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Small Data Scattering for Nonlinear Schrödinger Wave and Klein-Gordon Equations

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Dedicated to Professor Giorgio Velo on the occasion of his sixtieth birthday

Abstract. Small data scattering for nonlinear Schrödinger equations (**NLS**), nonlinear wave equations (**NLW**), nonlinear Klein-Gordon equations (**NLKG**) with power type nonlinearities is studied in the scheme of Sobolev spaces on the whole space $\mathbb{R}^n$ with order $s < n/2$. The assumptions on the nonlinearities are described in terms of power behavior p_1 at zero and p_2 at infinity such as $1+4/n \le p_1 \le p_2 \le 1+4/(n-2s)$ for **NLS** and **NLKG**, and $1+4/(n-1) \le p_1 \le p_2 \le 1+4/(n-2s)$ for **NLW**.

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1. – Introduction

We consider the small data scattering for nonlinear Schrödinger equations (**NLS**), nonlinear wave equations (**NLW**), nonlinear Klein-Gordon equations (**NLKG**) in the Sobolev space H^s for **NLS** and **NLKG**, and in the homogeneous Sobolev space $\dot{H}^s$ for **NLW**. The equations in this paper take the form

$$\begin{array}{llll} \textbf{NLS} & i\partial_t u - \Delta u = f(u), & n \ge 1, & 0 \le s < n/2, \\ \textbf{NLW} & \partial_t^2 u - \Delta u = f(u), & n \ge 2, & 1/2 \le s < n/2, \\ \textbf{NLKG} & \partial_t^2 u - \Delta u + u = f(u), & n \ge 2, & 1/2 \le s < n/2, \end{array}$$

where u is a complex valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, Δ is the Laplacian in $\mathbb{R}^n$, and f is a nonlinear interaction. A typical form of $f(u)$ is given by

$$f(u) = \begin{cases} c_1|u|^{p(0)} + c_2|u|^{p(s)} & \text{for } \textbf{NLS}, \ \textbf{NLKG} \\ c_1|u|^{p(1/2)} + c_2|u|^{p(s)} & \text{for } \textbf{NLW}, \end{cases} \tag{1.1}$$

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where $c_1, c_2 \in \mathbb{C}$, $p(s) \equiv 1 + 4/(n-2s)$. A scaling argument shows that $p(s)$ is the critical power at the level of H^s.

There is a large literature on the small data scattering for **NLS**, **NLW** and **NLKG**. See for instance [4], [7], [10], [13], [15], [16], [18], [19], [21] and [22], and references therein.

In [7], Kato has shown the small data scattering theory for **NLS** with f satisfying $f \in C^{\{s\}}(\mathbb{C};\mathbb{C})$, $f^{(j)}(0) = 0$,

$$f^{(j)}(z) = \begin{cases} O(|z|^{p-j}) & \text{as } |z| \to \infty\,, \\ O(|z|^{p(0)-j}) & \text{as } |z| \to 0\,, \end{cases}$$

for $0 \le j \le \{s\}$ and $\max(\{s\}, p(0)) \le p \le p(s)$, where $\{s\} = [s]+1$ for $s \notin \mathbb{Z}_+$, $\{s\} = s$ for $s \in \mathbb{Z}_+$, and $[s]$ denotes the largest integer less than or equal to s. It is also shown that the auxiliary spaces of solutions are removable when p satisfies

$$p < 1 + \frac{\min(2s+2, 4)}{n-2s} \quad \text{if} \quad n \ge 2, \qquad p \le \frac{2}{1-2s} \quad \text{if} \quad n = 1\,.$$

In [13], we have shown the small data scattering theory for **NLW** with f satisfying $f \in C^{\max(1,[s-1/2])}(\mathbb{C};\mathbb{C})$,

$$\begin{aligned} &f^{(j)}(z) = O(|z|^{p(s)-j}) \text{ for } \quad z \in \mathbb{C}\,, \\ &\left| f^{([s-1/2])}(z) - f^{([s-1/2])}(w) \right| \le C|z-w|^{p(s)-[s-1/2]} \\ &\qquad\qquad \text{if} \quad s - 1/2 < p(s) < [s+1/2] \end{aligned} \tag{1.2}$$

for $0 \le j \le \max(1, [s-1/2])$.

In [22], Wang has shown the small data scattering theory for **NLKG** with f satisfying one of two conditions

$$\begin{aligned} &(a) \quad f \in C^1(\mathbb{C};\mathbb{C}), |f'(z)| \le C|z|^{p-1}, p(0) \le p < p(1/2) \text{ for } 1/2 \le s \le 3/2\,, \\ &(b) \quad f \in C^{\kappa}(\mathbb{C};\mathbb{C}), |f^{(j)}(z)| \le C(|z|^{p_1-j} + |z|^{p_2-j}), 0 \le j \le \kappa\,, \\ &\qquad\qquad \text{for } p(1/2) \le p_1 \le p_2 \le p(s) \text{ and } 1/2 \le s < n/2\,, \end{aligned}$$

where κ is a constant greater than $[s]$. Thus the sum of two powers $|u|^{1+4/n} + |u|^{1+4/(n-2s)}$ is excluded.

In this paper, we extend all those results above to the full range of powers of the nonlinearities such as (1.1) with minimal regularity assumption on f using the property (1.2). For that purpose we exploit refined Strichartz estimates, Corollary 2.8 below, which is needed to show the uniqueness of solutions without auxiliary spaces, as well as to deal with the sum of two powers $|u|^{1+4/n}$ and $|u|^{1+4/(n-2s)}$ for **NLKG**. The proof of such refined Strichartz estimates is inspired by [5], [9], [11]. In order to state our results precisely, we introduce the following notation.

For any r with $1 \le r \le \infty$, $L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any r with $1 < r < \infty$, $H^s_r = (1-\Delta)^{-s/2}L^r$ denotes the Sobolev space defined in terms of Bessel potentials. For any $s \in \mathbb{R}$ and any r with $1 < r < \infty$, $\dot{H}^s_r = (-\Delta)^{-s/2}L^r$ denotes the homogeneous Sobolev space defined in terms of Riesz potentials. To introduce the Besov space and the homogeneous Besov space, let ϕ_0 be a nonnegative function on $\mathbb{R}^n$ with

$$\text{supp } \phi_0 \subset \{\xi \in \mathbb{R}^n \ ; \ 1/2 \le |\xi| \le 2\}$$

such that $\{\phi_0(2^{-j}\cdot)\}_{j=-\infty}^{\infty}$ forms the Littlewood-Paley dyadic decomposition on $\mathbb{R}^n \setminus \{0\}$. Let $\{\psi_j\}_{-\infty}^{\infty}$ and $\tilde{\psi}$ be functions defined by

$$\mathcal{F}\psi_j(\xi) \equiv \phi_0(2^{-j}\xi), \quad \mathcal{F}\tilde{\psi}(\xi) \equiv 1 - \sum_{j=1}^{\infty} \phi_0(2^{-j}\xi)\,,$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote Fourier transform and its inverse, respectively. For any $s \in \mathbb{R}$ and any r, m with $1 \le r, m \le \infty$, the Besov space $B^s_{r,m}$ and the homogeneous Besov space $\dot{B}^s_{r,m}$ are defined by

$$B^s_{r,m} \equiv \left\{ u \in \mathcal{S}'(\mathbb{R}^n); \|u; B^s_{r,m}\| \equiv \left\{ \|\tilde{\psi} * u;\, L^r\|^m + \sum_{j=1}^{\infty} (2^{sj}\|\psi_j * u; L^r\|)^m \right\}^{1/m} < \infty \right\}$$

$$\dot{B}^s_{r,m} \equiv \left\{ u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n);\ \|u;\, \dot{B}^s_{r,m}\| \equiv \left\{ \sum_{j=-\infty}^{\infty} (2^{sj}\|\psi_j * u;\, L^r\|)^m \right\}^{1/m} < \infty \right\}$$

where $*$ denotes the convolution in $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{P}(\mathbb{R}^n)$ denote the sets of tempered distributions and of polynomials on $\mathbb{R}^n$, respectively. We refer to [1], [6], [20] for general information on Besov and Triebel-Lizorkin spaces and their homogeneous counterparts. For simplicity we make abbreviations such as $H^s = H^s_2$, $\dot{H}^s = \dot{H}^s_2$, $B^s_r = B^s_{r,2}$, $\dot{B}^s_r = \dot{B}^s_{r,2}$. For any Banach spaces X and Y having a common dense subspace, we put

$$\|a;\, X \cap Y\| \equiv \max(\|a;\, X\|, \|a;\, Y\|)$$

for any $a \in X \cap Y$. For any interval $I \subset \mathbb{R}$ and any Banach space X we denote by $C(I; X)$ the space of strongly continuous functions from I to X, by $L^q(I; X)$ (or by L^qX, for simplicity) the space of measurable functions u from I to X such that $\|u(\cdot); X\| \in L^q(I)$. For any r with $1 \le r \le \infty$, r' is the exponent dual to r defined by $1/r + 1/r' = 1$. For $a, b \in \mathbb{R}$ we denote by $a \vee b$ and $a \wedge b$ the maximum and minimum of a and b, respectively, and we denote by $a \lesssim b$ the inequality $a \le Cb$ with some constant $C > 0$ which is independent on other constants and variables in question.

The behavior of nonlinearity f is described by the following assumptions $N(s, p)$ and $N(s, p_1, p_2)$ with $s \geq 0$, $1 \leq p, p_1, p_2 < \infty$.

$N(s,p)$: $f \in C^{[s]}(\mathbb{C};\mathbb{C})$, $f^{(j)}(z) = O(|z|^{p-j})$ for $0 \leq j \leq [s] \wedge p$. Moreover $f^{(j)}(z) = 0$ for $p < j \leq [s]$ if $p < [s]$.

$$|f^{([s])}(z) - f^{([s])}(w)| \lesssim \begin{cases} (|z| \vee |w|)^{p-[s]-1}|z-w| & \text{if } [s]+1 \leq p\,, \\ |z-w|^{p-[s]} & \text{if } s < p < [s]+1\,, \\ 0 & \text{if } p \leq s\,. \end{cases} \tag{1.3}$$

$N(s, p_1, p_2)$: f is written as a finite sum $\sum_{j=1}^{\ell} f_j$ with f_j satisfying $N(s, p_j^*)$ for some p_j^* with $p_1 \leq p_j^* \leq p_2$.

In this paper, $\tilde{H}^s, \tilde{\mathcal{H}}^s$, and $\bar{\mathcal{H}}^s$ denote the class of solutions, the class of data, and the class in which the data is needed to be small, respectively, which are defined by the following correspondence.

