Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. I (2002), pp. 1-11

Local Approximation of Semialgebraic Sets

MASSIMO FERRAROTTI – ELISABETTA FORTUNA –
LES WILSON

Abstract. Let A be a closed semialgebraic subset of Euclidean space of codimension at least one, and containing the origin O as a non-isolated point. We prove that, for every real $s \geq 1$, there exists an algebraic set V which approximates A to order s at O. The special case $s = 1$ generalizes the result of the authors that every semialgebraic cone of codimension at least one is the tangent cone of an algebraic set.

Mathematics Subject Classification (2000): 14P10.

Introduction

In this paper we investigate the possibility of locally approximating semialgebraic sets by algebraic ones. This question was already considered by L. Bröcker (see [B1], [B2]) from a point of view somewhat different from ours.

Evidently the answer depends on what approximating means. The notion of approximation we will use originates from combining the two following observations. First of all, one of the main tools to get information about the geometric behavior of a semialgebraic subset A of $\mathbb{R}^n$ near a singular point, say for instance the origin O, is the tangent cone $C(A)$ at O, i.e. the union of limits of secant half-lines $\overline{Ox_m}$ as $x_m \in A$ tends to O. Secondly, one method to "measure" how near $C(A)$ and A are is to consider the Hausdorff distance between the sections $A \cap S_r$ and $C(A) \cap S_r$ of A and $C(A)$ with the sphere S_r centered at O of radius r. We will check that this distance vanishes, as r tends to 0, of order > 1. So one can take as a "measure of proximity" between two sets A and B the order of vanishing at 0 of the Hausdorff distance $D(A \cap S_r, B \cap S_r)$ and call the sets A and B s-equivalent if this distance tends to 0 more rapidly than r^s.

This work was partially supported by GNSAGA (CNR) and MURST
Pervenuto alla Redazione il 2 maggio 2000 e in forma definitiva 1 gennaio 2001.

According to this definition, we will investigate the question whether any closed semialgebraic subset of $\mathbb{R}^n$ of codimension ≥ 1 can be s-approximated by (i.e. is s-equivalent to) an algebraic subset locally at a point, say O. For $s = 1$ a positive answer can be easily obtained by remarking that any closed semialgebraic subset is 1-equivalent to its tangent cone and by using a previous result (see [F-F-W]) showing that any closed semialgebraic cone of codimension ≥ 1 is 1-equivalent to an algebraic set.

In this article we generalize this result and prove (Theorem 1.4) that, for any real $s \geq 1$, every closed semialgebraic subset of $\mathbb{R}^n$ of codimension ≥ 1 can be s-approximated by an algebraic subset of $\mathbb{R}^n$.

We are grateful to the referee for his useful comments.

1. – s-equivalence and first properties

If A and B are non-empty compact subsets of $\mathbb{R}^n$, let us denote by $D(A, B)$ the classical Hausdorff distance, i.e.

$$D(A, B) = \inf \{\epsilon \mid A \subseteq N_\epsilon(B), \ B \subseteq N_\epsilon(A)\},$$

where $N_\epsilon(A) = \{x \in \mathbb{R}^n \mid d(x, A) < \epsilon\}$ and $d(x, A) = \inf_{y \in A} \|x - y\|$.

If we let $\delta(A, B) = \sup_{x \in B} d(x, A)$, then $D(A, B) = \max\{\delta(A, B), \ \delta(B, A)\}$.

Observe that $\delta(A, B) = 0$ if, and only if, $B \subseteq A$ and that, for any A, B, C subsets of $\mathbb{R}^n$, we have $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$.

Let A be a semialgebraic subset of $\mathbb{R}^n$, $O \in A$. The tangent cone $C(A)$ at O can be defined as the set of points $u \in \mathbb{R}^n$ such that there exist a sequence $x_m \in A$ converging to O and a sequence of real positive numbers t_m such that $\lim_{m \to \infty} t_m x_m = u$.

We will denote by $\mathrm{Der}(A)$ the set of non-isolated points of A. If $O \in \mathrm{Der}(A)$, let us estimate how much $C(A)$ approximates A locally at O by computing the Hausdorff distance between $A \cap S_r$ and $C(A) \cap S_r$ where S_r is the sphere of radius r centered at the origin. We point out that, by the Curve Selection Lemma, if $O \in \mathrm{Der}(A)$ the set $A \cap S_r$ is not empty for any r small enough, which we will always implicitly assume.

Lemma 1.1. *Let A be a closed semialgebraic subset of $\mathbb{R}^n$ with $O \in \mathrm{Der}(A)$. Then*

$$\lim_{r \to 0} \frac{D(A \cap S_r, C(A) \cap S_r)}{r} = 0 .$$

Proof. Let us set $C = C(A)$, $A_r = A \cap S_r$ and $C_r = C \cap S_r$.

