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Francesco Maddalena
Sergio Solimini

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Concentration and Flatness Properties of the Singular Set of Bisected Balls

FRANCESCO MADDALENA – SERGIO SOLIMINI

Abstract. In this paper we exploit the relation between the notion, introduced in [22], called splitness and the notion introduced in [9] (see [18]) called concentration. An immediate consequence is that the so called bisection property proved in [22] for the minimizers of functionals with free discontinuities implies the concentration property which in turn (see [16]) gives the lower semicontinuity of the Hausdorff measure with respect to the Hausdorff distance.

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Contents

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Introduction

The aim of this paper consists in establishing some properties of the minimizers of the functionals with free discontinuities which are useful for some approaches to both the existence and the regularity theory. In spite of this fact, the content of this paper can be set at a higher level of generality and we shall never need to mention any functional in the following. Therefore we shall not introduce the reader to this subject but we shall refer him to the existent literature (see the papers [2], [3], [4], [9], [10], [11], [12], [13], [20] and the books [6], [18]).

The existence approach introduced for the case of planar domains in [9] makes use of a property called *Concentration Property* which was proved therein in two dimensions. On the other hand, in [22] a related property called *Bisection Property* has been established in any dimension. We shall show here that the first property is actually implied by the second one, this will allow the extention of the approach of [9] which will be made in [16], where the results in this paper find a first application. Moreover in [17] other aspects, based on the notion of *flatness*, of the arguments which we are going to develop here are used in order to produce some regularity theorems, which improve and are based on some previous ones due to L. Ambrosio, N. Fusco and D. Pallara [5].

We shall give two different proofs of the main results of the paper, a more technical but faster one based on the use of Coarea Formula and a more elementary but longer one based on a discrete technique which also gives a further result needed in some applications ([17]).

The paper is organized as follows. In Section 1 we shall introduce the basic notation and we shall state the main final results. Section 2, which is essentially descriptive, introduces the main geometric tools. Section 3 is devoted to the formal proof of a very intuitive assertion, which will be after employed and which, roughly speaking, states that the most convenient way to split a ball B in parts with measure less than or equal to one half of the total measure consists in dividing the ball exactly in two parts. In Section 4 it is shown that the sets with almost minimal measure which divide the ball in two connected components with almost equal measure have a very small flatness, namely they are almost completely close to a suitable hyperplane through the center of the ball. In Section 5 we shall prove the main results by using the Coarea Formula. In Section 6 we state a discrete version of the geometric properties introduced in Section 2 and, in particular, the notion of some specific coverings, called *punchings*, which will be used to pass to a discretization of the properties. Sections 7 and 8 respectively are the discrete counterpart of Sections 3 and 4. In Section 9 we shall introduce some similarity arguments which will be needed for the Flatness Theorem which will then be deduced in the subsequent Section 10 and finally used in Section 11 for the alternative proof of the main results.

The results in this paper have been announced in May 1997 at the *Meeting on Free Discontinuity Problems*, held in Cortona and, in a more detailed form, in

October of the same year at the *Meeting on Differential Equations and Calculus of Variations* held at Isola d'Elba. Subsequently, in a paper of S. Rigot ([21]) a result on the existence of big C^1-pieces of K (see also [17]) has shown in particular the concentration property for minima of the Mumford-Shah functional, giving a different proof of the main application of this paper. The arguments of [21] only need to be used in a simpler form to this aim and lead back the concentration property of the minimizers to the uniform rectifiability estimates established by G. David and S. Semmes in [11]. We are grateful to Luigi Ambrosio for the indication of the reference [7] and to Guy David for several discussions on this topic.

1. – Notation and main results

Let X be a subset of $\mathbb{R}^N$, we shall denote by $\mathcal{H}^\alpha(X)$ the Hausdorff measure of dimension α of X and by $|X|$ its Lebesgue measure. $\mathbf{1}_X$ represents the characteristic function of X. If u is any real function defined on a subset X of $\mathbb{R}^N$, then $\mathrm{osc}_X u$ will represent the oscillation of a function u on X, that is $\mathrm{osc}_X u = \sup_X u - \inf_X u$. For $\varepsilon > 0$, the closed ε-neighborhood of X is

$$X(\varepsilon) = \{x \in \mathbb{R}^N \mid d(x, X) \leq \varepsilon\}.$$

Given a subset $X \subset \mathbb{R}^N$, we shall say that $\mathcal{N}$ is an *ε-net* of X if $\mathcal{N}$ is a finite set such that every point in X has a distance less than ε to some point of $\mathcal{N}$. The open ball with center $x \in \mathbb{R}^N$ and radius r, $0 < r < \infty$, will be denoted by $B(x,r)$. B_1 will represent the unitary ball $B(0,1)$ of $\mathbb{R}^N$. For $\lambda \in \mathbb{R}$, we shall use the notation $\lambda B = B(x, \lambda r)$ when $B = B(x,r)$. For every $k \in \mathbb{N}$, b_k denotes the measure of the unit ball of $\mathbb{R}^k$, thus kb_k is the $(k-1)$-dimensional measure of its boundary.

