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Fibered Microstructures for Some Nonlocal Dirichlet Forms

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Abstract. In this article we study the homogenization of some fibered microstructures in order to obtain prescribed nonlocal effects from strongly local conduction problems in a bounded open set Ω of $\mathbb{R}^3$. According to the Beurling-Deny formula these nonlocal effects are represented by a so-called jumping measure defined on the product $\Omega \times \Omega$. In particular we reach the measures of type $j(dx, dy) = \mathbf{1}_E(dx) \otimes \mathbf{1}_E(dy)$ where E is a smooth open subset of Ω. If the set E is connected the starting microstructure is only composed of high conductivity fibers. If the set E is not connected we also need a mixture of high and low conductivity fibers in the regions separating the components of E.

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1. – Introduction

This article is devoted to the asymptotic behaviour of quadratic strongly local forms on a bounded domain Ω of $\mathbb{R}^3$, of the type

$$F_\varepsilon(u) := \int_\Omega a_\varepsilon\, |\nabla u|^2, \quad \text{for } u \in H^1_0(\Omega)\,, \tag{1}$$

where a_ε is a positive sequence from $L^\infty(\Omega)$ which is not uniformly bounded. Using the asymptotic theory of the so-called Dirichlet forms established by Mosco [21], the sequence F_ε converges as ε tends to 0 (in the sense of the Γ-convergence, see [2], [13], [14] and more recently [18]) to the functional

$$F(u) := \int_\Omega A(dx)\nabla u \cdot \nabla u + \int_\Omega u^2\, k(dx) + \int_{\Omega\times\Omega} (u(x) - u(y))^2\, j(dx, dy)\,, \quad \text{for } u \in C^1_0(\Omega)\,, \tag{2}$$

where the first term is the strongly local diffusion part, the second term is local but not strongly and the third one is nonlocal, according to the Beurling-Deny

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representation formula of the Dirichlet form (see [5], [4], [16] and [11] for the complete theory). In (2), k is called the killing measure and j the jumping measure.

Khruslov [10], [15] has first given a quite general class of a_ε for which nonlocal terms arise (for other kinds of nonlocal effects see Khruslov [15] and the references therein, Tartar [22], Dal Maso, Gulliver, Mosco [12] and Bouchitté, Picard [7]). More recently, Bellieud and Bouchitté [3] deeply studied one of the examples from [10] (also extending it to the nonlinear framework of the p-laplacian). In this three-dimensional example, the conductivity law a_ε is equal to 1 except in a region ω_ε of very small measure where $a_\varepsilon = \alpha_\varepsilon \gg 1$. The set ω_ε is a ε-periodic lattice of very thin incrossing fibers which are parallel to one of three orthogonal directions of the space $\mathbb{R}^3$; each fiber from ω_ε is a long cylinder of radius $\varepsilon\, r_\varepsilon$, with $r_\varepsilon \ll 1$, which crosses the domain Ω. For this example, the limiting functional F of (2) obtained in [10] and [3] is defined by the following measures:

$$(3)\quad k(dx) = \gamma\left(1 - \int_\Omega G(x,y)\,dy\right) \quad\text{and}\quad j(dx,dy) = \frac{\gamma}{2}\,G(x,y)\,dx\,dy\,,$$

where $\gamma := \lim_{\varepsilon\to 0} \frac{6\pi}{\varepsilon^2|\log r_\varepsilon|}$ and the kernel G is the fundamental solution of the problem

$$(4)\quad \begin{cases} -\beta\,\Delta G(x,\cdot) + \gamma\,G(x,\cdot) = \gamma\,\delta_x & \text{in } \Omega \\ G(x,\cdot) = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with $\beta := \lim_{\varepsilon\to 0} \pi\alpha_\varepsilon r_\varepsilon^2$.

The motivation in the previous works [10] and [3], was to point out asymptotic nonlocal terms thanks to appropriate microstructures. Our aim is now to better understand the links between the microstructures and the nonlocal effects. More precisely, we are interested by the following inverse problem: starting from a prescribed nonlocal term with a given jumping measure j, we try to find a suitable microstructure which asymptotically yields the desired measure j.

From a general point of view, this inverse problem is far to be evident. First of all, it is not proved that any Dirichlet form associated with F in (2) is a Γ-limit of a sequence of strongly local forms of type (1). However, the Γ-closure of the Dirichlet Forms has been obtained by Mosco in [20]. But the general result from [20] does not provide any constructive method to obtain the limit forms. Indeed, it is divised in two quite independent steps: the asymptotic behaviour of the functional F_ε of (1) and the representation formula (2) which is proved thanks to the properties of the limit functional. This procedure is based on arguments of functional analysis and does not allow to directly express the measures k and j from (2) with respect to F_ε. In our work we give results for some specific jumping measures. The first jumping measure we want to achieve is the Lebesgue measure as in the first step of the domain relaxation problems (see [2], [8] and [19]).

In the first part of the article, we consider a microstructure which allows us to achieve the Lebesgue measure $j(dx, dy) = \frac{1}{2|\Omega|} dx\, dy$, as well as any measure of type

$$j_E(dx, dy) := \frac{1}{2|E|} \mathbf{1}_E(dx) \otimes \mathbf{1}_E(dy), \tag{5}$$

where E is a smooth connected open subset of Ω. In terms of the limiting behaviour of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where f is a given function of $L^2(\Omega)$, we prove (see Theorem 1) that there exists a sequence $a_\varepsilon \geq 1$ such that u_ε weakly converges in $H_0^1(\Omega)$ to the solution u_0 of the equation

$$-\Delta u_0 + u_0 - \overline{u_0} = f \quad \text{in } \Omega, \quad \text{where } \overline{u_0} := \frac{1}{|\Omega|} \int_\Omega u_0 . \tag{7}$$

In fact, the choosen microstructure (which gives the jumping measure $\frac{1}{2|\Omega|} dx\, dy$ or equivalently the limit equation (7)) is very similar to the fibered microstructure from [10] or [3], previously described. However, in our model the fibers of high conductivity are stopped at a very small distance $d_\varepsilon \ll 1$ of the boundary of Ω. We also assume that $\beta = +\infty$. In the framework of [3], the previous condition implies that the solution G of (4) is equal to 0, and thus leads (with $\gamma = 1$) to the killing measure $k(dx) = dx$ without jumping measure, or equivalently to the limiting equation

$$-\Delta u_0 + u_0 = f \quad \text{in } \Omega . \tag{8}$$

The result (7) is apparently surprising since a small modification of the microstructure (very close to the boundary of the domain) completely changes the limit behaviour of the Dirichlet problem (6) by passing from (8) to (7). In fact, the gap of conductivity ($1 \ll \alpha_\varepsilon$) at the fibers ends behaves as a Neumann boundary condition which leads to the constant term $\overline{u_0}$ in (7). On the contrary, when the fibers touch the boundary as in [3], the Dirichlet condition leads to the constant 0 in (8) (see Remark 2).

In the second part of the article, we extend the result to a jumping measure of type j_E (5) when E is no more connected, and in particular when $E := A \cup B$ has two connected components A and B. Contrary to the connected case, we did not succeed to achieve this jumping measure under the equicoerciveness condition $a_\varepsilon \geq 1$. But, by introducing new fibers of low conductivity $a_\varepsilon = \varepsilon \ll 1$ in the region between A and B, we prove (see Theorem 4) that the solution

u_ε of problem (6) weakly converges in $L^2(\Omega)$ to the solution $u_0 \in H_0^1(\Omega)$ of the equation

$$(9) \qquad -\Delta u_0 + \mathbf{1}_{A\cup B}\,(u_0 - \overline{u_0}^{A\cup B}) = f \quad \text{in } \Omega\,,$$

which also yields the jumping measure $j_{A\cup B}$ from (5).

However, we are far from attaining the initial aim, namely obtaining any prescribed measure j. By extending the previous results, we can approach any measure of type $\nu(dx) \otimes \nu(dy)$ where ν is a positive Borel measure on Ω absolutely continuous with respect to the capacity (see Remark 6). But another kinds of measure, even very simple, seem to be out of reach. In particular, we do not succeed to obtain any jumping measure of type

$$(10) \qquad \frac{1}{2|A||B|}\,(\mathbf{1}_A(dx) \otimes \mathbf{1}_B(dy) + \mathbf{1}_B(dx) \otimes \mathbf{1}_A(dy))\ ,$$

where A and B are two regular open subset of Ω with disjoint closure, which corresponds to the strange limit equation of (6)

$$(11) \qquad -\Delta u_0 + \frac{\mathbf{1}_A}{|A|}\,(u_0 - \overline{u_0}^B) + \frac{\mathbf{1}_B}{|B|}\,(u_0 - \overline{u_0}^A) = f \quad \text{in } \Omega\,.$$

2. – The case of a measure $\mathbf{1}_E \otimes \mathbf{1}_E$ with E a connected set

2.1. – Description of the geometry

POSITION OF THE PROBLEM. Let Ω be a bounded connected domain of $\mathbb{R}^3$, with a smooth boundary of class C^1. We consider, for $\varepsilon > 0$, a set ω_ε of three ε-periodic lattices of cylinders in Ω. Each lattice is composed of a large number of open cylinders distant from ε, of radius $\varepsilon\, r_\varepsilon$ ($r_\varepsilon \ll 1$), and parallel to the axis x_i, $i = 1\ldots3$. We assume that the axes of two different orthogonal lattices ε-periodically intersect. The period cell of ω_ε is thus composed of three incrossing cylinders of length ε, which are centered at the center of a small cube of side ε as shown in figure 1.
We also assume that the cylinders cross the whole domain but not intersect the boundary $\partial\Omega$, more precisely

$$(12) \qquad 0 < d_\varepsilon := \text{dist}(\omega_\varepsilon, \partial\Omega) \ll 1\,.$$

Finally, we suppose that the set ω_ε is composed of entire periodic cells as shown in figure 2.
The following figure 3 shows a section of Ω by a vertical plane which contains the axes of some fibers from ω_ε.

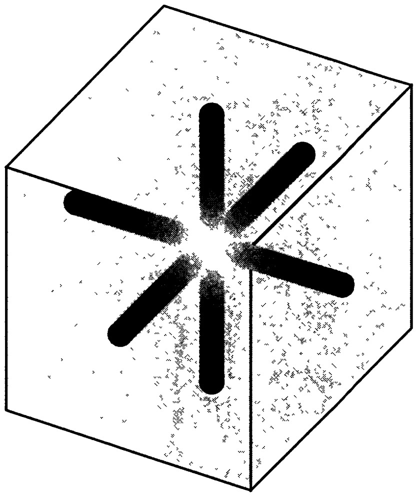
Fig. 1. The period cell

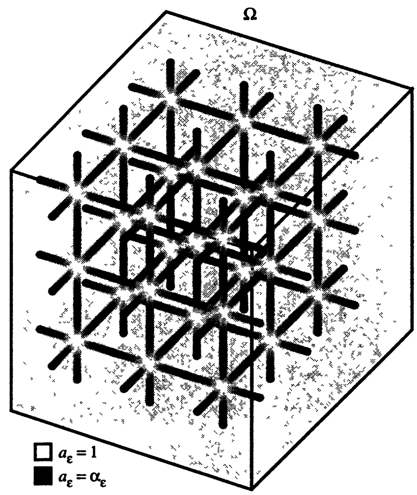

Fig. 2. The fibered structure

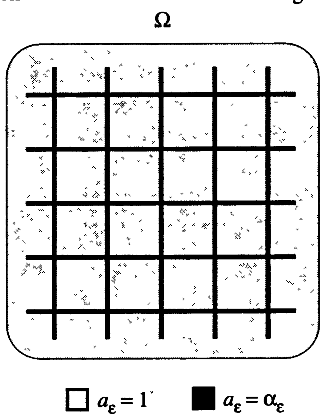

Fig. 3. A cross section of the fibered structure

2.2. – Statements of the results (and extensions)

Starting with this geometry we will consider a conduction problem with high conductivity fibers. Let a_ε be the function defined

$$a_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \backslash \omega_\varepsilon \\ \alpha_\varepsilon & \text{if } x \in \omega_\varepsilon , \end{cases} \tag{13}$$

where $\alpha_\varepsilon \gg 1$.

