

Higher Regularity for Nonlinear Oblique Derivative Problems in Lipschitz Domains

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Abstract. There is a long history of studying nonlinear boundary value problems for elliptic differential equations in a domain with sufficiently smooth boundary. In this paper, we show that the gradient of the solution of such a problem is continuous when a directional derivative is prescribed on the boundary of a Lipschitz domain for a large class of nonlinear equations under weak conditions on the data of the problem. The class of equations includes linear equations with fairly rough coefficients as well as Bellman equations.

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Introduction

It is well known that the boundary value problem

$$(0.1) \quad F(x, u, Du, D^2u) = 0 \text{ in } \Omega, \quad G(x, u, Du) = 0 \text{ on } \partial\Omega$$

has a smooth solution under appropriate conditions on the functions F and G for domains $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. The main features of interest are that F is elliptic, which means that the matrix derivative $\partial F(x, z, p, r)/\partial r$ is positive-definite, and that G is oblique, which means (for domains with smooth boundary) that the angle between $\partial G(x, z, p)/\partial p$ and the unit interior normal is less than $\pi/2$ on $\partial\Omega$. In [23] it was shown that this problem has a $C^{2,\alpha}$ solution for some $\alpha \in (0, 1)$ if $\partial\Omega$ is sufficiently smooth and if F and G satisfy natural conditions about the growth of certain combinations of the derivatives of F and G . (See this reference for a more complete description of the hypotheses.) Subsequently several authors [4], [34] improved that work by decreasing or modifying the hypotheses on (the derivatives of) F and G , but they continued to assume at least $\partial\Omega \in C^2$. For linear problems,

it is known [17], [28] that all solutions are $C^{1,\alpha}$ even if the domain is only Lipschitz. The main purpose of this paper is to show that the gradient of the solution of certain nonlinear problems is continuous up to the boundary when the domain is Lipschitz; we primarily study linear boundary conditions although some nonlinear conditions will be considered.

An important element of our investigation is the assumption of Dini continuity of certain coefficients rather than the usual assumption of Hölder continuity. There are several reasons for introducing this hypothesis. Except for a few simple technical calculations, there are very few places in the development of the theory in which the proofs for Hölder continuity are significantly easier. The Dini condition is known to be the optimal one for continuity of the gradient (or even boundedness) up to the boundary. In [17], a definition of the linear boundary condition $\beta \cdot Du = g$ was given that applies to (suitable) functions which are not globally C^1 , specifically, we define $\beta \cdot Du$ to be the scaled directional derivative

$$\beta(x) \cdot Du(x) = \lim_{h \rightarrow 0^+} \frac{u(x + h\beta(x)) - u(x)}{h}.$$

The introduction of Dini continuity will allow us to improve the results in [17] concerning this situation.

We recall first (see the discussion in [22]) that a vector β is oblique at $x_0 \in \partial\Omega$ if there is an open cone with axis β and vertex x_0 which lies inside Ω . A vector field β defined on some subset Σ of $\partial\Omega$ is oblique on Σ if $\beta(x_0)$ is oblique at x_0 for any $x_0 \in \Sigma$. We also say that a nonlinear function $G(x, z, p)$ is oblique on Σ if the vector field $\partial G(x, z, p)/\partial p$ is oblique on Σ for any $(z, p) \in \mathbb{R} \times \mathbb{R}^n$.

An interesting element of our results is the effect of the geometry of the domain on the regularity of the solution. If all other data of the problem are sufficiently smooth, then we show at the start of Section 2 that there is a positive constant σ_0 , determined only by the opening angle of an exterior cone to Ω and the modulus of ellipticity of the function F , such that the gradient of the solution is Hölder continuous with any exponent less than σ_0 ; in particular, if the domain is convex, the exponent can be taken arbitrarily in the range $(0, 1)$. On the other hand, the Hölder norm also depends on the domain through the opening angle of the cone in the definition of obliqueness. In particular, this norm depends on the full Lipschitz norm of $\partial\Omega$ and on the interaction between the boundary condition and the boundary. This behavior complements the Hölder continuity results in [22], which provide a Hölder exponent and norm for solutions determined only by the angle of the cone in the definition of obliqueness.

We begin in Section 1 with some properties of functions connected with our Dini hypothesis; for the most part, these properties are analogs of the corresponding properties for power functions. Using the results of that section and the definition of obliqueness given above, we prove a basic modulus of

continuity estimate for the gradient in Section 2 for a restricted class of nonlinear functions F and linear functions G . This result improves those in [17] and [28] by considering nonlinear equations, by removing the second derivatives estimates used in those works, and by assuming only Dini continuity of the appropriate coefficients. Next, we prove existence and regularity results for some mixed boundary value problems in Section 3. In Section 4, we use a perturbation argument based on the one in [3] to prove that our results can be applied to a wide class of nonlinear problems. Because we are unable at present to handle nonlinear boundary conditions or quasilinear equations with leading coefficient depending on the gradient, we compensate by considering equations with quadratic growth in the gradient for the lower order terms and with coefficients which lie in the class of function spaces used by Apushkinskaya and Nazarov (see [1]). It was shown in [17, Theorem 5.1] that solutions of linear problems are in $C^{2,\alpha}$ if $\partial\Omega \in C^{1,\alpha}$, and Safonov [30] proved a nonlinear version of this result. In Section 5 we use a variant of our argument to provide an alternative proof of Safonov's result. We return in Section 6 to a consideration of nonlinear equations with the linear boundary condition $\beta \cdot Du = g$ with g merely continuous. Under suitable hypotheses on F and β , we show that problem (0.1) has a unique solution which satisfies the boundary condition in the sense indicated above. Such a result was proved for linear problems (but under somewhat stronger hypotheses on the coefficients in the problem) in domains with $C^{1,\alpha}$ boundary by Giraud [8]; a version for linear problems in Lipschitz domains was asserted (but not proved correctly) by Nadirashvili [27] and a correct proof in this case appears in [17]. Finally, the analogous parabolic results are stated, and their proofs sketched in Section 7. These results improve those for linear equations in [19], [2], and [24].

We thank the referee for making many useful suggestions which improved the exposition. In particular, Section 3 was completely rewritten for clarity.

1. – Dini functions and properties of continuous functions

In this section, we study properties of various continuous functions (see also [16, Section 1] and the references therein). We recall that a continuous increasing function ζ defined on $[0, 1]$ with $\zeta(0) = 0$ is *Dini* if the function $I(\zeta)$ defined by

$$I(\zeta)(s) = \int_0^s \frac{\zeta(t)}{t} dt$$

is finite for $s \in (0, 1)$. In addition, we say that a continuous, increasing function defined on $[0, 1]$ is δ -*decreasing* for some $\delta \in (0, 1]$ if

$$\frac{\zeta(s)}{s^\delta} \leq \frac{\zeta(t)}{t^\delta}$$

for all s and t in $(0, 1]$ with $s \geq t$ and that ζ is δ -decaying if ζ is α -decreasing for some $\alpha \in (0, \delta)$. We note that if $\zeta(s) = As^\alpha$ for positive constants A and $\alpha \leq 1$, then ζ is Dini and $I(\zeta)(s) = (A/\alpha)s^\alpha$. Furthermore, this ζ is δ -decreasing for $\delta \in [\alpha, 1]$ and δ -decaying for $\delta \in (\alpha, 1]$.

We pause to compare the concept of δ -decaying functions to condition (*) from [9]:

$$(1.1) \quad \lim_{\tau \rightarrow 0^+} \sup_{0 < \rho \leq 1/2} \frac{\tau^\delta[\rho^\delta + \zeta(\rho)]}{(\tau\rho)^\delta + \zeta(\tau\rho)} = 0.$$

If ζ is δ -decaying, we have $\zeta(1) \leq \zeta(\rho)/\rho^\alpha$ for some $\alpha < \delta$, and therefore

$$\frac{\tau^\delta[\rho^\delta + \zeta(\rho)]}{(\tau\rho)^\delta + \zeta(\tau\rho)} \leq \left(1 + \frac{1}{\zeta(1)}\right) \tau^{\delta-\alpha},$$

which goes to zero, uniformly with respect to ρ , as $\tau \rightarrow 0$ so (1.1) holds. Conversely, if (1.1) holds, let $\tau \in (0, 2^{-1/\delta})$ be such that

$$\frac{\tau^\delta[\rho^\delta + \zeta(\rho)]}{(\tau\rho)^\delta + \zeta(\tau\rho)} \leq \frac{1}{2},$$

and choose α so that $\tau^\alpha = 2\tau^\delta$, noting that $\alpha \in (0, \delta)$. Then simple algebra implies that $\zeta(\rho) \leq \tau^{-\alpha}\zeta(\tau\rho)$, and hence

$$\zeta(\rho) \leq \tau^{-k\alpha}\zeta(\tau^k\rho)$$

for any nonnegative integer k . Now choose $s > t$ in $(0, 1/2]$ and let k be a nonnegative integer such that $\tau^{k+1} \leq t/s \leq \tau^k$. Then

$$(1.2) \quad \frac{\zeta(s)}{s^\alpha} \leq \frac{\zeta(\tau^{-k-1}t)}{(\tau^{-k}t)^\alpha} \leq \tau^{-\alpha} \frac{\zeta(t)}{t^\alpha}.$$

Now we define ζ_1 by

$$\zeta_1(s) = s^\alpha \sup_{t \geq s} \frac{\zeta(t)}{t^\alpha}.$$

It's easy to check that ζ_1 is continuous, α -decreasing, and increasing with $\zeta \leq \zeta_1$. Moreover, (1.2) implies that $\zeta_1 \leq \tau^{-\alpha}\zeta$.

We now perform some calculations which will simplify later proofs. We define the operator J acting on continuous, increasing functions ζ by

$$J(\zeta)(s) = \frac{1}{s} \int_0^s \zeta(t) dt.$$

For $\alpha \in (0, 1]$, we define the operator J_α acting on α -decreasing, Dini function by

$$J_\alpha(\zeta)(s) = I(J(\zeta))(s^{1/\alpha}).$$

LEMMA 1.1. *Let ζ be a continuous increasing function with $\zeta(0) = 0$ which is α -decreasing for some $\alpha \in (0, 1]$.*

- (a) *Then $J(\zeta)$ is continuous, increasing, and α -decreasing. In addition, $J(\zeta)(0) = 0$ and $J(\zeta) \leq \zeta \leq 2J(\zeta)$ on $[0, 1]$.*
- (b) *If ζ is Dini, then $J_\alpha(\zeta)$ is increasing and concave.*
- (c) *If ζ is Dini, then $I(\zeta)$ is α -decreasing.*

PROOF. Part (a) follows by elementary means; see, for example, [31, Section 5].

For part (b), we have that $\zeta_\alpha = J_\alpha(\zeta)$ is increasing on $[0, 1]$ and C^2 on $(0, 1]$ by construction. If $0 < s \leq 1$, then

$$\zeta'_\alpha(s) = \frac{1}{\alpha} \frac{J(\zeta)(s^{1/\alpha})}{(s^{1/\alpha})^\alpha},$$

so ζ'_α is positive and decreasing and hence ζ_α is increasing and concave. To prove (c), suppose $s > t$. Then the change of variables $\tau = su/t$ yields

$$I(\zeta)(s) = \int_0^t \frac{\zeta(su/t)}{u} du \leq \frac{t^\alpha}{s^\alpha} \int_0^t \frac{\zeta(u)}{u} du = \frac{t^\alpha}{s^\alpha} I(\zeta)(t),$$

which shows that $I(\zeta)$ is α -decreasing. □

We also introduce some weighted seminorms, analogous to those defined on page 96 of [7], for such a ζ . To keep the notation consistent with that for Hölder seminorms, we define the function Z (to be read “capital ζ ”) by $s^{Z(s)} = \zeta(s)$. Also for a domain $\Omega \in \mathbb{R}^n$, we set

$$\Omega[x, R] = \{y \in \Omega : |x - y| < R\}, \quad \Sigma[x, R] = \{y \in \partial\Omega : |x - y| < R\};$$

and we suppress x from the notation when $x = 0$. We also define $d^*(x) = R - |x|$. The seminorms are then defined by

$$[u]_1^* = \sup_{x \in \Omega[R]} \{d^*(x) |Du(x)|\},$$

$$[u]_{1+Z}^* = \sup_{\substack{x \neq y \text{ in } \Omega[R] \\ |x-y| \leq d^*(x)/2}} \left\{ d^*(x) \frac{|Du(x) - Du(y)|}{\zeta(|x - y|/d^*(x))} \right\}.$$

We also recall (see, for example, [7, (4.5)]) that

$$|u|_0 = \sup_{x \in \Omega[R]} |u(x)|.$$

With these definitions and the notation $x' = (x^1, \dots, x^{n-1})$, we have an interpolation inequality analogous to [7, Lemma 6.34].

LEMMA 1.2. *Suppose ζ is a continuous increasing function on $[0, 1]$ with $\zeta(0) = 0$. Suppose also that there are a function ω , defined for $|x'| < R$, and a nonnegative constant ω_0 such that*

$$(1.3) \quad \Omega[R] = \{x : x^n > \omega(x'), |x| < R\}, \quad |\omega(x') - \omega(y')| \leq \omega_0|x' - y'|$$

for all x' and y' with $|x'|, |y'| < R$, and set $\kappa = 4(1 + \omega_0^2)^{1/2}$. Then for any $\varepsilon \in (0, \zeta(1/2))$, we have

$$(1.4) \quad [u]_1^* \leq \varepsilon [u]_{1+Z}^* + \frac{\kappa}{\zeta^{-1}(\varepsilon)} |u|_0.$$

PROOF. If $[u]_1^* = 0$, there is nothing to prove. Otherwise, choose x_0 such that $d^*(x_0)|Du(x_0)| \geq (1/2)[u]_1^*$ and set $A = d^*(x_0)$. For $\mu \in (0, 1/2]$ to be further specified, set $y = (x', x'' + (A\mu/2))$, and

$$x_1 = y + \frac{\mu A}{\kappa} \frac{Du(x_0)}{|Du(x_0)|}, \quad x_2 = y - \frac{\mu A}{\kappa} \frac{Du(x_0)}{|Du(x_0)|}.$$

It follows that $|x_1 - x_2| = 2\mu A/\kappa$, that the line segment joining x_1 and x_2 lies inside $\Omega[R]$, and that $x_1 - x_2$ is parallel to $Du(x_0)$. The mean value theorem gives a point \bar{x} on the line segment joining x_1 and x_2 such that

$$Du(\bar{x}) \cdot (x_1 - x_2) = u(x_1) - u(x_2),$$

and therefore

$$\begin{aligned} |Du(x_0)| &= \frac{x_1 - x_2}{|x_1 - x_2|} \cdot Du(x_0) \\ &= \frac{x_1 - x_2}{|x_1 - x_2|} \cdot (Du(x_0) - Du(\bar{x})) + \frac{x_1 - x_2}{|x_1 - x_2|} \cdot Du(\bar{x}) \\ &\leq \frac{\zeta(|x_0 - \bar{x}|/d^*(x_0))}{d^*(x_0)} [u]_{1+\zeta}^* + \frac{2}{|x_1 - x_2|} |u|_0 \\ &\leq \frac{\zeta(\mu)}{A} [u]_{1+\zeta}^* + \frac{\kappa}{\mu A} |u|_0. \end{aligned}$$

The desired result follows from this inequality by simple rearrangement with $\mu = \zeta^{-1}(\varepsilon)$. \square

It will be useful in our application of Lemma 1.2 to note that, if the supremum in the definition of $[\cdot]_{1+\zeta}^*$ is only taken over all $x \neq y$ in $\Omega[R]$ with $|x - y| \leq \rho|x|$ for some $\rho \in (0, 1/2)$, then (1.4) holds for $\varepsilon < \zeta(\rho)$. In particular, we can use this observation in our next lemma on the modulus of continuity estimate for the gradient in terms of an oscillation estimate.