	$\tilde{H}^s$	$\tilde{\mathcal{H}}^s$	$\bar{\mathcal{H}}^s$
NLS	H^s	H^s	$\dot{H}^s$
NLW	$\dot{H}^{1/2} \cap \dot{H}^s$	$\dot{\mathcal{H}}^{1/2} \cap \dot{\mathcal{H}}^s$	$\dot{\mathcal{H}}^s$
NLKG	H^s	$\mathcal{H}^s$	$\mathcal{H}^s$

Here $\dot{\mathcal{H}}^s = \dot{H}^s \oplus \dot{H}^{s-1}$ and $\mathcal{H}^s = H^s \oplus H^{s-1}$. We denote by $\vec{\phi}$ the elements of $\tilde{\mathcal{H}}^s$ with $\vec{\phi} = \phi = u(0)$ for **NLS**, $\vec{\phi} = (\phi, \psi) = (u(0), \partial_t u(0))$ for **NLW** and **NLKG**. Accordingly, we define $\vec{u}(t) = u(t)$ for **NLS** and $\vec{u}(t) = (u(t), \partial_t u(t))$ for **NLW** and **NLKG**. For $s, s_1, s_2 \in \mathbb{R}$ and $\gamma > 0$, we put

$$A^s(s_1, s_2, \gamma) \equiv \{\vec{\phi} \in \tilde{\mathcal{H}}^s \ ; \ \|\vec{\phi}; \bar{\mathcal{H}}^{s_1} \cap \bar{\mathcal{H}}^{s_2}\| \leq \gamma\}\,,$$

so that the first index s denotes the required regularity of the data and the next two indices s_1 and s_2 are related to the smallness of the data, which is realized by the norm of $\bar{\mathcal{H}}^{s_1} \cap \bar{\mathcal{H}}^{s_2}$ of size less than or equal to γ. Let $\omega \equiv -\Delta$ for **NLS**, $\omega \equiv \sqrt{-\Delta}$ for **NLW**, $\omega \equiv \sqrt{1-\Delta}$ for **NLKG**. Let $U(t) \equiv \exp(it\omega)$. For $\vec{\phi} \in \tilde{\mathcal{H}}^s$, let

$$\Gamma(t)\vec{\phi} \equiv \begin{cases} U(t)\phi & \text{for } \textbf{NLS}\,, \\ (\cos t\omega)\phi + \dfrac{\sin t\omega}{\omega}\psi & \text{for } \textbf{NLW} \text{ and } \textbf{NLKG}\,. \end{cases}$$

For $t_0, t \in \mathbb{R}$, let

$$\widetilde{G_{t_0}}h(t) \equiv \begin{cases} -i\displaystyle\int_{t_0}^{t} U(t-\tau)h(\tau)d\tau & \text{for } \textbf{NLS}\,, \\ \displaystyle\int_{t_0}^{t} \frac{\sin(t-\tau)\omega}{\omega}h(\tau)d\tau & \text{for } \textbf{NLW} \text{ and } \textbf{NLKG}. \end{cases}$$

Then **NLS**, **NLW** and **NLKG** are rewritten by the following integral equation

$$\textbf{INT} \qquad u(t) = \Phi(u)(t) \equiv \Gamma(t)\vec{\phi} + \widetilde{G_0}(f(u))(t)\,.$$

The above integral equation is considered in the following auxiliary spaces. For $s \in \mathbb{R}$, we put

$$X^s \equiv L^\infty \tilde{H}^s \cap \begin{cases} L^{q_0} B^s_{q_0}, & q_0 \equiv \dfrac{2(n+2)}{n} \quad \text{for } \mathbf{NLS}, \\ L^{q_1}(\dot{B}^{s-1/2}_{q_1} \cap \dot{B}^0_{q_1}), & q_1 \equiv \dfrac{2(n+1)}{n-1} \quad \text{for } \mathbf{NLW}, \\ \cap_{q_0 \le q \le q_1} L^q B^{s-1/2}_q & \qquad\qquad\qquad \text{for } \mathbf{NLKG}, \end{cases}$$

endowed with the metric d given by

$$d(u, v) \equiv \begin{cases} \|u - v; X^0\| & \text{for } \mathbf{NLS}, \\ \|u - v; X^{1/2}\| & \text{for } \mathbf{NLW} \text{ and } \mathbf{NLKG}. \end{cases}$$

For $1 \le p < \infty$, let $s(p)$ be the critical regularity associated with the power p, given by

$$s(p) \equiv \begin{cases} n/2 - 2/(p-1) & \text{for } \mathbf{NLS} \text{ and } \mathbf{NLW}, \\ 1/2 \vee (n/2 - 2/(p-1)) & \text{for } \mathbf{NLKG}. \end{cases}$$

With the notation above, our main results read as:

THEOREM 1.1. *Let n, s, p_1, p_2 satisfy*

$$\begin{array}{lllll} n \ge 1, & 0 \le s < n/2, & s_0 = 0, & p(0) \le p_1 \le p_2 \le p(s) & \textit{for } \mathbf{NLS}, \\ n \ge 2, & 1/2 \le s < n/2, & s_0 = 1/2, & p(1/2) \le p_1 \le p_2 \le p(s) & \textit{for } \mathbf{NLW}, \\ n \ge 2, & 1/2 \le s < n/2, & s_0 = 1/2, & p(0) \le p_1 \le p_2 \le p(s) & \textit{for } \mathbf{NLKG}. \end{array}$$

Let f satisfy $N(s - s_0, p_1, p_2)$. Then there exists $\gamma > 0$ with the following property.

(1) *For any data $\vec{\phi} \in A^s(s(p_1), s(p_2), \gamma)$,* **INT** *has a unique global solution u in $C(\mathbb{R}; \tilde{H}^s) \cap X^s$.*

(2) *There exists unique two states $\vec{\phi}_+, \vec{\phi}_-$ in $\tilde{\mathcal{H}}^s$ such that*

$$\|\vec{u}(t) - \Gamma(t)\vec{\phi}_\pm; \tilde{\mathcal{H}}^s\| \to 0 \tag{1.4}$$

as $t \to \pm\infty$. More precisely,

$$\begin{aligned} &\|u(t) - U(t)\phi_\pm; H^s\| \to 0 && \textit{for } \mathbf{NLS}, \\ &\left\|\left(u(t) - (\cos t\omega)\phi_\pm - \frac{\sin t\omega}{\omega}\psi_\pm, \partial_t\left(u(t) - (\cos t\omega)\phi_\pm - \frac{\sin t\omega}{\omega}\psi_\pm\right)\right); \dot{\mathcal{H}}^{1/2} \cap \dot{\mathcal{H}}^s\right\| \to 0 && \textit{for } \mathbf{NLW}, \\ &\left\|\left(u(t) - (\cos t\omega)\phi_\pm - \frac{\sin t\omega}{\omega}\psi_\pm, \partial_t\left(u(t) - (\cos t\omega)\phi_\pm - \frac{\sin t\omega}{\omega}\psi_\pm\right)\right); \mathcal{H}^s\right\| \to 0 && \textit{for } \mathbf{NLKG} \end{aligned} \tag{1.5}$$

with $\vec{\phi}_\pm = (\phi_\pm, \psi_\pm)$ for **NLW** *and* **NLKG**.

(3) *For any data $\vec{\phi}_-$ in $A^s(s(p_1), s(p_2), \gamma)$, there exists a unique global solution u of* **INT** *in $C(\mathbb{R}; \tilde{H}^s) \cap X^s$ and a unique state $\vec{\phi}_+ \in \tilde{\mathcal{H}}^s$ such that* (1.5) *holds. Moreover the scattering operator $S : \vec{\phi}_- \mapsto \vec{\phi}_+$ is well-defined on $A^s(s(p_1), s(p_2), \gamma)$ to $\tilde{\mathcal{H}}^s$ and is continuous in the following sense. If $\{\vec{\phi}_-^j\}_{j=1}^\infty \subset A^s(s(p_1), s(p_2), \gamma)$ satisfies $\|\vec{\phi}_- - \vec{\phi}_-^j; \tilde{\mathcal{H}}^{s_0}\| \to 0$ as $j \to \infty$, then*

$$\|\vec{\phi}_+ - \vec{\phi}_+^j; \tilde{\mathcal{H}}^\mu\| \to 0 \tag{1.6}$$

as $j \to \infty$ for any μ with $s_0 \le \mu < s(p_2)$ where $\vec{\phi}_+^j = S(\vec{\phi}_-^j)$.

(4) *In parts* (1) *and* (3)*, the auxiliary space X^s is removable if p_2 satisfies*

$$\begin{cases} p_2 \le 1+(1+2s)/(1-2s) & \text{for } n=1\,, \\ p_2 < 1+(1+s)/(1-s) & \text{for } n=2\,, \\ p_2 \le 1+(2n+4(n-1)s)/n(n-2s) & \text{for } n\ge 3 \text{ and } 0\le s<n/2(n-1)\,, \\ p_2 < 1+4(n-1)/n(n-2) & \text{for } n\ge 3 \text{ and } s=n/2(n-1)\,, \\ p_2 \le 1+4/(n-2s) & \text{for } n\ge 3 \text{ and } n/2(n-1)<s<n/2 \end{cases} \tag{1.7}$$

for **NLS***,*

$$p_2 \le 1 + 4n/(n+1)(n-2s) \tag{1.8}$$

for **NLW** *and* **NLKG**. *In particular, uniqueness of solutions holds in $C(I; \tilde{H}^s)$ for any interval I in $\mathbb{R}$ without smallness condition on the data if $p_2 \ne p(s)$.*

Proposition 1.1. *Let us consider* **NLW***and* **NLKG**. *Let $n \ge 3$. Let*

$$1 \le s < 3/2, \text{ for } n = 3, \quad 2/3 \le s < 2 \text{ for } n = 4\,,$$
$$(n-2)/(n+1) \le s < n/2 \text{ for } n \ge 5\,.$$

Then the state (4) *in Theorem* 1.1 *holds with* (1.8) *replaced by $p_2 < p(s)$. In particular, uniqueness of solutions holds in $C(I; \tilde{H}^s)$ for any interval I in $\mathbb{R}$ without smallness condition on the data.*

Remark 1. The assumptions of the theorem above cover for instance nonlinearities of the form

(1.9)
- cu^p with $p \in \mathbb{Z}_+$,
- $c|u|^{p-1}u$ with $p = 2m+1$ and $m \in \mathbb{Z}_+$,
- $c|u|^p$ and $c|u|^{p-1}u$ with $p > s - s_0$,

where $c \in \mathbb{C}$, $p(0) \le p \le p(s)$ for **NLS** and **NLKG**, $p(1/2) \le p \le p(s)$ for **NLW**, and

- $c|u|^p \log|u|$, $c|u|^{p-1}u \log|u|$ with $p > [s - s_0] + 1$, where $p(0) < p < p(s)$ for **NLS** and **NLKG**, $p(1/2) < p < p(s)$ for **NLW**, and any finite sum of the above nonlinearities. In particular, for **NLS**, the quintic correction of cubic interaction

- $f(u) = c_1|u|^2u + c_2|u|^4u, \quad c_1, c_2 \in \mathbb{C}$

falls within the scope with $n \geq 2$, $(n-1)/2 \leq s < n/2$. The logarithmic correction of cubic interaction (see Section 3.5 in [17])

- $f(u) = c_1|u|^2u + c_2|u|^2u \log|u|^2$,

also, falls within the scope with $3 \leq n \leq 5$, $(n-2)/2 < s < 2 \wedge (n/2)$. On the other hand, the nonlinearity $f(u) = cu \log|u|^2$ requires a special treatment (see [2], [3]).

Remark 2. The regularity assumptions $C^{\{s\}}$ in [7] and C^{κ} in [22] for f, described above, are refined in the theorem as $C^{[s-s_0]}$ with the property (1.3). For **NLKG**, p_1 and p_2 can be taken in the full range $p(0) \leq p_1 \leq p_2 \leq p(s)$ as compared to $p(1/2) \leq p_1 \leq p_2 \leq p(s)$ in [22].

Remark 3. As for **NLKG** for $n = 1$, we have shown in [12] the scattering theory for small data at the level of $H^{1/2}$ for nonlinearities with exponential growth at infinity such as $f(u) = c|u|^{p(0)} \exp(\nu|u|^2)$, $c \in \mathbb{C}$, $\nu > 0$.