We consider the conduction problem:

$$\begin{cases} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega , \end{cases} \tag{14}$$

where f is a given function $f \in L^2(\Omega)$.

Our aim is to give the precise asymptotic behaviour of (14), according to the geometrical parameters $\varepsilon, r_\varepsilon, d_\varepsilon$ and α_ε.

THEOREM 1. *Let us assume that*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \,|\log r_\varepsilon| = 6\pi\,, \tag{15}$$

and

$$\lim_{\varepsilon\to 0} \alpha_\varepsilon r_\varepsilon^2 = +\infty\,. \tag{16}$$

Then, for a suitable choice of the distance d_ε in (12), *the solution u_ε of problem* (14) *weakly converges in $H^1(\Omega)$ to the solution u_0 of the problem*

$$\begin{cases} -\Delta u_0 + u_0 - \overline{u_0} = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{17}$$

where

$$\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0 := \fint_\Omega u_0\,.$$

In terms of the Γ-convergence of Dirichlet-forms, the previous result can be written as follows.

COROLLARY 1. *Let f_ε be the Dirichlet form defined by*

$$f_\varepsilon(u,v) := \int_\Omega a_\varepsilon \nabla u\cdot\nabla v\,,$$

for any $u, v \in D(f_\varepsilon) = H_0^1(\Omega) \subset L^2(\Omega)$. Let F_ε be the functional associated to f_ε and defined by

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega a_\varepsilon\,|\nabla u|^2 & \text{if} \quad u\in H_0^1(\Omega) \\ +\infty & \text{if} \quad u \in L^2(\Omega)\backslash H_0^1(\Omega)\,. \end{cases} \tag{18}$$

Then, F_ε Γ-converge to F_0 for the strong topology of $L^2(\Omega)$, where F_0 is defined by

$$F_0(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 + \frac{1}{2|\Omega|}\int_{\Omega\times\Omega}(u(x)-u(y))^2 & \text{if} \quad u\in H_0^1(\Omega) \\ +\infty & \text{if} \quad u\in L^2(\Omega)\backslash H_0^1(\Omega)\,, \end{cases} \tag{19}$$

(F_0 is the functional associated to the Dirichlet form

$$f_0(u,v) := \int_\Omega \nabla u\cdot\nabla v + \frac{1}{2|\Omega|}\int_{\Omega\times\Omega}(u(x)-u(y))(v(x)-v(y))\,,$$

for any $u, v \in H_0^1(\Omega)$.)

REMARK 1. In [3] Bellieud et Bouchitté, following a preceding result obtained by Khruslov [15], consider a similar lattice of fibers, but the fibers intersect the boundary of Ω. Their result is completely different. Indeed, in the case where

$$\lim_{\varepsilon\to 0}\alpha_\varepsilon r_\varepsilon^2 \in]0,+\infty[\,, \tag{20}$$

the limit functional is

$$F(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \int_{\Omega\times\Omega}(u(x)-u(y))^2 j(dx,dy) + \int_\Omega u^2 k(dx) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u\in H_0^1(\Omega) \\ +\infty \qquad\qquad\qquad\qquad\qquad\quad \text{if } u\in L^2(\Omega)\backslash H_0^1(\Omega)\,, \end{cases} \tag{21}$$

where k and j are non-negative measures, which are directly computed in function of the Green kernel $G(x,y)$ solution of

$$\begin{cases} -\Delta G(x,\cdot)+G(x,\cdot)=\delta_x & \text{in } \Omega \\ G(x,\cdot)=0 & \text{on } \partial\Omega\,. \end{cases} \tag{22}$$

Otherwise, in the case where $\lim_{\varepsilon\to 0}\alpha_\varepsilon r_\varepsilon^2 = +\infty$ (see (16)), there is no more any so-called jumping measure j in the limit and the limit functional is

$$F(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx & \text{if } u\in H_0^1(\Omega) \\ +\infty & \text{if } u\in L^2(\Omega)\backslash H_0^1(\Omega)\,, \end{cases} \tag{23}$$

In this context, the case with fibers of high conductivity is similar to that with Dirichlet condition on the boundary of fibers studied by Marchenko, Khruslov [17] and Cioranescu, Murat [8], and their asymptotic behaviour coincide.

In our context, under the same assumption (16), a jumping measure appears in the limit functional, namely

$$j(dx,dy) = \frac{1}{2|\Omega|}\,dx\otimes dy\,.$$

Therefore the asymptotic behaviour is completely modified even by the modification of the conductivity in a region of a very small measure: here a thin layer around the boundary of the domain Ω.

The following theorem states the asymptotic behaviour of (14) in the case of a medium conductivity.

THEOREM 2. *Let us assume that the condition* (15) *is satisfied and*

$$\lim_{\varepsilon\to 0} \alpha_\varepsilon r_\varepsilon^2 = \kappa \in]0, +\infty[\,. \tag{24}$$

Then, the solution u_ε *of the problem* (14) *weakly converges in* $H^1(\Omega)$ *to the solution* u_κ *of the coupled problem*

$$\begin{cases} -\Delta u_\kappa + u_\kappa - v_\kappa = f & \text{in} \quad \Omega \\ u_\kappa = 0 & \text{on} \quad \partial\Omega \\ -\Delta v_\kappa + \dfrac{1}{\kappa}(v_\kappa - u_\kappa) = 0 & \text{in} \quad \Omega \\ \dfrac{\partial v_\kappa}{\partial n} = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{25}$$

Theorem 2 is a simple adaption of Theorem 1.

REMARK 2. This limit behaviour is similar to that of [3] at the point of view of the equations in Ω. But the boundary conditions are quite different. In [3] the function v satisfies the same boundary condition than the function u_ε.

In our context, we pass from the Dirichlet boundary condition for u_ε to the Neumann condition for v_κ. The region of conductivity 1 around the boundary of Ω seems to an isolating one from the point of view of the fibers. This implies the Neumann condition for v_κ.

REMARK 3. Passing to the limit for $\kappa \to \infty$ in problem (25) yields:

$$\begin{cases} -\Delta u_\infty + u_\infty - v_\infty = f & \text{in} \quad \Omega \\ u_\infty = 0 & \text{on} \quad \partial\Omega \\ -\Delta v_\infty = 0 & \text{in} \quad \Omega \\ \dfrac{\partial v_\infty}{\partial n} = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{26}$$

This implies that v_∞ = constant. However, since $\overline{v_\kappa} = \overline{u_\kappa}$ by (25), we obtain that $v_\infty = \overline{u_\infty}$, and $u_\infty = u_0$ the solution of the limit problem (17), which corresponds to $\lim_{\varepsilon\to 0} \alpha_\varepsilon r_\varepsilon^2 = +\infty$, (see condition (16)) namely $\kappa = +\infty$. There is thus a continuity between boths problems through the values of $\kappa \in]0, +\infty]$.

The previous results can be extended in order to obtain a large family of jumping measures j in the limit functionals. For that let us consider a partition of Ω composed of m open regular set Ω^k, such that

$$\overline{\Omega} = \bigcup_{k=1}^{m} \overline{\Omega^k} \quad \text{and} \quad \Omega^k \cap \Omega^{k'} = \emptyset \text{ if } k \neq k'\,. \tag{27}$$

We now consider the high conductivity fibers' microstructure in each subdomain Ω^k, by always assuming that the fibers have a distance from $\partial\Omega^k$ equal to d_ε. We assume that the fibers contained in Ω^k are cylinders of radius $\varepsilon r_\varepsilon^k \ll 1$. Let us denote by ω_ε^k the set of the fibers included in Ω^k and define the conductivity by

$$a_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \backslash \bigcup_{k=1}^{m} \overline{\omega_\varepsilon^k} \\ \alpha_\varepsilon^k & \text{if } x \in \omega_\varepsilon^k , \end{cases} \tag{28}$$

where $\alpha_\varepsilon^k \gg 1$.

We obtain the following result:

THEOREM 3. *Let us assume that*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \, |\log r_\varepsilon^k| = 6\pi \, \gamma^k , \tag{29}$$

where $\gamma^k > 0$, *and*

$$\lim_{\varepsilon \to 0} \alpha_\varepsilon^k (r_\varepsilon^k)^2 = +\infty . \tag{30}$$

Then, the solution u_ε *of problem* (14) *weakly converges in* $H^1(\Omega)$ *to the solution* u_0 *of the problem*

$$\begin{cases} -\Delta u_0 + \sum_{k=1}^{m} \gamma^k \mathbf{1}_{\Omega^k}(u_0 - \overline{u_0^k}) = f & \text{in} \quad \Omega \\ u_0 = 0 & \text{on} \quad \partial\Omega , \end{cases} \tag{31}$$

where $\overline{u_0^k} := \frac{1}{|\Omega^k|} \int_{\Omega^k} u_0$.

REMARK 4. Theorem 3 can be written in term of Γ-convergence where the limit functional is:

$$F_0(u) = \begin{cases} \int_\Omega |\nabla u|^2 + \sum_{k=1}^{m} \frac{\gamma^k}{2|\Omega^k|} \int_{\Omega^k \times \Omega^k} (u(x) - u(y))^2 & \text{if } u \in H_0^1(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \backslash H_0^1(\Omega), \end{cases} \tag{32}$$

and the jumping measure of the Beurling-Deny decomposition formula is

$$j(dx, dy) = \sum_{k=1}^{m} \frac{\gamma^k}{2|\Omega^k|} \mathbf{1}_{\Omega^k}(dx) \otimes \mathbf{1}_{\Omega^k}(dy) . \tag{33}$$

2.3. – Proof of the results

In a general way, we shall use the symbol $\hat{}$ for functions which only depend on the geometrical structure.

PRELIMINARY RESULTS. Let us denote by $Y := [-\frac{1}{2}, \frac{1}{2}[^3$ the unit cube of $\mathbb{R}^3$ and by $Q_r^i \subset Y$ the cylinder of radius r, $0 < r < \frac{1}{2}$, of length 1 and of axis Ox_i, for $i = 1 \dots 3$. For a given $R_\varepsilon \in]r_\varepsilon, \frac{1}{2}[$, we define the function $\hat{V}_\varepsilon^i$, for $i = 1 \dots 3$, by

$$\hat{V}_\varepsilon^i(y) = \begin{cases} 0 & \text{if } r \le r_\varepsilon\,, \\ 1 & \text{if } r \ge R_\varepsilon\,, \\ \dfrac{\log r - \log r_\varepsilon}{\log R_\varepsilon - \log r_\varepsilon} & \text{if } r_\varepsilon < r < R_\varepsilon\,, \end{cases} \tag{34}$$

where $r = (\sum_{j \neq i} y_j^2)^{\frac{1}{2}}$. Let us denote for the sake of simplicity also $\hat{V}_\varepsilon^i$ its Y-periodic extension. We set

$$\hat{\delta}_\varepsilon = \int_Y |\nabla \hat{V}_\varepsilon^i|^2 \tag{35}$$

(independent from i).