Since A_r is compact and non-empty, there exists $x_r \in A_r$ such that $\delta(C_r, A_r) = d(x_r, C_r)$. In order to prove that $\lim_{r \to 0} \frac{\delta(C_r, A_r)}{r} = 0$, it is enough to prove that, for any sequence of real positive numbers $\{r_i\}$ converging to 0 and such that $\{\frac{x_{r_i}}{r_i}\}$ converges to a limit, say $u \in C$, we have

$\lim_{i\to\infty}\frac{\delta(C_{r_i},A_{r_i})}{r_i} = \lim_{i\to\infty}\frac{d(x_{r_i},C_{r_i})}{r_i} = 0$. This follows from the fact that $r_i u \in C_{r_i}$, so that

$$\frac{d(x_{r_i},C_{r_i})}{r_i} \leq \frac{\|x_{r_i}-r_i u\|}{r_i} = \left\|\frac{x_{r_i}}{r_i}-u\right\| .$$

Let us now prove that $\lim_{r\to 0}\frac{\delta(A_r,C_r)}{r} = 0$. As above, there is a $y_r \in C_r$ such that $\delta(A_r,C_r) = d(y_r,A_r)$. Let $u_r = y_r/r$. For any sequence $\{r_i\}$ converging to 0, consider the sequence $\{u_{r_i}\}$; we can assume that $\{u_{r_i}\}$ converges, say to u. There is an analytic curve γ such that $\gamma(0) = O$ and $\gamma(t) \in A$ for $t \in [0,\epsilon]$, and the image of γ is tangent to u at O (see e.g. [K-R]). Furthermore we can assume that γ intersects each sphere of sufficiently small radius r in one point. Let x_i be the point of intersection of γ with the sphere of radius r_i. Then

$$\lim_{i\to\infty}\frac{d(y_{r_i},A_{r_i})}{r_i} \leq \lim_{i\to\infty}\frac{\|y_{r_i}-x_i\|}{r_i} = 0. \qquad \square$$

By the previous lemma, the Hausdorff distance between $A\cap S_r$ and $C(A)\cap S_r$ vanishes, as r tends to 0, of order > 1. We can therefore introduce a sort of "measure of proximity" between two sets near a common non-isolated point, say the origin O, as follows:

DEFINITION 1.2. Let A and B be closed semialgebraic subsets of $\mathbb{R}^n$ with $O \in \mathrm{Der}(A)\cap\mathrm{Der}(B)$ and let s be a real number ≥ 1.

(1) We say that $A \leq_s B$ if $\lim_{r\to 0}\dfrac{\delta(B\cap S_r, A\cap S_r)}{r^s} = 0$.

(2) We say that A and B are s-equivalent (and we will write $A \sim_s B$) if $A \leq_s B$ and $B \leq_s A$, i.e. if $\lim_{r\to 0}\dfrac{D(A\cap S_r, B\cap S_r)}{r^s} = 0$.

It is easy to check that $\leq_s$ is transitive and that $\sim_s$ is an equivalence relationship. Using Definition 1.2, Lemma 1.1 says that $A \sim_1 C(A)$. Moreover we have

PROPOSITION 1.3. *Let A and B be closed semialgebraic subsets of $\mathbb{R}^n$, with $O \in \mathrm{Der}(A)\cap\mathrm{Der}(B)$. Then $A \sim_1 B$ if and only if $C(A) = C(B)$.*

PROOF. Assume that $A \sim_1 B$ and let us prove that $C(A) \subseteq C(B)$. Let $u \in C(A)$; without loss of generality we can suppose that $\|u\| = 1$. Let $\{x_i\}$ be a sequence of points of $A\setminus\{O\}$ converging to O and such that $\lim_{i\to\infty}\frac{x_i}{\|x_i\|} = u$.

Denote $r_i = \|x_i\|$, $A_{r_i} = A\cap S_{r_i}$ and $B_{r_i} = B\cap S_{r_i}$. So $x_i \in A_{r_i}$. By the hypothesis $\lim_{i\to\infty}\frac{\delta(B_{r_i},A_{r_i})}{r_i} = 0$ and consequently $\lim_{i\to\infty}\frac{d(x_i,B_{r_i})}{r_i} = 0$. For any i big enough, there exists $y_i \in B_{r_i}$ such that $d(x_i,B_{r_i}) = \|x_i - y_i\|$. So

$$\lim_{i\to\infty}\left\|\frac{x_i}{r_i}-\frac{y_i}{r_i}\right\| = \lim_{i\to\infty}\frac{\|x_i-y_i\|}{r_i} = 0\,,$$

which implies that $\frac{y_i}{r_i} \to u$, i.e. $u \in C(B)$. Exactly in the same way one may check that $C(B) \subseteq C(A)$.

The opposite implication is a trivial consequence of Lemma 1.1. $\square$

In particular Proposition 1.3 assures that two semialgebraic cones are 1-equivalent if and only if they are equal. Consequently any 1-equivalence class of closed semialgebraic sets contains exactly one semialgebraic cone. Let us recall that in [F-F-W] it is proved that every semialgebraic cone in $\mathbb{R}^n$ (of codimension ≥ 1) is the tangent cone to an algebraic subset of $\mathbb{R}^n$. So in any 1-equivalence class you can find an algebraic representative.