LEMMA 1.3. *Let $\tau < 1$ and $\rho_1 < 1$ be positive constants and let ζ be a Dini function which is 1-decreasing. Let Ω satisfy (1.3) and let u be a bounded, uniformly continuous function in $\Omega[R]$. Suppose that, for each $x \in \Sigma[R]$, there is a sequence of linear polynomials $(P_k(\cdot; x))$ such that*

$$(1.5) \quad \sup_{\Omega[x, \tau^k \rho_1 d^*(x)]} (u - P_k(\cdot; x)) \leq \zeta(\tau^k \rho_1) \tau^k \rho_1 d^*(x).$$

Then Du exists on $\Sigma[R]$ and

$$(1.6) \quad d^*(x)|Du(x) - Du(y)| \leq C(n, \tau, \omega_0, \rho_1) I(\zeta) \left(\frac{|x - y|}{d^*(x)} \right)$$

for all x and y in $\Sigma[R]$ with $|x - y| \leq \rho_1 d^*(x)$.

PROOF. Let φ be a nonnegative, $C^2(\mathbb{R}^n)$ function with L^1 norm equal to 1 supported in the ball of radius $1/(2(\omega_0^2 + 1)^{1/2})$ centered at $(0, \dots, 0, 1/2)$. For any $L^1(\mathbb{R}^n)$ function w , we define the mollification W by

$$W(x; \delta) = \int_{\mathbb{R}^n} w(x - \delta y)\varphi(y) dy.$$

for $x \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$. We extend u to be zero outside of $\Omega[R]$ and use U to denote the mollification of this extension.

For any $\mu \in (0, \rho_1 d^*(x))$, there is a nonnegative integer k such that $\tau^{k+1} \leq \mu/(\rho_1 d^*(x)) \leq \tau^k$, and, by noting that U and the corresponding mollification of $u - P_k$ have the same second derivatives, it follows that

$$(1.7) \quad |D^2U(x, \mu)| \leq C(n, \tau)\zeta \left(\frac{\mu}{d^*(x)} \right) \frac{1}{\mu d^*(x)}$$

for $x \in \Sigma[R]$ and $\mu \in (0, \rho_1 d^*(x))$. Elementary integration shows that U is C^1 with respect to x and μ for $x \in \Sigma[R]$ and $0 \leq \mu < \rho_1 d^*(x)$. In particular, if x and y are as in the conclusion to this theorem, we observe that

$$\begin{aligned} |Du(x) - Du(y)| &\leq |DU(x, 0) - DU(x, |x - y|)| \\ &\quad + |DU(x, |x - y|) - DU(y, |x - y|)| \\ &\quad + |DU(y, 0) - DU(y, |x - y|)|. \end{aligned}$$

From (1.7), we have

$$|DU(x, 0) - DU(x, |x - y|)| \leq C(n, \tau, \rho_1) \int_0^{|x-y|} \frac{\zeta(\mu/d^*(x))}{\mu d^*(x)} d\mu,$$

and the change of variables $s = \mu/d^*(x)$ shows that this integral is just $I(\zeta)(|x - y|/d^*(x))/d^*(x)$. The other two terms on the right hand side of this inequality are estimated similarly (noting that $\zeta \leq I(\zeta)$) to infer (1.6). \square

We remark that this lemma also follows from the Main theorem from [10], which is somewhat more general. See also Lemma 4 from [9].

2. – The main estimate

We begin by studying a simple model problem. Let F be a function defined on \mathbb{S}^n , the set of all $n \times n$ symmetric matrices, and suppose that there are positive constants Λ and λ such that

$$(2.1) \quad \lambda|\xi|^2 \leq \frac{\partial F}{\partial r_{ij}}(r)\xi_i\xi_j \leq \Lambda|\xi|^2.$$

For simplicity, we set $F^{ij} = \partial F / \partial r_{ij}$ and we write F_r for the matrix $[F^{ij}]$. In addition, we always assume $0 \in \partial\Omega$, and we consider the boundary value problem

$$(2.2) \quad F(D^2u) = 0 \text{ in } \Omega[R], \quad \beta \cdot Du = g(x) \text{ on } \Sigma[R],$$

where β is oblique on $\Sigma[R]$.

Our starting point is a barrier construction given in [26, Theorem 3] (see also [26, Section 8] for the case of two dimensions, which is not presented as part of Theorem 3). Miller determines a function σ_0 , defined on $\mathbb{Z}^+ \times [1, \infty) \times (0, \pi)$ such that, if $\sigma \in (0, \sigma_0(n, \mu, \theta_0))$ and K is an infinite cone with vertex 0 and semi-vertex angle $\theta_0 \in (0, \pi)$, then there is a function w_1 such that

$$a^{ij} D_{ij} w_1(x) \leq 0, \quad |x|^\sigma \leq w_1(x) \leq c_1(n, \mu, \theta_0, \sigma) |x|^\sigma$$

for any $x \in \mathbb{R}^n \setminus K$ and any matrix $[a^{ij}]$ such that

$$(2.3) \quad |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq \mu |\xi|^2.$$

(The function w_1 depends on the cone, but c_1 does not.) Moreover, for fixed n and μ , σ_0 is a continuous, strictly increasing function of θ_0 with $\sigma_0(n, \mu, \pi/2) = 1$. To simplify our terminology, we say that a domain Ω satisfies an *exterior θ_0 -cone condition* at $x_0 \in \partial\Omega$ if there is an infinite cone with vertex x_0 and semi-vertex angle θ_0 which does not intersect Ω . When n , μ and θ_0 are clear from the context, we just write σ_0 . Note that we can take $\sigma_0 = 1$ if Ω is convex or, more generally, if it satisfies a uniform exterior sphere condition; in fact, we can take $\sigma_0 = 1$ if $\partial\Omega \in C^1$.

It will be important in later sections to consider a special class of functions in connection with this constant σ_0 . We say that ζ is a D_1 function if ζ is a Dini function with $\zeta(1) = 1$ which is σ_0 -decaying. If ζ is a D_1 function and $I(\zeta)$ is also Dini, we say that ζ is a D_2 function.

The key observation (see [17, p. 1190]) in proving regularity is that $v = D_n u$ is the solution of a suitable Dirichlet problem when u is sufficiently smooth, so (compare with [17, Lemma 2.1]) it satisfies a Hölder estimate. We present this estimate (under minimal smoothness hypotheses on u) in a form which will be useful later.

LEMMA 2.1. *Suppose $u \in C^1(\overline{\Omega[R]}) \cap W_{loc}^{2,n}(\Omega[R])$ satisfies (2.2) for some constant unit vector β , and suppose $\Omega[R]$ satisfies an exterior θ_0 -cone condition at 0 . Let ζ be an α -decreasing function for some $\alpha \in (0, \sigma_0)$ with $\zeta(1) = 1$, and suppose there is a nonnegative constant G_0 such that*

$$(2.4) \quad |g(y) - g(0)| \leq G_0 \zeta(|y|/R)$$

for all $y \in \Sigma[R]$. Then there is a constant C , determined only by n , α , Λ/λ , and θ_0 , such that

$$(2.5) \quad |\beta \cdot Du(x) - g(0)| \leq C[G_0 + \sup |\beta \cdot Du|] \zeta(|x|/R)$$

for all $x \in \Omega[R]$.

PROOF. We set $v = \beta \cdot Du - g(0)$ and, for $h > 0$, we define

$$v_h(x) = \frac{u(x + h\beta) - u(x)}{h} - g(0).$$

Then, for each sufficiently small positive h , there is a matrix $[a_h^{ij}]$ satisfying (2.3) with $\mu = \Lambda/\lambda$ such that $a_h^{ij} D_{ij} v_h = 0$ in $\Omega[R - h]$. In addition, for any $\varepsilon > 0$, there is a positive h_ε such that, for any $h \in (0, h_\varepsilon)$, $|v_h - v| \leq \varepsilon$ in $\Omega[R - h]$. We now fix $\sigma = (\alpha + \sigma_0)/2$ and write c_1 and w_1 for this choice of σ .

Now fix ε , let $\rho \in (0, R)$, choose $h < \min\{h_\varepsilon, R - \rho\}$, and set

$$w_h^\pm = \pm v_h + \varepsilon + G_0 \zeta \left(\frac{\rho}{R} \right) + \left[\sup_{\Omega[\rho]} |\beta \cdot Du - g(0)| \right] \frac{w_1}{\rho^\sigma}.$$

It follows that

$$a_h^{ij} D_{ij} w_h^\pm \leq 0 \text{ in } \Omega[\rho], \quad w_h^\pm \geq 0 \text{ on } \partial(\Omega[\rho]),$$

by virtue of parts (a) and (c) of Lemma 1.1. The maximum principle implies that $w_h^\pm \geq 0$ in $\Omega[\rho]$. Sending $h \rightarrow 0$ and then $\varepsilon \rightarrow 0$ yields

$$\sup_{\Omega[\tau\rho]} |\beta \cdot Du - g(0)| \leq c_1 \tau^\sigma \sup_{\Omega[\rho]} |\beta \cdot Du - g(0)| + G_0 \zeta \left(\frac{\rho}{R} \right).$$

We now invoke [16, Lemma 5.3] with $\delta = (\alpha + \sigma)/2$ and τ chosen so that $c_1 \tau^\sigma \leq \tau^\delta$ to infer that

$$\sup_{\Omega[\rho]} |\beta \cdot Du - g(0)| \leq C \left[G_0 + \sup_{\Omega[R]} |\beta \cdot Du - g(0)| \right] \zeta \left(\frac{\rho}{R} \right)$$

and therefore

$$|\beta \cdot Du(x) - g(0)| \leq C \left[G_0 + \sup_{\Omega[R]} |\beta \cdot Du - g(0)| \right] \zeta \left(\frac{|x|}{R} \right).$$

The proof is completed by noting that $|g(0)| \leq \sup_{\Omega[R]} |\beta \cdot Du|$. □

Our next step is to use this estimate to infer a corresponding estimate for the full gradient. Such an estimate for linear equations (and ζ a power function) has been proved using some estimates for second derivatives of u ([17, Lemma 2.3] and [28, Section 3]), but it is important for applications to nonlinear problems that we not invoke such estimates. The proof is modeled on that of [17, Lemma 2.3] but it takes better advantage of first derivative estimates.

LEMMA 2.2. *Suppose, in addition to the hypotheses of Lemma 2.1, that there are constants $\omega_0 \geq 0$ and $\varepsilon \in (0, 1)$ such that*

$$(2.6a) \quad \{x : x^n > \omega_0|x'|, |x| < R\} \subset \Omega,$$

$$(2.6b) \quad |\beta'(x)| \leq \frac{1-\varepsilon}{\omega_0} \beta^n(x),$$

and that any line parallel to β intersects $\Sigma[R]$ at most once. Then there are positive constants $\eta(n, \Lambda/\lambda)$ and $C(n, \alpha, \theta_0, \Lambda/\lambda, \omega_0)$ and, for each $\rho \in (0, R)$ and $\tau \in (0, 1)$, a linear polynomial P_1 such that $\beta \cdot DP_1 = g(0)$ and

$$(2.7) \quad \text{osc}_{\Omega[\tau\rho]}(u - P_1) \leq C \left(\tau^{1+\eta} \text{osc}_{\Omega[\rho]} u + (G_0 + \sup_{\Omega[R]} |Du|) \rho \zeta(\rho/R) \right).$$

PROOF. We set $x_1 = (\rho/2)\beta$ and we use $B(r)$ to denote the ball of radius r centered at x_1 , observing that there is a constant $\kappa(\varepsilon, \omega_0)$ such that $B(2\kappa\rho) \subset \Omega[\rho]$. If $\tau \geq \kappa$, then (2.7) is clear with $P_1 \equiv 0$, so we may assume that $\tau \leq \kappa$.

In this case, a simple approximation argument involving difference quotients and [7, Corollary 9.24] gives a constant $\eta(n, \Lambda/\lambda)$ such that

$$(2.8) \quad \text{osc}_{B(\tau\rho)} Du \leq C(n, \Lambda/\lambda, \varepsilon, \omega_0) \tau^\eta \text{osc}_{B(\rho\kappa)} Du.$$

(This is just the simple version of the interior gradient estimate; see, for example, [32, Theorem 5.1].) Our next steps are to infer an upper bound for the right hand side of this estimate and then a lower bound for the left hand side.

First, we set $d'(x) = 2\rho\kappa - |x - x_0|$ and we define

$$U_1 = [u]_{1,\rho}^* = \sup_{x \in B(2\rho\kappa)} \{d'(x)|Du(x)|\},$$

$$U_2 = [u]_{1+Z,\rho}^* = \sup_{\substack{x \neq y \text{ in } B(2\rho\kappa) \\ |x-y| \leq d'(x)/2}} \left\{ d'(x) \frac{|Du(x) - Du(y)|}{\zeta(|x-y|/d'(x))} \right\}$$

(which are the same as the seminorms of Section 1 with d' in place of d^* and $B(2\rho\kappa)$ in place of $\Omega[R]$). The proof of (2.8) implies that $U_2 \leq CU_1$, and the proof of the interpolation inequality Lemma 1.2 gives $U_1 \leq C \text{osc}_{B(2\rho\kappa)} u$. It then follows from the definition of the norms that

$$\sup_{B(\rho\kappa)} |Du| \leq \frac{C}{\rho} \text{osc}_{B(2\rho\kappa)} u \leq \frac{C}{\rho} \text{osc}_{\Omega[\rho]} u.$$

Next we define u_1 by $u_1(x) = u(x) - Du(x_1) \cdot (x - x_1)$. It follows that

$$\text{osc}_{B(\tau\rho)} u_1 \leq \tau\rho \sup_{B(\tau\rho)} |Du_1| \leq \tau\rho \text{osc}_{B(\tau\rho)} Du_1 = \tau\rho \text{osc}_{B(\tau\rho)} Du$$

because $Du_1(x_1) = 0$. We now combine these last two inequalities with (2.8) to obtain

$$(2.9) \quad \operatorname{osc}_{B(\tau\rho)} u_1 \leq C\tau^{1+\eta} \operatorname{osc}_{\Omega[\rho]} u.$$

Our final step is to relate the oscillation of u_1 over $B(\tau\rho)$ to its oscillation over $\Omega[\tau\rho]$. Let $x \in \Omega[\tau\rho]$ and set $x_2 = x + (\rho/2)\beta$, so $x_2 \in B(\tau\rho)$. Then

$$|u_1(x) - u_1(0)| \leq |u_1(x) - u_1(x_2)| + |u_1(x_2) - u_1(x_1)| + |u_1(x_1) - u_1(0)|$$

by the triangle inequality, and the second term on the right is estimated via (2.9). The first and third terms are estimated via Lemma 2.1. We have

$$\begin{aligned} u_1(x_2) - u_1(x) &= u(x_2) - u(x) - \beta \cdot Du(x_1)\rho/2 \\ &= \int_0^{\rho/2} [\beta \cdot Du(x + s\beta) - \beta \cdot Du(x_1)] ds, \end{aligned}$$

and Lemma 2.1 gives

$$|\beta \cdot Du(x + s\beta) - g(0)| + |\beta \cdot Du(x_1) - g(0)| \leq CG_1\zeta(\rho/R)$$

(with $G_1 = G_0 + \sup_{\Omega[R]} |\beta \cdot Du|$) for $s \in [0, \rho/2]$, so

$$|u_1(x_2) - u_1(x)| \leq CG_1\rho\zeta(\rho/R),$$

and a similar estimate is valid for $|u_1(x_1) - u(0)|$. Combining all these estimates yields

$$\sup_{x \in \Omega[\tau\rho]} |u_1 - u_1(0)| \leq C \left(\tau^\eta \operatorname{osc}_{\Omega[\rho]} u + G_1\rho\zeta(\rho/R) \right),$$

and the final estimate follows easily from this one with

$$P_1(x) = Du(x_1) \cdot (x - (\beta \cdot x)\beta) + g(0)\beta \cdot x. \quad \square$$

We are now in a position to state and prove our first modulus of continuity estimate for the gradient.