We also define the rescaled function $\hat{v}_\varepsilon^i$, for $i = 1 \dots 3$, by

$$\hat{v}_\varepsilon^i(x) := \hat{V}_\varepsilon^i\left(\frac{x}{\varepsilon}\right), \quad \text{for } x \in \Omega\,, \tag{36}$$

and

$$\hat{v}_\varepsilon(x) := \hat{v}_\varepsilon^1(x)\, \hat{v}_\varepsilon^2(x)\, \hat{v}_\varepsilon^3(x), \quad \text{for } x \in \Omega\,. \tag{37}$$

Since $R_\varepsilon \to 0$, it is easy to check that

$$\hat{v}_\varepsilon \rightharpoonup 1 \quad * \text{ weakly in } H^1(\Omega)\,. \tag{38}$$

Let ω_ε^i, for $i = 1 \dots 3$, be the subset of ω_ε composed of the fibers which are parallel to the axis Ox_i. For any sequence $v_\varepsilon \in L^2(\Omega)$ we define the rescaled functions $\tilde{v}_\varepsilon^i$ by

$$\tilde{v}_\varepsilon^i := \frac{\mathbf{1}_{\omega_\varepsilon^i}}{\pi r_\varepsilon^2}\, v_\varepsilon, \quad \text{for } x \in \Omega\,, \tag{39}$$

where $\pi r_\varepsilon^2 \sim \frac{|\omega_\varepsilon^i|}{|\Omega|}$.

The proof of Theorem 1 is based on the following results.

LEMMA 1. *There exists a constant $C > 0$, such that for any $V \in H^1(Y)$,*

$$\begin{aligned} &\left| \int_Y \nabla \hat{V}_\varepsilon^i \cdot \nabla V - \hat{\delta}_\varepsilon \left(\fint_{Y \setminus Q_{R_\varepsilon}^i} V - \fint_{Q_{r_\varepsilon}^i} V \right) \right| \\ &\quad \le C\, \hat{\delta}_\varepsilon \left(\sqrt{|\log R_\varepsilon|}\, \|\nabla V\|_{L^2(Y \setminus Q_{R_\varepsilon}^i)} + \frac{1}{r_\varepsilon} \|\nabla V\|_{L^2(Q_{r_\varepsilon}^i)} \right). \end{aligned} \tag{40}$$

Estimate (40) provides a bound on structural functions $\hat{V}_\varepsilon^i$. We shall use this result in the following Lemma 2. This lemma states the asymptotic behaviour (41) for sequences of bounded energy which weakly converge in L^∞.

LEMMA 2. *Let v_ε be a sequence of $H_0^1(\Omega)$, such that*

$$v_\varepsilon \rightharpoonup v_0 \text{ in } L^\infty(\Omega) \ * \text{ weakly and } \int_\Omega a_\varepsilon |\nabla v_\varepsilon|^2 \le C\,. \tag{41}$$

Let us assume that limit (15) *holds. Then, for any function $\varphi \in C^1(\overline{\Omega})$, we have, for a suitable choice of R_ε,*

$$\int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon\, \varphi + \frac{1}{3}\int_\Omega (\tilde{v}_\varepsilon^1 + \tilde{v}_\varepsilon^2 + \tilde{v}_\varepsilon^3)\,\varphi \to \int_\Omega v_0\, \varphi\,, \tag{42}$$

where $\hat{v}_\varepsilon$ is defined in (37) *and $\tilde{v}_\varepsilon^i$, for $i = 1 \ldots 3$, is defined from v_ε by* (39).

Moreover, the sequence $a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon$ is bounded in $L^1(\Omega)$.

The previous result combined with limit (16) will allow us to prove that, for any sequence v_ε satisfying the assumptions of Lemma 2 (we will come to this case for u_ε by considering the small perturbation (45) of equation (14)), there exists a constant c_0 such that

$$\frac{1}{3}\,\frac{\mathbf{1}_{\omega_\varepsilon}}{\pi r_\epsilon^2}\, v_\varepsilon \rightharpoonup c_0 \quad * \text{weakly in } \mathcal{M}(\overline{\Omega})\,. \tag{43}$$

Then, we will use the following Lemma 3 to prove that, if the function v_ε is also solution of an equation of type (14), the constant c_0 from (43) is equal to the mean $\overline{v_0}$ of the weak limit of v_ε.

LEMMA 3. *Let v_ε be a function of $H_0^1(\Omega)$ such that* $\operatorname{div}(a_\varepsilon \nabla v_\varepsilon)$ *is bounded in $L^2(\Omega)$. Then, for a suitable choice of the distance d_ε in* (12), *we have*

$$\int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon \to 0\,. \tag{44}$$

PROOF OF THEOREM 1. First, let us remark that, assuming that $f \in L^\infty(\Omega)$, for any $\eta > 0$, the solutions u_ε^η of the following problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon^\eta) + \eta\, u_\varepsilon^\eta = f & \text{in } \Omega \\ u_\varepsilon^\eta = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{45}$$

are bounded in $L^\infty(\Omega)$ by $\frac{\|f\|_\infty}{\eta}$. Moreover, we have $\lim_{\eta\to 0} \|u_\varepsilon^\eta - u_\varepsilon\|_{L^2(\Omega)} = 0$ uniformly with respect to ε thanks to the equicoerciveness of $\int_\Omega a_\varepsilon |\nabla u_\varepsilon|^2$.

Let us define, as in (39), the rescaled functions

$$\tilde{u}_\varepsilon^{\eta,i} = \frac{\mathbf{1}_{\omega_\varepsilon^i}}{\pi r_\varepsilon^2}\, u_\varepsilon^\eta\,. \tag{46}$$

FIRST STEP. We want to prove that $\tilde{u}_\varepsilon^{\eta,i} \rightharpoonup \nu^\eta$ in $* \ \mathcal{M}(\overline{\Omega})$, the weak sense of measures. The uniform bound in ε: $\|u_\varepsilon^\eta\|_{L^\infty} \le \frac{\|f\|_\infty}{\eta}$ implies immediately that $\tilde{u}_\varepsilon^{\eta,i}$ is bounded in $L^1(\Omega)$. Therefore we have $\tilde{u}_\varepsilon^{\eta,i} \rightharpoonup \nu^{\eta,i}$ in $*\ \mathcal{M}(\overline{\Omega})$, up to a subsequence. On the other hand, by a result from [3](1.4,b) and since the axis form two orthogonal lattices ω_ε^i, ω_ε^j, $i \neq j$, periodically intersecting, one obtains $\nu^{\eta,1} = \nu^{\eta,2} = \nu^{\eta,3} = \nu^\eta$. More precisely, this result is based on the following one:

Let v_ε be a sequence in $H^1(\Omega)$, such that $\int_\Omega a_\varepsilon |\nabla v_\varepsilon|^2 \leq C$, then $\tilde{v}_\varepsilon^i - \tilde{v}_\varepsilon^j \rightharpoonup 0*$ weakly in $\mathcal{M}(\overline{\Omega})$ (see [3](1.4,b)).

SECOND STEP. We want to prove that $\nu^\eta = \overline{u_0^\eta}$, where u_0^η is the $*$ weak limit of u_ε^η in $L^\infty(\Omega)$, as $\varepsilon \to 0$ and for a fixed η. First we will prove that

$$\nu^\eta(dx) = \frac{\nu^\eta(\overline{\Omega})}{|\Omega|}\, dx\,,$$

where dx denotes the Lebesgue measure and $|\Omega|$ the Lebesgue measure of Ω.
We have, for any $i = 1 \ldots 3$,

$$\fint_{\omega_\varepsilon^i} \left(\frac{\partial u_\varepsilon^\eta}{\partial x_i}\right)^2 \leq \frac{1}{\alpha_\varepsilon\, |\omega_\varepsilon^i|} \int_{\omega_\varepsilon^i} a_\varepsilon |\nabla u_\varepsilon^\eta|^2 \leq \frac{C}{\alpha_\varepsilon\, |\omega_\varepsilon^i|} \to 0\,, \tag{47}$$

since $\alpha_\varepsilon r_\varepsilon^2 \to \infty$ by condition (16). Using the Cauchy-Schwarz inequality

$$\fint_{\omega_\varepsilon^i} \left|\frac{\partial u_\varepsilon^\eta}{\partial x_i}\right| \leq \left(\fint_{\omega_\varepsilon^i} \left(\frac{\partial u_\varepsilon^\eta}{\partial x_i}\right)^2\right)^{\frac{1}{2}} \to 0\,. \tag{48}$$

We thus obtain that

$$\frac{\mathbf{1}_{\omega_\varepsilon^i}}{|\omega_\varepsilon^i|} \frac{\partial u_\varepsilon^\eta}{\partial x_i} \to 0 \text{ strongly in } L^1(\Omega)\,. \tag{49}$$

Let $\Phi \in W^{2,p}(\Omega; \mathbb{R}^3)$, with $p > 3$, such that $\Phi = 0$ on the boundary of Ω. We recall that, thanks to the regularity of the boundary, $\Phi \in C^1(\overline{\Omega})$. By an integration by parts we have, for $i = 1 \ldots 3$

$$\fint_{\omega_\varepsilon^i} u_\varepsilon^\eta \frac{\partial \Phi_i}{\partial x_i} = -\fint_{\omega_\varepsilon^i} \frac{\partial u_\varepsilon^\eta}{\partial x_i}\, \Phi_i + \frac{1}{|\omega_\varepsilon^i|} \int_{\Gamma_\varepsilon^i} u_\varepsilon^\eta\, \Phi_i\, n_i\,, \tag{50}$$

where Γ_ε^i denotes the union of the bases of the cylinders composing ω_ε^i (the boundary terms on the side boundaries of the cylinders vanish since the outside normal is orthogonal to the axis Ox_i). We also have

$$\left|\frac{1}{|\omega_\varepsilon^i|} \int_{\Gamma_\varepsilon^i} u_\varepsilon^\eta\, \Phi_i\, n_i\right| \leq \frac{\|f\|_{L^\infty} |\Gamma_\varepsilon^i|}{\eta\, |\omega_\varepsilon^i|} \sup_{\Gamma_\varepsilon^i} |\Phi_i| \leq C_\eta \sup_{\Gamma_\varepsilon^i} |\Phi_i| \to 0\,, \tag{51}$$

since $\mathrm{dist}(\Gamma^i_\varepsilon, \partial\Omega) \to 0$ and Φ_i is a continuous function which vanishes on the boundary of Ω.

Moreover, by (49) we have

$$\fint_{\omega^i_\varepsilon} \Phi_i \frac{\partial u^\eta_\varepsilon}{\partial x_i} \to 0, \quad \text{whence} \quad \fint_{\omega^i_\varepsilon} \frac{\partial \Phi_i}{\partial x_i} u^\eta_\varepsilon \to 0.$$

Then, by the definition of the measure ν^η, we obtain

$$\int_{\overline{\Omega}} (\mathrm{div}\, \Phi)\, \nu^\eta = 0, \tag{52}$$

for any $\Phi \in W^{2,p}(\Omega; \mathbb{R}^3)$, with $\Phi = 0$ on $\partial\Omega$.

On the other side, thanks to the C^1-regularity of the boundary of Ω, we can apply the following result (see the Corollary 3.8 of [1] or [6]):

For any $\varphi \in W^{1,p}(\Omega)$, with $p > 3$, there exists $\Phi \in W^{2,p}(\Omega; \mathbb{R}^3)$ such that $\Phi = 0$ on $\partial\Omega$ and $\mathrm{div}\, \Phi = (\varphi - \overline{\varphi})$ *in Ω.*

Using (52) and the previous result, we have for any $\varphi \in C^1(\overline{\Omega})$,

$$0 = \int_{\overline{\Omega}} (\mathrm{div}\, \Phi)\, \nu^\eta = \int_{\overline{\Omega}} (\varphi - \overline{\varphi})\, \nu^\eta = \int_{\overline{\Omega}} \varphi\, \nu^\eta - \overline{\varphi}\, \nu^\eta(\overline{\Omega}). \tag{53}$$

Therefore $\nu^\eta(dx)$ is equal to $\frac{\nu^\eta(\overline{\Omega})}{|\Omega|} dx$. It thus remains to identify the constant $\frac{\nu^\eta(\overline{\Omega})}{|\Omega|}$. However, by Lemma 2 and 3 with $\varphi = 1$, we have

$$0 + \frac{1}{3} \int_{\overline{\Omega}} 3\, \nu^\eta = \nu^\eta(\overline{\Omega}) = \int_\Omega u^\eta_0, \tag{54}$$

whence $\nu^\eta(dx) = \overline{u^\eta_0}\, dx$.