The aim of this paper is to prove, generalizing the result of [F-F-W], that any s-equivalence class of closed semialgebraic sets contains an algebraic representative, precisely:

THEOREM 1.4 (Approximation Theorem). *For any real number $s \geq 1$ and for any closed semialgebraic set $A \subset \mathbb{R}^n$ of codimension ≥ 1 with $O \in \mathrm{Der}(A)$, there exists an algebraic subset V of $\mathbb{R}^n$ such that $A \sim_s V$.*

EXAMPLE 1.5. (i) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^3 - y^2 = 0, y \geq 0\}$. It is easy to check that, if n is an odd integer greater than $(4/3)s + 2$, $s \geq 1$, then the algebraic set $V = \{(x, y) \in \mathbb{R}^2 \mid (x^3 - y^2)^2 - y^n = 0\}$ is s-equivalent to A.
(ii) Let $A = \{x \geq 0, y \geq 0, z = 0\} \subset \mathbb{R}^3$. Fix $s \geq 1$. Let $V = \{f = 0\}$, where $f(x, y, z) = (z^2 - x^n)^2 - y^m$. If m and n are odd integers, $n > 2s$ and $m > 2ns$, then $A \sim_s V$. To see this, first note that $(1) : V \subset \{y \geq 0\}$. Now suppose $(x, y, z) \in V$. Then $z^2 = x^n \pm y^{m/2}$. Since $n > 2s$ and $m > 4s$, we have $(2) : |z| = o(\|(x, y)\|)^s)$. Finally, either $x^n = z^2 + y^{m/2}$ or $x^n = z^2 - y^{m/2}$. In the first case, $x \geq 0$. In the second case, $x \geq -y^{m/2n}$. Thus, in both cases we have $(3) : x \geq -y^{m/2n}$, and $m/2n > s$. Together, (1), (2) and (3) imply that $A \sim_s V$.

The proof of Theorem 1.4 will be achieved in Section 3.

2. – Some technical tools

Let us collect in this section the main ingredients which will be used in the proof of the Approximation Theorem. An essential tool will be Łojasiewicz' inequality, which we will use in the following slightly modified version:

LEMMA 2.1. *Let A be a compact semialgebraic subset of $\mathbb{R}^n$. Assume f and g are semialgebraic functions defined on A such that f is continuous, $V(f) \subseteq V(g)$, g continuous at the points of $V(g)$ and such that $|g| < 1$ on A. Then there exists a positive constant N such that $|g|^N \leq |f|$ on A and $|g|^N < |f|$ on $A \setminus V(f)$.*

PROOF. In the classical Łojasiewicz' inequality one assumes that g is continuous on A in order to get that the function $\frac{g^N}{f}$, extended to 0 on $V(f)$, is continuous on A. Following the proof given in [B-C-R] (Proposition 2.6.4 and Theorem 2.6.6), one realizes that our hypotheses are sufficient to get the boundedness of $\frac{g^N}{f}$ on A. Increasing N if necessary, we can further obtain that $|\frac{g^N}{f}| < 1$ on $A \setminus V(f)$. □

For notational simplicity, we will often denote by W_r the intersection of a set W with the sphere S_r of radius r centered at the origin. Moreover, for every $x \in \mathbb{R}^n$ and every $r > 0$, we will denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ of radius r centered at x.

One of the most important properties of semialgebraic sets is their local conical structure (see, for instance [B-C-R], Theorem 9.3.5), which guarantees that, if $X \subset \mathbb{R}^n$ is a closed semialgebraic set and $O \in \mathrm{Der}(X)$, then there exists a constant $R_X > 0$ such that for any $R < R_X$ there exists a semialgebraic homeomorphism $\phi : \overline{B(O, R)} \to \overline{B(O, R)}$ such that $\phi(X \cap \overline{B(O, R)})$ is the cone with vertex at O over $X \cap S_R$ and $\|\phi(x)\| = \|x\|$ for all $x \in \overline{B(O, R)}$.

Lemma 2.2. *Let A, B be closed semialgebraic subsets of $\mathbb{R}^n$ with $O \in \mathrm{Der}(A) \cap \mathrm{Der}(B)$. Then there exists $R > 0$ such that the function*

$$\rho(r) = \begin{cases} \delta(A_r, B_r) & \text{if } r \in (0, R) \\ 0 & \text{if } r = 0 \end{cases}$$

is continuous on $[0, R)$.

Proof. Since $\rho(r) \le 2r$, the function ρ is continuous at 0.

Let $0 < R < \min\{R_A, R_B\}$ and let $r_0 \in (0, R)$. There exists $y_0 \in B_{r_0}$ such that $\rho(r_0) = \delta(A_{r_0}, B_{r_0}) = d(y_0, A_{r_0})$. For any sequence $\{r_i\}$ converging to r_0, by the local conical structure of B, there exists a sequence of points $y_i \in B_{r_i}$ converging to y_0. Again by compactness, for any i there exists $x_i \in A_{r_i}$ such that $d(y_i, A_{r_i}) = \|y_i - x_i\|$ and we can assume that $\{x_i\}$ converges to $x_0 \in A_{r_0}$. Hence $\lim_{i\to\infty} d(y_i, A_{r_i}) = \|y_0 - x_0\| \ge d(y_0, A_{r_0})$. So we have

$$d(y_i, A_{r_i}) \le \delta(A_{r_i}, B_{r_i}) = \rho(r_i) \le \delta(A_{r_i}, A_{r_0}) + \delta(A_{r_0}, B_{r_0}) + \delta(B_{r_0}, B_{r_i}) .$$