The main results stated in the present paper make use of the notion of *ε-splitness* of a ball B, introduced in [22], which will be formally defined in Section 2. Roughly speaking, if K is any closed subset of B, the *ε-splitness* of B requires the existence of a function u having a suitably small gradient on $B \setminus K$ and a large oscillation, quantified in terms of ε, on all the subsets of B with a measure greater than $(\frac{1}{2} + \varepsilon)|B|$. Furthermore, the results concern some geometric properties of the set K, namely the *ε-closeness* to a disk, which will be introduced in Section 2, the flatness $\mathbf{A}_K(B)$ and the tilt $\mathbf{T}_K(B)$, which are well known in literature and will be recalled in the same section. The main results are the following.

THEOREM 1.1 (Concentration). *For every $\varepsilon > 0$ there exists δ such that, if K is a closed subset of a ball B of radius R and B is δ-split by K, then*

$$\mathcal{H}^{N-1}(K) \geq (1-\varepsilon) b_{N-1} R^{N-1}. \tag{1.1}$$

THEOREM 1.2 (Flatness - Tilt). *For every $\varepsilon > 0$ there exists δ such that, if a ball B of radius R is δ-split by K and*

$$\mathcal{H}^{N-1}(K) \leq (1+\delta) b_{N-1} R^{N-1}, \tag{1.2}$$

then the set K is ε-close to a disk D of B and $\mathbf{A}_K(B) < \varepsilon$, $\mathbf{T}_K(B) < \varepsilon$.

We shall give two different proofs of the above theorems. The first one will be based on the use of Coarea Formula in order to convert the splitness assumption in a real disconnection condition by filling the holes of K with a suitable small level set. The second proof will be deduced by a generalization of Theorem 1.2, namely Theorem 1.3 below which turns out to be useful for further developments of the regularity properties ([17]). The gain of generality consists in the replacement of the *absolute* minimality assumption (1.2) with a *relative* minimality condition (with respect to big enough sub-balls) which we are going to introduce.

We shall fix $\varepsilon' > 0$ and, consequently, $t \in \mathbb{R}$ and $n \in \mathbb{N}$ in the following way:

$$t = \min\left\{ \frac{\varepsilon'}{5^{N+1}}, \frac{1}{4}\left(\left(\frac{1+\varepsilon'}{1+\frac{2\varepsilon'}{3}} \right)^{\frac{1}{N-1}} - 1 \right) \right\} \tag{1.3}$$

and n is the least integer such that

$$(1+3t)^{N-1} + 2^{2N-1} n^{-1} < \frac{1+\varepsilon'}{1+\frac{\varepsilon'}{2}} \tag{1.4}$$

holds. So t and n are defined as functions of ε'. When ε' is given, for any $\delta > 0$, t and n as above, for a given ball B of radius R and a closed subset $K \subset B$, we introduce the following minimality condition

$$\begin{array}{ll} & \text{For every } \delta\text{-split ball } B' \subset B \text{ of radius } r \geq t^n R: \\ (\mathrm{M}_\delta) & R^{1-N}\mathcal{H}^{N-1}(K) \leq (1+\frac{\varepsilon'}{2}) r^{1-N} \mathcal{H}^{N-1}(K \cap B'). \end{array}$$

THEOREM 1.3 (Flatness). *For every $\varepsilon, \delta_1 > 0$, there exist ε' (only depending on ε) and δ_2 such that, if B is δ_2-split by K and satisfies (M_{δ_1}), then K is ε-close to some hyperplane P through the center of B.*

COROLLARY 1.4. *For every $\varepsilon, \delta_1, c_0 > 0$, there exist ε' (only depending on ε) and δ_2 such that, if B is δ_2-split by K, satisfies (M_{δ_1}) and $\mathcal{H}^{N-1}(K \cap B) \leq c_0 R^{N-1}$, then $\mathbf{A}_K(B) < \varepsilon$.*

The key idea of the proof relies in passing from the splitness of the ball B to a real disconnection property, through a sort of discretization of the problem. Such a discretization is achieved by substituting the set K with a finite set of small balls, which we shall call *punching*. Subsequently, a similarity argument, linking the configuration of K in B with that of its trace on the balls of the punching and based on (M_{δ_1}), will allow the proof of Theorem 1.3 and, consequently a longer but more elementary alternative proof of the main results.