THEOREM 2.3. *Let F satisfy (2.1) and let g be continuous on $\Sigma[R]$. Suppose there is a Lipschitz function ω such that (1.3) holds and suppose $\Omega[R]$ satisfies an exterior θ_0 -cone condition at each point of $\Sigma[R]$. Suppose also that there is $\varepsilon \in (0, 1)$ such that (2.6b) holds, and that there are nonnegative constants B_0 and G_0 along with a D_1 function ζ , which is η -decaying for η the constant from Lemma 2.2, such that*

$$(2.10a) \quad |\beta(x) - \beta(y)| \leq B_0\zeta \left(\frac{|x - y|}{d^*(x)} \right) \beta^n(x),$$

$$(2.10b) \quad |g(x) - g(y)| \leq G_0\zeta \left(\frac{|x - y|}{d^*(x)} \right) \beta^n(x)$$

for all x and y in $\Sigma[R]$ with $|x - y| \leq d^*(x)/2$. If also $u \in C^1(\overline{\Omega[R]}) \cap W_{\text{loc}}^{2,n}(\Omega[R])$ satisfies (2.2), then

$$(2.11) \quad [u]_{1+Z_2}^* \leq C(B_0, n, \theta_0, \Lambda/\lambda, \varepsilon, \omega_0, \zeta) \left[\sup_{\Omega[R]} |u| + G_0R \right]$$

for $\zeta_2 = I(\zeta)$.

PROOF. First, fix $x_0 \in \Sigma[R]$ and note that we can apply Lemma 2.2 to u in $\Omega \cap B(x_0, d^*(x_0)/2)$ with $R_0 = d^*(x_0)/2$ replacing R and $x - x_0$ replacing x . Specifically, we use $\bar{g}(x) = [(\beta(x) - \beta(x_0)) \cdot Du + g]/|\beta(x_0)|$ in place of g and $\beta(x_0)/|\beta(x_0)|$ in place of β , and we note that

$$|\bar{g}(x) - \bar{g}(x_0)| \leq \left(G_0 + B_0 \sup_{\Sigma[x_0, R_0]} |Du| \right) \zeta \left(\frac{|x - x_0|}{R_0} \right).$$

With $\tau \in (0, 1)$ to be further specified, we apply Lemma 2.2 first to u in $\Omega[x_0, R_0]$ and then to u_1 in $\Omega[x_0, \tau R_0]$, noting that u_1 satisfies the same differential equation as u , that

$$\sup_{\Omega[x_0, \tau R_0]} |Du_1| \leq \operatorname{osc}_{\Omega[x_0, R_0]} Du,$$

and that \bar{g}_1 defined by $\bar{g}_1(x) = [(\beta(x) - \beta(x_0)) \cdot Du_1 + g]/|\beta(x_0)|$ satisfies the estimate

$$|\bar{g}_1(x) - \bar{g}_1(x_0)| \leq \left(G_0 + B_0 \sup_{\Sigma[x_0, R_0]} |Du_1| \right) \zeta \left(\frac{|x - x_0|}{R_0} \right) \leq CH\zeta(\tau)$$

for $H = G_0 + \sup_{\Omega[x_0, R_0]} |Du|$. Proceeding in this fashion, we obtain a sequence of linear polynomials (P_k) such that

$$\operatorname{osc}_{\Omega[x_0, \tau^k R_0]} (u - P_k) \leq C \left(\tau^{1+\eta} \operatorname{osc}_{\Omega[x_0, \tau^{k-1} R_0]} (u - P_{k-1}) + H\zeta(\tau^{k-1})\tau^{k-1}R_0 \right).$$

We now set

$$v_k = \frac{\operatorname{osc}_{\Omega[x_0, \tau^k R_0]} (u - P_k)}{CH\tau^{-1-\sigma}\zeta(\tau^k)\tau^k R_0}$$

and recall that $\zeta(\tau^{k-1})\tau^{k-1}R_0 \leq \tau^{-1-\sigma}\zeta(\tau^k)\tau^k R_0$ because ζ is σ -decreasing. It follows that

$$v_k \leq C\tau^\eta \frac{\zeta(\tau^{k-1})}{\zeta(\tau^k)} v_{k-1} + 1.$$

Since ζ is η -decaying, there is $\theta \in (0, \eta)$ such that ζ is θ -decreasing. It follows that $v_k \leq C\tau^{\theta-\eta}v_{k-1} + 1$. Now choose $\tau < 1$ so that $C\tau^{\theta-\eta} \leq 1/2$ to infer that $v_k \leq (1/2)v_{k-1} + 1$. An easy induction argument now shows that $v_k \leq 2$ for all $k \geq 1$ because

$$\operatorname{osc}_{\Omega[x_0, R_0]} u \leq CHR_0.$$

Rewriting the inequality $v_k \leq 2$ in terms of u shows that u satisfies (1.5) with $C[G_0R + [u]_1^*]\zeta$ in place of ζ and hence

$$[u]_{1+Z_2}^* \leq C[G_0R + [u]_1^*].$$

An application of Lemma 1.2 with $C[G_0R + [u]_1^*]I(\zeta)$ in place of ζ (and μ sufficiently small) completes the proof. \square

Note that C depends on ζ only through the number $\alpha < \sigma_0$ such that ζ is α -decreasing.

3. – Existence of solutions to some mixed boundary value problems

In this section, we prove unique solvability for a class of mixed boundary value problems with a special form. Specifically, we use the abbreviation $\Omega^+[R]$ for the subset of $\overline{\Omega}$ on which $|x| = R$, and we study the problem

$$(3.1) \quad F(D^2u) = 0 \text{ in } \Omega[R], \quad \beta \cdot Du = 0 \text{ on } \Sigma[R], \quad u = \psi \text{ on } \Omega^+[R]$$

for a given function ψ . (Exact assumptions on β and $\partial\Omega$ will be given below.) If this problem admits a globally smooth (that is, $C^{2,\alpha}$ for some $\alpha \in (0, 1)$) solution for an appropriate class of ψ , then standard functional analysis (as in [12] or [7, Section 17.2]) reduces the solvability issue to establishing *a priori* estimates. Unfortunately, we need to consider this problem with ψ merely continuous. Moreover, it is well-known that even for smooth data, the problem need not have a smooth solution. (See, for example, [18, Theorem 2].)

We therefore use a more suitable approach. Our plan is simple: first, we prove existence when $\partial\Omega$ and ψ are sufficiently smooth and F is suitably approximated; then, we argue by approximation to consider the general problem.

Our starting point is a result on solvability of nonlinear equations in Banach spaces. To state this result, we recall that if P is a function defined on a Banach space X with values in another Banach space Y , then P has a *Gateaux variation* at $u \in X$ if the limit

$$P_u(g) = \lim_{\varepsilon \rightarrow 0} \frac{P(u + \varepsilon g) - Pu}{\varepsilon}$$

exists for any $g \in X$. Then [12, Lemma 4] gives us the following basic result.

LEMMA 3.1. *Suppose X and Y are Banach spaces and that $P: X \rightarrow Y$ has a Gateaux variation at each $u \in X$. If, for all $u \in X$, there is $g \in X$ such that $P_u(g) + Pu = 0$ and if $P(X)$ is a closed subset of Y , then there is $w \in X$ such that $Pw = 0$.*

This lemma allows us to prove an existence theorem for a related mixed problem with smooth data. The related problem differs from (3.1) in two ways: We assume that F , $\partial\Omega$, and ψ are sufficiently smooth; and we replace F by a somewhat more complicated function which was used before in [5, Section 7] and [32, Theorem 8.1] to study the Dirichlet problem for fully nonlinear elliptic equations without boundary estimates. In addition, we introduce some weighted norms and seminorms, which will be useful here. For $\alpha \in (0, 1)$, $b \in \mathbb{R}$, and k a nonnegative integer, we define

$$\begin{aligned} [f]_{k+\alpha}^{(b)} &= \sup_{\varepsilon > 0} (\varepsilon R)^{k+\alpha+b} [D^k f]_{\alpha; \{d^* > \varepsilon R\}}, \\ |f|_k^{(b)} &= \sum_{j \leq k} \sup (d^*)^{k+b} |D^j f|, \quad |f|_{k+\alpha}^{(b)} = |f|_k^{(b)} + [f]_{k+\alpha}^{(b)}, \end{aligned}$$

where $D^k f$ is the tensor of all k -th order derivatives of f . The space of all functions f with finite norm $|f|_{k+\alpha}^{(b)}$ will be denoted by $H_{k+\alpha}^{(b)}$. We note that $[f]_{k+\alpha}^{(0)} = [f]_{k+\alpha}^*$ from Section 2.

LEMMA 3.2. *Suppose there is a C^3 function ω such that condition (1.3) holds, let F be a concave (or convex), C^3 function defined on \mathbb{S}^n which satisfies (2.1), and let β be a constant vector which satisfies (2.6b) on $\Omega[R]$. If $\eta \in C^3(\overline{B(R)})$ has compact support in $B(R)$ and if $0 \leq \eta \leq 1$ in $B(R)$, then, for any $\psi \in C^4(\overline{\Omega[R]})$, there is a unique solution u of*

$$(3.2) \quad \begin{aligned} [1 - \eta(x)]\Delta u + \eta(x)F(D^2u) &= 0 \text{ in } \Omega[R], \\ \beta \cdot Du &= 0 \text{ on } \Sigma[R], \quad u = \psi \text{ on } \Omega^+[R]. \end{aligned}$$

PROOF. We first define \tilde{F} by

$$\tilde{F}(x, r) = [1 - \eta(x)](\text{tr } r + \Delta\psi(x)) + \eta(x)F(r + D^2\psi(x)),$$

where $\text{tr } r$ denotes the trace of r . Then we observe that u solves (3.2) if and only if $w = u - \psi$ satisfies the conditions

$$\tilde{F}(x, D^2w) = 0 \text{ in } \Omega[R], \quad \beta \cdot [Dw + D\psi] = 0 \text{ on } \Sigma[R], \quad w = 0 \text{ on } \Omega^+[R].$$

Hence, if

$$\begin{aligned} X &= \{u \in H_{2+\alpha}^{(-\delta)} : u = 0 \text{ on } \Omega^+[R]\}, \\ Y &= H_{\alpha}^{(2-\delta)}(\Omega[R]) \times H_{1+\alpha}^{(1-\delta)}(\Sigma[R]) \end{aligned}$$

(with $\delta \in (0, 1)$ to be determined and $\alpha \in (0, 1)$ arbitrary), and if we define $P: X \rightarrow Y$ by

$$Pw = (\tilde{F}(x, D^2w), \beta \cdot [Dw + D\psi]),$$

the existence part of the lemma is reduced to showing that there is $w \in X$ with $Pw = 0$. By Lemma 3.1, we only have to check the properties of the Gateaux variation of P in the hypotheses of that lemma and that $P(X)$ is closed.

By direct calculation, we have

$$P_w(g) = (\tilde{F}^{ij}(x, D^2w)D_{ij}g, \beta \cdot Dg),$$

and [15, Theorem 2] provides a constant δ so that, for each $w \in X$, there is a unique function $g \in X$ such that

$$\tilde{F}^{ij}(x, D^2w)D_{ij}g = -Pw \text{ in } \Omega[R], \quad \beta \cdot Dg = -\beta \cdot Dw \text{ on } \Sigma[R], \quad w = 0 \text{ on } \Omega^+[R].$$

In other words, $P_w(g) + Pw = 0$. (Note that for $w \in X$, $F^{ij}(D^2w)$ need not be in the correct space to apply the results of [15].)

To show that $P(X)$ is closed, we need to estimate $|w|_X$ in terms of $|Pw|_Y$ for any $w \in X$, and this estimate is relatively straightforward. First, we write v for the solution of

$$\Delta v = \tilde{F}(x, D^2w) \text{ in } \Omega[R], \quad \beta \cdot Dv = \beta \cdot Dw \text{ on } \Sigma[R], \quad v = 0 \text{ on } \Omega^+[R],$$

given by [15, Theorem 2], which also says that $|v|_X \leq C|Pw|_Y$. Hence there is a continuous, matrix-valued function $[a^{ij}]$ defined on $\text{supp } \eta \cap \Omega[R]$ such that $a^{ij} D_{ij}(w - v) = F(D^2w) - F(D^2v)$, and therefore, $g = \eta(x)[F(D^2w) - F(D^2v)] \in L^\infty(\Omega[R])$. If we set $b^{ij} = [1 - \eta]\delta^{ij} + \eta a^{ij}$, then $h = w - v$ satisfies the conditions

$$b^{ij} D_{ij}h = g \text{ in } \Omega[R], \quad \beta \cdot Dh = 0 \text{ on } \Sigma[R], \quad h = 0 \text{ on } \Omega^+[R].$$

Now [21, Lemma 4.5] gives an L^∞ estimate for h and hence for w . We then infer from [15, Theorem 1] that $|w| \leq C(d^*)^\delta$. Next, the interior second derivative Hölder estimates [7, Theorem 17.15] and the boundary second derivative Hölder estimates [30, (3.6)] show that we can estimate $|w|_X$ in terms of $|Pw|_Y$ and the data of the problem (such as the C^3 norm of $\partial\Omega$ but not the C^3 norm of F). The closedness of $P(X)$ now follows from this estimate by a standard argument. Let (w_n) be a sequence in X and suppose $Pw_n \rightarrow h$ in Y . It follows from the estimate just proved that (w_n) is a bounded sequence in X , and the compactness result [6, Lemma 4.2] gives a subsequence which converges uniformly on $\Omega[R]$ to a limit function $w \in X$, with second derivatives converging uniformly on compact subsets of $\Omega[R]$ to the corresponding second derivatives of x and first derivatives converging uniformly on compact subsets of $\Sigma[R]$ to the corresponding first derivatives of w . It follows that $Pw = h$, so $P(X)$ is closed. \square

Our next step is an existence theorem under weak hypotheses on the smoothness of the data of our problem along with an estimate which will be useful later as well.