By the local conical structure of A, we have that $\overline{A - A_{r_0}} \cap S_{r_0} = A_{r_0}$, which implies that $\lim_{i\to\infty} \delta(A_{r_i}, A_{r_0}) = 0$. An analogous argument shows that also $\lim_{i\to\infty} \delta(B_{r_0}, B_{r_i}) = 0$. So from the previous inequalities we get that

$$\lim_{i\to\infty} \rho(r_i) = \delta(A_{r_0}, B_{r_0}) = \rho(r_0)$$

and hence the thesis. □

Note that Lemma 2.2 in particular implies that the function $r \to D(A_r, B_r)$ is continuous on an interval $[0, R)$ for a suitable R.

In Definition 1.2 we introduced the notion of s-equivalence between two sets by means of limits; we are now interested in finding a geometric version of that condition, i.e. a topological tool to control if two sets are sufficiently close one to the other near the origin. To that purpose, given a semialgebraic subset $X \subset \mathbb{R}^n$, $O \in \mathrm{Der}(X)$, and a real positive constant σ, let us consider the set

$$\mathcal{U}(X, \sigma) = \{y \in \mathbb{R}^n \mid \exists\, z \in X \cap S_{\|y\|}, \|y - z\| < \|z\|^\sigma\} .$$

Observe that of course $X \setminus \{O\} \subseteq \mathcal{U}(X, \sigma)$, that $\mathcal{U}(X, \sigma) = \bigcup_{x \in X} \left(B(x, \|x\|^\sigma) \cap S_{\|x\|}\right)$ and that $\mathcal{U}(X, \sigma)$ is semialgebraic whenever σ is rational.

Lemma 2.3. *Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set, $O \in \mathrm{Der}(X)$. Then for any real positive σ, the set $\mathcal{U}(X, \sigma) \cap B(O, R)$ is open in $\mathbb{R}^n$ for every $0 < R < R_X$.*

Proof. Let $\mathcal{U} = \mathcal{U}(X, \sigma)$. Let y_0 be in $\mathcal{U} \cap B(O, R)$ and let $\rho = \|y_0\|$. So there exists $x_0 \in X$ such that $\|x_0 - y_0\| < \|x_0\|^\sigma$ and $\|x_0\| = \rho$.

Let $\epsilon > 0$ be such that $B(x_0, \epsilon) \subseteq B(O, R)$ and, if $x \in B(x_0, \epsilon)$ and $y \in B(y_0, \epsilon)$, then $\|x - y\| < \|x\|^\sigma$. The local conical structure of X assures that there exists a real number δ with $0 < \delta \leq \epsilon$ such that, for any $\eta \geq 0$ with $|\eta - \rho| < \delta$, the set $X \cap B(x_0, \epsilon) \cap S_\eta$ is not empty.

We prove that $B(y_0, \delta) \subseteq \mathcal{U} \cap B(O, R)$. Clearly $B(y_0, \delta) \subseteq B(O, R)$. Let $y \in B(y_0, \delta)$; since $|\|y\| - \rho| < \delta$, we have $X \cap B(x_0, \epsilon) \cap S_{\|y\|} \neq \emptyset$. Then there exists $x \in X$ such that $\|x\| = \|y\|$ and $\|x - y\| < \|x\|^\sigma$, which proves that $y \in \mathcal{U}$. □

The sets $\mathcal{U}(X, \sigma)$ can be used as a topological tool to test s-equivalence:

Proposition 2.4. *Let A, B be closed semialgebraic subsets of $\mathbb{R}^n$ and let $s \geq 1$. If $O \in \mathrm{Der}(A) \cap \mathrm{Der}(B)$, then $A \leq_s B$ if and only if there exist real constants $R > 0$ and $\sigma > s$ such that $A \cap B(O, R) \setminus \{O\} \subseteq \mathcal{U}(B, \sigma)$.*

Proof. Assume that $A \leq_s B$. By Lemma 2.2 there exists $R > 0$ such that the function $\rho : [0, R] \to \mathbb{R}$ defined by $\rho(0) = 0$ and $\rho(r) = \delta(B_r, A_r)$ if $r > 0$ is continuous. The function ρ is semialgebraic, hence, after decreasing R if necessary, we can assume that either $\rho \equiv 0$ on $[0, R]$ or $\rho > 0$ on $(0, R]$. In the first case, we have that $A \cap B(O, R) \subseteq B \cap B(O, R)$ and then our thesis holds trivially.

So assume that ρ is strictly positive on $(0, R]$ and consider the semialgebraic function $\mu : [0, R] \to \mathbb{R}$ defined by $\mu(0) = 0$ and $\mu(r) = \frac{\rho(r)}{r^s}$ if $r > 0$. Since by our hypothesis $\lim_{r\to 0} \mu(r) = 0$, the function $\mu(r)$ is continuous and $V(\mu) = \{0\}$. Hence, up to decreasing R, by Lemma 2.1 there exist $\alpha > 0$ such that $\mu(r) < r^\alpha$ for all $r \in (0, R]$, which yields that $\rho(r) < r^{s+\alpha}$ for all $r \in (0, R]$. Set $\sigma = s + \alpha > s$.