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2. – Strichartz estimates

Let λ_j, σ_j, $j = 1, 2$, satisfy

$$
\begin{aligned}
&\lambda_1 = \lambda_2 = 0, \quad \sigma_1 = \sigma_2 = n && \text{for } \textbf{NLS}, \\
&\lambda_1 = \lambda_2 = (n+1)/2, \quad \sigma_1 = \sigma_2 = n-1 && \text{for } \textbf{NLW}, \\
&(n+1)/2 \leq \lambda_j \leq (n+2)/2, \ \ 2n - 2\lambda_j \leq \sigma_j \leq 2\lambda_j - 2, \sigma_j > 0, j = 1, 2, \\
& && \text{for } \textbf{NLKG}.
\end{aligned} \tag{2.10}
$$

Our purpose in this section is to show the estimates

$$
\begin{aligned}
&\left\| Gh; L^q(\mathbb{R}; \bar{B}_r^{-\lambda_1 \alpha(r)}) \right\| \lesssim \left\| h; L^{\tilde{q}}(\mathbb{R}; \bar{B}_{\tilde{r}}^{-\lambda_2 \alpha(\tilde{r})}) \right\|, \\
&Gh(t) \equiv \int_J U(t-\tau) h(\tau) d\tau
\end{aligned} \tag{2.11}
$$

for various $r, q, \tilde{r}$ and $\tilde{q}$ in $[1, \infty)$, where $\alpha(r) \equiv 1/2 - 1/r$ and J denotes an interval in $\mathbb{R}$ of the form $[0, t]$, $[t, \infty)$ and $(-\infty, t]$, and $\bar{B}^s_r$ denotes $\dot{B}^s_r$ for **NLS** and **NLW**, B^s_r for **NLKG**. For that purpose we first recall a basic reasoning of the problem. We start from the following well-known decay estimate

$$\|U(t)\phi;\, \bar{B}^{-\lambda_1/2}_{\infty}\| \lesssim |t|^{-\sigma_1/2}\|\phi;\, \bar{B}^{\lambda_1/2}_{1}\|\,. \tag{2.12}$$

Interpolating the energy estimate and (2.12) and applying the Hardy-Littlewood-Sobolev inequality in the time variable, we have

$$\left\|\int_J U(\cdot - \tau)h(\tau)d\tau;\, L^q \bar{B}^{-\lambda_1\alpha(r)}_{r}\right\| \lesssim \|h;\, L^{q'} \bar{B}^{-\lambda_1\alpha(r')}_{r'}\|\,. \tag{2.13}$$

Applying the bilinear estimate associated with (2.13), we also have

$$\left\|\int_J U(\cdot - \tau)h(\tau)d\tau;\, L^\infty \bar{B}^{0}_{2}\right\| \lesssim \|h;\, L^{q'} \bar{B}^{-\lambda_1\alpha(r')}_{r'}\| \tag{2.14}$$

for $1 \le r$, $q \le \infty$ with $1/r + 2/\sigma_1 q = 1/2$ and $1/2 - 1/\sigma_1 < 1/r \le 1/2$. Considering the dual operator U' of U in $\mathbb{R}^{1+n}$ of the form $U'h \equiv \int_{\mathbb{R}} U(-\tau)h(\tau)d\tau$, (2.14) yields

$$\|U(\cdot)\phi;\, L^q \bar{B}^{-\lambda_1\alpha(r)}_{r}\| \lesssim \|\phi;\, \bar{B}^0_2\|\,. \tag{2.15}$$

Interpolating (2.13) and (2.14) and applying the duality argument, we obtain the following well-known lemma (see [5], [8], [22] and references therein).

LEMMA 2.1. *Let* $1 \le r, q, \tilde{r}, \tilde{q} \le \infty$ *satisfy*

$$\begin{gathered} 1/r + 2/\sigma_1 q = 1/2, \quad 1/\tilde{r} + 2/\sigma_1 \tilde{q} = 1/2 + 2/\sigma_1\,, \\ 1/2 - 1/\sigma_1 < 1/r \le 1/2 \le 1/\tilde{r} < 1/2 + 1/\sigma_1\,. \end{gathered}$$

Then (2.11) *and* (2.15) *hold with* $\lambda_1 = \lambda_2$.

To use the bilinear estimate, we consider the sequence

$$P(h, g) \equiv \left\{\left\langle \int_{J \cap I_{t,j}} U(t-\tau)h(\tau)d\tau,\, g(t)\right\rangle_{t,x}\right\}_{j=-\infty}^{\infty}, \qquad I_{t,j} \equiv \left[t + 2^j,\, t + 2^{j+1}\right],$$

in the following lemma. For $r, q, \tilde{r}$ and $\tilde{q}$, we put

$$\beta(r, q, \tilde{r}, \tilde{q}) \equiv \frac{\sigma_1}{2}\left(\frac{1}{r} + \frac{2}{\sigma_1 q} - \frac{1}{2}\right) - \frac{\sigma_2}{2}\left(\frac{1}{\tilde{r}} + \frac{2}{\sigma_2 \tilde{q}} - \frac{1}{2} - \frac{2}{\sigma_2}\right).$$

For $x \in \mathbb{R}$, $x \sim 0$ means that x is sufficiently close to 0.

LEMMA 2.2. *Let $r, q, \tilde{r}, \tilde{q}$ satisfy $q \geq \tilde{q}$,*

$$
\begin{aligned}
&1/r + 2/(\sigma_1+\sigma_2)q < 1/2, \quad \beta(r,q,\tilde{r},\tilde{q}) \sim 0, \\
&1/r' \leq 1/\tilde{r} \begin{cases} < 1-(\sigma_2-2)/\sigma_2 r & \text{if } \sigma_2 \geq 2, \\ \leq 1 & \text{if } \sigma_2 < 2. \end{cases}
\end{aligned} \tag{2.16}
$$

Then the following estimate holds.

$$
\begin{aligned}
&\left| \left\langle \int_{J\cap I_{t,j}} U(t-\tau)h(\tau)d\tau, g \right\rangle_{t,x} \right| \\
&\quad \lesssim 2^{\beta(r,q,\tilde{r},\tilde{q})j} \|h;\, L^{\tilde{q}} \bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})}\| \|g;\, L^{q'} \bar{B}_{r'}^{-\lambda_1\alpha(r')}\|.
\end{aligned} \tag{2.17}
$$

PROOF OF LEMMA 2.2. By the decay estimate (2.12) with $\sigma_1 \mapsto (\sigma_1+\sigma_2)/2$ and $\lambda_1 \mapsto (\lambda_1+\lambda_2)/2$, we have

$$
\left\| \int_{J\cap I_{t,j}} U(t-\tau)h(\tau)d\tau;\, \bar{B}_\infty^{-\lambda_1/2} \right\| \lesssim 2^{-(\sigma_1+\sigma_2)j/4} \|h;\, L^1(I_{t,j};\, \bar{B}_1^{\lambda_2/2})\|. \tag{2.18}
$$

Let $1/\tilde{r}_0 = 1 - r/2\tilde{r}'$. Then by (2.14) we have

$$
\left\| \int_{J\cap I_{t,j}} U(t-\tau)h(\tau)d\tau;\, \bar{B}_2^0 \right\| \lesssim \|h;\, L^{\tilde{q}_0}(I_{t,j};\, \bar{B}_{\tilde{r}_0}^{-\lambda_2\alpha(\tilde{r}_0)})\|, \tag{2.19}
$$

where $\tilde{q}_0$ is given by $1/\tilde{r}_0 + 2/\sigma_2\tilde{q}_0 = 1/2 + 2/\sigma_2$. Interpolating (2.18) and (2.19), we have

$$
\left\| \int_{J\cap I_{t,j}} U(t-\tau)h(\tau)d\tau; \bar{B}_r^{-\lambda_1\alpha(r)} \right\| \lesssim 2^{-(\sigma_1+\sigma_2)\alpha(r)j/2} \|h; L^\ell(I_{t,j}; \bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})})\|, \tag{2.20}
$$

where $1/\ell \equiv 1 - \sigma_2(1/r + 1/\tilde{r} - 1)/2$. Therefore by the Hölder inequality, we have

$$
\begin{aligned}
&\left| \left\langle \int_{J\cap I_{t,j}} U(t-\tau)h(\tau)d\tau, g \right\rangle_{t,x} \right| \\
&\quad \lesssim 2^{-(\sigma_1+\sigma_2)\alpha(r)j/2} \big\| \|h;\, L_\tau^\ell(I_{t,j};\, \bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})})\|;\, L_t^q \big\| \cdot \|g;\, L^{q'} \bar{B}_{r'}^{-\lambda_1\alpha(r')}\|.
\end{aligned}
$$

Applying the Hölder and Minkowski inequalities to the norm of h by $q \geq \tilde{q}$, we have

$$
\big\| \|h;\, L_\tau^\ell(I_{t,j};\, \bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})})\|;\, L_t^q \big\| \lesssim 2^{(1/\ell - 1/\tilde{q} + 1/q)j} \|h;\, L^{\tilde{q}} \bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})}\|
$$

if $\tilde{q} \geq \ell$. Then we have the required results. The condition $\tilde{q} \geq \ell$ is satisfied by (2.16). □

Lemma 2.2 is rewritten as the following corollary by the duality argument.

COROLLARY 2.1. *Let $r, q, \tilde{r}, \tilde{q}$ satisfy $q \geq \tilde{q}$,*

$$1/\tilde{r} + 2/(\sigma_1+\sigma_2)\tilde{q} > 1/2 + 2/(\sigma_1+\sigma_2), \quad \beta(r,q,\tilde{r},\tilde{q}) \sim 0\,,$$

$$1/\tilde{r} \leq 1/r' \begin{cases} < 1-(\sigma_1-2)/\sigma_1\tilde{r}' & \text{if } \sigma_1 \geq 2\,,\\ \leq 1 & \text{if } \sigma_1 < 2\,. \end{cases}$$

Then the estimate (2.17) *holds.*

To show Proposition 2.1, below, we recall the following bilinear interpolation due to O'Neil.

LEMMA 2.3 [1, Section 3.13.5 (b)], [9, Lemma 6.1]. *Let A_j, B_j, C_j, $j = 0, 1$, be Banach spaces. Let T be a bilinear operator with boundedness properties as*

$$T : \begin{cases} A_0 \times B_0 \to C_0\,,\\ A_0 \times B_1 \to C_1\,,\\ A_1 \times B_0 \to C_1\,. \end{cases}$$

Let $\theta_0, \theta_1, r_0, r_1, r$ satisfy $0 < \theta_0, \theta_1, \theta_0+\theta_1 < 1$, $1 \leq r_0, r_1, r \leq \infty$, $1 \leq 1/r_0 + 1/r_1$. Then T is a bounded bilinear operator as

$$T \;:\; (A_0, A_1)_{\theta_0, r_0 r} \times (B_0, B_1)_{\theta_1, r_1 r} \longrightarrow (C_0, C_1)_{\theta_0+\theta_1, r}\,,$$

where $(A_0, A_1)_{\theta,r}$ denotes the real interpolation space.

PROPOSITION 2.1. *Let $r, q, \tilde{r}, \tilde{q}$ satisfy $q > 1$, $\tilde{q} \leq q$, $\beta(r,q,\tilde{r},\tilde{q}) = 0$,*

$$1/r + 2/(\sigma_1+\sigma_2)q < 1/2 \;\; \text{if } 1/r' \leq 1/\tilde{r} < (1-(\sigma_2-2)/\sigma_2 r) \wedge 1\,,$$

$$\begin{aligned} &1/\tilde{r} + 2/(\sigma_1+\sigma_2)\tilde{q} > 1/2 + 2/(\sigma_1+\sigma_2)\\ &\text{if } 1/\tilde{r} \leq 1/r' < (1-(\sigma_1-2)/\sigma_1\tilde{r}') \wedge 1\,, \end{aligned} \tag{2.21}$$

where both conditions in (2.21) *are assumed when $r' = \tilde{r}$. Then* (2.11) *holds.*

PROOF OF PROPOSITION 2.1. Let $r, q, \tilde{r}, \tilde{q}$ satisfy the assumptions in the proposition, and let r_* and $\tilde{r}_*$ be sufficiently close to r and $\tilde{r}$, respectively. Then by Lemma 2.2 and Corollary 2.1, we have

$$P : L^{\tilde{q}}\bar{B}_{\tilde{r}_*}^{-\lambda_2\alpha(\tilde{r}_*)} \times L^{q'}\bar{B}_{r'_*}^{-\lambda_1\alpha(r'_*)} \longrightarrow \ell_\infty^{-\beta(r_*,q,\tilde{r}_*,\tilde{q})}\,, \tag{2.22}$$

where $\ell_q^s = L^q(\mathbb{Z}, 2^{sj}dj)$ for $s \in \mathbb{R}$, $1 \leq q \leq \infty$, where d_j denotes the point-mass 1 at j. Applying Lemma 2.3 with $r_0 = r_1 = 2$, $r = 1$ to (2.22), and using the facts that

- $(L^p, L^p)_{\theta,r} = L^p$ for $1 \leq p, r \leq \infty$,
- $(B_{p_0}^{s_0}, B_{p_1}^{s_1})_{\theta,2} \hookleftarrow (B_{p_0}^{s_0}, B_{p_1}^{s_1})_{\theta,p} = B_p^s$
  for $s_0, s_1 \in \mathbb{R}$, $1 \leq p_0, p_1 \leq 2$, $s = (1-\theta)s_0 + \theta s_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$,
- $(\ell_\infty^{s_0}, \ell_\infty^{s_1})_{\theta,1} = \ell_1^s$ for $s = (1-\theta)s_0 + \theta s_1$, $s_0 \neq s_1$,

$0 < \theta < 1$ (see [1, Theorems 3.4.1, 6.4.5. 5.6.1]), we have

$$P : L^{\tilde{q}}\bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})} \times L^{q'}\bar{B}_{r'}^{-\lambda_1\alpha(r')} \longrightarrow \ell_1^0\,.$$

Since $q', r' \neq \infty$, we obtain (2.11). $\square$

The next corollary follows from the above proposition and Lemma 2.1 immediately.