2. – Geometric preliminaries

We are going to set some geometric tools which will turn out to be useful in characterizing some properties, in terms of mean densities, of the singular set K of a given real function.

Let P denote any affine $(N-1)$-dimensional subspace of $\mathbb{R}^N$. Let K be a closed subset of a given ball B with radius R, if P is a hyperplane through the center of B, then $D = P \cap B$ is a disk of B and, by denoting by p the orthogonal projection onto P, we give the following definition.

DEFINITION 2.1. The set K is said to be ε-close to D if

$$\mathcal{H}^{N-1}(K \setminus P(\varepsilon R)) \leq \varepsilon \mathcal{H}^{N-1}(K), \tag{2.5}$$

$$\mathcal{H}^{N-1}(D \setminus p(K)) \leq \varepsilon R^{N-1}. \tag{2.6}$$

Let $X \subset \mathbb{R}^N$ and $u \in L^1_{\mathrm{loc}}(X)$ be a real function. We shall denote by $\|u\|^*_{2N}$ the least constant c such that the following weak summability condition of ∇u holds.

$$\text{(WS)} \qquad \textit{For every ball} \quad B \subset \mathbb{R}^N : \int_B |\nabla u| \leq c|B|^{\frac{2N-1}{2N}},$$

where ∇u is defined in the sense of distributions on $\overset{\circ}{X}$ and it is assumed to be extended by zero on the rest of $\mathbb{R}^N$. The above condition states that the distributional gradient ∇u, extended by zero on $\mathbb{R}^N \setminus \overset{\circ}{X}$, belongs to the Morrey Space $L^{1,\frac{2N-1}{2}}$. Let $B = B(x,r)$ be a given ball, $K \subset B$ any closed subset.

DEFINITION 2.2. Let $\varepsilon > 0$ be given. The ball B is said to be ε-split by a function $u : B \to \mathbb{R}$ if it does not contain any subset $\widetilde{B}$ such that

$$|\widetilde{B}| \geq \left(\frac{1}{2} + \varepsilon\right)|B|, \tag{2.7}$$

$$\operatorname{osc}_{\widetilde{B}} u \leq \varepsilon^{-1} r^{\frac{1}{2}}. \tag{2.8}$$

DEFINITION 2.3. The ball B is said to be ε-split by a closed set K if there exists $u : B \to \mathbb{R}$, satisfying (WS) on $B \setminus K$ with $\|u\|^*_{2N} = 1$, such that B is ε-split by u.

We shall refer to any subset $\widetilde{B}$ satisfying (2.7) and (2.8) as to a *region with ordinary oscillation*. In other words, for every real function u with $\|u\|^*_{2N} \leq 1$ we can find a subset with ordinary oscillation in a ball B if and only if B is not *ε-split*. We remark that if u satisfies (WS) on $B \setminus K$ and B is *ε-split* by u, then B is *ε'-split* by K, with $\varepsilon' = \max\{\varepsilon, \varepsilon\|u\|^*_{2N}\}$. Let B be a given ball with radius R, for every $\varepsilon > 0$ we shall denote by $\mathcal{S}_\varepsilon(B)$ the set of all closed subsets K of B such that B is *ε-split* by K. We put

$$s_\varepsilon(B) = \inf_{K \in \mathcal{S}_\varepsilon(B)} \mathcal{H}^{N-1}(K),$$
$$s(\varepsilon) = s_\varepsilon(B_1)$$

and

$$s_0 = \lim_{\varepsilon \to 0} s(\varepsilon).$$

We note that $s(\varepsilon)$ is a monotone decreasing function. By rescaling, we can easily see that for every $R > 0$

$$s_\varepsilon(B) = R^{N-1} s(\varepsilon).$$

The above quantities will be compared with the analogous ones related to disconnection conditions.

DEFINITION 2.4. Let $\varepsilon > 0$ be given. The ball B is said to be ε-disconnected by a set $K \subset B$ if there does not exist any connected component of $B \setminus K$ with measure greater than or equal to $(\frac{1}{2} + \varepsilon)|B|$.