THEOREM 3.3. *Suppose there is a Lipschitz function ω such that condition (1.3) holds, let β be a constant vector satisfying (2.6b) for some $\varepsilon > 0$, and let F be a concave (or convex) function defined on \mathbb{S}^n which satisfies (2.1). Then, for any $\psi \in C(\overline{\Omega[R]})$, there is a unique solution u of (3.1). Moreover, if Ω satisfies an exterior θ_0 -cone condition, then, for any $\delta \in (0, 1)$ and any $\sigma \in (0, \sigma_0)$, there is a constant $C(\delta, n, \Lambda/\lambda, \sigma, \omega_0)$ such that*

$$(3.3) \quad \sup_{\{d^* > \delta R\}} d^{1-\sigma} |D^2u| \leq CR^{-1-\sigma} \underset{\Omega[R]}{\text{osc}} u.$$

PROOF. We argue by approximation, from Lemma 3.2. Suppose first that F , $\partial\Omega$, and ψ are as smooth as in Lemma 3.2, and (as in [5] or [32]) let (η_m) be a sequence of compactly supported $C^3(B(R))$ such that $\eta_m \equiv 1$ on $B((1-1/m)R)$. We write u_m for the corresponding solution of (3.2) given by Lemma 3.2. Then [21, Lemma 4.5] implies an L^∞ bound for u_m which is independent of m , and a uniform modulus of continuity estimate follows from [22, Corollaries 3.5 and 8.5] (for points in $\overline{\Sigma[R]}$) and [7, Corollaries 9.24 and 9.28] (for all other points in $\overline{\Omega[R]}$). Hence, after extracting a suitable subsequence, we may assume that the sequence (u_m) converges uniformly to a continuous function u . On any compact subset of $\Omega[R] \cup \Sigma[R]$, there is a positive integer M such that

$\eta_m \equiv 1$ on this subset if $m \geq M$, so the second derivative Hölder estimates from the previous lemma imply that $Du_m \rightarrow Du$ and $D^2u_m \rightarrow D^2u$ uniformly on compact subsets of $\Omega[R] \cup \Sigma[R]$, so u is a solution of (3.1).

For general F , $\partial\Omega$, and ψ , we approximate these quantities by smooth ones and note that the uniform estimates described in the previous paragraph apply with only minor change: the second derivative Hölder estimate is only valid on compact subsets of $\Omega[R]$ so we use Theorem 2.3 to obtain a Hölder gradient estimate (with exponent σ) on compact subsets of $\Sigma[R]$.

Finally, we infer (3.3) from the combination of Theorem 2.3 and the interior second derivative Hölder estimate. □

4. – The perturbation argument

We are now ready to show that solutions of our problem have continuous gradient under fairly weak hypotheses on the functions F , β , and g . To state our result, we recall the notation

$$\|u\|_{p;S} = \left(\int_S |u|^p dx \right)^{1/p}$$

for S an open subset of \mathbb{R}^n , $p \geq 1$, and u measurable and finite almost everywhere in S .

THEOREM 4.1. *Let Ω satisfy (1.3) and also an exterior θ_0 -cone condition, suppose F is concave (or convex) with respect to r , and suppose F satisfies*

$$(4.1) \quad \lambda|\xi|^2 \leq F^{ij}(x, z, p, r)\xi_i\xi_j \leq \Lambda|\xi|^2$$

Suppose also that there are nonnegative functions b and f along with a nonnegative constant v_1 and a nonnegative, continuous increasing function ζ_0 with $\zeta_0(0) = 0$ such that

$$(4.2) \quad |F(x, z, p, 0)| \leq \lambda[v_1|p|^2 + b(x)|p| + f(x)]$$

for all $(x, z, p, r) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ and

$$(4.3) \quad |F(x, z, p, r) - F(y, w, 0, r)| \leq \lambda\zeta_0\left(\frac{|x-y|}{d^*(x)}\right)|r| + \lambda[v_1|p|^2 + b(x)|p| + f(x) + f(y)]$$

for all x and y in $\Omega[R]$ with $|x-y| < d^(x)/2$, all z and w in \mathbb{R} , all p and q in \mathbb{R}^n and all $r \in \mathbb{S}^n$. Suppose also that there are nonnegative constants B_0 and G_0 and a D_1 function ζ such that β and g satisfy (2.10) and*

$$(4.4) \quad |g(x)| \leq G_0|\beta(x)|$$

for all x and y in $\Sigma[R]$ with $|x - y| < d^*(x)/2$. Suppose finally that there are nonnegative functions b_1, b_2, f_1 , and f_2 along with a D_2 function ζ_1 and nonnegative constants B and F_0 such that $b = b_1 + b_2$ and $f = f_1 + f_2$ and

$$(4.5a) \quad \|b_1\|_{n; \Omega[x, \rho]} \leq B \zeta \left(\frac{\rho}{d^*(x)} \right),$$

$$(4.5b) \quad b_2 \leq B \frac{\zeta_1(d/d^*)}{d},$$

$$(4.5c) \quad \|f_1\|_{n; \Omega[x, \rho]} \leq F_0 \zeta \left(\frac{\rho}{d^*(x)} \right),$$

$$(4.5d) \quad f_2 \leq F_0 \frac{\zeta_1(d/d^*)}{d},$$

for all $x \in \Omega[R]$ and all $\rho \in (0, d^*(x)/2)$. If $u \in C^0(\overline{\Omega[R]}) \cap W_{loc}^{2,n}(\Omega[R])$ satisfies

$$(4.6) \quad F(x, u, Du, D^2u) = 0 \text{ in } \Omega[R], \quad \beta \cdot Du = g \text{ on } \Sigma[R],$$

then u has continuous gradient. Specifically, if we set

$$(4.7) \quad H = \sup_{\Omega[R]} u + (F_0 + G_0)R,$$

then

$$(4.8) \quad [u]_{1+Z_2}^* \leq C(B_0, B, \theta_0, n, \Lambda/\lambda, \Lambda_0, \nu_1 H, \varepsilon, \zeta, \zeta_0, \zeta_1, \omega_0)H,$$

where $\zeta_2 = I(\zeta + I(\zeta_1))$.

PROOF. Fix $x_0 \in \Sigma[R]$, set $R_0 = d^*(x_0)/2$ and $\zeta_3 = \zeta + I(\zeta_1)$, and define \bar{u} by

$$\bar{u}(x) = u(x) - \frac{g(x_0)}{|\beta(x_0)|^2} \beta(x_0) \cdot (x - x_0).$$

Our main step is to show that there are constants $\tau \in (0, 1/2)$ and $\bar{\rho} \in (0, 1)$ along with a sequence of polynomials $(P_k)_{k=0}^\infty$ such that the oscillations

$$M_k = \operatorname{osc}_{\Omega[x_0, \tau^k \bar{\rho} R_0]} (\bar{u} - P_k)$$

satisfy the relation

$$(4.9) \quad M_k \leq CH \zeta_3(\tau^k \bar{\rho}) \tau^k \bar{\rho}.$$

To prove the existence of such polynomials, we also show that they satisfy the additional properties

$$(4.10a) \quad \beta(x_0) \cdot DP_k = 0,$$

for $k \geq 0$ and

$$(4.10b) \quad |DP_k - DP_{k-1}| \leq C \frac{M_{k-1}}{\tau^{k-1} \bar{\rho} R_0}$$

for $k \geq 1$. When $k = 0$, condition (4.10a) is clearly satisfied for $P_0 \equiv 0$.

It will be convenient to define

$$S_k = \frac{M_k}{H \zeta_3(\tau^k \bar{\rho}) \tau^k \bar{\rho}}, \quad S_k^* = \max_{j \leq k} S_j,$$

$\rho_k = \tau^k \bar{\rho} R_0$, and

$$G_k = G_0 + \sum_{j=0}^{k-1} \frac{M_j}{\rho_j}.$$

We then note that

$$\zeta_2(b) - \zeta_2(a) = \int_a^b \frac{\zeta_3(s)}{s} ds \geq (b-a) \frac{\zeta_3(b)}{b}$$

for $0 < a < b \leq 1$. It follows that

$$R_0 \sum_{j=0}^{k-1} \frac{M_j}{\rho_j} = H \sum_{j=0}^{k-1} S_j \zeta_3(\tau^j \bar{\rho}) \bar{\rho} \leq C H S_k^* k \zeta_2(\bar{\rho}) \bar{\rho}.$$

Moreover, $G_0 R_0 \leq H$, so

$$(4.11) \quad G_k R_0 \leq C(1 + S_k^* \zeta_2(\bar{\rho})) H.$$

Here, we use C to denote any constant determined by the same quantities as for C in (4.8) (but not, of course, determined by τ or $\bar{\rho}$) and C_k for any constant determined also by $S_k^* \zeta_2(\bar{\rho})$.

Suppose we have the polynomials P_0, \dots, P_m for some nonnegative integer m . Then $u_m = \bar{u} - P_m$ satisfies the equation $F^m(x, Du_m, D^2u_m) = 0$ in $\Omega[x_0, \rho_m]$ with

$$F^m(x, p, r) = F \left(x, u(x), p + \frac{g(x_0)}{|\beta(x_0)|^2} \beta(x_0) + DP_m, r \right),$$

so

$$(4.12) \quad |F^m(x, p, 0)| \leq 2\nu_1 \lambda [|p|^2 + G_m^2] + \lambda b [|p| + G_m] + \lambda f.$$

In addition, u_m satisfies the boundary condition

$$\beta \cdot Du_m = (g - g(x_0)) + g(x_0) \frac{\beta \cdot (\beta(x_0) - \beta)}{|\beta(x_0)|^2} + (\beta(x_0) - \beta) \cdot DP_m,$$

so

$$(4.13) \quad |\beta \cdot Du_m| \leq C\beta^n(x_0)G_m\zeta(\tau^m\bar{\rho}).$$

Now note that there is a point $y_m \in \Omega[x_0, \rho_m/2]$ such that $d(y_m) \in (\rho_m/4, \rho_m/2)$ and $|f_1(y_m)| \leq CF_1\zeta(\rho_m/R_0)/\rho_m$. We then let v_m be the solution of the boundary value problem

$$F(y_m, u(x_0), 0, D^2v_m) = 0 \text{ in } \Omega[x_0, \rho_m/2], \quad \beta(x_0) \cdot Dv_m = 0 \text{ on } \Sigma[x_0, \rho_m/2], \\ v_m = u_m \text{ on } \Omega^+[x_0, \rho_m/2]$$

given by Theorem 3.3. Because v_m must attain its maximum and minimum over $\Omega[\rho]$ on $\Omega^+[\rho]$, it follows that $\text{osc}_{\Omega[\rho_m/2]} v_m \leq M_m$. Taking (4.12) into account, we can apply [21, Corollary 4.3] and the usual interior Hölder estimate [7, Corollary 9.25] (which holds for $\|b_1\|_n$ sufficiently small by the argument in [22, Lemma 3.1 and Corollary 3.5]) to \bar{u} at each point of $\Omega^+[x_0, \rho_m/2]$ and then we apply [21, Corollary 4.4] and [7, Corollary 9.28] to v at each point of $\Omega^+[x_0, \rho_m/2]$. It follows that

$$|u_m - v_m| \leq C_m\delta^\theta [M_m + (F_0 + G_m)\zeta_3(\tau^m\bar{\rho})\rho_m + v_1G_m^2\rho_m^2]$$

in $E' = \Omega[x_0, \rho_m/2] \setminus \Omega[x_0, (1-\delta)\rho_m/2]$ for any $\delta \in (0, 1)$. Now (4.11) implies that

$$v_1G_m\rho_m \leq C_m\tau^m\bar{\rho} \leq C_n\zeta_3(\tau^m\bar{\rho}).$$

It follows that

$$(4.14) \quad |u_m - v_m| \leq C_m\delta^\theta [M_m + H\tau^m\bar{\rho}\zeta_3(\tau^m\bar{\rho})]$$

in E' .

To proceed, we choose $\alpha < \sigma_0$ so that ζ and ζ_1 are α -decreasing and we set $\sigma = (\alpha + \sigma_0)/2$. Then Theorem 2.3 with $\zeta(s)$ replaced by s^σ and the usual interior second derivative estimate imply that, for any $\delta \in (0, 1)$, there is a constant C_δ , determined only by $\alpha, \theta_0, n, \varepsilon, \omega_0$, and δ , such that

$$|Dv_m| \leq C_\delta \frac{M_m}{\rho_m}, \quad |D^2v_m| \leq C_\delta \frac{M_m}{d\rho_m} \frac{(d/R_0)^\sigma}{(\rho_m/R_0)^\sigma}$$

in $E = \Omega[x_0, (1-\delta)\rho_m/2]$. In addition, there is a matrix-valued function $[a^{ij}]$ satisfying (2.3) with $\mu = \Lambda/\lambda$ such that

$$\lambda a^{ij} D_{ij}(u_m - v_m) = F(x, u, Du, D^2u) - F(x, u, Du, D^2v_m) \\ = F(y_m, u(x_0), 0, D^2v_m) - F(x, u, Du, D^2v_m).$$

Now, let $\xi(x)$ be a unit vector which is parallel to $D(u_m - v_m)(x)$ whenever $D(u_m - v_m) \neq 0$, and define L by $Lw = a^{ij} D_{ij}w + b\xi^i D_i w$. Then simple calculations show that

$$L(u_m - v_m) + 2\nu_1 |D(u_m - v_m)|^2 \geq -A_1 b_1 - A_2 \frac{\zeta_1(d/R_0)}{d} - A_3 \frac{(d/R_0)^\sigma}{d} - f_1$$

in E for

$$\begin{aligned} A_1 &= C_\delta \frac{M_m}{\rho_m} + G_m, \\ A_2 &= C_\delta B \frac{M_m}{\rho_m} + 2\nu_1 A_1^2 \frac{\rho_m}{\zeta_1(\tau^m \bar{\rho})} + F_0 + G_m, \\ A_3 &= C_\delta \zeta_0(\tau^m \bar{\rho}) M_m \rho_m^{-1-\sigma} R_0^\sigma. \end{aligned}$$

In addition, (4.13) implies that

$$\beta \cdot D(u_m - v_m) \geq -A_4 \beta^n(x_0)$$

on $E_0 = \Sigma[x_0, (1 - \delta)\rho_m/2]$ for

$$A_4 = \left[C_\delta \frac{M_m}{\rho_m} + C G_m \right] \zeta(\tau^m \bar{\rho}).$$

We now set

$$U = [\exp(2\nu_1(u_m - v_m)) - 1]/(2\nu_1)$$

and note that

$$LU \geq -A_1 b_1 - A_2 \frac{\zeta_1(d/R_0)}{d} - A_3 \frac{(d/R_0)^\sigma}{d} - f_1$$

in E and $\beta \cdot DU \geq -A_4 \beta^n(x_0)$ on E_0 . Thus we conclude from [21, Lemma 4.5] that

$$\begin{aligned} \sup_E U &\leq \sup_{\Omega^+[x_0, (1-\delta)\rho_m]} U \\ &\quad + C[A_1 \zeta(\tau^m \bar{\rho}) + A_2 I(\zeta_1)(\tau^m \bar{\rho}) + A_3 \rho_m^{1+\sigma} R_0^{-\sigma} + A_4 + F_0 \zeta_3(\tau^m \bar{\rho})] \rho_m, \end{aligned}$$

which implies an upper bound for $u_m - v_m$. Using also the similar lower bound and recalling that $\Omega^+[x_0, (1 - \delta)\rho_m/2] \subset \bar{E}'$, we infer that

$$\begin{aligned} \sup_E |u_m - v_m| \\ \leq C \left(\sup_{E'} |u_m - v_m| + [A_1 + A_2 + F_0] \rho_m \zeta_3(\tau^m \bar{\rho}) + A_3 \rho_m^{1+\sigma} R_0^{-\sigma} + A_4 \rho_m \right). \end{aligned}$$

