lemma 8.13 concludes the proof that A is sectorial, with spectral angle ω. In particular the following *a priori* estimate holds.

THEOREM 8.14. *There exists $C > 0$ such that $\forall u \in \mathcal{D}(A)$*

$$\|u\|_{W^{2m,p}(\mathbb{R}^n\times\mathbb{R}^+)} \le C\left(\|Au\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} + \|u\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\right).$$

PROOF. Let $u \in \mathcal{D}(A)$. Then for $|\alpha| + \ell \le 2m$ we have, by Lemmas 8.8 and 8.12,

$$D_x^\alpha D_t^\ell u = D_x^\alpha D_t^\ell R_{-1}(D_x)\,(-1-A)u = G_{-1,\alpha,\ell}(D_x)\,(-1-A)u.$$

Therefore by Lemma 8.7

$$\begin{aligned}\|u\|_{W^{2m,p}(\mathbb{R}^n\times\mathbb{R}^+)} &= \left(\sum_{|\alpha|+\ell\le 2m} \|D_x^\alpha D_t^\ell u\|^p_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\right)^{1/p} \\ &\le \left(\sum_{|\alpha|+\ell\le 2m} \|G_{-1,\alpha,\ell}(D_x)\|^p\right)^{1/p}\left(\|Au\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} + \|u\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\right). \quad \square\end{aligned}$$

In order prove the boundedness of the H^∞ functional calculus for A, we consider the function $G : (\Sigma_{\varphi(\theta)})^n \to \mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+))$, $G(z) = \widehat{\left(h(A_z)\right)}$. We know that G is holomorphic and R-bounded on $(\Sigma_{\varphi(\theta)})^n$ with R_p-bound $\le C\,\|h\|_\infty$ (see Theorems 6.37, 6.38 and Lemma 4.7). Moreover by Lemma 8.3 (a) the operators $G(z)$ commute with the resolvents of $D_1, \dots, D_n$; therefore $G(D_x) \in \mathcal{L}(L^p(\mathbb{R}^n \times \mathbb{R}^+))$, with $\|G(D_x)\| \le C\,\|h\|_\infty$ (Theorem 4.18). Therefore the following lemma concludes the proof of Theorem 8.1.

LEMMA 8.15. *Let $h \in H_0^\infty(S_\theta)$. Then $h(A) = G(D_x)$.*

PROOF. As in Subsection 6.4, we take the curve γ parametrized by $\mathbb{R}\setminus\{0\} \ni t \mapsto |t|\,e^{-i\eta\,\mathrm{sgn}\,t}$ (with $\omega < \eta < \theta$). $R_\mu(D_x)$ is a bounded operator (Lemma 8.7) and is defined by

$$R_\mu(D_x) = \Psi(D_x)^{-1}\int_{\widetilde{\Gamma}} \Psi(z) R_\mu(z) \prod_{r=1}^n (z_r - D_r)^{-1}\,dz$$

so that the range of $\int_{\widetilde{\Gamma}} \Psi(z) R_\mu(z) \prod_{r=1}^n (z_r - D_r)^{-1} dz$ is contained in $\mathcal{D}(\Psi(D_x)^{-1})$. Now for some $s \in \mathbb{R}^+$

$$\left\|\Psi(z)\,h(\mu)\,R_\mu(z)\prod_{r=1}^n (z_r - D_r)^{-1}\right\| \le C\prod_{r=1}^n |1 + z_r|^{-2}\,|\mu|^{-1}\,\min\left\{|\mu|^{-s}, |\mu|^s\right\}$$

and hence $\int_\gamma \int_{\widetilde{\Gamma}} \Psi(z)\, h(\mu)\, R_\mu(z) \prod_{r=1}^n (z_r - D_r)^{-1}\, dz\, d\mu$ converges in the operator norm and equals $\int_{\widetilde{\Gamma}} \Psi(z) \int_\gamma h(\mu)\, R_\mu(z)\, d\mu \prod_{r=1}^n (z_r - D_r)^{-1}\, dz$. Then

$$
\begin{aligned}
h(A) &= \frac{1}{2\pi i} \int_\gamma h(\mu)\, R_\mu(D_x)\, d\mu \\
&= \frac{1}{2\pi i} \int_\gamma (2\pi i)^{-n}\, \Psi(D_x)^{-1} \int_{\widetilde{\Gamma}} \Psi(z)\, h(\mu)\, R_\mu(z) \prod_{r=1}^{n} (z_r - D_r)^{-1}\, dz\, d\mu
\end{aligned}
$$

(as $\Psi(D_x)^{-1}$ is closed)

$$
\begin{aligned}
&= \frac{1}{2\pi i}\, \Psi(D_x)^{-1} \int_\gamma (2\pi i)^{-n} \int_{\widetilde{\Gamma}} \Psi(z)\, h(\mu)\, R_\mu(z) \prod_{r=1}^{n} (z_r - D_r)^{-1}\, dz\, d\mu \\
&= (2\pi i)^{-n}\, \Psi(D_x)^{-1} \int_{\widetilde{\Gamma}} \Psi(z)\, \frac{1}{2\pi i} \int_\gamma h(\mu)\, R_\mu(z) d\mu \prod_{r=1}^{n} (z_r - D_r)^{-1} dz \\
&= G(D_x),
\end{aligned}
$$

since

$$
\frac{1}{2\pi i} \int_\gamma h(\mu)\, R_\mu(z)\, d\mu = \widehat{\left(\frac{1}{2\pi i} \int_\gamma h(\mu)\, (\mu - A_z)^{-1}\, d\mu \right)} = \widehat{\big(h(A_z) \big)} = G(z). \quad \square
$$

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During the preparation of this paper the authors became aware that R. Denk, M. Hieber and J. Prüss have obtained the L^p-maximal regularity for the solution of the Cauchy problem (1.1) when A is an elliptic operator of arbitrary order in a domain, acting on Banach space valued functions, with minimal assumptions on the regularity of the coefficients. In the case of constant coefficients on a half-space they have also obtained the boundedness of the H^∞ functional calculus. However their methods are quite different from ours.

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Dipartimento di Matematica
Università di Bologna
Piazza di Porta San Donato, 5
40127 Bologna, Italy
dore@dm.unibo.it
venni@dm.unibo.it