We shall denote by $\mathcal{D}_\varepsilon(B)$ the set of all closed subsets K of B such that B is ε-disconnected by K. We put

$$\sigma_\varepsilon(B) = \inf_{K \in \mathcal{D}_\varepsilon(B)} \mathcal{H}^{N-1}(K),$$
$$\sigma(\varepsilon) = \sigma_\varepsilon(B_1)$$

and

$$\sigma_0 = \lim_{\varepsilon \to 0} \sigma(\varepsilon).$$

By rescaling, we get for every $R > 0$

$$\sigma_\varepsilon(B) = R^{N-1} \sigma(\varepsilon).$$

We remark that if B is *ε-disconnected* by K, then it is also *ε-split* by K, thus

$$\mathcal{D}_\varepsilon(B) \subset \mathcal{S}_\varepsilon(B)$$

(indeed it can be seen by taking a constant function u on the connected components of $B \setminus K$). Therefore, for every $\varepsilon > 0$ we have

$$s_\varepsilon(B) \le \sigma_\varepsilon(B), \tag{2.9}$$

$$s_0 \le \sigma_0 \le b_{N-1}. \tag{2.10}$$

The last inequality is really an equality (see [7]), nevertheless in order to get this information we need to prove a very intuitive property, which constitutes the subject of the next section.

If $B' \subset B$ is any ball and $K \in \mathcal{D}_\varepsilon(B)$, we define in the usual way the *restriction* K' of K to B'. An *extension* K'' of a given $K \in \mathcal{D}_\varepsilon(B)$ to a ball $B'' \supset B$ with radius $R'' > R$ is defined as any set containing K and such that any two different connected components of $B \setminus K$ are never contained in the same connected component of $B'' \setminus K''$, namely they are not merged in the bigger ball. We notice that such operations do not keep the property of being in $\mathcal{D}_\varepsilon$ because they do not preserve the estimate of the relative measure of the connected components with respect to the measure of the new ball. Such an estimate is just what we have to prove if we want to claim that $K' \in \mathcal{D}_\varepsilon(B')$ or $K'' \in \mathcal{D}_\varepsilon(B'')$.

Let B be any ball, with radius R. By denoting with ρ the supremum among the radii of the balls contained in B, which are *ε-split* by K, we shall call the ratio $\nu_b = \dfrac{\rho}{R}$ *bisection factor.* It represents the scale transition needed to reach an *ε-split* ball in B; obviously $\nu_b = \nu_b(B, K, \varepsilon)$. We shall say that a closed subset K of an open set $\Omega \subset \mathbb{R}^N$ satisfies the *Bisection Property* on Ω when for every $\varepsilon > 0$ there exists $\alpha(\varepsilon)$ such that, for every ball $B \subset \Omega$ centered on K, with radius $R \leq 1$, the lower estimate $\nu_b(B, K, \varepsilon) \geq \alpha(\varepsilon)$ holds.

Given a ball B centered on K, with radius R, we say that the set K is *ε-concentrated* on B if the mean density of K on B is bigger than $1 - \varepsilon$, namely

$$\mathcal{H}^{N-1}(K \cap B) > (1 - \varepsilon) b_{N-1} R^{N-1}.$$

Given any ball B, with radius R, we denote with $\nu_c = \nu_c(B, K, \varepsilon)$ the *concentration factor*, that is the ratio between the supremum among the radii of the balls contained in B on which K is *ε-concentrated* and R. We shall say that a closed subset K of an open set $\Omega \subset \mathbb{R}^N$ satisfies the *Concentration Property* on Ω when for every $\varepsilon > 0$ there exists $\alpha(\varepsilon)$ such that, for every ball $B \subset \Omega$ centered on K, with radius $R \leq 1$, the lower estimate $\nu_c(B, K, \varepsilon) \geq \alpha(\varepsilon)$ holds.

We remark that Theorem 1.1 allows us to claim the following statement.

COROLLARY 2.1. *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that, for every B and K, $\nu_c(\varepsilon) \geq \nu_b(\delta)$.*

In particular, in [22] it is proved that condition (WS) is satisfied by a minimum (u, K) of the N-dimensional Mumford-Shah functional. So the bisection property proved therein with respect to all the minimal pairs (u, K) falls in the definition given above. Therefore, by Corollary 2.1 we can assert that the concentration property easily follows. Indeed, for every ball, $\nu_c(\varepsilon) \geq \nu_b(\delta) \geq \alpha(\delta(\varepsilon))$. The concentration property was introduced for the first time in [9], its importance relies on the circumstance that the Hausdorff measure is lower semicontinuous along sequences of sets uniformly enjoying such a property and converging for the Hausdorff distance (see [9] and [18, chap. 10]). Furthermore, Theorem 1.2 already provides a reinforcement to the uniform concentration property, indeed we can claim that, at a scale bounded below, there exists a ball contained in B where the density of K is greater or equal to $1 + \varepsilon$, or K is *ε-close* to a suitable hyperplane, while Theorem 1.3

provides the tool towards further reinforcements (see [17] or [21] under a different approach). As in the two dimensional case, this argument allows to prove directly the existence of a closed set minimizing the Mumford-Shah functional. For arbitrary dimension this theme is fully developed in [16] where the results of the present paper find a first application.