THIRD STEP. Determination of the limit problem. Let $\varphi \in \mathcal{D}(\Omega)$, by plugging the test function $\varphi \hat{v}_\varepsilon$ in (45), we have

$$\int_\Omega a_\varepsilon \nabla u^\eta_\varepsilon \cdot \nabla(\varphi \hat{v}_\varepsilon) + \eta \int_\Omega u^\eta_\varepsilon \varphi = \int_\Omega f \varphi + o(1), \tag{55}$$

since $|\{\hat{v}_\varepsilon \neq 1\}| = O(R^2_\varepsilon) = o(1)$.

By Lemma 2 and the result of the second step, we have

$$\int_\Omega a_\varepsilon \nabla u^\eta_\varepsilon \cdot \nabla \hat{v}_\varepsilon\, \varphi \to \int_\Omega (u^\eta_0 - \overline{u^\eta_0})\, \varphi.$$

On the other side, by convergence (38), we have

$$\int_\Omega a_\varepsilon \nabla u^\eta_\varepsilon \cdot \nabla \varphi\, \hat{v}_\varepsilon = \int_{\Omega \setminus \omega_\varepsilon} \nabla u^\eta_\varepsilon \cdot \nabla \varphi + o(1) = \int_\Omega \nabla u^\eta_\varepsilon \cdot \nabla \varphi + o(1) \tag{56}$$

since $\nabla u_\varepsilon^\eta$ is bounded in $L^2(\Omega)$ and $|\omega_\varepsilon| \to 0$. Whence, for $\varepsilon \to 0$,

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\varphi\, \hat{v}_\varepsilon \to \int_\Omega \nabla u_0^\eta \cdot \nabla\varphi\,, \tag{57}$$

and

$$\eta \int_\Omega u_\varepsilon^\eta \varphi \to \eta \int_\Omega u_0^\eta \varphi\,, \tag{58}$$

since $u_\varepsilon^\eta \to u_0^\eta$ weakly in $H^1(\Omega)$. Finally, we obtain

$$\int_\Omega \nabla u_0^\eta \cdot \nabla\varphi + \int_\Omega (u_0^\eta - \overline{u_0^\eta})\,\varphi + \eta \int_\Omega u_0^\eta \varphi = \int_\Omega f\,\varphi\,. \tag{59}$$

Moreover, it is easy to deduce from (59) that, for $\eta \to 0$, the sequence u_0^η converges strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$ to the solution u_0 of

$$\int_\Omega \nabla u_0 \cdot \nabla\varphi + \int_\Omega (u_0 - \overline{u_0})\,\varphi = \int_\Omega f\,\varphi, \quad \varphi \in \mathcal{D}(\Omega)\,,$$

which is the varational formulation of (17).

The statement is now proved remarking that $u_\varepsilon - u_0 = u_\varepsilon - u_\varepsilon^\eta + u_\varepsilon^\eta - u_0^\eta + u_0^\eta - u_0$ and using the convergence $\lim_{\eta\to 0} \|u_\varepsilon^\eta - u_\varepsilon\|_{L^2(\Omega)} = 0$ which is uniform with respect to ε.

3. – The case of a measure $\mathbf{1}_E \otimes \mathbf{1}_E$ with E a non-connected set

In this section, we give a microstructure which allows us to obtain any jumping measure of the type

$$j(dx, dy) = \frac{1}{2|A \cup B|} \mathbf{1}_{A\cup B}(dx) \otimes \mathbf{1}_{A\cup B}(dy) \tag{60}$$

where A and B are two disjoint connected open subsets of Ω such that $A, B \subset\subset \Omega$. Contrary of the first problem with one connected component we need regions with low conductivity to establish the result.

3.1. – Description of the geometry

Let A and B two smooth connected open subsets of Ω such that $\overline{A} \cap \overline{B} = \emptyset$. For the sake of simplicity we assume that there exists an open cylinder D parallel to the x_1-axis (x_1-parallel in the following) which joins two plane faces of A and B, and moreover $\overline{D} \cap \partial\Omega = \emptyset$.

In both sets A and B we consider the same fibered structure as in the first problem; we denote by ω_ε^A and ω_ε^B the set of the fibers of radius $\varepsilon\, r_\varepsilon$ in A and B such that $\mathrm{dist}(\omega_\varepsilon^A, \partial A)$ and $\mathrm{dist}(\omega_\varepsilon^B, \partial B) \geq d_\varepsilon$.

We extend to the set D the fibers ω_ε^A and ω_ε^B which are x_1-parallel and we only keep the fibers which are distant of d_ε from the side boundary of D; we denote by ω_ε^D this set of fibers. We envelopp each fiber from ω_ε^D by a fiber of same length, of inner radius $\varepsilon\, r_\varepsilon$ and outer radius $\varepsilon\, R_\varepsilon$, where $r_\varepsilon \ll R_\varepsilon \ll 1$ define the function $\hat{V}_\varepsilon^i$ (34) in the first problem; we denote by $\bar{\omega}_\varepsilon^D$ this set of fibers, which corresponds to the region of low conductivity around ω_ε^D as shown in figure 4.

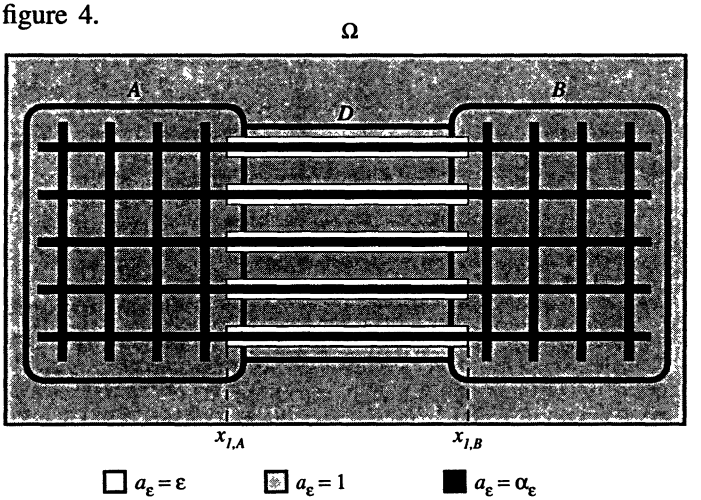

Fig. 4. Fibers of low and high conductivity

3.2. – Statements of the results

Now, let us define the conductivity coefficients of the previous microstructure by

$$(61) \qquad a_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \backslash (\omega_\varepsilon^A \cup \omega_\varepsilon^B \cup \omega_\varepsilon^D \cup \bar{\omega}_\varepsilon^D) \\ \alpha_\varepsilon \gg 1 & \text{if } x \in \omega_\varepsilon^A \cup \omega_\varepsilon^B \cup \omega_\varepsilon^D \\ \varepsilon \ll 1 & \text{if } x \in \bar{\omega}_\varepsilon^D , \end{cases}$$

and we consider the following conduction problem

$$(62) \qquad \begin{cases} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{in } \partial\Omega , \end{cases}$$

where f is a given function $f \in L^2(\Omega)$.

We obtain the following result.

THEOREM 4. *Let us assume that*

$$\lim_{\varepsilon\to 0} \varepsilon^2 |\log r_\varepsilon| = 6\pi \quad \text{and} \quad \lim_{\varepsilon\to 0} \alpha_\varepsilon r_\varepsilon^2 = +\infty, \tag{63}$$

as well as

$$\lim_{\varepsilon\to 0}\left(\varepsilon^2 |\log R_\varepsilon| + \frac{R_\varepsilon^2}{\varepsilon}\right) = 0, \tag{64}$$

Then, the solution u_ε of problem (14) *weakly converges in $L^2(\Omega)$ to the solution u_0 of the problem*

$$\begin{cases} -\Delta u_0 + \mathbf{1}_{A\cup B}(u_0 - \overline{u_0}^{A\cup B}) = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{65}$$

where

$$\overline{u_0}^{A\cup B} := \frac{1}{|A\cup B|}\int_{A\cup B} u = \fint_{A\cup B} u.$$

In terms of the Γ-convergence of Dirichlet-forms, the previous result can be written as follows.

COROLLARY 2. *Let F_ε be the functional associated to these problems as in* (18). *Then, F_ε Γ-converges to F_0 for the weak topology of $L^2(\Omega)$, where F_0 is defined by*

$$F_0(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 + \frac{1}{2|A\cup B|}\int_{\Omega\times\Omega} (u(x)-u(y))^2 \mathbf{1}_{A\cup B}(dx)\otimes \mathbf{1}_{A\cup B}(dy) & \\ & \text{if } u\in H_0^1(\Omega) \\ +\infty & \text{if } u\in L^2(\Omega)\setminus H_0^1(\Omega). \end{cases} \tag{66}$$

REMARK 5. The role of the low conductivity ($a_\varepsilon = \varepsilon$ in $\bar{\omega}_\varepsilon^D$) is essential in order to cancel the effects of the strong conductivity in the region D between A and B. In return, we loose the equicoerciveness of $\int_\Omega a_\varepsilon |\nabla u|^2$ and hence the strong convergence of the sequence u_ε. We thus have to work with the weak topology of $L^2(\Omega)$.

REMARK 6. The previous result can be also extended by considering a family of m regular connected open subsets of Ω: Ω^k, for $k = 1\ldots m$, with disjoint closure. In each subset Ω^k we put a lattice of high conductivity fibers as in A or B, and we join Ω^k to any neighbouring Ω^j by a lattice of parallel and high conductivity fibers surrounded by fibers of low conductivity as in the region D. Morever we consider n_k incrossing fibers in the period cell of the lattice corresponding to Ω^k ($n_k = 3$ in the previous cases). By this construction we can obtain the functional

$$F(u) := \int_\Omega |\nabla u|^2 + \frac{1}{6\sum_{k=1}^m n_k |\Omega^k|} \times \int_{\Omega\times\Omega} (u(x)-u(y))^2 \left(\sum_{k=1}^m n_k \mathbf{1}_{\Omega^k}\right)(dx) \otimes \left(\sum_{k=1}^m n_k \mathbf{1}_{\Omega^k}\right)(dy).$$

Then, we can approach any jumping measure of type

$$j(dx,dy) := \nu(dx)\otimes\nu(dy),$$

where ν is a positive Borel measure on Ω absolutely continuous with respect to the capacity, using the same method as Dal Maso and Mosco in [9] (see Theorems 4.16 and 4.17).

3.3. – Proof of Theorem 4 and Corollary 2

Proof of Theorem 4. The proof of Theorem 4 is divided in four steps. In the first step, we prove that u_ε weakly converge in $L^2(\Omega)$ to a function $u_0 \in H_0^1(\Omega)$ and we precise the weak convergence of $\mathbf{1}_{\Omega\setminus\bar\omega_\varepsilon^D}\nabla u_\varepsilon$ in $L^2(\Omega;\mathbb{R}^3)$. In the second step, we study the modified equation (45) satisfied by u_ε^η and we prove that the weak limit u_0^η of u_ε^η, is solution of a problem of type (65), with the zero order term equal to $\mathbf{1}_A(u_0^\eta - c_A^\eta) + \mathbf{1}_B(u_0^\eta - c_B^\eta)$. In the third step, we prove that c_A^η and c_B^η are equal to the mean of u_0^η over $A\cup B$. The fourth step is devoted to the passing to the limit $\eta\to 0$.

First step. We want to prove that u_ε weakly converge in $L^2(\Omega)$ to a function $u_0\in H_0^1(\Omega)$.