COROLLARY 2.2. *Let* $r, q, \tilde{r}, \tilde{q}$ *satisfy*

$$1/r+2/\sigma_1 q = 1/2,\ \sigma_1/2(\sigma_1+2) \le 1/r \le 1/2,\ 1/\tilde{r} = 1/\tilde{q} = (\sigma_2+4)/2(\sigma_2+2)\,.$$

Then (2.11) *and* (2.15) *hold.*

We put a, b as follows.

$$a \equiv (\sigma_1+\sigma_2-4)/2(\sigma_1+\sigma_2-2), \quad b \equiv (\sigma_1+\sigma_2-4)/2(\sigma_1+\sigma_2)\,.$$

COROLLARY 2.3. *Let* $\sigma_1, \sigma_2, r, q, \tilde{r}, \tilde{q}$ *satisfy*

$$\sigma_1, \sigma_2 \ge 2,\ \ 1/r' \le 1/\tilde{r},\ \ b \le 1/r < 1/2\,, \tag{2.23}$$

$$1/r+2/\sigma_1 q = 1/2,\ \ 1/\tilde{r}+2/\sigma_2\tilde{q} = 1/2+2/\sigma_2\,, \tag{2.24}$$

$$1/\tilde{r} \begin{cases} < 1-(\sigma_2-2)/\sigma_2 r & \text{if } a \le 1/r, \\ \le 1/2+2/\sigma_2-\sigma_1\alpha(r)/\sigma_2 & \text{if } 1/r < a\,. \end{cases} \tag{2.25}$$

Then (2.11) *holds. Moreover* $q = \tilde{q}$ *if* $1/\tilde{r}$ *equals to the RHS of the last inequality, and* $q > \tilde{q}$ *otherwise.*

PROOF OF COROLLARY 2.3. The condition $b \le 1/r$ is needed for the existence of $\tilde{r}$ in (2.25) when $1/r < a$. The conditions $\beta(r, q, \tilde{r}, \tilde{q}) = 0$ and (2.21) follow from (2.24). The index r which satisfies (2.24) with $q = \tilde{q}$ and $1/\tilde{r} = 1-(\sigma_2-2)/\sigma_2 r$ is given by $1/r = a$. If $a \le 1/r$ and $1/\tilde{r} < 1-(\sigma_2-2)/\sigma_2 r$, then $q > \tilde{q}$. If $1/r < a$ and $1/\tilde{r} \le 1/2+2/\sigma_2-\sigma_1\alpha(r)/\sigma_2$, then $q \ge \tilde{q}$. □

COROLLARY 2.4. *Let* $2 < \sigma_2 \le \sigma_1$. *Let* $r, q, \tilde{r}, \tilde{q}$ *be given by*

$$(1/r, 1/q) = (1/2-1/\sigma_1, 1/2),\ \ (1/\tilde{r}, 1/\tilde{q}) = (1/2+1/\sigma_2, 1/2)\,.$$

Then (2.11) *holds.*

PROOF OF COROLLARY 2.4. Since r satisfies $b \le 1/r < a$ by $2 < \sigma_2 \le \sigma_1$ and $\tilde{r}$ satisfies $1/\tilde{r} = 1/2+2/\sigma_2-\sigma_1\alpha(r)/\sigma_2$, (2.11) follows from Corollary 2.3. □

Applying the duality argument described in the proof of Lemma 2.1 to Corollary 2.4 with $\sigma_1 = \sigma_2$, we have the following endpoint estimates for Lemma 2.1.

COROLLARY 2.5. *Let* $\sigma_1 > 2$. *Let* $r, q, \tilde{r}$ *and* $\tilde{q}$ *satisfy*

$$1/r+2/\sigma_1 q = 1/2,\ 1/\tilde{r}+2/\sigma_1\tilde{q} = 1/2+2/\sigma_1\,,$$
$$1/2-1/\sigma_1 \le 1/r \le 1/2 \le 1/\tilde{r} \le 1/2+1/\sigma_1\,.$$

Then (2.11) *and* (2.15) *hold with* $\lambda_1 = \lambda_2$.

REMARK 4. Corollary 2.5 has been shown in [9] for Schrödinger and wave equations, and is applied to show the local well-posedness of **NLW** with

$$\begin{aligned} &|f^{(j)}(u)| = O(|u|^{p-j}),\ \ j=0,1,\ \ p = 1+4(n-1)/(n^2-2n+5),\\ &\qquad s=(n-3)/2(n-1),\ \ n\geq 4\,, \end{aligned} \tag{2.26}$$

where $r, q, \tilde{r}, \tilde{q}$ are taken as

$$1/\tilde{r} = 1/2+1/(n-1),\ \ 1/\tilde{q} = 1/2,\ \ 1/r = 1/\tilde{r}-2/(n+1),\ \ 1/q = (n-1)\alpha(r)/2\,.$$

By Corollary 2.5, we can show the local well-posedness of **NLKG** with (2.26) analogously.

COROLLARY 2.6. *The conclusion of Corollary* 2.3 *holds with* (2.23) *replaced by the following conditions*

$$\sigma_1 > 2,\ \sigma_2 \geq 2,\ 1/2 \leq 1/\tilde{r},\ b \vee (1/2 - 1/\sigma_1) \leq 1/r < 1/2\,.$$

PROOF OF COROLLARY 2.6. By Corollary 2.5, we have

$$\|Gh;\, L^q\bar{B}_r^{-\lambda_1\alpha(r)}\| \lesssim \|h;\, L^1\bar{B}_2^0\|\,.$$

Interpolating this estimate and the conclusion of Corollary 2.3, we may replace $1/r' \leq 1/\tilde{r}$ in (2.23) with $1/2 \leq 1/\tilde{r}$. $\square$

In the following lemma we show (2.11) for $1/\tilde{r} = 1-(\sigma_2-2)/\sigma_2 r$ when $a < 1/r$ in Corollary 2.3. However, we were not able to obtain the estimate when $a = 1/r$.

LEMMA 2.4. *Let* $\sigma_1 > 2$. *Let* $r, q, \tilde{r}, \tilde{q}$ *satisfy* $\beta(r,q,\tilde{r},\tilde{q}) = 0$,

$$\begin{aligned} &(\sigma_1-2)/\sigma_1\tilde{r}' \leq 1/r \leq 1/\tilde{r}'\,,\\ &1/\tilde{r} + 2/(\sigma_1+\sigma_2)\tilde{q} > 1/2 + 2/(\sigma_1+\sigma_2),\ 0 \leq 1/q < 1/\tilde{q} < 1\,. \end{aligned} \tag{2.27}$$

Then (2.11) *holds.*

PROOF OF LEMMA 2.4. Let r_*, q_* be given by

$$1/r_* = \tilde{r}'/2r,\quad 1/r_* + 2/\sigma_1 q_* = 1/2\,.$$

By (2.12) and (2.15), we have

$$\big\||t|^{(\sigma_1+\sigma_2)/4}U(t)\phi;\, L^\infty(\bar{B}_\infty^{-\lambda_1/2}, dw)\big\| \lesssim \|\phi;\, \bar{B}_1^{\lambda_2/2}\|\,, \tag{2.28}$$

$$\big\||t|^{(\sigma_1+\sigma_2)/4}U(t)\phi;\, L^{q_*}(\bar{B}_{r_*}^{-\lambda_1\alpha(r_*)}, dw)\big\| \lesssim \|\phi;\, \bar{B}_2^0\|\,, \tag{2.29}$$

where dw is the measure given by

$$dw(t) = |t|^{-(\sigma_1+\sigma_2)q_*/4}dt$$

and $L^p(X, dw)$ denotes the weighted Lebesgue space given by

$$\|u;\, L^p(X, dw)\| = \left\{ \int \|u(t);\, X\|^p dw(t) \right\}^{1/p} .$$

Interpolating (2.28) and (2.29), we have

$$\left\| |t|^{(\sigma_1+\sigma_2)/4} U(t)\phi;\, L^\ell(\bar{B}_r^{-\lambda_1\alpha(r)}, dw) \right\| \lesssim \|\phi;\, \bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})}\| ,$$

which is rewritten by

$$\left\| |t|^\gamma U(t)\phi;\, L^\ell \bar{B}_r^{-\lambda_1\alpha(r)} \right\| \lesssim \|\phi;\, \bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})}\| , \tag{2.30}$$

where $1/\ell \equiv \sigma_1(1 - 1/\tilde{r} - 1/r)/2$, $\gamma \equiv -(\sigma_1 + \sigma_2)\alpha(\tilde{r})/2$. Applying (2.30), we have

$$\begin{aligned} |\langle Gh, g\rangle| &= \int ds \int d\tau |\langle U(\tau)h(s), g(\tau+s)\rangle| \\ &\lesssim \|h;\, L^{\tilde{q}} \bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})}\| \cdot \| \| |\tau|^{-\gamma} g(\tau+s);\, L_\tau^{\ell'} \bar{B}_{r'}^{-\lambda_1\alpha(r')} \|;\, L_s^{\tilde{q}'} \| . \end{aligned}$$

If $\tilde{q}, \ell$ satisfy

$$0 < \ell'/\tilde{q}' < \ell'/q' < 1 , \tag{2.31}$$

then applying the Hardy-Littlewood-Sobolev inequality with $\beta(r, q, \tilde{r}, \tilde{q}) = 0$, we have

$$|\langle Gh, g\rangle| \lesssim \|h;\, L^{\tilde{q}} \bar{B}_{\tilde{r}}^{-\lambda_2\alpha(\tilde{r})}\| \|g;\, L^{q'} \bar{B}_{r'}^{-\lambda_1\alpha(r')}\| .$$

Therefore we obtain (2.11). The condition (2.31) is satisfied by (2.27). □

Applying the duality argument to Lemma 2.4, we have the following Corollary.

COROLLARY 2.7. *Let $\sigma_2 > 2$. Let $r, q, \tilde{r}, \tilde{q}$ satisfy $\beta(r, q, \tilde{r}, \tilde{q}) = 0$, $\tilde{q} < q < \infty$,*

$$1/r' \le 1/\tilde{r} \le 1 - (\sigma_2 - 2)/\sigma_2 r, \quad 1/r + 2/(\sigma_1 + \sigma_2)q < 1/2 .$$

Then (2.11) *holds.*

Combining Corollary 2.6 with Corollary 2.7, we obtain the following corollary.