Now let $x \in A \cap B(O, R)$, $x \neq O$ and let $r = \|x\|$. Then $d(x, B_r) \leq \delta(B_r, A_r) = \rho(r) < r^\sigma$. So there exists $y \in B_r$ such that $\|x - y\| < \|y\|^\sigma$, that is $x \in \mathcal{U}(B, \sigma)$. Hence $A \cap B(O, R) \setminus \{O\} \subseteq \mathcal{U}(B, \sigma)$.

Now let us prove the opposite implication. For any sufficiently small positive r, as A_r and B_r are compact and non-empty, there exist $x_r \in A_r$ and $y_r \in B_r$ such that $\delta(B_r, A_r) = d(x_r, B_r) = \|x_r - y_r\|$.

By hypothesis there exist constants $R > 0$ and $\sigma > s$ such that $A \cap B(O, R) \setminus \{O\} \subseteq \mathcal{U}(B, \sigma) = \{y \in \mathbb{R}^n | \exists x \in B \cap S_{\|y\|}, \|y - x\| < \|x\|^\sigma\}$.

If $r < R$, then $x_r \in A \cap B(O, R)$, so there exists a point $z_r \in B_r$ such that $\|x_r - z_r\| < \|z_r\|^\sigma = r^\sigma$. Since $\|x_r - y_r\| = \inf_{y \in B_r} \|x_r - y\|$, we have $\|x_r - y_r\| \leq \|x_r - z_r\| < r^\sigma$. Hence, if $r < R$,

$$\frac{\|x_r - y_r\|}{r^s} < r^{\sigma - s},$$

which implies that $\lim_{r\to 0} \frac{\delta(B_r, A_r)}{r^s} = 0$, i.e. $A \leq_s B$. □

The second main tool in the proof of Theorem 1.4 will be obtained as a consequence of the following:

LEMMA 2.5. *Let $X \subset Y \subseteq \mathbb{R}^n$ be closed semialgebraic sets with $O \in \mathrm{Der}(X)$ and $\overline{Y \setminus X} = Y$. Then there exist positive constants $\beta, R, \overline{q} \in \mathbb{R}$, with $\beta < 1$ and $R < 1$, such that, for any r with $0 < r \leq R$, for any $q \geq \overline{q}$ and for every $x \in X \cap S_r$, we have*

$$B\left(x, r^{q\beta}\right) \cap Y \cap S_r \not\subseteq \bigcup_{z \in X \cap S_r} B\left(z, r^q\right) \cap Y \cap S_r \, .$$

PROOF. By local conical structure, there exists a positive real number R such that, for any $0 < r \leq R$, we have $\overline{Y_r \setminus X_r} = Y_r$. After taking R small enough and anyhow $R < 1/2$, we can assume that the semialgebraic function $\rho : [0, R] \to \mathbb{R}$, defined by $\rho(r) = \delta(X_r, Y_r)$ for any $r \in (0, R]$ and $\rho(0) = 0$, is smaller than 1 and, by Lemma 2.2, continuous.

Consider the closed semialgebraic set

$$K = \{(r, t) \in \mathbb{R}^2 \mid 0 \leq r \leq R, \ \ 0 \leq t \leq \rho(r)\} \, ;$$

for $(r, t) \in K \setminus \{(0, 0)\}$ the sets

$$W(r, t) = Y_r \setminus \bigcup_{x \in X_r} B(x, t) = \{y \in Y_r \mid d(y, X_r) \geq t\}$$

are non-empty and semialgebraic. Let us define the function $\Phi : \ K \to \mathbb{R}$ by setting $\Phi(0, 0) = 0$ and

$$\Phi(r, t) = \delta(W(r, t), X_r) \qquad \forall\, (r, t) \neq (0, 0) \, .$$

The function Φ is semialgebraic and $V(\Phi) = [0, R] \times \{0\}$; moreover $\Phi(r, t) \leq 2r$ for every $(r, t) \in K$, which implies both that Φ is bounded on K by $2R < 1$ and that $\lim_{(r,t) \to (0,0)} \ \Phi(r, t) = 0$.

Since we want to apply Lemma 2.1 to the functions t and $\Phi(r, t)$ on K, we have to prove that Φ is continuous at the points of $[0, R] \times \{0\}$.

So let us check that, for any $r_0 \in (0, R]$, we have $\lim_{(r,t) \to (r_0,0)} \ \Phi(r, t) = 0$.

If not, then there exist $\epsilon > 0$ and a sequence $\{(r_i, t_i)\}$ of points in K converging to $(r_0, 0)$ such that for all $i \in \mathbb{N}$ there exists $x_i \in X_{r_i}$ such that $d(x_i, W(r_i, t_i)) > \epsilon$. After choosing a suitable subsequence of $\{x_i\}$ if necessary, we can suppose that $\{x_i\}$ converges to a point $x_0 \in X_{r_0}$.