Now we introduce the following quantities which play a key role in studying the regularity of the set K in the ball $B = B(x, r)$ (see [5] and [6] for more details).

SCALED FLATNESS

$$\mathbf{A}_K(B) = \mathbf{A}_K(x, r) = r^{-N-1} \min_{P \in \mathcal{A}} \int_{B \cap K} d^2(y, P) d\mathcal{H}^{N-1}(y),$$

where $\mathcal{A}$ denotes the set of the affine hyperplanes in $\mathbb{R}^N$. $\mathbf{A}_K(B)$ measures the scaled flatness of K into the ball B.

SCALED TILT

$$\mathbf{T}_K(B, T) = \mathbf{T}_K(x, r, T) = r^{1-N} \int_{B \cap K} \| S_y - T \|^2 \, d\mathcal{H}^{N-1}(y),$$

where T is a given hyperplane of $\mathbb{R}^N$ and $\| S_y - T \|$ denotes the distance of the directions of the approximate tangent space S_y to a rectifiable set K at y from T. It measures the oscillation of S_y in B with respect to a given hyperplane. We will use the same notation also in the case of an unrectifiable set K, by assuming the integrand equal to one when S_y is not defined. Moreover, we put

$$\mathbf{T}_K(B) = \inf_{T \in \mathcal{A}} \mathbf{T}_K(B, T).$$

3. - Bisection properties

The aim of the definition of $\mathcal{D}_\varepsilon(B)$ is to get a disconnection of the ball B in parts whose measures are not much greater than one half of the measure of B. Although it is evident that the way to reach such a disconnection through a set with minimal measure is to create exactly two connected components, the rigorous proof of this fact is not completely obvious. The crucial point consists in showing that, if a set has almost minimal measure in $\mathcal{D}_\varepsilon(B)$, then at least one of the connected components has a measure close to $\frac{1}{2}|B|$, as stated in the following theorem. The present section is all devoted to the proof of Theorem 3.1, so, if the reader believes the assertion true, he can skip to the next section.

THEOREM 3.1. *For every $\varepsilon > 0$ there exists δ such that, if B is a ball of radius R and $K \in \mathcal{D}_\delta(B)$ with $\mathcal{H}^{N-1}(K) \leq (\sigma_0 + \delta)R^{N-1}$, then there exists a connected component C of $B \setminus K$, such that $|C| > (\frac{1}{2} - \varepsilon)|B|$.*

The above property is obviously scale invariant, so we can work on a fixed ball B of radius R. In order to prove Theorem 3.1 by contradiction, we assume that a constant $\gamma < \frac{1}{2}$ can be fixed in such a way that, for δ arbitrarily small, there exists a set $K \in \mathcal{D}_\delta(B)$ with $\mathcal{H}^{N-1}(K) < (\sigma_0 + \delta)R^{N-1}$ such that, for every C connected component of $B \setminus K$, we have $|C| < \gamma|B|$. Therefore, by continuity, we can fix R' and R'' depending on γ (but constant with respect to δ), with $R' < R < R''$, such that, if B' and B'' are any pair of balls, $B' \subset B \subset B''$, of radius respectively greater than or equal to R' and less than or equal to R'', K' is the restriction of K to B' and K'' is any extension of K' from B' to B'', then K' and K'' are respectively in $\mathcal{D}_\delta(B')$ and $\mathcal{D}_\delta(B'')$.

For $\rho > 0$ let $B(\rho) = \frac{\rho}{R}B$ be the ball with the same center of B and radius ρ; we define the following density function

$$d(\rho) = \frac{\mathcal{H}^{N-2}(K \cap \partial B(\rho))}{\rho^{N-2}}. \tag{3.11}$$

In order to simplify the computations, we will work up to quantities which are infinitesimal as δ tends to zero. To this aim, we will denote by $h(\delta)$ any function which tends to zero as δ tends to zero, and we shall omit to specify it more. So, in general, different occurrences of the symbol $h(\delta)$ will stand for different quantities with the common feature of being infinitesimal, analogously to the Landau symbols O and o, which we are also going to use. Our first aim consists in proving some lemmas whose final goal relies in stating that, modulo an infinitesimal error, the density $d(\rho)$ is a constant function for $R' \leq \rho \leq R$. This result will represent the main ingredient used in the proof of Theorem 3.1 and can be stated as follows.

LEMMA 3.1. *There exists a positive constant $\overline{d}$, such that*

$$\int_{R'}^{R} |d(\rho) - \overline{d}|\rho^{N-2}d\rho = h(\delta), \tag{3.12}$$

$$\mathcal{H}^{N-1}\left(K \setminus \frac{R'}{R}B\right) = \overline{d}\int_{R'}^{R} \rho^{N-2}d\rho + h(\delta). \tag{3.13}$$

We shall give the proof of Lemma 3.1 after some preliminary steps.