Let Q_R be the cylinder of fixed radius $R > R_\varepsilon$, of length 1 and of axis Ox_1. It is easy to prove by contradiction that there exists a constant $C>0$ such that, for any function $V\in H^1(Y)$,

$$\int_{Q_R} V^2 \le C\left(\int_{Y\setminus Q_R} V^2 + \int_Y |\nabla V|^2\right),$$

and since $Q_{R_\varepsilon}\subset Q_R$,

$$\int_{Q_{R_\varepsilon}} V^2 \le C\left(\int_{Y\setminus Q_{R_\varepsilon}} V^2 + \int_Y |\nabla V|^2\right).$$

Then, by rescaling the previous estimate and summing over each cell of size ε around $\overline{\omega}_\varepsilon^D$, we obtain for any $v\in H_0^1(\Omega)$,

$$\int_{\overline{\omega}_\varepsilon^D} v^2 \le C\left(\int_{\Omega\setminus\overline{\omega}_\varepsilon^D} v^2 + \varepsilon^2\int_\Omega |\nabla v|^2\right) \le C\left(\int_{\Omega\setminus\overline{\omega}_\varepsilon^D} v^2 + \int_\Omega a_\varepsilon |\nabla v|^2\right).$$

On the other hand, since $v = 0$ on $\partial\Omega$ and $\overline{\omega}_\varepsilon^D$ is a set of parallel cylinders, it is easy to see that

$$\int_{\Omega\setminus\overline{\omega}_\varepsilon^D} v^2 \le C \int_{\Omega\setminus\overline{\omega}_\varepsilon^D} |\nabla v|^2$$

for an appropriate constant C. Therefore, both previous estimates show that there exists a constant $C > 0$ such that, for any $v \in H_0^1(\Omega)$,

$$\int_\Omega v^2 \le C \int_\Omega a_\varepsilon |\nabla v|^2 = C\, F_\varepsilon(v)\,. \tag{67}$$

Since u_ε is a solution of the problem (62), inequality (67) implies that u_ε is bounded in $L^2(\Omega)$ and, up to a subsequence, u_ε weakly converges in $L^2(\Omega)$ to a function $u_0 \in L^2(\Omega)$.

Let $\Phi \in \mathcal{D}(\Omega; \mathbb{R}^3)$, we have

$$\int_\Omega u_\varepsilon \mathrm{div}\Phi = -\int_\Omega \nabla u_\varepsilon \cdot \Phi = -\int_{\bar{\omega}_\varepsilon^D} \nabla u_\varepsilon \cdot \Phi - \int_\Omega \xi_\varepsilon \cdot \Phi \tag{68}$$

where $\xi_\varepsilon := \mathbf{1}_{\Omega\setminus\bar{\omega}_\varepsilon^D} \nabla u_\varepsilon$. By the Cauchy-Schwarz inequality we have

$$\left| \int_{\bar{\omega}_\varepsilon^D} \nabla u_\varepsilon \cdot \Phi \right| \le \|\Phi\|_{L^\infty(\Omega)} \left(\frac{|\bar{\omega}_\varepsilon^D|}{\varepsilon} \int_{\bar{\omega}_\varepsilon^D} \varepsilon\, |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}} = O\left(\frac{R_\varepsilon}{\sqrt{\varepsilon}} \right) \to 0 \tag{69}$$

by (64), whence by passing to the limit in the previous equality

$$\int_\Omega u \,\mathrm{div}\, \Phi = -\int_\Omega \xi_0 \cdot \Phi\,, \tag{70}$$

where ξ_0 is the $L^2(\Omega, \mathbb{R}^3)$-weak limit of ξ_ε which is bounded in $L^2(\Omega, \mathbb{R}^3)$ since $|\xi_\varepsilon|^2 \le a_\varepsilon\, |\nabla u_\varepsilon|^2$.

Therefore $u_0 \in H^1(\Omega)$ and $\xi_0 = \nabla u_0$. Since ∇u_ε is bounded in a neighbourhood of $\partial\Omega$, u_ε weakly converges in $H_0^1(\Omega)$ in such a neighbourhood and hence $u_0 \in H_0^1(\Omega)$. We thus have, up to a subsequence,

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 \in H_0^1(\Omega) & \text{weakly in } L^2(\Omega) \\ \mathbf{1}_{\Omega\setminus\bar{\omega}_\varepsilon^D} \nabla u_\varepsilon \rightharpoonup \nabla u_0 & \text{weakly in } L^2(\Omega, \mathbb{R}^3)\,. \end{cases} \tag{71}$$

Second step. As in the first problem and in order to have a L^∞-estimate for the solution, we consider the modified problem

$$\begin{cases} -\,\mathrm{div}\,(a_\varepsilon \nabla u_\varepsilon^\eta) + \eta\, u_\varepsilon^\eta = f & \text{in } \Omega \\ u_\varepsilon^\eta = 0 & \text{on } \partial\Omega\,, \end{cases}$$

We shall prove that there exist two constants c_A^η and c_B^η such that u_0^η is a solution of

$$-\Delta u_0^\eta + \mathbf{1}_A\, (u_0^\eta - c_A^\eta) + \mathbf{1}_B\, (u_0^\eta - c_B^\eta) = f \quad \text{in } \Omega\,. \tag{72}$$

Let $x_{1,A}$ be the largest x_1-coordinate of ω_ε^A and $x_{1,B}$ be the smallest x_1-coordinate of ω_ε^B in the cylinder of Ω supported by D (see figure 4). Between the plane $x_1 = x_{1,A}$ and the plane $x_1 = x_{1,B}$, there are only the x_1-parallel fibers of ω_ε^D and $\bar{\omega}_\varepsilon^D$. We again consider the functions $\hat{v}_\varepsilon^i$, for $i = 1\dots 3$, from the first problem, associated to the x_i-parallel fibers with r_ε and R_ε. We redefine the function $\hat{v}_\varepsilon$ in Ω by

$$\hat{v}_\varepsilon(x) := \begin{cases} \hat{v}_\varepsilon^1\,\hat{v}_\varepsilon^2\,\hat{v}_\varepsilon^3 & \text{if } x_1 \notin [x_{1,A}, x_{1,B}] \\ \hat{v}_\varepsilon^1 & \text{if } x_1 \in [x_{1,A}, x_{1,B}]\,. \end{cases} \tag{73}$$

Remark that the function $\hat{v}_\varepsilon$ belongs to $H^1(\Omega)$ since, by definition of $\hat{v}_\varepsilon^i$, we have $\hat{v}_\varepsilon^2 = \hat{v}_\varepsilon^3 = 1$ if $x_1 \in \{x_{1,A}, x_{1,B}\}$, and similarly to (38), the sequence $\hat{v}_\varepsilon$ also satisfies the convergence

$$\hat{v}_\varepsilon \rightharpoonup 1 \quad * \text{ weakly in } H^1(\Omega)\,. \tag{74}$$

By proceeding as in the first problem and similarly to the weak convergence (43), we can prove that there exist two constants c_A^η and c_B^η such that

$$\frac{\mathbf{1}_{\omega_\varepsilon^A}}{\pi r_\varepsilon^2}\, u_\varepsilon^\eta \rightharpoonup c_A^\eta \quad \text{and} \quad \frac{\mathbf{1}_{\omega_\varepsilon^B}}{\pi r_\varepsilon^2}\, u_\varepsilon^\eta \rightharpoonup c_B^\eta \quad * \text{ weakly in } \mathcal{M}(\overline{\Omega})\,, \tag{75}$$

since $\lim_{\varepsilon\to 0} \alpha_\varepsilon r_\varepsilon^2 = +\infty$ by (16), and for any $\varphi \in C^1(\overline{\Omega})$,

$$\int_A \nabla u_\varepsilon^\eta \cdot \nabla \hat{v}_\varepsilon\, \varphi \to \int_A (u_0^\eta - c_A^\eta)\,\varphi \quad \text{and} \quad \int_B \nabla u_\varepsilon^\eta \cdot \nabla \hat{v}_\varepsilon\, \varphi \to \int_B (u_0^\eta - c_B^\eta)\,\varphi\,. \tag{76}$$

We consider a function $\varphi_\varepsilon \in C_0^1(A \cup B \cup D)$ such that

$$0 \le \varphi_\varepsilon \le 1, \quad \varphi_\varepsilon = \begin{cases} 1 \text{ if } \operatorname{dist}(x, \partial(A\cup B\cup D)) \ge d_\varepsilon \\ 0 \text{ if } x \in \partial(A\cup B\cup D) \end{cases} \quad \text{and } |\nabla\varphi_\varepsilon| \le \frac{c}{d_\varepsilon}\,, \tag{77}$$

where $\operatorname{dist}\big(\omega_\varepsilon^A \cup \omega_\varepsilon^B \cup \omega_\varepsilon^D, \partial(A \cap B \cap D)\big) \ge d_\varepsilon$ by construction. We also assume that $d_\varepsilon \gg R_\varepsilon^2$.

Let $\varphi \in \mathcal{D}(\Omega)$, we consider the function $\varphi_\varepsilon \hat{v}_\varepsilon \varphi + (1 - \varphi_\varepsilon)\varphi$, where $\hat{v}_\varepsilon$ is defined by (73) and φ_ε by (77), as test function in problem (62). Then, using the equation satisfied by u_ε^η and the convergence (74) satisfied by $\hat{v}_\varepsilon$, we have

$$\begin{aligned} \int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla(\varphi_\varepsilon \hat{v}_\varepsilon \varphi + (1-\varphi_\varepsilon)\varphi) &= \int_\Omega f(\varphi_\varepsilon \hat{v}_\varepsilon \varphi + (1-\varphi_\varepsilon)\varphi) - \int_\Omega \eta u_\varepsilon^\eta \varphi \\ &\to \int_\Omega f\,\varphi - \int_\Omega \eta\, u_0^\eta\, \varphi\,, \end{aligned} \tag{78}$$

and

$$\begin{aligned} &\int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla(\varphi_\varepsilon \hat{v}_\varepsilon \varphi + (1-\varphi_\varepsilon)\varphi) \\ &= \int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\varphi(\varphi_\varepsilon \hat{v}_\varepsilon + (1-\varphi_\varepsilon)) + \int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\varphi_\varepsilon(\hat{v}_\varepsilon - 1)\varphi + \int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\hat{v}_\varepsilon\, \varphi_\varepsilon\, \varphi \end{aligned}$$

Firstly, by the definition of (61), we have

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla \varphi(\varphi_\varepsilon \hat{v}_\varepsilon + (1-\varphi_\varepsilon)) = \int_{\Omega\setminus(A\cup B\cup D)} \nabla u_\varepsilon^\eta \cdot \nabla\varphi$$
$$+ \int_{A\cup B\cup D\setminus\bar{\omega}_\varepsilon^D} \nabla u_\varepsilon^\eta \cdot \nabla\varphi\,(\varphi_\varepsilon \hat{v}_\varepsilon + (1-\varphi_\varepsilon)) + \int_{\bar{\omega}_\varepsilon^D} \varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\varphi\, \hat{v}_\varepsilon$$
$$\to \int_\Omega \nabla u_0^\eta \cdot \nabla\varphi \quad \text{by the convergence (71) extended to } u_\varepsilon^\eta .$$

Secondly, by proceeding as in the first problem, we have

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\varphi_\varepsilon\,(\hat{v}_\varepsilon - 1)\,\varphi = \int_\Omega \nabla u_\varepsilon^\eta \cdot \nabla\varphi_\varepsilon\,(\hat{v}_\varepsilon - 1)\,\varphi \to 0 \tag{79}$$

since $d_\varepsilon \gg R_\varepsilon^2$.
Thirdly, we have

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\hat{v}_\varepsilon\, \varphi_\varepsilon\, \varphi = \int_{A\cup B\cup D} a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\hat{v}_\varepsilon\, \varphi + o(1)$$
$$= \int_A \nabla u_\varepsilon^\eta \cdot \nabla\hat{v}_\varepsilon\, \varphi + \int_B \nabla u_\varepsilon^\eta \cdot \nabla\hat{v}_\varepsilon\, \varphi + \int_{\bar{\omega}_\varepsilon^D} \varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\hat{v}_\varepsilon\, \varphi + o(1),$$

since, between the planes $x_1 = x_{1,A}$ and $x_1 = x_{1,B}$, we have $\nabla\hat{v}_\varepsilon\,\varphi = \nabla\hat{v}_\varepsilon^1\,\varphi = 0$ outside $\bar{\omega}_\varepsilon^D$. We also have

$$\int_{\bar{\omega}_\varepsilon^D} \varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\hat{v}_\varepsilon\, \varphi = O(\sqrt{\varepsilon}) \tag{80}$$

whence by (76),

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\hat{v}_\varepsilon\, \varphi_\varepsilon\, \varphi \to \int_A (u_0^\eta - c_A^\eta)\,\varphi + \int_B (u_0^\eta - c_B^\eta)\,\varphi . \tag{81}$$

Finally, we obtain, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega \nabla u_0^\eta \cdot \nabla\varphi + \int_\Omega \eta\, u_0^\eta\, \varphi + \int_A (u_0^\eta - c_A^\eta)\,\varphi + \int_B (u_0^\eta - c_B^\eta)\,\varphi = \int_\Omega f\,\varphi , \tag{82}$$

which is equivalent to the desired equation (72).