Since $\overline{Y_{r_0} \setminus X_{r_0}} = Y_{r_0}$, there exists $y_0 \in Y_{r_0} \setminus X_{r_0}$ such that $\|y_0 - x_0\| < \epsilon/4$.

Let $t_0 = d(y_0, X_{r_0})/2$ and let $\eta < t_0$ be such that $B(y_0, \eta) \cap X = \emptyset$; in particular $\eta < \epsilon/4$ and $d(y_0, X_{r_0}) > t_0 + \eta$.

We claim there exists $a > 0$ such that, if $|r - r_0| < a$, then $d(y_0, X_r) \geq t_0 + \eta$. In fact otherwise there would exist sequences $\{r_i\}$ converging to r_0 and $\{z_i\}$, with $z_i \in X_{r_i}$, such that $\|y_0 - z_i\| < t_0 + \eta$ for any $i \in \mathbb{N}$. As usual we can assume that z_i converges to a point $z_0 \in X_{r_0}$. Hence $\|y_0 - z_0\| \leq t_0 + \eta < d(y_0, X_{r_0})$, which is absurd.

Now, by the local conical structure of Y and our choice of η, we may assume that, after decreasing a if necessary, for every $r \in (r_0 - a, r_0 + a)$ there exists $y_r \in Y_r \setminus X_r$ such that $\|y_0 - y_r\| < \eta$.

Then, for any $r \in (r_0 - a, r_0 + a)$ and for any $x \in X_r$, we have

$$\|x - y_r\| \geq | \ \|x - y_0\| - \|y_0 - y_r\| \ | \geq | \ d(y_0, X_r) - \|y_0 - y_r\| \ | \geq t_0 ,$$

which yields that $y_r \in W(r, t_0)$ and therefore $y_r \in W(r, t)$ for all $t \leq t_0$.

Moreover $\|y_r - x_0\| \leq \|y_r - y_0\| + \|y_0 - x_0\| < \eta + \epsilon/4 < \epsilon/2$.

For i sufficiently large, $t_i < t_0$, $r_i \in (r_0 - a, r_0 + a)$ and $\|x_i - x_0\| < \epsilon/2$. So

$$\epsilon < d(x_i, W(r_i, t_i)) \leq \|x_i - y_{r_i}\| \leq \|x_i - x_0\| + \|x_0 - y_{r_i}\| < \epsilon$$

which is a contradiction.

As announced, we can now apply Lemma 2.1 to the functions t and $\Phi(r, t)$ on K; so there exists a positive constant $\beta < 1$ such that

$$\Phi(r, t) < t^\beta \qquad \forall\, (r, t) \in K, \ \ t \neq 0 .$$

It follows from the inequality immediately above that, for any $(r, t) \in K$, $t \neq 0$, and for any $x \in X_r$, the set $B(x, t^\beta) \cap W(r, t)$ is not empty, so that

$$B(x, t^\beta) \cap Y_r \not\subseteq \bigcup_{z \in X_r} B\,(z, t) \cap Y_r . \tag{2.5.1}$$

Since $V(\rho) = \{0\}$, by Lemma 2.1 we find a positive constant $\overline{q}$ such that

$$\rho(r) \geq r^{\overline{q}} \qquad \forall\, r \in [0, R] .$$

Therefore $(r, r^q) \in K$ for any $q \geq \overline{q}$; substituting $t = r^q$ in (2.5.1) yields the desired result. □

As a consequence of the previous result we get the possibility of finding a closed semialgebraic subset of Y sufficiently "close" to Y, but meeting the fixed subset X only in the origin. In fact:

Corollary 2.6. *Let $X \subset Y \subseteq \mathbb{R}^n$ be closed semialgebraic sets with $O \in$* Der(X) *and $\overline{Y \setminus X} = Y$. For any $s \in \mathbb{R}$, $s \geq 1$, there exists a positive R and a closed semialgebraic set $\Gamma \subseteq Y$ such that $\Gamma \cap X \cap B(O, R) = \{O\}$ and $Y \leq_s \Gamma$.*

Proof. Consider the positive constants β, R and $\overline{q}$ obtained by applying Lemma 2.5 to X and Y, and choose a rational $q > \max\{\overline{q}, \frac{s+1}{\beta}\}$. Let

$$\Omega = \bigcup_{x \in X} \left(B(x, \|x\|^q) \cap Y \cap S_{\|x\|}\right) = \mathcal{U}(X, q) \cap Y \qquad \text{and} \qquad \Gamma = Y \setminus \Omega .$$

Clearly Γ is a closed semialgebraic set such that $\Gamma \cap X \cap B(O, R) = \{O\}$. Let us prove that $Y \leq_s \Gamma$, i.e. for all $y \in Y \cap B(O, R)$, $y \neq O$, there exists $z \in \Gamma$ with $\|z\| = \|y\|$ and $\|z - y\| < \|z\|^\sigma$ for some $\sigma > s$. This is evident if $y \in \Gamma$, so assume $y \in \Omega$ and set $\|y\| = r$. Then there exists $x \in X \cap S_r$ such that $\|y - x\| < r^q$. By Lemma 2.5 there exists $z \in B(x, r^{q\beta}) \cap S_r \cap Y$ such that $z \in \Gamma$. Hence, if r is small enough, we get

$$\|y - z\| \leq \|y - x\| + \|x - z\| < r^q + r^{q\beta} < 2r^{q\beta} < r^{s+1} ,$$

so our thesis holds with $\sigma = s + 1$. □

Let us conclude this section by remarking that s-equivalence commutes with set unions, which will enable us to start the proof of the Approximation Theorem by a simplifying reduction step:

LEMMA 2.7. *Let A, A', B and B' be closed semialgebraic subsets of $\mathbb{R}^n$, and assume that O is not isolated in any of them.*

(1) *If $A \leq_s B$ and $A' \leq_s B'$, then $A \cup A' \leq_s B \cup B'$.*
(2) *If $A \sim_s B$ and $A' \sim_s B'$, then $A \cup A' \sim_s B \cup B'$.*

PROOF. The result is an immediate consequence of Proposition 2.4. □

3. – Proof of the Approximation Theorem

This section is entirely devoted to prove Theorem 1.4.

By Lemma 2.7, it will be sufficient to prove our result for semialgebraic subsets of $\mathbb{R}^n$ of the kind

$$A = \{f = 0, h_1 \geq 0, \dots, h_k \geq 0\}$$

with $\dim A < n$.

Since s-equivalence depends only on the germs at O, we can omit from the presentation of A all the inequalities $h_i \geq 0$ such that $h_i(O) > 0$, i.e. we can suppose that $h_i(O) = 0$. We can also assume that $h_i|_A \not\equiv 0$ for each i.

For the same reason we are allowed to identify a semialgebraic set with a realization of its germ at the origin in a suitable ball $B(O, R)$ which we will shrink without mention whenever necessary.

The proof will be done by induction on k. If $k = 0$, the result is trivial. So let $k > 0$ and assume the theorem holds for any closed semialgebraic subset of $\mathbb{R}^n$ which can be presented as above using at most $k-1$ polynomial inequalities.

Let us consider at first the case when $k \geq 2$.

Let $Z = \bigcup_{i=2}^k V(h_i)$ and consider the semialgebraic set $A' = \overline{A \setminus Z}$. Since $A' = \overline{A' \setminus Z}$, Corollary 2.6 applied to $X = A' \cap Z$ and $Y = A'$ shows that there exists a closed semialgebraic subset $\Gamma \subseteq A'$ such that $\Gamma \cap Z = \{O\}$ and $A' \leq_s \Gamma$.

Let $\zeta : A \to \mathbb{R}$ be the function defined by $\zeta(x) = d(x, Z)$ for every $x \in A$. The function ζ is semialgebraic, continuous and $V(\zeta) = A \cap Z$; moreover, if $x \in A$ and $h_i(x) > 0$ for each $i = 2, \dots, k$, then $\zeta(x) > 0$ and $h_i|_{B(x,\zeta(x))} > 0$ for each $i = 2, \dots, k$. Since $V(\zeta) \cap \Gamma = A \cap Z \cap \Gamma = \{O\}$, by Lemma 2.1 there exists a positive $l \in \mathbb{Q}$ such that $\zeta(x) \geq \|x\|^l$ for all $x \in \Gamma$. Hence, for each $i = 2, \dots, k$,

$$h_i|_{B(x,\|x\|^l)} > 0 \qquad \forall\, x \in \Gamma \setminus \{O\}\,. \tag{1}$$

Let $A_1 = \{f = 0, h_1 \geq 0\}$. Corollary 2.6, applied to $X = A_1$ and $Y = \mathbb{R}^n$, assures that there exists a closed semialgebraic subset $\Gamma_1 \subseteq \mathbb{R}^n$ such that $\Gamma_1 \cap A_1 = \{O\}$ and $\mathbb{R}^n \leq_{s+l} \Gamma_1$.

In the case $k = 1$, evidently $Z = \emptyset$, hence $A = A' = A_1$ and we just need the second application of Corollary 2.6, where conventionally we set $l = 0$.

Set $\Lambda = \{h_2 \geq 0, \dots, h_k \geq 0\}$ if $k \geq 2$ and $\Lambda = \mathbb{R}^n$ if $k = 1$.

We want to construct a polynomial function g on $\mathbb{R}^n$ such that

$$A' \leq_s V(g) \cap \Lambda \leq_s A\,.$$

We will need to consider the open semialgebraic set $\mathcal{U} = \mathcal{U}(A, \sigma)$, where $\sigma \in \mathbb{Q}$ and $\sigma > s$. Evidently

$$V(f) \cap \{h_1 \geq 0\} \cap \Gamma_1 = A_1 \cap \Gamma_1 = \{O\}$$

and

$$V(f) \cap \{h_1 \geq 0\} \cap \Lambda \cap \mathcal{C}\mathcal{U} = A \cap \mathcal{C}\mathcal{U} = \{O\}\,,$$

where $\mathcal{C}\mathcal{U} = \mathbb{R}^n \setminus \mathcal{U}$. Therefore, if we set

$$W = (\Gamma_1 \cup (\Lambda \cap \mathcal{C}\mathcal{U})) \cap \{h_1 \geq 0\}\,,$$

W is a closed semialgebraic set such that $V(f) \cap W = \{O\}$.