LEMMA 3.2. *For every $r \in [R', R]$ let $B' = \frac{r}{R}B$, then the following inequality holds.*

$$\mathcal{H}^{N-1}(K \setminus B') \leq \sigma_0(R^{N-1} - r^{N-1}) + h(\delta). \tag{3.14}$$

PROOF. As we have already noted, K' is in $\mathcal{D}_\delta(B')$, so we have by definition

$$\mathcal{H}^{N-1}(K') \geq \sigma(\delta)r^{N-1} = (\sigma_0 + h(\delta))r^{N-1}. \tag{3.15}$$

Moreover, by hypothesis we know that

$$\mathcal{H}^{N-1}(K) \leq (\sigma_0 + \delta)R^{N-1} = (\sigma_0 + h(\delta))R^{N-1}. \tag{3.16}$$

By combining (3.15) and (3.16), we get (3.14). □

By (3.14) we obtain a first integral estimate on $d(\rho)$, indeed for every $r \in [R', R]$ the following inequality holds

$$\int_r^R d(\rho)\rho^{N-2}d\rho \leq \mathcal{H}^{N-1}(K \setminus B') \leq \sigma_0(R^{N-1} - r^{N-1}) + h(\delta)\,. \tag{3.17}$$

Now we want to get a uniform lower estimate on $d(\rho)$ which matches (3.17) and shows that the two inequalities are actually equalities. To this aim, we are going to consider for a fixed $r \in [R', R]$ the restriction K' of K to the concentric ball B' to B of radius r and then the radial extension K'' of K' to the ball B'' of radius R''. To this aim we set

$$\overline{d} = \inf_{\rho \in [R', R]} d(\rho)$$

and we take $\overline{\rho} \in [R', R]$ such that

$$d(\overline{\rho}) \leq (1+\delta)\overline{d}\,. \tag{3.18}$$

Lemma 3.3. *The following lower bound holds true*

$$\overline{d} \geq \sigma_0(N-1) + h(\delta)\,. \tag{3.19}$$

Proof. If B' is the ball with the same center of B and radius $\overline{\rho}$, by taking K'' as above, we get

$$\mathcal{H}^{N-1}(K'') = \mathcal{H}^{N-1}(K) - \mathcal{H}^{N-1}(K \setminus B') + d(\overline{\rho}) \int_{\overline{\rho}}^{R''} r^{N-2}dr\,. \tag{3.20}$$

So

$$\begin{aligned} \mathcal{H}^{N-1}(K'') &\leq \mathcal{H}^{N-1}(K) - \int_{\overline{\rho}}^{R} d(\rho)\rho^{N-2}d\rho + d(\overline{\rho})\int_{\overline{\rho}}^{R''} \rho^{N-2}d\rho \\ &\leq (\sigma_0+\delta)R^{N-1} - \int_{\overline{\rho}}^{R} \overline{d}\rho^{N-2}d\rho + (1+\delta)\overline{d}\int_{\overline{\rho}}^{R''} \rho^{N-2}d\rho \\ &= (\sigma_0+\delta)R^{N-1} + \overline{d}\int_{R}^{R''} \rho^{N-2}d\rho + h(\delta)\,. \end{aligned} \tag{3.21}$$

Since K'' is in $\mathcal{D}_\delta(B'')$ and therefore

$$\mathcal{H}^{N-1}(K'') \geq \sigma(\delta)(R'')^{N-1} = (\sigma_0 + h(\delta))(R'')^{N-1}\,, \tag{3.22}$$

by (3.21) and (3.22) we finally get

$$\begin{aligned} \overline{d} &\geq \left(\int_R^{R''} \rho^{N-2}d\rho\right)^{-1} \left((\sigma_0 + h(\delta))(R'')^{N-1} - (\sigma_0+\delta)R^{N-1}\right) + h(\delta) \\ &= \sigma_0(N-1) + h(\delta)\,. \end{aligned} \tag{3.23}$$

□

Proof of Lemma 3.1. By integrating (3.19) we get

$$\int_{R'}^{R} \overline{d}\rho^{N-2}d\rho \geq \sigma_0(R^{N-1} - (R')^{N-1}) + h(\delta)\,,$$

so, by using (3.17) for $r = R'$, since $d(\rho) - \overline{d} \geq 0$, we finally have (3.12). So the inequalities in (3.17) are equalities and the first one of them, combined with (3.12) gives (3.13). □