COROLLARY 2.8. *Let $\sigma_1 > 2$, $\sigma_2 > 2$. Let $r, q, \tilde{r}, \tilde{q}$ satisfy*

$$1/r + 2/\sigma_1 q = 1/2, \ 1/\tilde{r} + 2/\sigma_2\tilde{q} = 1/2 + 2/\sigma_2, \ b \vee ((\sigma_1 - 2)/2\sigma_1) \le 1/r < 1/2 ,$$

$$1/2 \le 1/\tilde{r} \begin{cases} \le 1 - (\sigma_2 - 2)/\sigma_2 r & \text{if } a < 1/r , \\ < 1 - (\sigma_2 - 2)/\sigma_2 r & \text{if } 1/r = a , \\ \le 1/2 + 2/\sigma_2 - \sigma_1\alpha(r)/\sigma_2 & \text{if } 1/r < a . \end{cases}$$

Then (2.11) *holds. Here $q = \tilde{q}$ if $1/\tilde{r}$ equals to the RHS of the last inequality, and $q > \tilde{q}$ otherwise.*

3. – Proof of Theorem 1.1

(1) Since f is written as a finite sum such that $f = \sum_{j=1}^{\ell} f_j$ with f_j satisfying $N(s - s_0, p_j)$ with $p_1 \le p_j \le p_2$, we define σ_j and λ_j, $1 \le j \le \ell$, as $\sigma_j = \sigma_1$, $\lambda_j = \lambda_1$ for **NLS** and **NLW**,

$$\sigma_j = 4/(p_j - 1) \text{ if } p(0) \le p_j \le p(1/2), \quad \sigma_j = n - 1 \text{ if } p(1/2) < p_j \le p(s)$$

and $\lambda_j = (\sigma_j + 2)/2$ for **NLKG**. Then $\sigma_j, \lambda_j, 1 \le j \le \ell$, satisfy (2.10). For $\mu \ge 0$, we define

$$Y^\mu \equiv \begin{cases} L^\infty \dot{H}^\mu \cap L^{q_1} \dot{B}^\mu_{q_1} & \text{for } \textbf{NLS}\,, \\ L^\infty \dot{H}^\mu \cap L^{q_1} \dot{B}^{\mu-1/2}_{q_1} & \text{for } \textbf{NLW}\,, \\ L^\infty H^\mu \cap \bigcap_{j=1}^{\ell} L^{q_j} B^{\mu-1/2}_{q_j} & \text{for } \textbf{NLKG}\,, \end{cases}$$

where $q_j \equiv 2(\sigma_j + 2)/\sigma_j$, $1 \le j \le \ell$. By Corollary 2.2, in particular for **NLKG**, we have

$$\max_{1 \le j \le \ell} \|Gh;\, L^{q_j} B^{-1/2}_{q_j}\| \lesssim \min_{1 \le j \le \ell} \|h;\, L^{q_j'} B^{1/2}_{q_j'}\|\,.$$

Therefore we have

$$\text{(3.32)}\quad \|\Phi(u);\, Y^\mu\| \lesssim \|\vec{\phi};\, \bar{\mathcal{H}}^\mu\| + \sum_{j=1}^{\ell} \begin{cases} \|f_j(u); L^{q_1'} \dot{B}^\mu_{q_1'}\| & \text{for } \textbf{NLS}\,, \\ \|f_j(u); L^{q_j'} \bar{B}^{\mu-1/2}_{q_j'}\| & \text{for } \textbf{NLW},\ \textbf{NLKG}\,, \end{cases}$$

where $\bar{\mathcal{H}}^s$ denotes $\dot{H}^\mu$ for **NLS**, $\dot{\mathcal{H}}^\mu$ for **NLW**, $\mathcal{H}^\mu$ for **NLKG**, and we have used the property that $(-\Delta)^{\mu/2}$ [resp. $(1 - \Delta)^{\mu/2}$] is an isomorphism on $\dot{B}^0_r$ [resp. B^0_r] to $\dot{B}^{-\mu}_r$ [resp. $B^{-\mu}_r$], $1 \le r \le \infty$.

LEMMA 3.1 [14, Lemma 2.2]. *Let $s > 0$, $1 \le p < \infty$. Let f satisfy $N(s, p)$. Let $\tilde{r}, m, r$ satisfy $1 \le \tilde{r} < \infty$, $2 \le m, r < \infty$,*

$$1/\tilde{r} = (p - 1)/m + 1/r\,.$$

Then

$$\|f(u);\, \dot{B}^s_{\tilde{r}}\| \lesssim \|u;\, \dot{B}^0_m\|^{p-1} \|u;\, \dot{B}^s_r\|\,,$$

where $\dot{B}$ may be replaced by B.

There exist (r_j^*, q_j^*) such that

$$1/r_j^* + 2/\sigma_j q_j^* = 1/2, \quad 0 < 1/q_j^* \le 1/q_j, \quad 1/q_j' = (p_j - 1)/q_j^* + 1/q_j\,,$$

$$1/q_j^* = \begin{cases} 1/r_j^* - s(p_j)/n & \text{for } \textbf{NLS}\,, \\ 1/r_j^* - (s(p_j) - \lambda_j \alpha(r_j^*))/n & \text{for } \textbf{NLW},\ \textbf{NLKG}\,. \end{cases}$$

Indeed, the above q_j^* is given by $(p_j - 1)(\sigma_j + 2)/2$. Therefore by Lemma 3.1 with the embedding $\bar{B}_r^\mu \hookrightarrow \bar{B}_m^0$ for $1/r - \mu/n = 1/m$, $\mu \geq 0$, and the Hölder inequality in time variable, the second terms on the RHS of (3.32) are estimated by

$$
(3.33)\qquad
\begin{aligned}
&\sum_{j=1}^{\ell} \|u;\, L^{q_j^*} \dot{B}_{r_j^*}^{s(p_j)}\|^{p_j-1} \|u;\, L^{q_1} \dot{B}_{q_1}^{\mu}\| && \text{for } \mathbf{NLS}\,,\\
&\sum_{j=1}^{\ell} \|u;\, L^{q_j^*} \bar{B}_{r_j^*}^{s(p_j)-\lambda_j\alpha(r_j^*)}\|^{p_j-1} \|u;\, L^{q_j} \bar{B}_{q_j}^{\mu-1/2}\| && \text{for } \mathbf{NLW},\ \mathbf{NLKG}\,.
\end{aligned}
$$

For $\delta > 0$, $M > 0$, $R > 0$, we put

$$
\begin{aligned}
A^s(s(p_1), s(p_2), \gamma, \delta) &\equiv \left\{ \vec{\phi} \in A^s(s(p_1), s(p_2), \gamma)\ ;\ \|\vec{\phi};\, \bar{\mathcal{H}}^s\| \leq \delta \right\},\\
X^s(R, M) &\equiv \left\{ u \in X^s\ ;\ \max_{1\leq j\leq \ell} \|u;\, Y^{s(p_j)}\| \leq R,\ \|u;\, Y^s\| \leq M \right\}.
\end{aligned}
$$

Then by (3.32), (3.33) and applying an analogous argument to d, we have

$$
(3.34)\qquad
\begin{cases}
\|\Phi(u);\, Y^s\| \lesssim \delta + \displaystyle\sum_{j=1}^{\ell} R^{p_j-1} M\,,\\[2ex]
\displaystyle\max_{1\leq j\leq \ell} \|\Phi(u);\, Y^{s(p_j)}\| \lesssim \gamma + \sum_{j=1}^{\ell} R^{p_j-1} R\,,\\[2ex]
d(u, v) \lesssim \displaystyle\sum_{j=1}^{\ell} R^{p_j-1} d(u, v)
\end{cases}
$$

for $\vec{\phi} \in A^s(s(p_1), s(p_2), \gamma, \delta)$ and $u, v \in X^s(R, M)$. Φ is a contraction map on $X^s(R, M)$ if the RHS of (3.34) are dominated by M, R, and $d(u, v)/2$, respectively. Indeed, this is realized by putting $M = 2\delta$ and $R = 2\gamma$ with γ satisfying

$$
(3.35)\qquad C \sum_{j=1}^{\ell} (2\gamma)^{p_j-1} \leq 1/2\,.
$$

Since $X^s(R, M)$ endowed with metric d is a complete metric space, we obtain the global solution u of **INT** if γ is sufficiently small. The continuity of u in time variable follows from the unitarity of U and Lebesgue's convergence theorem.

To show the uniqueness of solutions in $C(\mathbb{R};\, \tilde{H}^s) \cap X^s$, let u and v be two solutions with the same data. Then we have

$$
d(u, v\chi_T) \leq C \sum_{j=1}^{\ell} \left(\|u;\, Y^{s(p_j)}\| \vee \|v\chi_T;\, Y^{s(p_j)}\| \right)^{p_j-1} d(u, v\chi_T)\,,
$$

where $\chi_T(t) = 1$ for $|t| \le T$, $\chi_T(t) = 0$ for $|t| \ge T$. Since the sum on the RHS is smaller than $1/2$ for sufficiently small T, we have $u(t) = v(t)$ for $|t| \le T$. Repeating this procedure, we obtain $u(t) = v(t)$ for all t.

(2) The asymptotic states $\vec{\phi}_\pm = \phi_\pm$ for **NLS**, $\vec{\phi}_\pm = (\phi_\pm, \psi_\pm)$ for **NLW** and **NLKG** are given by

$$
\begin{aligned}
&\phi_\pm = \phi + \int_0^{\pm\infty} U(-\tau) f(u(\tau)) d\tau \qquad \text{for} \quad \mathbf{NLS}\,,\\
&\phi_\pm = \phi - \int_0^{\pm\infty} \frac{\sin \tau\omega}{\omega} f(u(\tau)) d\tau,\\
&\psi_\pm = \psi + \int_0^{\pm\infty} \cos \tau\omega f(u(\tau)) d\tau \qquad \text{for } \mathbf{NLW}, \mathbf{NLKG}\,.
\end{aligned}
\tag{3.36}
$$

Let us consider **NLKG**. The required regularity $\phi_\pm \in H^s$ follows from

$$\|\phi_\pm; H^s\| \le \|\phi; H^s\| + \left\| \int_t^{\pm\infty} \frac{\sin(t-\tau)\omega}{\omega} f(u(\tau)) d\tau; L^\infty H^s \right\|$$

since the second term on RHS is finite by (3.32) and (3.33). Similarly we have $\psi_\pm \in H^{s-1}$. Since $\vec{\phi}_\pm$ satisfies

$$\Gamma(t)\vec{\phi}_\pm = \Gamma(t)\vec{\phi} + \int_0^{\pm\infty} \frac{\sin(t-\tau)\omega}{\omega} f\big(u(\tau)\big) d\tau\,,$$

we have

$$
\begin{aligned}
\|u(t) - \Gamma(t)\vec{\phi}_+; H^s\| &\le \left\| \int_\nu^\infty \frac{\sin(\nu-\tau)\omega}{\omega} f(u(\tau)) d\tau; L_\nu^\infty([t,\infty); H^s) \right\| \\
&\lesssim \sum_{j=1}^{\ell} \left\| f_j(u); L^{q_j'}\big([t,\infty); B_{q_j'}^{s-1/2}\big) \right\|.
\end{aligned}
$$

Therefore we have $\|u(t) - \Gamma(t)\vec{\phi}_+; H^s\| \to 0$ as $t \to \infty$. We obtain (1.5) by an analogous argument for **NLS** and **NLW**. The uniqueness of states $\vec{\phi}_\pm$ in $\tilde{\mathcal{H}}^s$ with (1.5) follows from the unitarity of $U(t)$.

(3) The proof of (1) shows that there exists $\gamma > 0$ such that for any $\delta > 0$ and any data $\vec{\phi}_0 \in A^s(s(p_1), s(p_2), \gamma, \delta)$, the operator Φ_{t_0}, $t_0 \in [-\infty, \infty]$, given by

$$
\begin{aligned}
\Phi_{t_0}(u)(t) &\equiv \Gamma(t-t_0)\vec{\phi}_0 + \widetilde{G_{t_0}}(f(u))(t) \qquad \text{for} \qquad t_0 \in \mathbb{R}\,,\\
\Phi_{\pm\infty}(u)(t) &\equiv \Gamma(t)\vec{\phi}_0 + \widetilde{G_{\pm\infty}}(f(u))(t)\,,
\end{aligned}
$$

has a unique fixed point in $C(\mathbb{R}; \tilde{H}^s) \cap X^s(2\gamma, 2\delta)$, and if $t_0 \ne \pm\infty$, then the fixed point is also unique in $C(\mathbb{R}; \tilde{H}^s) \cap X^s$. Let $\vec{\phi}_- \in A^s(s(p_1), s(p_2), \gamma/2)$.