We now estimate the terms on the right hand side of this estimate. First,

$$A_1 \rho_m \zeta_3(\tau^m \bar{\rho}) \leq C_\delta \zeta_3(\tau^m \bar{\rho}) M_m + C_m H \tau^m \bar{\rho} \zeta_3(\tau^m \bar{\rho}),$$

and (for future reference)

$$\begin{aligned} A_1 \rho_m \zeta_3(\tau^m \bar{\rho}) &\leq C_m (1 + C_\delta) \tau^m \bar{\rho} \\ &\leq C_m (1 + C_\delta) \bar{\rho}^{1-\alpha} \zeta_1(\tau^m \bar{\rho}). \end{aligned}$$

Then

$$\begin{aligned} A_2 \rho_m \zeta_3(\tau^m \bar{\rho}) &\leq C C_\delta \zeta_3(\bar{\rho}) M_m + C_m (1 + C_\delta) \bar{\rho}^{1-\alpha} A_1 \rho_m \zeta_3(\tau^m \bar{\rho}) + C_m H \tau^m \bar{\rho} \zeta_3(\tau^m \bar{\rho}) \\ &\leq C_m (1 + C_\delta) \zeta_3(\bar{\rho}) M_m + C_m (1 + C_\delta \bar{\rho}^{1-\alpha}) H \tau^m \bar{\rho} \zeta_3(\tau^m \bar{\rho}). \end{aligned}$$

Finally,

$$A_3 \rho_m^{1+\sigma} R_0^{-\sigma} = C_\delta \zeta_0(\bar{\rho}) M_m$$

and

$$A_4 \rho_m \leq C_\delta \zeta(\bar{\rho}) + C_m H \tau^m \bar{\rho} \zeta_3(\tau^m \bar{\rho}).$$

Combining these estimates with (4.14) gives

$$\begin{aligned} |u_m - v_m| &\leq C_m ((1 + C_\delta) \zeta_3(\bar{\rho}) + \delta^\theta) M_m \\ &\quad + C_m [1 + \delta^\theta + C_\delta \bar{\rho}^{1-\alpha}] H \tau^m \bar{\rho} \zeta_3(\tau^m \bar{\rho}). \end{aligned}$$

in $\Omega[\rho_m/2]$. If we now take $\bar{\rho}$ so small (determined by δ) that $C_\delta \bar{\rho}^{1-\alpha} \leq 1$ and $(1 + C_\delta) \zeta_3(\bar{\rho}) \leq \delta^\theta$, we see that

$$(4.15) \quad |u_m - v_m| \leq 2C_m \delta^\theta M_m + C_m H \tau^m \bar{\rho} \zeta_3(\tau^m \bar{\rho}).$$

in $\Omega[\rho_m/2]$.

Next, we apply Theorem 2.3 to v_m and set $P_{m+1}(x) = Dv_m(x_0) \cdot (x - x_0)$, noting also that the constant η from Lemma 2.1 can be taken equal to 1 because F is concave. In this way, we conclude that

$$\operatorname{osc}_E(v_m - P_{m+1}) \leq C \tau^{1+\sigma} \operatorname{osc}_{\Omega[x_0, \rho_m/2]} v_m \leq C \tau^{1+\sigma} M_m$$

for any $\tau \in (0, 1)$. We now fix τ so that $C \tau^{1+\sigma} \leq \frac{1}{4} \tau^{1+\alpha}$. (Recall that ζ and ζ_1 are α -decreasing and $\alpha < \sigma < \sigma_0$.) Then, in conjunction with (4.15), this inequality gives

$$M_{m+1} \leq \left(2C_m \delta^\theta + \frac{1}{4} \tau^{1+\alpha} \right) M_m + C_m H \tau^m \bar{\rho} \zeta_3(\tau^m \bar{\rho}).$$

Now we write K for the value of C_m when $S_m^* \zeta_2(\bar{\rho}) = 1$. Then we take first δ and then $\bar{\rho}$ so small that

$$2K \delta^\theta \leq \frac{1}{4} \tau^{1+\alpha}.$$

If $\bar{\rho}$ is so small that $S_m^* \zeta_2(\bar{\rho}) \leq 1$, then this inequality can be rewritten as

$$M_{m+1} \leq \frac{1}{2} \tau^{1+\alpha} M_m + KH \tau^m \bar{\rho} \zeta_3(\tau^m \bar{\rho})$$

and then

$$S_{m+1} \leq \frac{1}{2} S_m + K \tau^{-\alpha}.$$

Now we note that there is a constant K_0 such that $S_0 \leq K_0 \tau^{-\alpha}$, and hence

$$S_m \leq \left(2 - \frac{1}{2^m}\right) (K + K_0) \tau^{-\alpha}$$

provided $\zeta_2(\bar{\rho}) 2(K + K_0) \tau^{-\alpha} \leq 1$, which means that we can choose $\bar{\rho}$ independent of m . This uniform bound for S_m yields (4.9) while (4.10a) and (4.10b) are clear.

The estimate (4.8) then follows from (4.9) and Lemma 1.3. \square

A simple bootstrap argument shows that Theorem 4.1 holds if we replace the argument $|x - y|/d^*(x)$ of ζ_0 by $|x - y|/d^*(x) + |z - w|$ and if we replace $g(x)$ by a nonlinear function $g(x, z)$ such that

$$|g(x, z)| \leq G_0 |\beta(x)|, \quad |g(x, z) - g(y, w)| \leq G_0 \beta^n(x) \zeta \left(\frac{|x - y|}{d^*(x)} + |z - w| \right)$$

for $|x - y| \leq d^*(x)/2$.

Furthermore, a simple variant of this theorem gives a sharp gradient modulus of continuity estimate once any modulus of continuity is known for Du , even for nonlinear boundary conditions.

COROLLARY 4.2. *Let Ω be as in Theorem 4.1. Suppose F satisfies (4.1) and is concave (or convex) with respect to r . Suppose also that there are nonnegative constants μ_0 and F_0 , a D_1 function ζ , a D_2 function ζ_1 , a continuous increasing function ζ_0 with $\zeta_0(0) = 0$, and a nonnegative function $f = f_1 + f_2$ such that*

$$(4.16a) \quad |F(x, z, p, 0)| \leq \lambda \left(f(x) + \mu_0 \frac{\zeta_1(d(x)/d^*(x))}{d(x)} \right),$$

$$(4.16b) \quad |F(x, z, p, r) - F(y, w, q, r)| \leq \lambda \zeta_0 \left(\frac{|x - y|}{d^*(x)} + |z - w| + |p - q| \right) |r| \\ + \lambda [f(x) + f(y)],$$

and (4.5c,d) for all x and y in $\Omega[R]$ with $|x - y| \leq d^*(x)/2$, z and w in \mathbb{R} , p and q in \mathbb{R}^n and $r \in \mathbb{S}^n$. Let $G \in C(\Sigma[R] \times \mathbb{R} \times \mathbb{R}^n)$ such that

$$(4.17) \quad \left| \frac{\partial G}{\partial p} \right| \leq \frac{1 - \varepsilon}{\omega_0} \chi,$$

where $\chi = \partial G / \partial p_n$ is assumed to be positive on $\Sigma[R]$. Suppose also that there is a nonnegative constant G_0 with

$$(4.18a) \quad |G(x, z, p) - G(y, w, p)| \leq G_0 \chi(x, z, p) \zeta \left(\frac{|x - y|}{d^*(x)} + |z - w| \right),$$

$$(4.18b) \quad \left| \frac{\partial G}{\partial p}(x, z, p) - \frac{\partial G}{\partial p}(x, z, q) \right| \leq \zeta_0(|p - q|)$$

for all x and Y in $\Sigma[R]$ with $|x - y| \leq d^*(x)/2$, all z and w in \mathbb{R} and all p and q in \mathbb{R}^n . If $u \in C^1(\overline{\Omega[R]}) \cap W_{loc}^{2,n}(\Omega[R])$ satisfies

$$(4.19) \quad F(x, u, Du, D^2u) = 0 \text{ in } \Omega[R], \quad G(x, u, Du) = 0 \text{ on } \Sigma[R],$$

and if there is a continuous, increasing function ζ_3 with

$$(4.20) \quad |Du(x) - Du(y)| \leq \zeta_3 \left(\frac{|x - y|}{d^*(x)} \right)$$

for all x and y in $\Omega[R]$ with $|x - y| \leq d^*(x)/2$, then (4.8) holds with H defined by (4.7) and C determined also by ζ_3 .

PROOF. The main step is an appropriate modification of the proof of Theorem 4.1, so we assume first that the geometry of that proof is in force. We now set

$$\bar{\beta} = \frac{\partial G}{\partial p}(0, u(0), Du(0)),$$

$\bar{u}(x) = u(x) - Du(0) \cdot x$, and take v to be the solution of

$$\begin{aligned} F(x_2, u(x_2), Du(x_2), D^2v) &= 0 && \text{in } \Omega[\rho/2], \\ \bar{\beta} \cdot Dv &= 0 && \text{on } \Sigma[\rho/2], \quad v = \bar{u} && \text{on } \Omega^+[\rho/2]. \end{aligned}$$

A simple calculation shows that

$$|G(x, u, Dv + Du(0))| \leq \zeta_0(|Dv|)|Dv| + G_1 \zeta \left(\frac{|x|}{R} \right),$$

and $M_0 \leq C \zeta_1(\rho/R)\rho$, so

$$\zeta_0(|Dv|)|Dv| \leq C_\delta \zeta_0 \left(C_\delta \zeta_1 \left(\frac{\rho}{R} \right) \right) \frac{M_0}{\rho}.$$

Next, we define the vector field β by

$$\beta(x) = \int_0^1 \frac{\partial G}{\partial p}(x, u, (1-t)Du + tDv) dt,$$

and observe that β is oblique on $\Sigma[\rho/2]$ with $|\beta| \leq (1 - \varepsilon)\beta^n/\omega_0$. Since $\beta \cdot D(\bar{u} - v) = G(x, u, Dv + Du(0))$, we infer that

$$|\beta \cdot D(\bar{u} - v)| \leq \tau^{1+\sigma} M_0/\rho + G_1 \zeta \left(\frac{|x|}{R} \right)$$

provided $\bar{\rho}$ is sufficiently small. With this estimate, we can imitate the proof of Theorem 4.1. □

As an application of this corollary, we assume that $\partial\Omega \in C^{1,\delta}$ for some $\delta > 0$ and that $\zeta(s) = \zeta_1(s) = s^\alpha$ for some $\alpha \in (0, 1)$. It follows from the argument in [20, Lemma 13.21] that the solution of (4.19) has Hölder continuous gradient with some exponent less than δ . Since σ (as described at the beginning of Section 2) in this case can be taken any number less than one, it follows that the gradient is Hölder continuous with exponent equal to α .

5. – Second derivative estimates

We now provide an alternative proof of Safonov’s result [30, Theorem 3.3] based on the techniques developed above. The first step is a variant of Krylov’s theorem [11, Theorem 4.1] on boundary Hölder gradient estimates for solutions of the Dirichlet problem. Most of the necessary modifications for dealing with Dini functions appear in [16, Section 5], but there are some important changes which must be made to handle curved boundaries directly. In addition, there is a typographical error in [16, Lemma 5.2]: the function ζ in (5.5) should be replaced by $I(\zeta)$.

We shall modify the notation from [16] slightly. We suppose now that Ω satisfies (1.3) for some function ω which is C^1 . In particular, we assume that there are nonnegative constants ω_0 and ω_1 along with a Dini function ζ_1 such that

$$(5.1) \quad |D\omega| \leq \omega_0, \quad [\omega]_{1+\zeta_1}^* \leq \omega_1 .$$

Then we write ρ^* for the regularized distance from [14, Theorem 2.1] (denoted by ρ in that reference) and note that there is a constant such that $|D^2\rho^*| \leq C\zeta_1(\rho/R)/\rho$ and $|D\rho^*| \geq 1$ in $\Omega[\rho]$ for $\rho \leq R$. Next, for μ a fixed constant and $\rho \in (0, R)$, we define

$$\Gamma(\rho) = \left\{ x \in \mathbb{R}^n : 0 < \rho^* < \frac{\rho}{2\mu(n+1)}, \quad |x'| < \rho \right\} ,$$

$$\Gamma'(\rho) = \left\{ x \in \mathbb{R}^n : \frac{\rho}{4\mu(n+1)} < \rho^* < \frac{3\rho}{4\mu(n+1)}, \quad |x'| < \rho \right\} .$$

We then have the following variation of [16, Lemma 5.1].

LEMMA 5.1. *Let $[a^{ij}]$ be a positive definite matrix satisfying (2.3), and define the operator L by $Lu = a^{ij}D_{ij}u$. Then there is a constant ρ_2 , determined only by $n, \mu, \zeta_1, \omega_0$, and ω_1 such that if $u \in W_{loc}^{2,n}(\Gamma(\rho)) \cap C^0(\overline{\Gamma(\rho)})$ for some $\rho \in (0, \rho_2R)$ is nonnegative with $Lu \leq 0$ in $\Gamma(\rho)$, then*

$$(5.2) \quad \inf_{\Gamma'(\rho)} \frac{u}{\rho^*} \leq 3 \inf_{\Gamma(\rho/2)} \frac{u}{\rho^*} .$$

PROOF. Throughout this proof, C denotes a constant determined only by $n, \mu, \zeta_1, \omega_0,$ and ω_1 . Set $A = \inf_{\Gamma'(\rho)} \frac{u}{\rho^*}$ and define functions w_0 and w_2 by

$$w_0(x) = \left(1 - \frac{\rho^*}{4\mu(n+1)R} + \frac{|x'|^2}{R^2} \right) \rho^*,$$

$$w_2(x) = I(\zeta_1) \left(\frac{\rho}{R} \right) \rho^*(x) - \int_0^{\rho^*(x)} I(\zeta_1) \left(\frac{s}{R} \right) ds.$$

Then

$$Lw_2 \leq \left[CI(\zeta_1) \left(\frac{\rho^*}{R} \right) - 1 \right] \frac{\zeta_1(\rho^*/R)}{\rho^*},$$

and hence $L(\pm w_0 + Cw_2) \leq 0$ provided ρ/R is sufficiently small. It follows that $L(u - A[\rho^* - Cw_2]) + \frac{A}{4}[w_0 + Cw_2] \leq 0$ in $\Gamma(\rho)$ and $u - A[\rho^* - Cw_2] + \frac{A}{4}[w_0 + Cw_2] \geq 0$ on $\partial\Gamma(\rho)$ so the maximum principle gives $u - A[\rho^* - Cw_2] + \frac{A}{4}[w_0 + Cw_2] \geq 0$ in $\Gamma(\rho)$. Evaluating this inequality in $\Gamma(\rho/2)$ and noting that $0 \leq w_2 \leq I(\zeta_1)(\rho/R)\rho^*$ in $\Gamma(\rho/2)$ completes the proof. \square

Our next step is the analog of [16, Lemma 5.2]. Here, it is crucial to note that, despite the typographical error in [16], the ζ_1 in our (5.3) is correct.