THIRD STEP. We want to prove that $c_A^\eta = c_B^\eta = \overline{u_0^\eta}^{A\cup B} := \frac{1}{|A\cup B|}\int_{A\cup B} u_0^\eta$.

Let Q be a small cylinder which extends the cylinder D in the set A. For the sake of simplicity we can assume that D is symmetric with respect to the plane $x_1 = 0$ and we denote by Q' the symmetrized of Q in B. Let ω_ε^1 be the set of x_1-parallel fibers, and for any $v \in C_0^1(\Omega)$ denote by $\tilde{v}_\varepsilon^1$, the rescaled function $\tilde{v}^1_\varepsilon := \frac{\mathbf{1}_{\omega^1_\varepsilon}}{\pi r^2_\varepsilon} v$, let $x := (x_1, x_2, x_3) \in Q$ and $x' := (-x_1, x_2, x_3) \in Q'$, by writing

$$v(x) - v(x') = \int_{-x_1}^{x_1} \frac{\partial v}{\partial x_1}(t, x_2, x_3)\, dt$$

and by using the Cauchy-Schwarz inequality we obtain

$$\text{(83)} \qquad \begin{aligned} \left| \int_Q (\tilde{v}^1_\varepsilon(x) - \tilde{v}^1_\varepsilon(x'))\, dx \right| &\leq C \frac{|\omega^1_\varepsilon|^{\frac{1}{2}}}{r^2_\varepsilon} \left(\int_{\omega^1_\varepsilon} \left(\frac{\partial v}{\partial x_1} \right)^2 dt \right)^{\frac{1}{2}} \\ &\leq \frac{C}{\sqrt{\alpha_\varepsilon r^2_\varepsilon}} \left(\int_\Omega a_\varepsilon\, |\nabla v|^2 \right)^{\frac{1}{2}} . \end{aligned}$$

This implies that, for $v := u^\eta_\varepsilon$ (after a density argument),

$$\text{(84)} \qquad \left| \int_Q (\tilde{u}^{1,\eta}_\varepsilon(x) - \tilde{u}^{1,\eta}_\varepsilon(x'))\, dx \right| \leq \frac{C}{\sqrt{\alpha_\varepsilon r^2_\varepsilon}} \to 0, \text{ by (63)}.$$

On the other hand, by the weak convergence (75), we have

$$\text{(85)} \qquad \int_Q (\tilde{u}^{1,\eta}_\varepsilon(x) - \tilde{u}^{1,\eta}_\varepsilon(x'))\, dx \to \frac{|Q|}{3} (c^\eta_A - c^\eta_B),$$

since, for any $i, j \in \{1, 2, 3\}$, $\tilde{u}^{i,\eta}_\varepsilon - \tilde{u}^{j,\eta}_\varepsilon \rightharpoonup 0$ in $* \mathcal{M}(\overline{A})$ and thus $\frac{\mathbf{1}_{\omega^A_\varepsilon}}{\pi r^2_\varepsilon} u_\varepsilon - 3\, \tilde{u}^{1,\eta}_\varepsilon \rightharpoonup 0$ in $* \mathcal{M}(\overline{A})$. Therefore we obtain $c^\eta_A = c^\eta_B$.

We take again the function φ_ε defined by (77) in the second step and we put the function $\varphi_\varepsilon\, (1 - \hat{v}_\varepsilon)$ as test function in problem (62).

Then, we obtain

$$\text{(86)} \quad \int_\Omega a_\varepsilon \nabla u^\eta_\varepsilon \cdot \nabla(\varphi_\varepsilon(1 - \hat{v}_\varepsilon)) = \int_\Omega f\, \varphi_\varepsilon\, (1 - \hat{v}_\varepsilon) - \int_\Omega \eta\, u^\eta_\varepsilon\, \varphi_\varepsilon\, (1 - \hat{v}_\varepsilon) \to 0,$$

On the other hand, by limits (79) and (81) we have

$$\begin{aligned} \int_\Omega a_\varepsilon \nabla u^\eta_\varepsilon \cdot \nabla(\varphi_\varepsilon(1 - \hat{v}_\varepsilon)) &= \int_\Omega \nabla u^\eta_\varepsilon \cdot \nabla \varphi_\varepsilon\, (1 - \hat{v}_\varepsilon) - \int_\Omega a_\varepsilon \nabla u^\eta_\varepsilon \cdot \nabla \hat{v}_\varepsilon\, \varphi_\varepsilon \\ \to - \int_A (u^\eta_0 - c^\eta_A) - \int_B (u^\eta_0 - c^\eta_B) &= - \int_{A \cup B} u^\eta_0 + |A \cup B|\, c^\eta_A , \end{aligned}$$

whence $c^\eta_A = c^\eta_B = \overline{u^\eta_0}^{A \cup B}$.

We have just proved that u^η_0 is solution of the problem

$$\text{(87)} \qquad \begin{cases} -\Delta u^\eta_0 + \eta\, u^\eta_0 + \mathbf{1}_{A \cup B} (u^\eta_0 - \overline{u^\eta_0}^{A \cup B}) = f & \text{in } \Omega \\ u^\eta_0 = 0 & \text{on } \partial\Omega . \end{cases}$$

FOURTH STEP. We want to pass to the limit $\eta \to 0$.

This step is more delicate than in the first problem since we have no equicoerciveness because of the regions of low conductivity.

By applying the estimate (67) of the first step to the sequence u_ε^η, it is easy to see that u_ε^η is uniformly bounded in $L^2(\Omega)$ with respect to ε and η. Then, by putting $u_\varepsilon^\eta - u_\varepsilon$ in equations (14) and (45), we obtain

$$\int_\Omega a_\varepsilon\, |\nabla(u_\varepsilon^\eta - u_\varepsilon)|^2 = -\int_\Omega \eta\, u_\varepsilon^\eta\, (u_\varepsilon^\eta - u_\varepsilon) = O(\eta)\,. \tag{88}$$

Since $a_\varepsilon \geq 1$ in $\Omega \setminus D$ and $u_\varepsilon^\eta = u_\varepsilon = 0$ on the exterior boundary of $\Omega \setminus D$, the previous estimate combined to the Poincaré inequality imply that $u_\varepsilon^\eta - u_\varepsilon$ strongly converges to 0 in $H^1(\Omega \setminus D)$ as $\eta \to 0$ and uniformly with respect to ε. We deduce from this uniform convergence that the weak limit u_0^0 of u_0^η (as $\eta \to 0$) in $H^1(\Omega \setminus D)$ is almost everywhere equal to the weak limit u_0 of u_ε (as $\varepsilon \to 0$) in $H^1(\Omega \setminus D)$. Let $\varphi \in \mathcal{D}(\Omega)$. By the results of the second step and in particular by limit (81), we have, for any $\eta > 0$,

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla \hat{v}_\varepsilon\, \varphi \underset{\varepsilon\to 0}{\longrightarrow} \int_{A\cup B} (u_0^\eta - \overline{u_0^\eta}^{A\cup B})\, \varphi\,. \tag{89}$$

Then, by using the uniform estimate (88) with respect to ε combined with the boundedness of $\nabla \hat{v}_\varepsilon$ in $L^2(\Omega; \mathbb{R}^3)$ and the equality $u_0^0 = u_0$ in $A \cup B$, we deduce from limit (89) the new one

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon \cdot \nabla \hat{v}_\varepsilon\, \varphi \underset{\varepsilon\to 0}{\longrightarrow} \int_{A\cup B} (u_0 - \overline{u_0}^{A\cup B})\, \varphi\,. \tag{90}$$

On the other hand, by the strong convergence of $\hat{v}_\varepsilon$ to 1 in $L^2(\Omega)$ and by the second weak limit of (71), we have

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi\, \hat{v}_\varepsilon = \int_\Omega \xi_\varepsilon \cdot \nabla\varphi\, \hat{v}_\varepsilon + o(1) \to \int_\Omega \nabla u_0 \cdot \nabla\varphi\,. \tag{91}$$

Finally, passing to the limit in the variational equality

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon \cdot \nabla(\varphi \hat{v}_\varepsilon) = \int_\Omega f\, \varphi\, \hat{v}_\varepsilon\,,$$

thanks to (90) and (91), yields

$$\int_\Omega \nabla u_0 \cdot \nabla\varphi + \int_{A\cup B} (u_0 - \overline{u_0}^{A\cup B})\, \varphi = \int_\Omega f\, \varphi\,,$$

which is the variational formulation of problem (65). This concludes the proof of Theorem 4.

PROOF OF COROLLARY 2. By estimate (67) F_ε defined in (18) is equicoercive with respect to the $L^2(\Omega)$-norm. Then, by a well known compactness result of Γ-convergence (see [18]) there exists a quadratic functional F defined in $L^2(\Omega)$ such that F_ε Γ-converges to F, up to a subsequence, for the $L^2(\Omega)$- weak topology. By the convergence of the minimizers in the Γ-convergence and by Theorem 4, we obtain, for any sequence of minimizers u_ε solutions of (62), i.e. which minimize the quadratic form $F_\varepsilon(v) - 2\int_\Omega fv$, the equality

$$F(u) - 2\int_\Omega fu = \lim_{\varepsilon\to 0}\left(F_\varepsilon(u_\varepsilon) - 2\int_\Omega fu_\varepsilon\right) = -\lim_{\varepsilon\to 0}\int_\Omega f\,u_\varepsilon\,, \tag{92}$$

whence

$$F(u) = \int_\Omega fu = \int_\Omega |\nabla u|^2 + \frac{1}{2|A\cup B|}\int_{\Omega\times\Omega}(u(x)-u(y))^2\,\mathbf{1}_{A\cup B}(dx)\otimes\mathbf{1}_{A\cup B}(dy)\,.$$

Therefore F satisfies equality (66).