In particular, we have $\vec{\phi}_- \in A^s(s(p_1), s(p_2), \gamma, 2\|\vec{\phi}_-; \bar{\mathcal{H}}^s\|)$. Then there exists a unique fixed point u of $\Phi_{-\infty}$ with $\vec{\phi}_0$ replaced by $\vec{\phi}_-$ in

$$W \equiv C(\mathbb{R}; \tilde{H}^s) \cap X^s\left(2\gamma, 4\|\vec{\phi}_-; \bar{\mathcal{H}}^s\|\right).$$

Let $\vec{\phi}, \vec{\phi}_+$ be defined by (3.36). Then it follows that $\vec{\phi}, \vec{\phi}_+ \in \tilde{\mathcal{H}}^s$, and u is a global solution of **INT** which satisfies (1.5). The above u is also unique in $C(\mathbb{R}; \tilde{H}^s) \cap X^s$ by the following argument. Let v be another solution in $C(\mathbb{R}; \tilde{H}^s) \cap X^s$ such that (1.5) holds for $\vec{\phi}_-$. Let

$$\vec{\phi}_{t_0} \equiv v(t_0) \quad \text{for} \quad \textbf{NLS}, \quad \vec{\phi}_{t_0} \equiv (v(t_0), \partial_t v(t_0)) \quad \text{for} \quad \textbf{NLW}, \ \textbf{NLKG}. \tag{3.37}$$

Then we have $\vec{\phi}_{t_0} \in A^s(s(p_1), s(p_2), \gamma, 2\|\vec{\phi}_-; \bar{\mathcal{H}}^s\|)$ for sufficiently small t_0 by the convergence of the norms $\|\vec{\phi}_{t_0}; \bar{\mathcal{H}}^\mu\| \longrightarrow \|\vec{\phi}_-; \bar{\mathcal{H}}^\mu\|$ as $t_0 \to -\infty$ for $\mu = s(p_1), s(p_2), s$. Therefore Φ_{t_0} with $\vec{\phi}_0$ replaced by $\vec{\phi}_{t_0}$ has a unique fixed point in W, and the fixed point is also unique in $C(\mathbb{R}; \tilde{H}^s) \cap X^s$. Since v satisfies $v = \Phi_{t_0}(v)$ with $\vec{\phi}_0$ replaced by $\vec{\phi}_{t_0}$, we have $v \in W$. Moreover since v satisfies $v = \Phi_{-\infty}(v)$ with $\vec{\phi}_0$ replaced by $\vec{\phi}_-$, we obtain $u = v$ by the uniqueness of the fixed point of $\Phi_{-\infty}$ in W.

The correspondence between $\vec{\phi}_-$ and $\vec{\phi}_+$ are given by

$$\begin{aligned} &\phi_+ = \phi_- + \int_{-\infty}^{\infty} U(-\tau) f(u(\tau)) d\tau \qquad \text{for} \quad \textbf{NLS}, \\ &\phi_+ = \phi_- - \int_{-\infty}^{\infty} \frac{\sin \tau\omega}{\omega} f(u(\tau)) d\tau, \\ &\psi_+ = \psi_- + \int_{-\infty}^{\infty} \cos \tau\omega f(u(\tau)) d\tau \qquad \text{for} \quad \textbf{NLW} \text{ and } \textbf{NLKG}, \end{aligned} \tag{3.38}$$

via global solutions with (1.5). The continuity of the scattering operator S : $\vec{\phi}_- \mapsto \vec{\phi}_+$ is shown as follows. Let u^k, $k \geq 1$, be global solutions with asymptotic states $\vec{\phi}^k_\pm$. In particular u^k is the fixed point of $\Phi_{-\infty}$ with $\vec{\phi}_0$ replaced by $\vec{\phi}^k_-$. Let us consider **NLKG**. Then we have

$$d(u, u^k) \leq C\|\vec{\phi} - \vec{\phi}_k; \mathcal{H}^{1/2}\| + C \sum_{j=1}^{\ell} \|f_j(u) - f_j(u^k); L^{q_j'} B^0_{q_j'}\|.$$

Since the last term is dominated by $d(u, u^k)/2$, we have $d(u, u^k) \to 0$ as $k \to \infty$. By (3.38), we have

$$\begin{aligned} \|\phi_+ - \phi^k_+; H^{1/2}\| \leq\ & \|\phi_- - \phi^k_-; H^{1/2}\| \\ & + \left\| \int_{-\infty}^{\infty} \frac{\sin(t-\tau)\omega}{\omega} (f(u(\tau)) - f(u^k(\tau))) d\tau; L^\infty H^{1/2} \right\| \end{aligned}$$

Since the last term is dominated by $d(u, u^k)/2$, we obtain $\phi^k_+ \to \phi_+$ in $H^{1/2}$ as $k \to \infty$. Similarly we also have $\psi^k_+ \to \psi_+$ in $H^{-1/2}$ as $k \to \infty$, and (1.6) follows from the complex interpolation $H^\mu = [H^{1/2}, H^{s(p_2)}]$. The results for **NLS** and **NLW** are obtained by an analogous argument.

(4) We use the following argument to show the uniqueness of solutions. Let $u, v \in C([-T,T]; \tilde{H}^s)$, $T > 0$, be two solutions with the same data. Then u and v satisfy

$$u - v = \sum_{j=1}^{\ell} \widetilde{G_0}\big(f_j(u) - f_j(v)\big)\,.$$

If there exist $r, q, \tilde{r}_j, \tilde{q}_j, r_j^*$, $1 \le j \le \ell$, such that

$$\|\widetilde{G_0}h; L^q L^r\| \lesssim \|h; L^{\tilde{q}_j} L^{\tilde{r}_j}\|\,, \tag{3.39}$$

$$1/\tilde{r}_j = (p_j - 1)/r_j^* + 1/r,\ \ q \ge \tilde{q}_j,\ \ \tilde{H}^s \hookrightarrow L^r \cap L^{r_j *}\,, \tag{3.40}$$

then by the Hölder inequality we have

$$\|u - v; L^q L^r\| \lesssim \sum_{j=1}^{\ell} T^{1/\tilde{q}_j - 1/q} (\|u; L^\infty \tilde{H}^s\| \vee \|v; L^\infty \tilde{H}^s\|)^{p_j - 1} \|u - v; L^q L^r\|\,.$$

Therefore taking T sufficiently small if $q > \tilde{q}_j$ for $1 \le j \le \ell$, and moreover taking $\|u; L^\infty \tilde{H}^s\|$ sufficiently small if there exists j with $q = \tilde{q}_j$, we have $u(t) = v(t), t \in [-T, T]$, in $\tilde{H}^s$. Repeating this procedure, we obtain the uniqueness in $C(\mathbb{R}; \tilde{H}^s)$.

LEMMA 3.2. *Let us consider* **NLS**. *Let s, p_1 and p_2 satisfy $0 \le s < n/2$, $1 + \min(2s/n, 2/n) \le p_1 \le p_2$,*

$p_2 \le 1 + (1+2s)/(1-2s)$ *if* $n = 1$, $\;p_2 \le 1 + (1+s)/(1-s)$ *if* $n = 2$,

$p_2 \le 1 + (2n + 4(n-1)s)/n(n-2s)$ *if* $n \ge 3$ *and* $0 \le s < n/2(n-1)$,

$p_2 < 1 + 4(n-1)/n(n-2)$ *if* $n \ge 3$ *and* $s = n/2(n-1)$,

$p_2 \le 1 + 4/(n-2s)$ *if* $n \ge 3$ *and* $n/2(n-1) < s < n/2$.

Then there exist $r, q, \tilde{r}_j, \tilde{q}_j, r_j^*$, $1 \le j \le \ell$, with (3.39) *and* (3.40). *Moreover $q = \tilde{q}_j$ if and only if $p_j = 1 + 4/(n-2s)$.*

PROOF OF LEMMA 3.2. Let r, q satisfy $1/r + 2/nq = 1/2$ and

$$1/r = \big((n-2s)/2n\big) \vee \big((n-2)/2n\big)\,.$$

Let $\tilde{r}_j, \tilde{q}_j$, $1 \le j \le \ell$, satisfy $1/\tilde{r}_j + 2/n\tilde{q}_j = 1/2 + 2/n$, $1 \le \tilde{r}_j \le 2$ if $n = 1$, $1 < \tilde{r}_j \le 2$ if $n = 2$,

$$1/2 \le 1/\tilde{r}_j \begin{cases} \le 1 - (n-2)/nr & \text{if } n \ge 3,\ (n-2)/2(n-1) < 1/r \le 1/2, \\ < (n^2 + 2n - 4)/2n(n-1) & \text{if } n \ge 3,\ 1/r = (n-2)/2(n-1), \\ \le 1/r + 2/n & \text{if } n \ge 3,\ (n-2)/2n \le 1/r < (n-2)/2(n-1). \end{cases}$$

Then (3.39) follows from Lemma 2.1 for $n = 1, 2$ and Corollary 2.8 for $n \ge 3$. Let r_j^* satisfy

$$(n-2s)/2n \le 1/r_j^* \le 1/2\,.$$

We now take $\tilde{r}_j$ and r_j^* such that (3.40) holds. It happens $q = q_j$ if and only if $r = 2n/(n-2)$, $\tilde{r}_j = r'$, $r_j^* = 2n/(n-2s)$, namely $p_j = 1 + 4/(n-2s)$. □

LEMMA 3.3. *Let us consider* **NLW** *and* **NLKG**. *Let* s, p_1 *satisfy* $1/2 \le s < n/2$, $p_1 \ge 1 + 4n/(n^2-1)$ *for* **NLW**, $p_1 \ge 1 + 4/(n+1)$ *for* **NLKG**, *and let* p_2 *satisfy*

$$p_1 \le p_2 \le 1 + 4n/(n+1)(n-2s)\,.$$

Then there exist $r, q, \tilde{r}_j, \tilde{q}_j, r_j^*$, $1 \le j \le \ell$ *with* (3.39) *and* (3.40). *Moreover* q *and* $\tilde{q}_j$ *satisfy* $q > \tilde{q}_j$ *for* $1 \le j \le \ell$.

PROOF OF LEMMA 3.3. Let r, q satisfy $1/r + 2/(n-1)q = 1/2$ and

$$1/r = \big((n-2s)/2n\big) \vee \big((n-1)/2(n+1)\big)\,.$$

Let $\tilde{r}_j, \tilde{q}_j$ satisfy $1/\tilde{r}_j + 2/(n-1)\tilde{q}_j = 1/2 + 2/(n-1)$ and

$$1/\tilde{r}_j = 1/r + 2/(n+1) \tag{3.41}$$

for $1 \le j \le \ell$. Then (3.39) follows from Lemma 2.1 for $n \le 3$ and Corollary 2.8 for $n \ge 4$. Let r_j^* satisfy

$$(n-2s)/2n \le 1/r_j^* \le \begin{cases} (n-1)/2n & \text{for } \mathbf{NLW}\,, \\ 1/2 & \text{for } \mathbf{NLKG}\,. \end{cases}$$

We now take r_j^* such that (3.40) holds. Note that q and q_j always satisfy $q > q_j$ by (3.41). □

If p_1 and p_2 satisfy the assumption in Theorem 1.1 with (1.7) and (1.8), then by Lemmas 3.2 and 3.3 solutions of **INT** with the same data are unique in $C(\mathbb{R}; \tilde{H}^s)$. Therefore the space X^s in (1) in the theorem is redundant.

To show that X^s in (3) in the theorem is removable, we use a similar argument in (3) in this section. Let v be another solution in $C(\mathbb{R}; \tilde{H}^s)$ such that (1.5) holds for $\vec{\phi}_-$. Then for sufficiently small t_0, v is a fixed point of Φ_{t_0} with $\vec{\phi}_0$ replaced by $\vec{\phi}_{t_0}$ which is defined by (3.37). Since the fixed point of Φ_{t_0} is unique in $C(\mathbb{R}; \tilde{H}^s)$ by Lemma 3.2 and 3.3, we have $v \in X^s$. Therefore we obtain $u = v$ by (3) in the theorem. So that we may remove X^s in (3) in the theorem. □

4. – Proof of Proposition 1.1

In Lemma 4.6, below, we show the uniqueness of solutions of **NLW** and **NLKG** with the same data in $C(I; \tilde{H}^s)$ for any interval I in $\mathbb{R}$. We follow the method of the proof of Proposition 3.1 in [6]. Throughout this section, we consider **NLW** and **NLKG** with $n \ge 3$ and $1/2 \le s < n/2$. We use Lemma 2.1 for $n = 3$, Corollary 2.8 for $n \ge 4$ with $\sigma \equiv \sigma_1 = \sigma_2 = n-1$, $\lambda \equiv \lambda_1 = \lambda_2 = (n+1)/2$. We put $\delta(r) \equiv n\alpha(r)$.

LEMMA 4.1. *Let r satisfy*

$$1/2(n-2) \le \alpha(r) \le 1/(n-1) \ \textit{for} \ \ n \ge 4, \ \ 0 \le \alpha(r) < 1/2 \ \textit{for} \ \ n = 3\,.$$

Let p satisfy

$$\frac{3n+1}{n^2-1} \le p-1 \begin{cases} \le \dfrac{2}{n-2s} \min\left(\dfrac{2n}{n+1}, 2 - \frac{n-1}{2}\alpha(r)\right) & \textit{for} \ \ n \ge 4\,, \\ < \dfrac{3}{3-2s} & \textit{for} \ \ n = 3\,. \end{cases}$$

Let f satisfy $N(s-1/2, p)$. Then

$$\left\| \int_{t_0}^{t} \frac{U(t-\tau)}{\omega} f(u(\tau))d\tau \,:\, L^q(I; \bar{B}_r^{s-\lambda\alpha(r)}) \right\| \lesssim |I|^{\nu} \|u; L^\infty(I; \tilde{H}^s)\|^p$$

for any $u \in L^\infty(I; \tilde{H}^s)$, where $q \equiv 2/(n-1)\alpha(r)$ and $\nu > 0$ is some constant, and t_0 is any point with $t_0 \in \bar{I} \subset \mathbb{R}$.