LEMMA 5.2. *Let $L, \zeta_1, \Omega,$ and ρ be as in Lemma 5.1. Then there are constants $\alpha_0 \in (0, 1)$ and $C,$ determined only by $n, \mu, \zeta_1, \omega_0,$ and ω_1 such that if ζ_1 is α_0 -decaying and if $u \in W_{loc}^{2,n}(\Gamma(\rho)) \cap C^0(\overline{\Gamma(\rho)})$ satisfies $Lu = 0$ in $\Omega[\rho]$ with u/ρ^* bounded there, then for any $\tau \in (0, 1),$*

$$(5.3) \quad \text{osc}_{\Omega[\tau\rho]} \frac{u}{\rho^*} \leq C \sup_{\Omega[\rho]} \frac{u}{\rho^*} \zeta_1 \left(\frac{\tau\rho}{R} \right).$$

PROOF. For $s > 0$ sufficiently small, we write m_s and M_s for the infimum and supremum, respectively of u/ρ^* over $\Gamma(s)$. Then we apply the weak Harnack inequality [7, Theorem 9.22] to $u - m_{4s}\rho^*$ and to $M_{4s}\rho^* - u$ in $\Gamma(2s),$ noting that $|L\rho^*| \leq C\zeta_1(s/R)/s$ in $\Gamma(2s).$ Using Lemma 5.1 (with $4s$ in place of ρ) as in the proof of [16, Lemma 5.2] gives a constant α_0 such that

$$\text{osc}_{\Gamma(s)} \frac{u}{\rho^*} \leq \left(\frac{1}{4} \right)^{\alpha_0} \text{osc}_{\Gamma(4s)} \frac{u}{\rho^*} + C\zeta_1 \left(\frac{4s}{R} \right) \sup_{\Gamma(4s)} \frac{u}{\rho^*},$$

and then [16, Lemma 5.3] gives the desired result. \square

From this result, we obtain an estimate for constant coefficient problems.

LEMMA 5.3. *Suppose F is a concave function of r and satisfies (2.1) and let condition (1.3) hold for some function $\omega \in C^1$ with $\omega(0) = 0$ and $D\omega(0) = 0$ which satisfies (5.1) for a Dini function ζ_1 which is α_0 -decaying for α_0 the constant*

from Lemma 5.2. Let $\rho \in (0, \rho_2 R)$ for ρ_2 the constant from Lemma 5.1. Suppose finally that $u \in C^1(\overline{\Omega[\rho]}) \cap W_{\text{loc}}^{2,n}(\Omega[\rho])$ satisfies

$$(5.4) \quad F(D^2u) = 0 \text{ in } \Omega[\rho], \quad \beta \cdot Du = 0 \text{ on } \Sigma[\rho].$$

for some constant vector β which satisfies (2.6b). Then there are positive constants C and θ , determined only by $n, \varepsilon, \Lambda/\lambda, \zeta_1, \omega_0,$ and ω_1 such that, for any $\tau \in (0, 1)$, there is a quadratic polynomial P_2 such that $DP_2(0) = 0$ and

$$(5.5) \quad \text{osc}_{\Omega[\tau\rho]}(u - P_2) \leq C \left(\tau^{2+\theta} \text{osc}_{\Omega[\rho]} u + \rho \zeta_1 \left(\frac{\rho}{R} \right) \sup_{\Omega[\rho]} \left| \frac{\beta}{|\beta|} \cdot Du \right| \right).$$

PROOF. Suppose first that $\beta^n = 1$ and $\beta' = 0$. Let us set $H = D_{nn}u(0)$ and $M = \sup |D_n u|$. Then a simple difference quotient argument shows that we can apply Lemma 5.2 with $D_n u$ in place of u . It follows that

$$(5.6) \quad |D_n u(x) - H[x^n - \omega(x')]| \leq C \frac{|x|}{R} \zeta_1 \left(\frac{|x|}{R} \right) M$$

for $x \in \Omega[R/2]$ and a straightforward barrier argument (see [16, Section 2]) gives us $|H| \leq CM/\rho$.

As in the proof of Lemma 2.2, we may assume that $\tau \leq \kappa$ for a suitable constant κ determined only by ω_0 . If we use the second derivative Hölder estimate from [7, Section 17.4] in place of the first derivative estimate (2.8), the proof of Lemma 2.2 provides a constant $\theta(n, \Lambda/\lambda)$ such that

$$\text{osc}_{B(\tau\rho)}(u - P_{0,2}) \leq C \tau^{2+\theta} \text{osc}_{\Omega[\rho]} u,$$

for

$$P_{0,2}(x) = Du(x_1) \cdot (x - x_1) + \frac{1}{2} D_{ij}u(x_1)(x - x_1)^i(x - x_1)^j$$

and $B(\tau\rho)$ is centered at $x_1 = (0, \dots, 0, \rho/2)$. Next, we define the matrix $[b_{ij}]$ by

$$b_{ij} = \begin{cases} D_{ij}u(x_1) & \text{if } i, j < n \\ D_{nn}u(0) & \text{if } i = j = n \\ 0 & \text{otherwise} \end{cases}$$

and set

$$P_2(x) = Du(x_1) \cdot (x - x_1) + \frac{1}{2} b_{ij}(x - x_1)^i(x - x_1)^j.$$

Then $P_2(x) - P_{0,2}(x) = (1/2)[D_{ij}u(x_1) - b_{ij}](x - x_1)^i(x - x_1)^j$, and the only nonzero terms in this sum are those with $i = n$ or $j = n$. Writing e_j for the j -th standard unit vector, we have

$$D_{nj}u(x_1) - b_{nj} = \left(D_{nj}u(x_1) - \frac{D_nu(x_1 + \tau\rho e_j) - D_nu(x_1)}{\tau\rho} \right) + \left(\frac{D_nu(x_1 + \tau\rho e_j) - D_nu(x_1)}{\tau\rho} - b_{nj} \right).$$

The first term on the right-hand side of this equation is estimated easily. The mean value theorem implies that $(D_n u(x_1 + \tau \rho e_j) - D_n u(x_1))/(\tau \rho) = D_{nj} u(y)$ for some y on the line segment joining x_1 and $x_1 + \tau \rho e_j$. Since $y \in B(\tau \rho)$, we have

$$\left| D_{nj} u(x_1) - \frac{D_n u(x_1 + \tau \rho e_j) - D_n u(x_1)}{\tau \rho} \right| \leq \operatorname{osc}_{B(\tau \rho)} D_{nj} u \leq C \tau^\theta \rho^{-2} \operatorname{osc}_{\Omega[\rho]} u.$$

To estimate the second term, we consider separately the cases $j < n$ and $j = n$. If $j < n$, then (5.6) implies that

$$\left| \frac{D_n u(x_1 + \tau \rho e_j) - D_n u(x_1)}{\tau \rho} \right| \leq \left| \frac{H}{\tau \rho} [(x_1 + \tau \rho e_j)^n - \omega((x_1 + \tau \rho e_j)') - (x_1)^n + \omega(x_1')] \right| + C \frac{M}{\tau \rho} \zeta_1 \left(\frac{\rho}{R} \right).$$

Now we note that $(x_1 + \tau \rho e_j)^n = x_1^n$, $\omega(x_1') = 0$, and $|\omega((x_1 + \tau \rho e_j)')| \leq C \zeta_1(\rho/R) \rho$ (because $\tau \leq 1$) to infer that

$$\left| \frac{D_n u(x_1 + \tau \rho e_j) - D_n u(x_1)}{\tau \rho} \right| \leq C \frac{M}{\tau R} \zeta_1 \left(\frac{\rho}{R} \right).$$

On the other hand, if $j = n$, then

$$\left| D_{nn} u(0) - \frac{D_n u(x_1 + \tau \rho e_n) - D_n u(x_1)}{\tau \rho} \right| \leq C \frac{M}{\tau R} \zeta_1 \left(\frac{\rho}{R} \right)$$

because

$$D_{nn} u(0) - \frac{H}{\tau \rho} [(x_1 + \tau \rho e_n)^n - x_1^n] = 0.$$

It follows that

$$(5.7) \quad \operatorname{osc}_{B(\tau \rho)} (u - P_2) \leq C \left(\tau^{2+\theta} \operatorname{osc}_{\Omega[\rho]} u + \rho \zeta_1 \left(\frac{\rho}{R} \right) \sup |D_n u| \right).$$

Now we set $w = u - P_2$ and estimate the oscillation of w over $\Omega[\tau \rho]$ in terms of the left hand side of (5.7). For $x \in \Omega[\tau \rho]$, we set $x_2 = (x', x^n + \rho/2)$. Then

$$\begin{aligned} w(x) - w(0) &= w(x) - w(x_2) + w(x_2) - w(x_1) + w(x_1) - w(0) \\ &= w(x_2) - w(x_1) + \int_0^{\rho/2} [D_n w(x', x^n + s) - D_n w(0, s)] ds. \end{aligned}$$

By definition, $|w(x_2) - w(x_1)| \leq \operatorname{osc}_{B(\tau \rho)} w$, so we only have to estimate the integral. For this estimate, we first note that

$$D_n w(y) = D_n u(y) - D_n u(x_1) - D_{nn} u(0)[y^n - \rho/2]$$

for any y . Therefore

$$D_n w(x', x^n + s) - D_n w(0, s) = D_n u(x', x^n + s) - D_n u(0, s) - Hx^n.$$

We estimate the right hand side of this equation via (5.6) to see that

$$\begin{aligned} |D_n w(x', x^n + s) - D_n w(0, s)| &\leq |H\omega(x')| + C\zeta_1 \left(\frac{\rho}{R}\right) M \\ &\leq C\zeta_1 \left(\frac{\rho}{R}\right) M. \end{aligned}$$

Combining this estimate with (5.7) yields the desired result when $\beta' = 0$.

To remove the assumption that $\beta' = 0$, we note that there are a constant $C(n, \varepsilon, \Lambda/\lambda, \omega_0)$ and a function \bar{F} satisfying

$$|\xi|^2 \leq \bar{F}^{ij} \xi_i \xi_j \leq C|\xi|^2$$

such that u^* , defined by

$$u^*(x, x^n) = u(x' + (\beta'/\beta^n)x^n, x^n),$$

satisfies the equation $\bar{F}(D^2 u^*) = 0$ in $\Omega[C\rho]$ and the boundary condition $D_n u^* = 0$ on $\Omega[C\rho]$. The estimate (5.5) for u^* easily implies the corresponding one for u . \square

A straightforward perturbation argument then gives the full regularity result.

THEOREM 5.4. *Let Ω satisfy (1.3) and (5.1) for some Dini function ζ , which is α -decaying for $\alpha = \min\{\alpha_0, \eta\}$ (the constants from Lemma 5.3). Suppose F is defined on $\Omega[R] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ and is concave (or convex) with respect to r . Suppose also that condition (4.1) holds and that*

$$(5.8) \quad |F(x, z, p, r) - F(y, w, q, r)| \leq \lambda \Lambda_3 \zeta \left(\frac{|x - y|}{d^*(x)} \right) (1 + |r|)$$

for x and y in $\Omega[R]$ with $|x - y| \leq d^*(x)/2$, z and w in \mathbb{R} , p and q in \mathbb{R}^n and $r \in \mathbb{S}^n$. Suppose also that G is defined on $\Sigma[R] \times \mathbb{R} \times \mathbb{R}^n$ and is C^1 with respect to (x, z, p) . Suppose also that $\chi = \partial G / \partial p_n > 0$ and that

$$(5.9a) \quad \left| \frac{\partial G}{\partial p}(x, z, p) \right| \leq \mu_1 \chi(x, z, p)$$

$$(5.9b) \quad \left| \frac{\partial G}{\partial x}(x, z, p) \right| \leq \frac{G_1}{d^*(x)} \chi(x, z, p)$$

$$(5.9c) \quad \left| \frac{\partial G}{\partial z}(x, z, p) \right| \leq G_1 \chi(x, z, p)$$

$$(5.9d) \quad \left| \frac{\partial G}{\partial p}(x, z, p) - \frac{\partial G}{\partial p}(y, w, q) \right| \leq G_1 \zeta \left(\frac{|x - y|}{d^*(x)} + |z - w| + |p - q| \right) \chi(x, z, p)$$

$$(5.9e) \quad \left| \frac{\partial G}{\partial x}(x, z, p) - \frac{\partial G}{\partial x}(y, w, p) \right| \leq \frac{G_1}{d^*(x)} \zeta \left(\frac{|x - y|}{d^*(x)} + |z - w| \right) \chi(x, z, p)$$

$$(5.9f) \quad \left| \frac{\partial G}{\partial z}(x, z, p) - \frac{\partial G}{\partial z}(y, w, p) \right| \leq G_1 \zeta \left(\frac{|x - y|}{d^*(x)} + |z - w| \right) \chi(x, z, p)$$

for x and y in $\Sigma[R]$ with $|x - y| \leq d^*(x)/2$, z and w in \mathbb{R} , and p and q in \mathbb{R}^n . If $u \in C^{1,\delta}(\overline{\Omega[R]}) \cap W_{loc}^{2,n}(\Omega[R])$ (for some $\delta > 0$) satisfies (4.19), then $u \in C^2(\Omega[R])$ and there is a constant C determined only by $G_1, n, \mu_1, \omega_0, \omega_1, \Lambda/\lambda, \Lambda_3, \zeta$, and $|u|_{1+\delta}$ such that

$$(5.10) \quad |D^2u(x) - D^2u(y)| \leq CI(\zeta) \left(\frac{|x - y|}{d^*(x)} \right) \frac{1}{d^*(x)},$$

for $|x - y| \leq d^*(x)/2$.

PROOF. The proof is similar to that of Theorem 4.1 (see also [20, Theorem 14.22]). Similarly to the proof of Corollary 4.2, we set $\bar{\beta} = \frac{\partial G}{\partial p}(0, u(0), Du(0))$ and we define \bar{u} by

$$\bar{u}(x) = u(x) - u(0) - Du(0) \cdot x - Q_{ij}x^i x^j,$$

where Q is a matrix such that

$$\bar{\beta}^i Q_{ij} = \frac{\partial G}{\partial x^j}(0, u(0), Du(0)) + \frac{\partial G}{\partial z}(0, u(0), Du(0)) D_j u(0).$$

Then we write v for the solution of

$$\begin{aligned} F(0, u(0), Du(0), D^2v) &= 0 && \text{in } E(\rho/2), \\ \bar{\beta} \cdot Dv &= 0 && \text{on } E_0(\rho/2), \quad v = \bar{u} && \text{on } E_+(\rho/2), \end{aligned}$$

and we note that

$$|G(x, u, Dv + D\bar{u} - Du)| \leq C \left[\zeta(|Dv|)|Dv| + \zeta \left(\frac{|x|}{R} \right) \right].$$

In this way, we obtain

$$\text{osc}_{\Omega[\tau\rho]}(\bar{u} - P_2) \leq C\tau^{2+\alpha} \text{osc}_{\Omega[\rho]} + C\zeta \left(\frac{\rho}{R} \right) \frac{\rho^2}{R},$$

and the proof is completed as before. □

6. – Problems with non-Dini, continuous boundary data

In [17], we showed that solutions of linear problems exist and take on the boundary values in a suitable way even if g is not Hölder continuous. (Of course, if g has a Dini modulus of continuity, then Du will be continuous by the arguments of this work.) Here, we prove a similar result for nonlinear problems with linear boundary conditions. Our starting point is the estimate (4.9), which, when specialized to linear equations, gives an alternative proof of [17, Proposition 4.2] with conditions (4.2) of that paper, which require that the coefficient

matrix $[a^{ij}]$ be Hölder continuous up to $\partial\Omega$, relaxed to the more general (3.3); however, our full existence theorem will require a nonlinear analog of (4.2) from [17].