PROOF OF THEOREM 3.1. We assume by contradiction that the thesis is false, so we can use all the construction made in this section. We set $R^* = \frac{R+R'}{2}$ and for every $y \in \partial B$ we denote by B^y the ball with radius R^* contained in B and tangent in y to ∂B, whereas B^* is the ball with the same center of B and radius R^*. We set

$$p(x) = \fint_{\partial B_R} \mathbf{1}_{B^y}(x) d\mathcal{H}^{N-1}(y) \leq 1 . \tag{3.24}$$

Because of the radial symmetry of the mappings $\mathbf{1}_{B^*}$ and p, we can deal with $\mathbf{1}_{B^*}(\rho)$ and $p(\rho)$. We can claim that

$$\int_{\mathbb{R}^N} p(x)dx = |B^*| \tag{3.25}$$

and therefore

$$\int_{\mathbb{R}^N} (\mathbf{1}_{B^*} - p)dx = 0 , \quad \text{namely} \int_0^\infty (\mathbf{1}_{B^*} - p)(\rho)\rho^{N-1}d\rho = 0 . \tag{3.26}$$

We note that, since

$$(\mathbf{1}_{B^*} - p)(\rho) \begin{cases} = 0 & \text{if } 0 < \rho < R' \\ > 0 & \text{if } R' < \rho < R^* \\ < 0 & \text{if } R^* < \rho < R , \end{cases}$$

then we can find a positive constant $\overline{c}$ such that

$$\int_0^R (\mathbf{1}_{B^*} - p)(\rho)\rho^{N-2}d\rho > \overline{c} . \tag{3.27}$$

Since $\mathbf{1}_{B^*} = p$ for $\rho \leq R'$,

$$\int_{R'}^R p(\rho)\rho^{N-2}d\rho \leq \int_{R'}^{R^*} \rho^{N-2}d\rho - \overline{c} . \tag{3.28}$$

Now we are going to estimate the mean trace t_m, defined as

$$\begin{aligned} t_m &= \fint_{\partial B} \mathcal{H}^{N-1}(K \cap B^y) d\mathcal{H}^{N-1}(y) \\ &= \fint_{\partial B} \left(\int_K \mathbf{1}_{B^y} d\mathcal{H}^{N-1} \right) d\mathcal{H}^{N-1}(y) = \int_K p \, d\mathcal{H}^{N-1} \end{aligned}$$

and to this aim we evaluate t_m by using (3.12) and (3.13).

$$\begin{aligned} t_m &= \int_{K \cap B'} p \, d\mathcal{H}^{N-1} + \int_{K \setminus B'} p \, d\mathcal{H}^{N-1} \\ &= \mathcal{H}^{N-1}(K \cap B') + \int_{R'}^R \overline{d}\rho^{N-2} p(\rho) d\rho + h(\delta) . \end{aligned}$$

By (3.12), (3.13), (3.28) and (3.19) we get

$$
\begin{aligned}
t_m &\leq \mathcal{H}^{N-1}(K \cap B') + \int_{R'}^{R^*} \overline{d}\rho^{N-2} d\rho + h(\delta) - \overline{d}\,\overline{c} \\
&= \mathcal{H}^{N-1}(K \cap B') + \int_{R'}^{R} \overline{d}\rho^{N-2} d\rho - \int_{R^*}^{R} \overline{d}\rho^{N-2} d\rho + h(\delta) - \overline{d}\,\overline{c} \\
&= \mathcal{H}^{N-1}(K \cap B) - \frac{\overline{d}}{N-1}(R^{N-1} - (R^*)^{N-1}) + h(\delta) - \overline{d}\,\overline{c} \\
&\leq \sigma_0 R^{N-1} - \sigma_0(R^{N-1} - (R^*)^{N-1}) + h(\delta) - \overline{c}\sigma_0(N-1) \\
&< \sigma(\delta)(R^*)^{N-1},
\end{aligned}
$$

for a small δ. Then we can find $y \in \partial B$ such that $\mathcal{H}^{N-1}(K \cap B^y) < \sigma(\delta)(R^*)^{N-1}$, which leads to a contradiction, since $K \cap B^y \in \mathcal{D}_\delta(B^y)$. □

4. – Flatness of minimal sets

In this section we will show how, thanks to the property by which the set with minimal measure which splits a ball B of radius R in two connected components of equal measure is a disk, we can claim that $\sigma_0 = b_{N-1}$ and the sets in $\mathcal{D}_\delta(B)$ with almost minimal measure are close to a hyperplane through the center of B. More precisely, we shall prove the following theorem.