So by Lemma 2.1 there exists an odd $m \in \mathbb{N}$ such that $f(x)^2 \geq |h_1(x)|^m$ for all $x \in W$ and $f(x)^2 > |h_1(x)|^m$ for all $x \in W \setminus \{O\}$.

Define $g = f^2 - {h_1}^m$.

By construction g is strictly positive on $W \setminus \{O\}$ and on $\{h_1 < 0\}$, hence g is strictly positive on Γ_1 and on $\Lambda \cap \mathcal{C}\mathcal{U}$. This latter fact implies that $V(g) \cap \Lambda \subseteq \mathcal{U} \cup \{O\}$ and therefore that $V(g) \cap \Lambda \leq_s A$.

In order to prove that $A' \leq_s V(g) \cap \Lambda$, since $A' \leq_s \Gamma$, we need only to prove that $\Gamma \leq_s V(g) \cap \Lambda$.

Let $x \in \Gamma$ and set $\|x\| = r$. We assume at first that $h_1(x) > 0$, so that $g(x) < 0$. Since $\mathbb{R}^n \leq_{s+l} \Gamma_1$, there exists $z \in \Gamma_1$ such that $\|z\| = r$ and $\|x - z\| < r^\eta$ with $\eta > s + l$.

As g is strictly positive on Γ_1, $g(z) > 0$. So, by the Intermediate Value Theorem on $B\,(x, r^\eta) \cap S_r$, there exists $w \in B\,(x, r^\eta) \cap S_r$ such that $g(w) = 0$.

Moreover if $k \geq 2$, as $\eta > l$, by (1) one has that $h_i(w) > 0$ for any $i = 2, \dots, k$, which means that $w \in V(g) \cap \Lambda \cap S_r$; hence $x \in \mathcal{U}(V(g) \cap \Lambda, \eta)$.

The same conclusion holds also if $h_1(x) = 0$, because in this case $g(x) = 0$ and hence $x \in V(g) \cap \Lambda$.

We have thus proved that $\Gamma \subseteq \mathcal{U}(V(g) \cap \Lambda, \eta)$ and therefore, since $\eta > s$, that $\Gamma \leq_s V(g) \cap \Lambda$ by Proposition 2.4.

Now, if $k = 1$ the theorem is proved since we have $A = A' \leq_s V(g) \leq_s A$, i.e. $A \sim_s V(g)$.

If $k \geq 2$, note that $V(g) \cap \Lambda$ is described by means of $k-1$ polynomial inequalities; so, using the inductive hypothesis, there exists a polynomial function g' such that $V(g) \cap \Lambda \sim_s V(g')$ and therefore $A' \leq_s V(g') \leq_s A$.

In order to conclude the proof for A, note that $A = A' \cup (A \cap Z)$ and that $A \cap Z = \bigcup_{i=2}^{k} B_i$, where

$$B_i = \{f^2 + {h_i}^2 = 0, h_1 \geq 0, \dots, h_{i-1} \geq 0, h_{i+1} \geq 0, \dots, h_k \geq 0\}.$$

By the inductive hypothesis, for any $i = 2, \dots, k$ there exists a polynomial function φ_i such that $B_i \sim_s V(\varphi_i)$; hence by Lemma 2.7

$$A \cap Z \sim_s V(\varphi) \qquad \text{where } \varphi = \varphi_2 \cdot \ldots \cdot \varphi_k .$$

Consequently

$$A = A' \cup (A \cap Z) \leq_s V(g') \cup V(\varphi) = V(g'\varphi).$$

Conversely $V(g') \cup V(\varphi) \leq_s A \cup (A \cap Z) = A$.

So the algebraic set $V = V(g'\varphi)$ is s-equivalent to A. □

REFERENCES

[B-C-R] J. Bochnak – M. Coste – M. F. Roy, “Géométrie algébrique réelle”, Springer-Verlag, 1987.

[B1] L. Bröcker, *Families of semialgebraic sets and limits*, In: “Real Algebraic Geometry” (Rennes 1991), M. Coste – L. Mahé – M.-F. Roy (eds.), Lecture Notes in Math., 1524, Springer-Verlag, Berlin, 1992, pp. 145-162.

[B2] L. Bröcker, *On the reduction of semialgebraic sets by real valuations*, in: “Recent advances in real algebraic geometry and quadratic forms”, Contemp. Math., 155, Amer. Math. Soc., Providence, RI, 1994, pp. 75-95.

[F-F-W] M. Ferrarotti – E. Fortuna – L. Wilson, *Real algebraic varieties with prescribed tangent cones*, Pacific J. Math. **194** (2000), 315-323.

[K-R] K. Kurdyka – G. Raby, *Densité des ensembles sous-analytiques*, Ann. Inst. Fourier (Grenoble) **39** (1989), 753-771.

Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino, Italy
ferraro@dm.unipi.it

Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2
56127 Pisa, Italy
fortuna@dm.unipi.it

Department of Mathematics
University of Hawaii
Honolulu, HI 96822, USA
les@math.hawaii.edu