To present our result, we introduce one further operator. For any continuous, increasing function ζ defined on $(0, 1)$, we define

$$\tilde{I}(\zeta)(s) = \int_s^1 \frac{\zeta(t)}{t} dt.$$

THEOREM 6.1. *Let Ω satisfy (1.3), and suppose F satisfies (4.1) and is concave (or convex) with respect to r . Suppose also that there are nonnegative constants ν_1 and μ_0 along with a nonnegative, continuous increasing function ζ_0 with $\zeta_0(0) = 0$ and a D_1 function ζ_1 such that*

$$(6.1a) \quad |F(x, z, p, 0)| \leq \lambda \nu_1 |p|^2 + \lambda \mu_0 \frac{\zeta_1(d(x))}{d(x)},$$

$$(6.1b) \quad |F(x, z, p, r) - F(y, w, 0, r)| \leq \lambda \zeta_0(|x - y|) |r| + \lambda \left[\nu_1 |p|^2 + \mu_0 \frac{\zeta_1(d(x))}{d(x)} \right]$$

for all x and y in $\Omega[R]$ with $|x - y| \leq d(x)/2$, all z and w in \mathbb{R} , all p and q in \mathbb{R}^n and all $r \in \mathbb{S}^n$. Suppose further that β satisfies (2.6b) and that there are nonnegative constants B_0 and G_0 , a D_1 function ζ_2 , and a continuous increasing function ζ with $\zeta(0) = 0$ and $\zeta \geq \zeta_2 + I(\zeta_1)$ such that (4.4) holds, $\zeta/(\zeta_2 + I(\zeta_1))$ is decreasing, and

$$(6.2a) \quad |\beta(x) - \beta(y)| \leq B_0 \zeta_2(|x - y|) \beta^n(x),$$

$$(6.2b) \quad |g(x) - g(y)| \leq G_0 \zeta(|x - y|) \beta^n(x),$$

for all x and y in $\Sigma[R]$ and all $\rho \in (0, \text{diam } \Omega)$. Suppose moreover that there is a Dini function ζ_3 which is θ -decaying (for θ the constant in Lemma 5.3) such that

$$(6.3) \quad |F(x, z, p, r) - F(y, w, q, r)| \leq \lambda \left[\zeta_3 \left(\frac{|x - y|}{d(x)} + |z - w| + |p - q| \right) |r| + \mu_0 d(x)^{\theta-1} \right]$$

for all x and y in $\Omega[R]$ with $|x - y| \leq d(x)/2$, all z and w in \mathbb{R} , all p and q in \mathbb{R}^n , and all $r \in \mathbb{S}^n$. Suppose finally that there are a nonnegative constant μ_2 and a D_1 function ζ_4 such that

$$(6.4) \quad \zeta(s) \zeta_0(s) + \zeta_2(s) \tilde{I}(\zeta)(s) \leq \zeta_4(s) \text{ for } s \in (0, 1),$$

and

$$(6.5a) \quad \left| \frac{\partial F}{\partial p} \right| \leq \mu_2 \lambda [|r| + |p| + \zeta_2(d)/d]$$

$$(6.5b) \quad \left| \frac{\partial F}{\partial x} \right| \leq \mu_2 \lambda \left[|r| \frac{\zeta_0(d)}{d} + \frac{|p| \zeta_2(d)}{d} + \zeta_2(d) d^{-2} \right],$$

$$(6.5c) \quad 0 \geq \frac{\partial F}{\partial z} \geq -\mu_2 \lambda \left[|r| + \frac{|p| \zeta_2(d)}{d} + \frac{\zeta_4(d)}{d^2} \right].$$

If $u \in W_{\text{loc}}^{2,n}(\Omega[R]) \cap C^1(\overline{\Omega[R]})$ is a solution of (4.6), then for any $\varepsilon > 0$, there is a $\delta > 0$ determined only by $B_0, G_0, \nu_1, \varepsilon, \mu_0, \mu_2, \Omega, n, \Lambda/\lambda, \zeta, \zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \sup |u|$ and R such that

$$(6.6) \quad \left| \frac{u(y + \beta(y)t) - u(y)}{t} - g(y) \right| \leq \varepsilon \text{ whenever } 0 < t < \delta$$

for any $y \in \Sigma[R/4]$.

PROOF. To prove this estimate, we first note that the proof of Theorem 4.1 can be modified to prove the estimates

$$|u(x) - u(y)| \leq C|x - y|^{(\sigma+1)/2}, \quad |Du(x)| \leq Cd^{(\sigma-1)/2}.$$

Specifically, we imitate that proof with u_m replaced by \bar{u} . Then the first inequality follows by using the easier estimate

$$\text{osc}_{\Omega[\tau\rho]} v_m \leq C\tau \text{osc}_{\Omega[\rho]} v_m,$$

which is an immediate consequence of the interior derivative estimate for v_m , and the second estimate follows from this one by the usual interior estimates. Note that these estimates only use the hypotheses (6.1) on F .

Furthermore the proof of Theorem 4.1 gives, for each $y \in \partial\Omega$, a sequence of linear polynomials $(P_k(\cdot; y))$ and constants $\tau < 1$ and ρ such that

$$\text{osc}_{\Omega[y, \tau^k \rho]} (u - P_k(\cdot; y)) \leq C\zeta(\tau^k \rho)\tau^k \rho.$$

The main modification in the proof is the estimate of $G_k \bar{\zeta}(\tau^m \bar{\rho})$, where $\bar{\zeta} = \zeta_2 + I(\zeta_1)$. For this estimate, we note that

$$\bar{\zeta}(\tau^m \bar{\rho}) \frac{M_j}{\rho_j} = HS_j \zeta(\tau^j \bar{\rho}) \bar{\zeta}(\tau^m \bar{\rho}) \leq HS_j \zeta(\tau^m \bar{\rho}) \bar{\zeta}(\tau^j \bar{\rho})$$

because $\zeta/\bar{\zeta}$ is decreasing. Therefore, for $x_0 \in \Omega[3R/4]$ (which will remain fixed for now) and $\rho = d(x_0)/2$, there is a linear polynomial P such that

$$\text{osc}_{B(x_0, \rho)} (u - P) \leq C\rho\zeta(\rho).$$

Using our pointwise estimate on Du , we then infer from the interior Hölder gradient estimate (proved in much the same way as Theorem 4.1) that

$$|Du(x) - Du(y)| \leq C \frac{\zeta_1(\rho)}{\rho} \left(\frac{|x - y|}{\rho} \right)^\sigma$$

for x and y in $B(x_0, 3\rho/4)$. It follows that $\bar{u} = u - P$ is a solution of the equation $\bar{F}(x, D^2\bar{u}) = 0$ in $\Omega[R/2]$ with $\bar{F}(x, r) = F(x, u(x), Du(x), r)$ and \bar{F} satisfies the structure condition

$$|\bar{F}(x, r) - \bar{F}(y, r)| \leq C\lambda\zeta_5\left(\frac{|x - y|}{\rho}\right)(|r| + 1)$$

with $\zeta_5(s) = \zeta_3(s^\sigma(\rho + \zeta(\rho))/\rho)$. The interior second derivative estimates (which are obtained via interpolation from the second derivative modulus of continuity estimates in [9]) then imply that

$$|D^2\bar{u}(x)| \leq C\left(\zeta_5(1) + \frac{\zeta(\rho)}{\rho}\right) \leq C\frac{\zeta(\rho)}{\rho}$$

for $x \in B(x_0, \rho/2)$ because $t = (\rho + \zeta(\rho))/\rho \geq 1$ and $\zeta_5(1) = \zeta_3(t) \leq \zeta_3(1)t$. It then follows that

$$|D^2u| \leq \frac{\zeta(d)}{d}, \quad |Du| \leq \tilde{I}(\zeta)(d).$$

Next, [14, Theorem 4.2(a)] gives a function $\bar{\beta} \in C^2(\Omega[R]) \cap C(\overline{\Omega[R]})$ such that $\bar{\beta} = \beta/\beta^n$ on $\partial\Omega$ and $|D\bar{\beta}|/d + |D^2\bar{\beta}| \leq C(B_0, \Omega, R)\zeta_2(d)/d^2$. We wish to prove a modulus of continuity estimate for $w = \bar{\beta} \cdot Du$ at points of $\partial\Omega$. This estimate will be proved in steps.

The first step is a pointwise bound on w which does not become infinite near $\Sigma[R]$. To prove this estimate, we define

$$\begin{aligned} a^{ij}(x) &= F^{ij}(x, u, Du, D^2u)/\lambda, \quad b^i(x) = F^i(x, u, Du, D^2u)/\lambda, \\ c(x) &= F_z(x, u, Du, D^2u)/\lambda, \end{aligned}$$

and write L for the operator given by $Lv = a^{ij}D_{ij}v + b^iD_iv + cv$. It then follows that

$$Lw = F^{ij}D_{ij}\bar{\beta}^k D_k u + 2F^{ij}D_i\bar{\beta}^k D_{jk}u + F^iD_i\bar{\beta}^k D_k u - \frac{\partial F}{\partial x^k}\bar{\beta}^k,$$

and hence w satisfies the differential inequality $|Lw| \leq C\zeta_4(d)d^{-2}$. Since ζ_4 is Dini, there are positive constants $R_0 \leq 1/2$ and C_1 such that $w_2 = J_\sigma(\zeta_4)(w_1)$ (see Lemma 1.1 for a discussion of J_σ) satisfies the inequality $Lw_2 \leq -C_1\zeta_4(d)d^{-2}$ in $\Omega' = \{x \in \Omega[3R/4] : d(x) < R_0\}$. Now, we let φ be a nonnegative C^2 function with compact support in $B(0, 3R/4)$ which is identically 1 in $B(0, 5R/8)$. Then $\bar{w} = \varphi w$ satisfies the inequality

$$\begin{aligned} |L\bar{w}| &\leq C\left(\frac{\zeta_4(d)}{d} + \frac{\zeta_4(d)}{d^2} + \frac{\zeta(d)}{d} + \tilde{I}(\zeta)(d)\left[\frac{\zeta(d)}{d} + 1 + \tilde{I}(\zeta)(d) + \frac{\zeta_2(d)}{d^2}\right]\right) \\ &\leq C\frac{\zeta_4(d)}{d^2} \end{aligned}$$

in Ω' because $\tilde{I}(\zeta)(d) \leq C |\ln d|$ for $d \leq 1/2$. Moreover, $|\bar{w}| \leq G_0$ on $\Sigma[3R/4]$ and there is a constant C_0 such that $|\bar{w}(x)| \leq C_0$ for $x \in \Omega[3R/4]$ with $d(x) = R_0$. It follows that there is a constant A so that

$$\begin{aligned} L(\pm\bar{w} + C_0 + G_0 + Aw_2) &\leq 0 \quad \text{in } \Omega', \\ \pm\bar{w} + C_0 + G_0 + Aw_2 &\geq 0 \quad \text{on } \partial\Omega'. \end{aligned}$$

The maximum principle then implies that $|\bar{w}| \leq C_0 + G_0 + Aw_2$ in Ω' and hence $|w(x)| \leq C$ if $x \in \Omega[5R/8]$ and $d(x) < R_0$.

The next step is the modulus of continuity estimate for w . Let us fix $\varepsilon > 0$ and choose $\delta_1 > 0$ so that $\zeta(\delta_1) < \varepsilon/2$. We also note that there is a constants A_1 and A_2 such that $L(A_1w_2 + |x - y|^2) \leq 0$ and $L(\pm(w - g(y)/\beta^n(y))) \leq A_2\zeta_4(d)/d^2$ for any $y \in \Sigma[R/2]$. It follows that

$$\begin{aligned} L\left(\pm\left(w - \frac{g(y)}{\beta^n(y)}\right) + \frac{\varepsilon}{2} + \frac{\zeta(1)}{\delta_1^2}(A_1w_2 + |x - y|^2) + A_2w_2\right) &\leq 0 \quad \text{in } \Omega'', \\ \left(\pm\left(w - \frac{g(y)}{\beta^n(y)}\right) + \frac{\varepsilon}{2} + \frac{\zeta(1)}{\delta_1^2}(A_1w_2 + |x - y|^2)\right) + A_2w_2 &\geq 0 \quad \text{on } \partial\Omega'', \end{aligned}$$

and hence there is a δ_2 (determined by the same quantities as δ) so that $|w(x - g(y)/\beta^n(y))| < 3\varepsilon/4$ if $|x - y| < \delta_2$.

Next, as in the proof of [17, Theorem 4.2], we note that for $y \in \partial\Omega$ and t positive and sufficiently small, there is $s \in (0, t)$ such that

$$(6.7) \quad \frac{u(y + t\beta(y)) - u(y)}{t} = \beta(y) \cdot Du(y + s\beta(y))$$

and from our modulus of continuity estimate for w , we have that

$$\left|\bar{\beta}(y') \cdot Du(y') - \frac{g(y)}{\beta^n(y)}\right| \leq \frac{3\varepsilon}{4\beta^n(y)}$$

for $y' = y + s\beta(y)$ and $s < \delta_2\beta^n(y)$ because $d(y') \leq Cs$. Therefore

$$\begin{aligned} |\beta(y) \cdot Du(y') - g(y)| &\leq |\beta(y) - \bar{\beta}(y')||Du(y')| + |\bar{\beta}(y') \cdot Du(y') - g(y)| \\ &\leq C\zeta_2(s)\tilde{I}(\zeta)(s) + 3\varepsilon/4, \end{aligned}$$

and the proof is completed by combining this estimate with (6.7). □

Let us note first that the hypothesis $u \in C^1(\overline{\Omega[R]})$ in Theorem 6.1 can be relaxed to $u \in C^0(\overline{\Omega[R]})$. To see why this is so, we first note that if F and g are smooth (in particular g is Dini continuous and F is globally Lipschitz with respect to its variables), then the modulus of continuity estimate for Du in Section 4 shows that $u \in C^1(\overline{\Omega[\rho]})$ for any $\rho < R$. Let us fix $\rho \in (0, R)$. Now let (F_k) and (g_k) be sequences of smooth functions converging uniformly

to F and g , respectively, and satisfying the hypotheses of Theorem 6.1 with constants independent of k . It is not hard to show that the problems

$$\begin{aligned} F_k(x, u_k, Du_k, D^2u_k) &= 0 && \text{in } \Omega[\rho], \\ \beta \cdot Du_k + u - u_k &= g_k && \text{on } \Sigma[\rho], \quad u = u_k && \text{on } E^+(\rho) \end{aligned}$$

are uniquely solvable for u_k and that $u_k \rightarrow u$ uniformly. Because the constant δ does not depend on k , we see that u also satisfies estimate (6.6).

Moreover, we can infer an existence theorem from our uniform estimate.

COROLLARY 6.2. *Let Ω be a bounded, Lipschitz domain, and suppose F satisfies (4.1) and is concave (or convex) with respect to r . Suppose also that there are nonnegative constants ν_1 and μ_0 along with a continuous, increasing function ζ_0 with $\zeta_0(0) = 0$, a D_1 function ζ_1 , and a θ -decaying, Dini function ζ_3 such that conditions (6.1) and (6.3) are satisfied. Suppose also that β is a continuous, oblique vector field defined on $\partial\Omega$ with modulus of obliqueness $\delta < 1$ and that are nonnegative constants B_0 and G_0 along with a D_1 function ζ_2 and a continuous, increasing function ζ with $\zeta(0) = 0$ and $\zeta \geq I(\zeta_1)$ such that (4.4) holds and*

$$(6.8a) \quad |\beta(x) - \beta(y)| \leq B_0 \zeta_2(|x - y|) |\beta(x)|,$$

$$(6.8b) \quad |g(x) - g(y)| \leq G_0 \zeta(|x - y|) |\beta(x)|$$

for all x and y in $\partial\Omega$. Suppose finally that conditions (6.4) and (6.5) hold. then there is a unique solution $u \in W_{\text{loc}}^{2,n}(\Omega) \cap C^0(\overline{\Omega})$ of

$$(6.9) \quad F(x, u, Du, D^2u) = 0 \text{ in } \Omega, \quad \beta \cdot Du - u = g \text{ on } \partial\Omega.$$

PROOF. We first note that uniqueness is an easy consequence of the maximum principle.