PROOF OF LEMMA 4.1. By the conditions on p, there exists s_1 with $1/2 \le s_1 \le s$,

$$\frac{n+\lambda}{\lambda} \le (p-1)(n-2s_1) \begin{cases} \le \min\left(\dfrac{2n}{\lambda}, 4-(n-1)\alpha(r)\right) & \text{for} \quad n \ge 4\,, \\ < 3 & \text{for} \quad n = 3\,. \end{cases}$$

Let $\tilde{r}$ be given by $\alpha(\tilde{r}) = (2-(p-1)(n-2s_1))/(n-1)$. Then $\tilde{r}$ satisfies the condition $\alpha(r) - 2/(n-1) < \alpha(\tilde{r}) \le 0$ for $n \ge 4$ since $2 \le (p-1)(n-2s_1) \le 4-(n-1)\alpha(r)$, the condition $-1/2 < \alpha(\tilde{r}) \le 0$ for $n = 3$ since $2 \le (p-1)(n-2s_1) < 3$. Therefore by Lemma 2.1 and Corollary 2.8, we have

$$\left\| \int_{t_0}^{t} \frac{U(t-\tau)}{\omega} f(u(\tau))d\tau;\, L^q(I; \bar{B}_r^{s-\lambda\alpha(r)}) \right\| \lesssim \|f(u); L^{\tilde{q}}(I; \bar{B}_{\tilde{r}}^{\rho_*})\|\,,$$

where $q, \tilde{q}, \rho_*$ are given by

$$q = 2/(n-1)\alpha(r), \quad 1/\tilde{q} = (n-1)\alpha(\tilde{r})/2 + 1, \quad \rho_* = s - \lambda\alpha(\tilde{r}) - 1\,.$$

Here ρ_* satisfies $\rho_* \ge 0$ since $s \ge 1/2$ and $(n+\lambda)/\lambda \le (p-1)(n-2s_1)$. Let r_*, r_{**} be numbers with $\delta(r_*) = s_1$, $1/\tilde{r} = (p-1)/r_* + 1/r_{**}$. Then r_{**} satisfies $\delta(r_{**}) = \lambda\alpha(\tilde{r}) + 1$, $0 \le \alpha(r_{**}) < 1/2$ since the last inequality is rewritten by $n(5-n)/(n+1) < (p-1)(n-2s_1) \le 4n/(n+1)$, which is satisfied by the condition on s_1. By Lemma 3.1 and the embeddings $\tilde{H}^{s_1} \hookrightarrow \bar{B}_{r_*}^0$, $\tilde{H}^s \hookrightarrow \bar{B}_{r_{**}}^{\rho_*}$, we have

$$\|f(u); \bar{B}_{\tilde{r}}^{\rho_*}\| \lesssim \|u; \bar{B}_{r_*}^0\|^{p-1} \|u; \bar{B}_{r_{**}}^{\rho_*}\| \lesssim \|u; \tilde{H}^{s_1}\|^{p-1} \|u; \tilde{H}^s\|\,,$$

so that we obtain the required result as $\nu = 1/\tilde{q} > 0$. $\square$

LEMMA 4.2. *Let r, s, μ satisfy $2 \le r < \infty$, $0 \le s \le \mu < s + \delta(r)$. Let r_* satisfy*

$$s \le \delta(r_*) \le \min\left(\frac{n}{2}, \mu, \frac{s\delta(r)}{s+\delta(r)-\mu}\right).$$

Then

$$\|\phi; B^0_{r_*}\| \lesssim \|\phi; H^s\|^{1-\theta}\|\phi; B^{\mu-\delta(r)}_r\|^\theta$$

for any $\phi \in H^s \cap B^{\mu-\delta(r)}_r$, where $\theta = (\delta(r_*)-s)/(\mu-s)$, B and H may be replaced with $\dot{B}$ and $\dot{H}$, respectively.

PROOF OF LEMMA 4.2. By $s \le \delta(r_*) \le \mu$, θ satisfies $0 \le \theta \le 1$. Let ℓ be a number given by $\alpha(\ell) = \theta\alpha(r)$. Then ℓ satisfies $\delta(m) - \delta(\ell) = (1-\theta)s + \theta(\mu - \delta(r))$, and $1 \le \ell \le m$ by $s \le \delta(r_*) \le s\delta(r)/(s+\delta(r)-\mu)$. Therefore by the embedding and interpolation, $B^0_{r_*} \hookleftarrow B^{\delta(r_*)-\delta(\ell)}_\ell = (H^s, B^{\mu-\delta(r)}_r)_{[\theta]}$, we obtain the conclusion. □

LEMMA 4.3. *Let s satisfy the condition in Proposition 1.1. Let r, q satisfy*

$$\max\left(\frac{1}{2(n-2)}, \frac{4}{n^2-1}, \frac{n-2s}{n^2-1}\right) \le \alpha(r) \le \min\left(\frac{1}{n-1}, \frac{2+(n-1)s}{n^2-1}\right) \quad \text{for} \quad n \ge 4, \tag{4.42}$$

$$\alpha(r) = \frac{1-\varepsilon_0}{2} \qquad\qquad \text{for} \quad n = 3, \tag{4.43}$$

$q = 2/(n-1)\alpha(r)$, where $\varepsilon_0 > 0$ is a sufficiently small number. Let ε be a number with $0 < \varepsilon \le (n-\lambda)\alpha(r)$ for $n \ge 4$, $\varepsilon_0 \le \varepsilon \le \alpha(r)$ for $n = 3$. Let p satisfy $p = 1+4(1-\varepsilon)/(n-2s)$. Let f satisfy $N(s-1/2, p)$. Then the following estimate holds

$$\left\|\int_{t_0}^t \frac{U(t-\tau)}{\omega} f(u(\tau))d\tau; L^q(I; \bar{B}^{\mu+\varepsilon-\delta(r)}_r)\right\| \lesssim |I|^\varepsilon \|u; L^\infty(I; \tilde{H}^s)\|^{p-1}\|u; L^q(I; \bar{B}^{\mu-\delta(r)}_r)\|$$

for any $u \in L^\infty(I; \tilde{H}^s) \cap L^q(I; \bar{B}^{\mu-\delta(r)}_r)$, and any μ with $s \le \mu \le s+(n-\lambda)\alpha(r)-\varepsilon$, where t_0 is any point with $t_0 \in \bar{I} \subset \mathbb{R}$.

PROOF OF LEMMA 4.3. First we note that there exists r with (4.42) for $n \ge 4$ by the conditions on s. Let r be any number with (4.42), (4.43), where $\varepsilon_0 > 0$ is a sufficiently small number determined later. Let $\tilde{r}, \tilde{q}$ be given by

$$\alpha(\tilde{r}) = \alpha(r) - \frac{1-\varepsilon}{n-\lambda}, \quad \frac{1}{\tilde{q}} = \frac{\sigma}{2}\alpha(\tilde{r}) + 1.$$

Then by Lemma 2.1 for $n = 3$, Corollary 2.8 for $n \ge 4$, we have

$$\left\|\int \frac{U(t-\tau)}{\omega} f(u(\tau))d\tau; L^q(I; \bar{B}^{\mu+\varepsilon-\delta(r)}_r)\right\| \lesssim \|f(u); L^{\tilde{q}}(I; \bar{B}^{\rho_*}_{\tilde{r}})\|,$$

where $\rho_* = \mu + \varepsilon - (n-\lambda)\alpha(r) - \lambda\alpha(\tilde{r}) - 1$. Here ρ_* satisfies $\rho_* \geq 0$ by $\mu \geq s$, $\varepsilon \leq (n-\lambda)\alpha(r)$ and

$$\alpha(r) \leq (2 + (n-1)s)/(n^2-1)\,. \tag{4.44}$$

Let $\eta = (s - \rho_*)/(s + \delta(r) - \mu)$. Then η satisfies $0 \leq \eta \leq 1$ by $\mu \leq s + (n-\lambda)\alpha(r) - \varepsilon$ and

$$4(1-\varepsilon)/(n^2-1) \leq \alpha(r)\,. \tag{4.45}$$

Let r_*, r_{**} satisfy

$$\alpha(r_{**}) = \eta\alpha(r), \quad 1/\tilde{r} = (p-1)/r_* + 1/r_{**}\,.$$

Then r_* satisfies $r_* < \infty$ by $2 < r$, and

$$s \leq \delta(r_*) \leq \mu_* \equiv \min\left(\frac{n}{2}, \mu, \frac{s\delta(r)}{s+\delta(r)-\mu}\right).$$

Indeed, the above inequalities are rewritten as

$$\begin{aligned}
&(s + (n-\lambda)\alpha(r) - \mu + 1 - \varepsilon)\alpha(r) - (s + \delta(r) - \mu)(p-1)(n-2s)/2n\\
\leq&(s + (n-\lambda)\alpha(r) - \mu)\alpha(\tilde{r})\\
\leq&(s + (n-\lambda)\alpha(r) - \mu + 1 - \varepsilon)\alpha(r) - (s + \delta(r) - \mu)(p-1)(n-2\mu_*)/2n\,.
\end{aligned}$$

The first inequality is easily shown, while the second is rewritten as $s + (n-1)\alpha(r) - \mu \geq a_*$, where a_* is given by

$$a_* = 0 \text{ for } \mu \geq n/2,\ \ a_* = \frac{(s+\delta(r)-\mu)(n-1)(n-2\mu)}{n(n-2s)} \quad \text{for } \delta(r) \leq \mu < n/2\,,$$

$$a_* = \frac{(n-1)(n(s-\mu) + (n-2s)\delta(r))}{n(n-2s)} \qquad \text{for} \quad s \leq \mu < \delta(r)\,.$$

The cases for $\mu \geq n/2$ and $s \leq \mu < \delta(r)$ follow from $\mu \leq s + (n-\lambda)\alpha(r) - \varepsilon$ and $s \geq 1/2$, respectively. The case for $\delta(r) \leq \mu < n/2$ is rewritten as

$$g(\mu) \equiv (n-1)(s + \delta(r) - \mu)(n - 2\mu) - n(n-2s)(s + (n-1)\alpha(r) - \mu) \leq 0\,.$$

Since $g(s) = 0$ and $g''(\mu) > 0$, $g(\mu) \leq 0$ holds for $s \leq \mu \leq s + (n-\lambda)\alpha(r) - \varepsilon$ if $g(s + (n-\lambda)\alpha(r) - \varepsilon) \leq 0$, which is rewritten as

$$g_1(\varepsilon) \equiv \left(\varepsilon - \frac{n-1}{2}\alpha(r)\right)\left(2\varepsilon + 2\lambda\alpha(r) - \frac{n-2s}{n-1}\right) \leq 0\,.$$

Since $g_1((n-1)\alpha(r)/2) = 0$ and $g_1''(\varepsilon) > 0$, $g_1(\varepsilon) \leq 0$ holds for $0 < \varepsilon \leq (n-\lambda)\alpha(r)$ if $g_1(0) \leq 0$, which is satisfied by the condition

$$(n-2s)/(n^2-1) \leq \alpha(r)\,. \tag{4.46}$$

By Lemma 3.1, Lemma 4.2 and the interpolation $\bar{B}^{\rho_*}_{r_{**}} = (\tilde{H}^s, \bar{B}^{\mu-\delta(r)}_r)_{[\eta]}$, we have

$$\|f(u); \bar{B}^{\rho_*}_{\tilde{r}}\| \lesssim \|u; \tilde{H}^s\|^{(1-\theta)(p-1)+1-\eta}\|u; \bar{B}^{\mu-\delta(r)}_r\|^{\theta(p-1)+\eta}\,,$$

where $\theta = (\delta(r_*) - s)/(\mu - s)$. Therefore we have

$$\|f(u); L^q(I; \bar{B}^{\rho_*}_{\tilde{r}})\| \lesssim |I|^\varepsilon \|u; L^\infty(I; \tilde{H}^s)\|^{p-1}\|u; L^q(I; \bar{B}^{\mu-\delta(r)}_r)\|\,,$$

where we have used the properties $\theta(p-1) + \eta = 1$ and $\varepsilon = 1/\tilde{q} - 1/q$. So that we obtain the required result for $n \geq 4$. For $n = 3$, the conditions (4.44), (4.45), (4.46) follow from $s \geq 1$, $\varepsilon \geq \varepsilon_0$, $\varepsilon_0 \leq 2/(n+1)$, respectively. □