3.4. – Proof of Lemma 1

Let $i = 1\ldots 3$ and $V \in C^1(\overline{Y})$. Since

$$\Delta\hat{V}^i_\varepsilon = 0 \quad \text{in} \quad Q_{R_\varepsilon}\backslash Q_{r_\varepsilon}\,, \tag{93}$$

we have by an integration by parts

$$\begin{aligned}&\int_Y \nabla\hat{V}^i_\varepsilon\cdot\nabla\left[V - \overline{V}^{Y\backslash Q^i_{R_\varepsilon}}\hat{V}^i_\varepsilon - \overline{V}^{Q^i_{r_\varepsilon}}(1-\hat{V}^i_\varepsilon)\right]\\ &= \int_{\Gamma_{R_\varepsilon}}\frac{\partial\hat{V}^i_\varepsilon}{\partial n}(V - \overline{V}^{Y\backslash Q^i_{R_\varepsilon}}) - \int_{\Gamma_{r_\varepsilon}}\frac{\partial\hat{V}^i_\varepsilon}{\partial n}(V - \overline{V}^{Q^i_{r_\varepsilon}})\,,\end{aligned} \tag{94}$$

where $\overline{V}^A := \fint_A V$ and Γ_r denotes the side boundary of the cylinder Q_r. By definition (35), the first term of (34) is equal to

$$\int_Y \nabla\hat{V}^i_\varepsilon\cdot\nabla V - \hat{\delta}_\varepsilon\,(\overline{V}^{Y\backslash Q^i_{R_\varepsilon}} - \overline{V}^{Q^i_{r_\varepsilon}}) \tag{95}$$

It is thus enough to estimate the second term of (34).

ESTIMATE OF $\int_{\Gamma_{R_\varepsilon}}\frac{\partial\hat{V}^i_\varepsilon}{\partial n}(V - \overline{V}^{Y\backslash Q^i_{R_\varepsilon}})$. By definition (34) of $\hat{V}^i_\varepsilon$ we have

$$\left.\frac{\partial\hat{V}^i_\varepsilon}{\partial n}\right|_{\Gamma^i_r} = \frac{1}{r}\,\frac{1}{\log(R_\varepsilon)-\log(r_\varepsilon)} = \frac{\hat{\delta}_\varepsilon}{2\pi r}\,. \tag{96}$$

Set $W = V - \overline{V}^{Y\backslash Q^i_{R_\varepsilon}}$, and let R be a fixed number of $]0, \frac{1}{2}[$. With the cylindrical coordinates we have, for any $r \in [R_\varepsilon, R]$,

$$r\, W(R_\varepsilon) = r\, W(r) + r \int_r^{R_\varepsilon} \frac{\partial W}{\partial \rho} d\rho \tag{97}$$

whence by integrating over $]R_\varepsilon, R[\times P$ where $P :=]0, 2\pi] \times] - \frac{1}{2}, \frac{1}{2}[$,

$$\begin{aligned} \frac{R^2 - R_\varepsilon^2}{2} \int_P |W(R_\varepsilon)| &\le \|W\|_{L^1(Y\backslash Q^i_{R_\varepsilon})} + \int_P \int_{R_\varepsilon}^R r \left(\int_{R_\varepsilon}^r \left| \frac{\partial W}{\partial \rho} \right| d\rho \right) \\ &\le \|W\|_{L^1(Y\backslash Q^i_{R_\varepsilon})} + \int_P R \int_{R_\varepsilon}^R \sqrt{\log(\tfrac{r}{R_\varepsilon})} \left(\int_{R_\varepsilon}^R \rho \left(\frac{\partial W}{\partial \rho} \right)^2 d\rho \right)^{\frac{1}{2}}, \end{aligned} \tag{98}$$

by the Cauchy-Schwarz inequality. We thus obtain

$$\int_P |W(R_\varepsilon)| \le C\, \|W\|_{L^1(Y\backslash Q^i_{R_\varepsilon})} + C \sqrt{|\log R_\varepsilon|} \left(\int_P \int_{R_\varepsilon}^R \rho \left(\frac{\partial W}{\partial \rho} \right)^2 d\rho \right)^{\frac{1}{2}}, \tag{99}$$

again by the Cauchy-Schwarz inequality, or equivalently

$$\int_P |W(R_\varepsilon)| \le C\, \|W\|_{L^1(Y\backslash Q^i_{R_\varepsilon})} + C \sqrt{|\log R_\varepsilon|}\, \|\nabla W\|_{L^2(Y\backslash Q^i_{R_\varepsilon})}. \tag{100}$$

This inequality combined with (96) yields

$$\begin{aligned} \left| \int_{\Gamma_{R_\varepsilon}} \frac{\partial \hat{V}^i_\varepsilon}{\partial n} (V - \overline{V}^{Y\backslash Q^i_{R_\varepsilon}}) \right| &\le \frac{\hat{\delta}_\varepsilon}{2\pi R_\varepsilon} R_\varepsilon \int_P |W(R_\varepsilon)| \\ &\le C\, \hat{\delta}_\varepsilon \left(\|W\|_{L^1(Y\backslash Q^i_{R_\varepsilon})} + \sqrt{|\log R_\varepsilon|}\, \|\nabla W\|_{L^2(Y\backslash Q^i_{R_\varepsilon})} \right). \end{aligned} \tag{101}$$

Moreover, by the Poincaré-Wirtinger inequality we have

$$\|W\|_{L^2(Y\backslash Q^i_{R_\varepsilon})} \le C\, \|\nabla W\|_{L^2(Y\backslash Q^i_{R_\varepsilon})}. \tag{102}$$

Indeed, if $\Gamma^i_\varepsilon := \{y \in \overline{Y\backslash Q^i_{R_\varepsilon}} \ / \ y_i = \frac{1}{2}\}$, by considering the projections of $y \in Y\backslash Q^i_{R_\varepsilon}$ on Γ^i_ε, we easily obtain

$$\left\| W - ⨍_{\Gamma^i_\varepsilon} W \right\|_{L^2(Y\backslash Q^i_{R_\varepsilon})} \le C \left\| \frac{\partial W}{\partial y_i} \right\|_{L^2(Y\backslash Q^i_{R_\varepsilon})}, \tag{103}$$

and similarly

$$\left| \fint_{\Gamma^i_\varepsilon} W - \fint_{Y\setminus Q^i_{R_\varepsilon}} W \right| \leq C \left\| \frac{\partial W}{\partial y_i} \right\|_{L^2(Y\setminus Q^i_{R_\varepsilon})},$$

which implies the desired estimate (102) since $\overline{W}^{Y\setminus Q^i_{R_\varepsilon}} = 0$.

Finally, we obtain

$$\left| \int_{\Gamma_{R_\varepsilon}} \frac{\partial \hat{V}^i_\varepsilon}{\partial n} (V - \overline{V}^{Y\setminus Q^i_{R_\varepsilon}}) \right| \leq C\, \hat{\delta}_\varepsilon \sqrt{|\log R_\varepsilon|}\; \|\nabla W\|_{L^2(Y\setminus Q^i_{R_\varepsilon})}. \tag{104}$$

ESTIMATE OF $\int_{\Gamma_{r_\varepsilon}} \frac{\partial \hat{V}^i_\varepsilon}{\partial n}(V - \overline{V}^{Q^i_{r_\varepsilon}})$. Set $W := V - \overline{V}^{Q^i_{r_\varepsilon}}$. Proceeding as for the previous estimate, but integrating this time over $]0, r_\varepsilon[\times P$, we have

$$\begin{aligned} \frac{r_\varepsilon^2}{2} \int_P |W(r_\varepsilon)| &\leq \|W\|_{L^1(Q^i_{r_\varepsilon})} + \int_P \left(\int_0^{r_\varepsilon} r \sqrt{\log\left(\frac{r_\varepsilon}{r}\right)}\, dr \right) \left(\int_0^{r_\varepsilon} \rho \left(\frac{\partial W}{\partial \rho}\right)^2 d\rho \right)^{\frac{1}{2}} \\ &\leq \|W\|_{L^1(Q^i_{r_\varepsilon})} + C\, r_\varepsilon^2 \int_P \left(\int_0^{r_\varepsilon} \rho \left(\frac{\partial W}{\partial \rho}\right)^2 d\rho \right)^{\frac{1}{2}} \\ &\leq C\, r_\varepsilon\, \|W\|_{L^2(Q^i_{r_\varepsilon})} + C\, r_\varepsilon^2\, \|\nabla W\|_{L^2(Q^i_{r_\varepsilon})}, \end{aligned} \tag{105}$$

by the Cauchy-Schwarz inequality.

This inequality combined with (96) yields

$$\begin{aligned} \left| \int_{\Gamma_{r_\varepsilon}} \frac{\partial \hat{V}^i_\varepsilon}{\partial n} (V - \overline{V}^{Q^i_{r_\varepsilon}}) \right| &\leq C\, \hat{\delta}_\varepsilon \int_P |W(r_\varepsilon)| \\ &\leq C\, \hat{\delta}_\varepsilon \left(\frac{1}{r_\varepsilon} \|W\|_{L^2(Q^i_{r_\varepsilon})} + \|\nabla W\|_{L^2(Q^i_{r_\varepsilon})} \right). \end{aligned} \tag{106}$$

It remains to estimate $\|W\|_{L^2(Q^i_{r_\varepsilon})}$. For that set

$$\tilde{W}(y) := \int_{-\frac{1}{2}}^{\frac{1}{2}} W(y)\, dy_i, \quad \text{independent of } y_i. \tag{107}$$

It is easy to check that

$$\|W - \tilde{W}\|_{L^2(Q^i_{r_\varepsilon})} \leq C \left\| \frac{\partial W}{\partial y_i} \right\|_{L^2(Q^i_{r_\varepsilon})}, \tag{108}$$

On the other side, $\tilde{W}$ has a zero mean in the disk of radius r_ε denoted by D_{r_ε}, the by r_ε-rescaling the Poincaré-Wirtinger inequality in the disk of radius 1, we obtain

$$\|\tilde{W}\|_{L^2(D_{r_\varepsilon})} \leq C\, r_\varepsilon\, \|\nabla \tilde{W}\|_{L^2(D_{r_\varepsilon})} \leq C\, r_\varepsilon\, \|\nabla W\|_{L^2(Q^i_{r_\varepsilon})}, \tag{109}$$

which implies that

$$\|W\|_{L^2(Q^i_{r_\varepsilon})} \leq C\, \|\nabla W\|_{L^2(Q^i_{r_\varepsilon})}. \tag{110}$$

This combined with (110) imply the desired estimate (40) and Lemma 1 is proved.

3.5. – Proof of Lemma 2

We assume that

$$\lim_{\varepsilon \to 0} \left(\frac{\log R_\varepsilon}{\log r_\varepsilon} \right) = 0, \tag{111}$$

by definition (37) of $\hat{v}_\varepsilon$ we have

$$\nabla \hat{v}_\varepsilon = \nabla \hat{v}^1_\varepsilon\, \hat{v}^2_\varepsilon\, \hat{v}^3_\varepsilon + \nabla \hat{v}^2_\varepsilon\, \hat{v}^3_\varepsilon\, \hat{v}^1_\varepsilon + \nabla \hat{v}^3_\varepsilon\, \hat{v}^1_\varepsilon\, \hat{v}^2_\varepsilon, \tag{112}$$

whence $|\nabla \hat{v}_\varepsilon| \leq |\nabla \hat{v}^1_\varepsilon| + |\nabla \hat{v}^2_\varepsilon| + |\nabla \hat{v}^3_\varepsilon|$, and by definition of $\hat{v}^i_\varepsilon$, for $i = 1 \ldots 3$,

$$\int_\Omega a_\varepsilon\, |\nabla \hat{v}^i_\varepsilon|^2 = \int_\Omega |\nabla \hat{v}^i_\varepsilon|^2 \leq \frac{C}{\varepsilon^2} \int_Y |\nabla \hat{V}^i_\varepsilon|^2 \leq C\, \frac{\hat{\delta}_\varepsilon}{\varepsilon^2} \leq C, \tag{113}$$

by assumption (15) and equation (111). Then, we have

$$\int_\Omega a_\varepsilon\, |\nabla \hat{v}^i_\varepsilon|^2 = \int_\Omega |\nabla \hat{v}^i_\varepsilon|^2 \leq C, \tag{114}$$

Let v_ε be a sequence $v_\varepsilon \in H^1_0(\Omega)$, such that (41) holds, and $\varphi \in C^1(\Omega)$. Since $\hat{v}_\varepsilon = 0$ in ω_ε (the set of fibers) and $a_\varepsilon = 1$ outside the fibers, we have

$$\int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon\, \varphi = \int_\Omega \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon\, \varphi. \tag{115}$$