Our next step is existence when ζ is Dini. In this case, we use an approximation argument. When $\partial\Omega \in C^4$, $F \in C^4$, and β and g are C^2 , we can use [23, Theorem 7.9] to infer existence provided conditions $F1$ – $F5$ of [23] are satisfied, and these conditions hold for functions F_k which converge uniformly to F and which satisfy conditions (6.1) and (6.3) uniformly. From these conditions, we infer that the solutions of the approximating problems converge in $C^1(\overline{\Omega})$ to a solution of the limit problem. Note that conditions (6.5a,b,c) are not needed for this existence result.

Finally, existence for arbitrary ζ satisfying our hypotheses follows by letting g_k be a sequence of Dini continuous functions which converge uniformly to g and noting that the corresponding solutions u_k converge uniformly to a solution of (6.9). The boundary condition for u is verified by using Theorem 6.1 and observing that we have a uniform C^σ estimate for u_k . \square

We point out that the boundary condition $\beta \cdot Du - u = g$ can be replaced by $\beta \cdot Du = g(x, u)$ provided $\partial g / \partial z$ is continuous and strictly negative. Moreover, the existence part of this corollary is true under much weaker conditions (see, for example [23, Section 7]); the main point is that suitable *a priori* estimates must be available.

For linear equations, we can remove the conditions on the derivatives of the coefficients entirely.

COROLLARY 6.3. *Let L be the operator defined by $Lu = a^{ij} D_{ij}u + b^i D_i u + cu$ and suppose there are positive constants μ and μ_2 along with a continuous, increasing function ζ_0 and D_1 functions ζ_2 and ζ_3 such that condition (2.3) holds,*

$$(6.10a) \quad |a^{ij}(x) - a^{ij}(y)| \leq \zeta_0(|x - y|),$$

$$(6.10b) \quad |b(x)| + |c(x)| + |f(x)| \leq \mu_2 \zeta_2(d(x))/d$$

for all x and y in $\Omega[R]$ and

$$(6.11a) \quad |a^{ij}(x) - a^{ij}(y)| \leq \zeta_3 \left(\frac{|x - y|}{d(x)} \right),$$

$$(6.11b) \quad |b^i(x) - b^i(y)| \leq \zeta_3 \left(\frac{|x - y|}{d(x)} \right) \frac{1}{d(x)},$$

$$(6.11c) \quad |c(x) - c(y)| \leq \zeta_3 \left(\frac{|x - y|}{d(x)} \right) \frac{1}{d(x)^2}$$

for all x and y in $\Omega[R]$ with $|x - y| \leq d(x)/2$. Suppose also that β and g satisfy conditions (2.6b), (4.4), and (6.2). If $u \in W_{loc}^{2,n}(\Omega[R]) \cap C^0(\overline{\Omega[R]})$ is a solution of $Lu = f$ in $\Omega[R]$, $\beta \cdot Du = g$ on $\Sigma[R]$, then for any $\varepsilon > 0$, there is $\delta > 0$ determined only by $\zeta, \zeta_0, \zeta_2, \zeta_3, B_1, G_0$, and μ_2 such that (6.6) holds.

PROOF. The crucial step is to notice that there is a positive-definite matrix-valued function A^{ij} such that $|A^{ij} - a^{ij}| \leq \zeta_0(d)$ and $|\partial A^{ij} / \partial x| \leq C \zeta_0(d)/d$. Then (as in [17]), we let v solve

$$A^{ij} D_{ij}v = 0 \text{ in } \Omega[3R/4], \quad \beta \cdot Dv = g \text{ on } \Sigma[3R/4], \quad v = u \text{ on } E_+(3R/4).$$

We easily obtain that $\sup|v| \leq \sup|u|$ and that, for any $\varepsilon > 0$, there is a δ_0 such that

$$\left| \frac{v(y + \beta(y)t) - v(y)}{t} - g(y) \right| \leq \frac{1}{2} \varepsilon \text{ whenever } 0 < t < \delta_0$$

for any $y \in \Sigma[R/2]$. Then $U = u - v$ satisfies $A^{ij} D_{ij}U = f^*$ in $\Omega[3R/4]$, $\beta \cdot DU = 0$ on $\Sigma[3R/4]$ for

$$f^* = f + [A^{ij} - a^{ij}] D_{ij}u - b^i D_i u - cu + f.$$

The interior estimates for Du and D^2u imply that $|f^*| \leq C \zeta_4(d)/d$ and hence we have a modulus of continuity estimate for DU up to $\Sigma[R/2]$. Combining the resultant estimate for $||U(y + t\beta(y)) - U(y)||/t$ with our corresponding estimate for v yields the desired result. □

Note that, if $c < 0$, then we obtain an existence theorem along the lines of [17, Theorem 4.2]. We leave its statement and proof to the reader.

Our hypotheses (specifically (6.4)) connect the functions ζ and ζ_2 . If we want a condition on ζ_2 which is independent of the particular function ζ , we note first that $\tilde{I}(\zeta)(d) \leq C |\ln d|$. Next, we take advantage of the following relations between Dini functions and the logarithm function.

LEMMA 6.4. *Let ζ be a Dini function. Then*

$$(6.12) \quad \lim_{t \rightarrow 0^+} \zeta(t) \ln t = 0,$$

and ζ_1 defined by

$$(6.13) \quad \zeta_1(s) = -J(\zeta)(s) \ln s + I(\zeta)(s)$$

is increasing. If $I(\zeta)$ is Dini, then so is ζ_1 , and if ζ is α -decreasing for some $\alpha \in (0, 1]$, then so is ζ_1 .

PROOF. Let $a \in (0, 1)$ and note that $-J(\zeta)(a) \ln a \geq 0$. Then integration by parts yields

$$\int_a^1 \frac{J(\zeta)(s)}{s} ds = -J(\zeta)(a) \ln a - \int_a^1 J(\zeta)'(s) \ln s ds.$$

Since the integrand on the right hand side of this inequality is negative and the integral on the left hand side converges, it follows that the integral on the right hand side also converges. Hence, the limit $K = \lim_{t \rightarrow 0^+} J(\zeta)(s) \ln s$ exists and it is clearly nonpositive. If $K < 0$, then there is a $\delta \in (0, 1)$ such that $J(\zeta)(s) \geq |K|/(2|\ln s|)$ for $0 < s \leq \delta$. But this inequality contradicts the hypothesis that ζ is Dini, so $K = 0$ and hence the limit in (6.12) must be zero.

A direct calculation shows that $\zeta_1' \geq 0$, so ζ_1 is increasing.

Now suppose $I(\zeta)$ is Dini and let $a \in (0, 1)$. Then integration by parts yields

$$\int_a^1 \frac{\zeta(s) \ln s}{s} ds = -I(\zeta)(a) \ln a - \int_a^1 \frac{I(\zeta)(s)}{s} ds.$$

Since $I(\zeta)$ is Dini, the right hand side of this equation has a limit as $a \rightarrow 0^+$, and hence ζ_1 is Dini.

Suppose finally that ζ is α -decreasing. Then, $I(\zeta)$ is α -decreasing by Lemma 1.1(c). Moreover the function ζ_2 defined by $\zeta_2(s) = -J(\zeta)(s) \ln s$ is the product of an α -decreasing function and a decreasing function, so it is also α -decreasing. Hence ζ_1 , being the sum of two α -decreasing functions, is also α -decreasing. \square

From this lemma, it follows that condition (6.4) is satisfied for any ζ provided ζ_2 is a D_2 function and ζ_0 is Dini. These hypotheses were used in [24]. On the other hand, (6.4) allows combinations of $\zeta, \zeta_0, \dots, \zeta_4$ with ζ not Dini and ζ_2 not D_2 . In particular, suppose there are constants δ and ε in the interval $(0, 1)$ such that

$$\zeta(s) = \zeta_0(s) = \frac{2}{[\ln(1/s)]^\delta}, \quad \zeta_2(s) = \frac{1}{[\ln(1/s)]^{1+\varepsilon}}$$

for $s \in (0, 1/2)$. In this case, ζ and ζ_0 are not Dini, and ζ_2 is in D_1 but not D_2 . Elementary calculation shows that

$$\tilde{I}(\zeta)(s) = \frac{1}{1-\delta} \left([\ln(1/s)]^{1-\delta} - [\ln 2]^{1-\delta} \right),$$

so condition (6.4) holds if $\delta + \varepsilon > 1$ and $\delta > 1/2$. Moreover, if $\zeta_1 = \varepsilon\zeta_2$, it is easy to check that $\zeta/(\zeta_2 + I(\zeta_1))$ is decreasing and greater than or equal to 1.

7. – Parabolic oblique derivative problems

The underlying principle of converting from elliptic to parabolic problems, in this context, is relatively simple (see the discussion in [20] compared to that in [7] for a more detailed description of the procedure). The crucial differences are that we use Ω to denote a domain in \mathbb{R}^{n+1} and we label points $X = (x, t)$. We refer to [20, pp. 7, 13] for the definitions of the parabolic boundary $\mathcal{P}\Omega$, the lateral boundary $S\Omega$, and the initial surface $B\Omega$. The parabolic distance between $X = (x, t)$ and $Y = (y, s)$ is then

$$|X - Y| = \max\{|x - y|, |t - s|^{1/2}\},$$

and we replace the ball $B(x, R)$ by the cylinder

$$Q(X, R) = \{(y, s) \in \mathbb{R}^{n+1} : |X - Y| < R, s < t\}.$$

We then write $\Omega[X, R]$ and $\Sigma[X, R]$ for the intersections of Ω and $S\Omega$, respectively, with $Q(X, R)$. In addition, we define

$$d(X) = \inf\{|X - Y| : Y \in S\Omega, s < t\},$$

$$d^*(X) = \inf\{|X - Y| : Y \in \mathcal{P}(\Omega[R]) \setminus S\Omega, s < t\}.$$

The analog of condition (1.3) is that there is a function ω such that

$$(7.1) \quad \Omega[R] = \{X : x^n > \omega(X') : |X| < R, t < 0, |\omega(X') - \omega(Y')| \leq \omega_0\}.$$

We then say (compare with [19, p. 26]) that Ω satisfies an exterior θ_0 -tusk condition at $X_0 \in \partial\Omega$ (for $\theta_0 \in (0, \pi/2)$) if there is a vector $x_1 \in \mathbb{R}^n$ such that

$$(t_1 - t)^{1/2} < \tan \theta_0 \left| x - x_0 - \frac{|X - X_0|}{2^{1/2}|x_1|} x_1 \right|$$

for $X \in \Omega$.

The parabolic analogs of [26, Theorem 3] and [25, Theorem 3.7] are [19, Lemma 12.1] and the remarks following [19, Lemma 13.1], respectively, and the parabolic analog of Aleksandrov’s maximum principle is [20, Theorem 7.1], the appropriate weak interior Harnack inequality is [20, Theorem 7.22] (but the correct Morrey space for b/λ should be $m^{n+1,1}$ there; see [22, Section 7] for details), and the boundary weak Harnack inequality is [22, Theorem 7.5]. With these results and noting that a linear polynomial is one of the form $P(x, t) = A \cdot x$, we can follow the program in Sections 1-4 to prove the following parabolic version of Theorem 4.1.

THEOREM 7.1. *Let Ω satisfy (7.1) and an exterior θ_0 -tusk condition, and suppose F satisfies*

$$(7.2) \quad \lambda|\xi|^2 \leq F^{ij}(X, z, p, r)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for all $(X, z, p, r) \in \Omega[R] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ and all $\xi \in \mathbb{R}^n$, and is concave (or convex) with respect to r . Suppose also that there are nonnegative constants ν_1 and μ_0 along with a nonnegative, continuous increasing function ζ_0 with $\zeta_0(0) = 0$, a D_2 function ζ_1 and a nonnegative function b such that

$$(7.3a) \quad |F(X, z, p, 0)| \leq \lambda[\nu_1|p|^2 + b|p| + \mu_0\zeta_1(d(X)/d^*(X))/d(X) + f],$$

$$(7.3b) \quad |F(X, z, p, r) - F(Y, w, 0, r)| \leq \lambda\zeta_0(|X - Y|/d^*(X))|r| + \lambda[\nu_1|p|^2 + b|p| + f(X) + f(Y)]$$

for all X and Y in $\Omega[R]$ with $|X - Y| < d^*(X)/2$, all z and w in \mathbb{R} , all p and q in \mathbb{R}^n and all $r \in \mathbb{S}^n$. Suppose finally that there are nonnegative functions $b_1, b_2, f_1,$ and f_2 along with nonnegative constants B and F_0 , a D_1 function ζ , and a D_2 function ζ_1 such that $b = b_1 + b_2, f = f_1 + f_2,$ and

$$(7.4a) \quad \|b_1\|_{n+1; \Omega[X, \rho]} \leq B\rho\zeta\left(\frac{\rho}{d^*(X)}\right), \quad b_2 \leq B\frac{\zeta(d/d^*)}{d^*},$$

$$(7.4b) \quad \|f_1\|_{n+1; \Omega[X, \rho]} \leq F_0\rho\zeta\left(\frac{\rho}{d^*(X)}\right), \quad f_2 \leq F_0\frac{\zeta(d/d^*)}{d^*},$$

for all $X \in \Omega[R]$ and $\rho \leq R$ Suppose finally that there are nonnegative constants B_0 , G_0 , and ε such that

$$(7.5a) \quad |\beta(X)| \leq \frac{1 - \varepsilon}{\omega_0} \beta^n(X),$$

$$(7.5b) \quad |\beta(X) - \beta(Y)| \leq B_0 \zeta \left(\frac{|X - Y|}{d^*(X)} \right) \beta^n(X),$$

$$(7.5c) \quad |g(X) - g(Y)| \leq G_0 \zeta \left(\frac{|X - Y|}{d^*(X)} \right) \beta^n(X),$$

$$(7.5d) \quad |g(X)| \leq G_0 |\beta(X)|$$

for all X and Y in $\Sigma[R]$ with $|X - Y| < d^*(X)/2$ and all $\rho \in (0, d^*(X)/2)$. If $u \in C^0(\overline{\Omega[R]}) \cap W_{n+1, \text{loc}}^{2,1}(\Omega[R])$ satisfies

$$(7.6) \quad F(X, u, Du, D^2u) = 0 \text{ in } \Omega[R], \quad \beta \cdot Du = g \text{ on } \Sigma[R],$$

then u has continuous gradient. Specifically, if we define H by (4.7), then (4.8) holds with $\zeta_2 = I(\zeta + I(\zeta_1))$.

The parabolic analog of Theorem 5.4 is left to the reader. We note here only that, in place of (5.10), the resulting estimate is that

$$|D^2u(X) - D^2u(Y)| + |u_t(X) - u_t(Y)| \leq CI(\zeta) \left(\frac{|X - Y|}{d^*(X)} \right) \frac{1}{d^*(X)}$$

for $|X - Y| \leq d^*(X)/2$.

Finally, the results of Section 6 have their obvious parabolic analogs, which extend the results in [24] and [19, Section 14].

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