LEMMA 4.4. *Let s satisfy the conditions in Proposition* 1.1. *Let r satisfy* (4.42), (4.43), *where $\varepsilon_0 > 0$ be sufficiently small, and let $q \equiv 2/(n-1)\alpha(r)$. Let p_1, p_2 satisfy*

$$\frac{3n+1}{n^2-1} \le p_1 - 1 \le p_2 - 1 \begin{cases} \le \dfrac{4(1-\varepsilon_0)}{3-2s} & \text{for} \quad n = 3\,, \\ < \dfrac{4}{n-2s} & \text{for} \quad n \ge 4\,. \end{cases}$$

Let f satisfy $N(s-1/2, p_1, p_2)$. Then any solution u of **NLW**, *or* **NLKG**, *in $C(I; \tilde{H}^s)$ satisfies $u \in L^q(I; \bar{B}_r^{s-\lambda\alpha(r)})$, where I is any bounded interval in $\mathbb{R}$.*

PROOF OF LEMMA 4.4. Let t_0 be any point with $t_0 \in \bar{I}$. The solution u is rewritten as

$$u(t) = (\cos(t-t_0)\omega)u(t_0) + \frac{\sin(t-t_0)\omega}{\omega}\partial_t u(t_0) + \sum_{j=1}^{k} \widetilde{G_{t_0}} f_j(u) + \sum_{j=k+1}^{\ell} \widetilde{G_{t_0}} f_j(u)\,,$$

where f_j satisfies $N(s-1/2, p_j^*)$ with

$$p_1 \le p_1^* \le \cdots \le p_k^* < 1 + \frac{4(1-(n-\lambda)\alpha(r))}{n-2s} \le p_{k+1}^* \le \cdots \le p_\ell^* \le p_2\,.$$

Let ε_j, $k+1 \le j \le \ell$, be numbers given by

$$p_j - 1 = 4(1-\varepsilon_j)/(n-2s)\,.$$

We note that $\varepsilon_{k+1} \ge \cdots \ge \varepsilon_\ell$. By Lemma 2.1 and Lemma 4.1, we have

$$u(t) - \sum_{j=k+1}^{\ell} \widetilde{G_{t_0}} f_j(u) \in L^q(I; \bar{B}_r^{s-\delta(r)} \cap \bar{B}_r^{s-\lambda\alpha(r)})\,.$$

By Lemma 4.3, we have

$$\widetilde{G_{t_0}} f_j(u) \in L^q\big(I; \bar{B}_r^{s-\delta(r)} \cap \bar{B}_r^{s+\varepsilon_j-\delta(r)}\big) \hookrightarrow L^q\big(I; \bar{B}_r^{s-\delta(r)} \cap \bar{B}_r^{s+\varepsilon_\ell-\delta(r)}\big)$$

for $k+1 \le j \le \ell$, so that we have $u \in L^q(I; \bar{B}_r^{s-\delta(r)} \cap \bar{B}_r^{s+\varepsilon_\ell-\delta(r)})$. Again by Lemma 4.3, we have

$$\widetilde{G_{t_0}} f_j(u) \in L^q\big(I; \bar{B}_r^{s-\delta(r)} \cap \bar{B}_r^{\rho_1}\big) \hookrightarrow L^q\big(I; \bar{B}_r^{s-\delta(r)} \cap \bar{B}_r^{\rho_2}\big)\,,$$

where $\rho_1 = (s+\varepsilon_\ell+\varepsilon_j-\delta(r)) \wedge (s-\lambda\alpha(r))$, $\rho_2 = (s+2\varepsilon_\ell-\delta(r)) \wedge (s-\lambda\alpha(r))$, so that we have $u \in L^q(I; \bar{B}_r^{s-\delta(r)} \cap \bar{B}_r^{\rho_2})$. Repeating this procedure, we obtain $u \in L^q(I; \bar{B}_r^{s-\lambda\alpha(r)})$. $\square$

LEMMA 4.5. *Let s satisfy $1 \le s < 3/2$ for $n = 3$, $5/8 \le s < 2$ for $n = 4$, $1/2 \le s < n/2$ for $n \ge 5$. Let p satisfy $p-1 \le 4/(n-2s)$, and $p-1 \ge 4n/(n^2-1)$ for* **NLW**, *$p-1 \ge 4/(n+1)$ for* **NLKG**. *Let f satisfy $N(s-1/2, p)$. Let r satisfy*

$$\begin{array}{ll} \max(1/2(n-2), (n-2s)/(n^2-1)) \le \alpha(r) \le \min(1/(n-1), s/\lambda) & \text{for } n \ge 4, \\ \alpha(r) = (1-\varepsilon_0)/2 & \text{for } n = 3, \end{array} \tag{4.47}$$

where $\varepsilon_0 > 0$ is some sufficiently small number. Then

$$\left\| \int_{t_0}^t \frac{U(t-\tau)}{\omega}(f(u(\tau)-f(v(\tau)))d\tau; L^q(I;\bar{B}_r^0)\right\|$$
$$\lesssim \begin{cases} |I|^{2/(n+1)} \max\limits_{w=u,v} \|w; L^\infty(I;\tilde{H}^s)\|^{p-1}\|u-v; L^q(I;\bar{B}_r^0)\| \\ \qquad\qquad \text{for} \quad p-1 < 4n/(n+1)(n-2s)\,, \\ |I|^{2-(p-1)(n-2s)/2} \max\limits_{w=u,v} \|w; L^\infty(I;\tilde{H}^s)\cap L^q(I;\bar{B}_r^{s-\lambda\alpha(r)})\|^{p-1}\|u-v; L^q(I;\bar{B}_r^0)\| \\ \qquad\qquad \text{for} \quad p-1 \ge 4n/(n+1)(n-2s)\,, \end{cases}$$

where $q \equiv 2/(n-1)\alpha(r)$, t_0 is any number with $t_0 \in \bar{I} \subset \mathbb{R}$.

PROOF OF LEMMA 4.5. First we note that there exists r with (4.47) by the conditions on s. Let $\tilde{r}$ satisfy $1/\tilde{r} = 1/r + 2/(n+1)$. Then $\tilde{r}$ satisfies

$$\alpha(r) - 2/(n-1) < \alpha(\tilde{r}) \le 0 \quad \text{for} \quad n \ge 4, \quad -1/2 < \alpha(\tilde{r}) \le 0 \quad \text{for} \quad n = 3\,.$$

By Lemma 2.1 and Corollary 2.8, we have

$$\left\| \int_{t_0}^t \frac{U(t-\tau)}{\omega}(f(u(\tau)) - f(v(\tau)))d\tau; L^q(I;\bar{B}_r^0)\right\| \lesssim \left\| f(u) - f(v); L^{\tilde{q}}(I;\bar{B}_{\tilde{r}}^0)\right\|,$$

where $1/\tilde{q} = (n-1)\alpha(\tilde{r})/2+1$. Let r_* satisfy $r_* = (p-1)(n+1)/2$. If $p-1 < 4n/(n+1)(n-2s)$, then r_* satisfies $1/2 \le \delta(r_*) < s$ for **NLW**, $0 \le \delta(r_*) < s$ for **NLKG**. By the Hölder inequality and the embedding $\tilde{H}^s \hookrightarrow \dot{H}^{\delta(r_*)} \hookrightarrow \bar{B}_{r_*}^0$, we have

$$\|f(u) - f(v); \bar{B}_{\tilde{r}}^0\| \lesssim \max_{w=u,v} \|w; \tilde{H}^s\|^{p-1}\|u-v; \bar{B}_r^0\|\,.$$

So that we obtain the required result. If $p-1 \ge 4n/(n+1)(n-2s)$, then r_* satisfies

$$s \le \delta(r_*) \le \min(n/2, s + (n-\lambda)\alpha(r), ns/\lambda)$$

since the above inequalities are satisfied if p and s satisfy

$$\frac{4n}{(n+1)(n-2s)} \le p - 1 \le \frac{4}{n-2s},$$
$$\frac{n}{2} - \min\left(\frac{n}{2}, s + (n-\lambda)\alpha(r), \frac{ns}{\lambda}\right) \le \frac{n(n-2s)}{2(n+1)}\,,$$

where the last inequality is satisfied by $1/2 \le s < n/2$ and $(n-2s)/(n^2-1) \le \alpha(r)$. By Lemma 4.2, we have

$$\|f(u)-f(v);\bar{B}^0_{\tilde r}\| \lesssim \max_{w=u,v}\|w;\tilde{H}^s\|^{(1-\theta)(p-1)}\|w;\bar{B}^{s-\lambda\alpha(r)}_r\|^{\theta(p-1)}\|u-v;\bar{B}^0_r\|\,,$$

where $\theta = (\delta(r_*)-s)/(\mu-s)$. By the Hölder inequality in time variable, we obtain the required result, where we use the property

$$1/\tilde{q} - \theta(p-1)/q - 1/q = 2-(p-1)(n-2s)/2\,. \qquad \square$$

Lemma 4.6. *Let n, s satisfy the conditions in Proposition* 1.1. *Let p_1, p_2 satisfy $p_1 \le p_2 < p(s)$, and $p_1 \ge p(1/2)$ for* **NLW**, *$p_1 \ge p(0)$ for* **NLKG**. *Let f satisfy $N(s-1/2, p_1, p_2)$. Let u, v be solutions of* **NLW**, *or* **NLKG**, *in $C(I;\tilde{H}^s)$ with $(u(t_0), \partial_t u(t_0)) = (v(t_0), \partial_t v(t_0))$ for some $t_0 \in I$. Then $u = v$ in $C(I;\tilde{H}^s)$.*

Proof of Lemma 4.6. By the condition on s, there exists r which satisfies (4.42), (4.43) with $\alpha(r) \le s/\lambda$, where $\varepsilon_0 > 0$ is taken as $p_2 - 1 \le 4(1-\varepsilon_0)/(n-2s)$ for $n = 3$. By Lemma 4.4, we have $u, v \in L^q(I;\bar{B}^{s-\lambda\alpha(r)}_r)$ with $q = 2/(n-1)\alpha(r)$. Since $u - v$ is rewritten by

$$u - v = \sum_{j=1}^{k}\int_{t_0}^{t}\frac{U(t-\tau)}{\omega}\big(f_j(u(\tau)) - f_j(v(\tau))\big)d\tau\,,$$

where f_j, $1 \le j \le k$, satisfies $N(s-1/2, p_j^*)$ with $p_1 \le p_j^* \le p_2$, by Lemma 4.5 we have

$$\|u-v;L^q(I_1;\bar{B}^0_r)\| \lesssim \sum_{j=1}^{k}|I_1|^{\nu_j}\max_{w=u,v}\|w;L^\infty(I_1;\tilde{H}^s)\cap L^q(I_1;\bar{B}^{s-\lambda\alpha(r)}_r)\|^{p_j^*-1}\|u-v;L^q(I_1;\bar{B}^0_r)\|$$

for any $I_1 \subset \mathbb{R}$ with $t_0 \in \bar{I}_1$, $|I_1| < 1$, where $\nu_j = \min(2/(n+1), 2-(p_j^*-1)(n-2s)/2)$, we note that $u, v \in L^q(I_1;\bar{B}^0_r)$ by the embeddings $B^{s-\lambda\alpha(r)}_r \hookrightarrow B^0_r$ for **NLKG**, and $\dot{H}^{1/2}\cap\dot{B}^{s-\lambda\alpha(r)}_r \hookrightarrow \dot{B}^0_r$ for **NLW**. Taking I_1 sufficiently small, we have $u = v$ in $C(I_1;\tilde{H}^s)$. Repeating this procedure, we obtain the required results. $\square$

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