We also have

$$\int_\Omega \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon\, \varphi = I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon, \tag{116}$$

where

$$I^i_\varepsilon := \int_\Omega \nabla v_\varepsilon \cdot \nabla \hat{v}^i_\varepsilon\, \varphi \prod_{j \neq i} \hat{v}^j_\varepsilon.$$

Let Ω_ε^i be the set of the open cylinders of same axis than that of ω_ε^i but of radius $\varepsilon R_\varepsilon$. Let us consider for instance I_ε^1. In particular $\omega_\varepsilon^i \subset \Omega_\varepsilon^i$ and $\{\hat{v}_\varepsilon^2 \hat{v}_\varepsilon^3 \neq 1\} \subset \Omega_\varepsilon^2 \cup \Omega_\varepsilon^3$. We have by the Cauchy-Schwarz inequality and by using the εY-periodicity of $\hat{v}_\varepsilon^1$

$$\left|\int_\Omega \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon^1 (\hat{v}_\varepsilon^2 \hat{v}_\varepsilon^3 - 1)\varphi\right| \leq C_\varphi \left(\int_{\Omega_\varepsilon^2 \cup \Omega_\varepsilon^3} |\nabla \hat{v}_\varepsilon^1|^2\right)^{\frac{1}{2}}$$
$$\leq C_\varphi \left(\frac{1}{\varepsilon^2}\int_{-R_\varepsilon}^{R_\varepsilon} dy_1 \int_{r_\varepsilon \leq r \leq R_\varepsilon} |\nabla \hat{V}_\varepsilon^1|^2 \, dy_2 \, dy_3\right)^{\frac{1}{2}} = O(R_\varepsilon^{1/2}), \tag{117}$$

The previous estimate implies that

$$I_\varepsilon^1 - \int_\Omega \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon^1 \varphi \to 0. \tag{118}$$

Moreover, we have

$$\int_\Omega \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon^1 \varphi = \int_\Omega \nabla \hat{v}_\varepsilon^1 \cdot \nabla(\varphi v_\varepsilon) - \int_\Omega \nabla \hat{v}_\varepsilon^1 \cdot \nabla \varphi \, v_\varepsilon, \tag{119}$$

and since v_ε is uniformly bounded by (41),

$$\left|\int_\Omega \nabla \hat{v}_\varepsilon^1 \cdot \nabla \varphi v_\varepsilon\right| \leq C_\varphi \int_\Omega |\nabla \hat{v}_\varepsilon^1| \leq C_\varphi |\omega_\varepsilon|^{\frac{1}{2}} \left(\int_\Omega |\nabla \hat{v}_\varepsilon^1|^2\right)^{\frac{1}{2}} = O(r_\varepsilon) \to 0. \tag{120}$$

This combined with limit (118) imply that for $\varepsilon \to 0$

$$I_\varepsilon^1 - \int_\Omega \nabla \hat{v}_\varepsilon^1 \cdot \nabla(\varphi v_\varepsilon) \to 0. \tag{121}$$

On the other hand, rescaling estimate (40) of Lemma 1, in each period cell of size ε and by summing over all the cells which compose ω_ε^1, we obtain the new estimate

$$\left|\int_\Omega \nabla \hat{v}_\varepsilon^1 \cdot \nabla(\varphi v_\varepsilon) - \frac{\hat{\delta}_\varepsilon}{\varepsilon^2}\int_\Omega \left(v_\varepsilon - \frac{\mathbf{1}_{\omega_\varepsilon^1}}{|Q_\varepsilon^1|} v_\varepsilon\right)\varphi\right|$$
$$\leq C\frac{\hat{\delta}_\varepsilon}{\varepsilon}\left(\sqrt{|\log R_\varepsilon|}\, \|\nabla(\varphi v_\varepsilon)\|_{L^2(\Omega)} + \frac{1}{r_\varepsilon}\|\nabla(\varphi v_\varepsilon)\|_{L^2(\omega_\varepsilon^1)}\right). \tag{122}$$

By (41), φv_ε is bounded in $H^1(\Omega)$ and, since v_ε is bounded in $L^\infty(\Omega)$ we have

$$\|\nabla(\varphi v_\varepsilon)\|_{L^2(\omega_\varepsilon^1)} \leq C_\varphi |\omega_\varepsilon^1|^{\frac{1}{2}} + C_\varphi \frac{1}{\sqrt{\alpha_\varepsilon}}\left(\int_{\omega_\varepsilon^1} \alpha_\varepsilon |\nabla v_\varepsilon|^2\right)^{\frac{1}{2}}$$
$$\leq C_\varphi \left(r_\varepsilon + \frac{1}{\sqrt{\alpha_\varepsilon}}\right), \tag{123}$$

whence, by assumptions (15) and (16),

$$(124)\quad \frac{\hat{\delta}_\varepsilon}{\varepsilon}\left(\sqrt{|\log R_\varepsilon|}\|\nabla(\varphi v_\varepsilon)\|_{L^2(\Omega)}+\frac{1}{r_\varepsilon}\|\nabla(\varphi v_\varepsilon)\|_{L^2(\omega_\varepsilon^1)}\right)\le C_\varphi\left(\varepsilon\sqrt{|\log R_\varepsilon|}+\varepsilon\right).$$

We choose R_ε such that

$$(125)\qquad \lim_{\varepsilon\to 0}\varepsilon^2\,|\log R_\varepsilon| = 0,$$

which implies condition (111) since r_ε satisfies (15).

Then, the previous estimates yield

$$(126)\qquad \int_\Omega \nabla\hat{v}_\varepsilon^1\cdot\nabla(\varphi v_\varepsilon)-\frac{\hat{\delta}_\varepsilon}{\varepsilon^2}\int_\Omega(v_\varepsilon-\tilde{v}_\varepsilon^1)\,\varphi\to 0,$$

which combined with limit (121) give

$$(127)\qquad I_\varepsilon^1-\frac{\hat{\delta}_\varepsilon}{\varepsilon^2}\int_\Omega(v_\varepsilon-\tilde{v}_\varepsilon^1)\,\varphi\to 0.$$

Moreover, by (125) and (15) we have

$$(128)\qquad \frac{\hat{\delta}_\varepsilon}{\varepsilon^2}=\frac{2\pi}{\varepsilon^2(\log R_\varepsilon-\log r_\varepsilon)}\underset{\varepsilon\to 0}{\sim}\frac{2\pi}{\varepsilon^2\,|\log r_\varepsilon|}\to\frac{1}{3}.$$

By estimate (114) and using (121), it is easy to remark that I_ε^1 is bounded, then the limit (127) implies that

$$(129)\qquad I_\varepsilon^1+\frac{1}{3}\int_\Omega\tilde{v}_\varepsilon^1\,\varphi\to\frac{1}{3}\int_\Omega v_0\,\varphi.$$

Similar limits holds true for I_ε^2 and I_ε^3 which yields the desired limit (42). Lemma 2 is proved.

3.6. – Proof of Lemma 3

By construction the set of the fibers ω_ε is contained in an open subset of Ω denoted by Ω_ε such that $\operatorname{dist}(\Omega_\varepsilon,\partial\Omega)\sim d_\varepsilon$. Let us consider $\varphi_\varepsilon\in C^1(\overline{\Omega})$ such that

$$(130)\qquad 0\le\varphi_\varepsilon\le 1,\quad \varphi_\varepsilon=\begin{cases}1 & \text{in }\Omega_\varepsilon\\ 0 & \text{on }\partial\Omega\end{cases}\quad\text{and}\quad |\nabla\varphi_\varepsilon|\le\frac{C}{d_\varepsilon}\ \text{in }\Omega.$$

Since v_ε belongs to $H_0^1(\Omega)$ and $g_\varepsilon := \operatorname{div}(a_\varepsilon \nabla v_\varepsilon)$ is bounded in $L^2(\Omega)$, the energy $\int_\Omega a_\varepsilon |\nabla v_\varepsilon|^2$ is bounded, and we by the Cauchy-Schwarz inequality

$$
(131)\qquad
\begin{aligned}
\left| \int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon \,(1-\varphi_\varepsilon) \right| &\le \left(\int_\Omega a_\varepsilon |\nabla v_\varepsilon|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega\setminus\Omega_\varepsilon} |\nabla \hat{v}_\varepsilon|^2 \right)^{\frac{1}{2}} \\
\left| \int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon \,(1-\varphi_\varepsilon) \right| &\le C \left(\int_{\Omega\setminus\Omega_\varepsilon} |\nabla \hat{v}_\varepsilon|^2 \right)^{\frac{1}{2}} .
\end{aligned}
$$

Then, by dividing $\Omega\setminus\Omega_\varepsilon$ in period cells of size ε and by assumption (15), we obtain

$$
(132)\qquad \left| \int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon \,(1-\varphi_\varepsilon) \right| \le C\, |\Omega\setminus\Omega_\varepsilon|^{\frac{1}{2}} \left(\frac{1}{\varepsilon^2} \int_Y |\nabla \hat{V}_\varepsilon|^2 \right)^{\frac{1}{2}} \le C\sqrt{d_\varepsilon}\,,
$$

whence

$$
(133)\qquad \int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon = \int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon \,\varphi_\varepsilon + o(1)\,.
$$

Moreover, by integrating by parts we have

$$
(134)\qquad
\begin{aligned}
\int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon \,\varphi_\varepsilon &= \int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla((\hat{v}_\varepsilon - 1)\varphi_\varepsilon) + \int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \varphi_\varepsilon \,(1-\hat{v}_\varepsilon) \\
&= \int_\Omega g_\varepsilon \,(\hat{v}_\varepsilon - 1)\,\varphi_\varepsilon + \int_{\Omega\setminus\Omega_\varepsilon} a_\varepsilon \nabla v_\varepsilon \cdot \nabla \varphi_\varepsilon \,(1-\hat{v}_\varepsilon)\,.
\end{aligned}
$$

On the first hand, since $|\{\hat{v}_\varepsilon \neq 1\}| = O(R_\varepsilon^2)$ and g_ε is bounded in $L^2(\Omega)$, we have

$$
\int_\Omega g_\varepsilon (\hat{v}_\varepsilon - 1)\,\varphi_\varepsilon \to 0\,.
$$

On the other hand, since $a_\varepsilon = 1$ in $\Omega\setminus\Omega_\varepsilon$, we have by the Cauchy-Schwarz inequality and by (130)

$$
(135)\qquad
\begin{aligned}
\left| \int_{\Omega\setminus\Omega_\varepsilon} a_\varepsilon \nabla v_\varepsilon \cdot \nabla \varphi_\varepsilon \,(1-\hat{v}_\varepsilon) \right| &= \left| \int_{\Omega\setminus\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla \varphi_\varepsilon (1-\hat{v}_\varepsilon) \right| \\
&\le \frac{C}{d_\varepsilon} \left(\int_{\Omega\setminus\Omega_\varepsilon} |\nabla v_\varepsilon|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega\setminus\Omega_\varepsilon} (1-\hat{v}_\varepsilon)^2 \right)^{\frac{1}{2}} \\
&\le \frac{C}{d_\varepsilon} \left| (\Omega\setminus\Omega_\varepsilon) \cap \{\hat{v}_\varepsilon \neq 1\} \right|^{\frac{1}{2}} \le \frac{C R_\varepsilon}{\sqrt{d_\varepsilon}}\,.
\end{aligned}
$$

We now choose d_ε such that $R_\varepsilon^2 \ll d_\varepsilon$, whence

$$\int_\Omega a_\varepsilon \nabla v_\varepsilon \cdot \nabla \hat{v}_\varepsilon \varphi_\varepsilon \to 0. \tag{136}$$

This limit, combined with (133), thus yield the desired limit. Lemma 3 is proved.

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