THEOREM 4.1. *$\sigma_0 = b_{N-1}$ and for every $\varepsilon > 0$ there exists δ such that every set $K \in \mathcal{D}_\delta(B)$, with $\mathcal{H}^{N-1}(K) \leq (\sigma_0 + \delta)R^{N-1} = (b_{N-1} + \delta)R^{N-1}$, turns out to be ε-close to some disk through the center of B.*

PROOF. Let B be a given ball and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of sets such that, for every $n \in \mathbb{N}$, $\mathcal{H}^{N-1}(K_n) \leq (\sigma_0 + \frac{1}{n})R^{N-1} \leq (b_{N-1} + \frac{1}{n})R^{N-1}$ and let C_n be the connected component of B, relative to K_n, having maximal measure. By Theorem 3.1 we get $|C_n| \to \frac{1}{2}|B|$ as $n \to \infty$, furthermore we note that $(C_n)_{n\in\mathbb{N}}$ is a sequence of sets with finite perimeter in B (Caccioppoli sets), that is $\mathbf{1}_{C_n} \in \mathrm{BV}(B)$ for every n, $\mathrm{BV}(B)$ denoting the space of functions of bounded variation defined on B (see [14], [15]). Since the perimeters of the sets C_n are bounded, $(\mathbf{1}_{C_n})_{n\in\mathbb{N}}$ is a bounded sequence in $\mathrm{BV}(B)$ and, by the compactness theorem on BV functions ([15, Theorem 1.19]), we know that there exists a subsequence $(\mathbf{1}_{C_{n_k}})_{k\in\mathbb{N}}$, by which we shall replace the whole $(\mathbf{1}_{C_n})_{n\in\mathbb{N}}$, converging in $L^1(B)$ to a function which can be identified as the characteristic function of a measurable set $\overline{C}$. Then

$$
|\overline{C}| = \lim_n |C_n| = \frac{1}{2}|B|. \tag{4.29}
$$

By virtue of a suitable relative isoperimetric inequality (see [7, Theorem 18.3.4]), we can say that (4.29) implies

$$P(\overline{C}, B) \geq b_{N-1} R^{N-1}, \tag{4.30}$$

where $P(A, B) = |D\mathbf{1}_A|(B)$ denotes the perimeter of A in B and, if the equality holds, that the reduced boundary $\partial^*\overline{C}$ can be identified with a disk of B. The reverse inequality follows by semicontinuity and enables us to conclude that $\partial^*\overline{C}$ is a disk and that $\sigma_0 = b_{N-1}$. The closeness of K_n to ∂C follows then by simple arguments (further details will be given in Section 8 in the more complicated discrete case). □

5. – Proof of the main results via Coarea Formula

It is well known ([19, Theorem 3.5.2]) that if $\Omega \subset \mathbb{R}^N$ is connected and suitably regular and $u : \Omega \to \mathbb{R}$ satisfies condition (WS), then u is Hölder continuous with index $\frac{1}{2}$. Therefore, for $u : B \to \mathbb{R}$ satisfying (WS) on $B \setminus K$, the condition that B is *ε-split* by u leads to guess that the singular set K should disconnect B in at least two parts of almost equal measure. This is not true, indeed one can easily give examples of sets K which *split* B, leaving it connected. On the other hand, what really happens is not so far from a disconnection, analogously to the situation considered in [11] about the failure of Poincaré Inequality. Indeed we shall prove that the sets K of minimal measure, among those which split B, must be close to some disk of B, in the sense of Definition 2.1. The main argument used to prove this result is Coarea Formula, whose application requires a preliminary regularization result which is proved in the following lemma.

According to Definition 2.3, if $K \in \mathcal{S}_\varepsilon(B)$ we can find a function u splitting B and such that $\|u\|_{2N}^* = 1$ which implies, in particular, that

$$\int_{B\setminus K} |\nabla u| \leq R^{N-\frac{1}{2}} \tag{5.31}$$

and this is the only summability property which we are going to use in this section.

Lemma 5.1. *Let $\varepsilon > 0$ be given and B be ε'-split by K for some $\varepsilon' < \varepsilon$. Then there exists a function $u \in C^\infty(B \setminus K)$, satisfying* (5.31) *and such that B is ε-split by u.*

Proof. Since B is ε'-split by K then there exists a function $u' \in L^1_{\mathrm{loc}}(B\setminus K)$ as in Definition 2.2. By truncating u' and applying Meyers-Serrin theorem ([1, Theorem 3.16]) we can take a sequence $(\varphi_n)_{n\in\mathrm{N}}$ with $\varphi_n \in C^\infty(B\setminus K)$, such that $\varphi_n \to u'$ in $L^1_{\mathrm{loc}}(B \setminus K)$ and $\nabla\varphi_n \to \nabla u'$ in $L^1(B \setminus K)$. For every $\lambda \in]0, 1[$ and for $n \in \mathbb{N}$ large enough, $\lambda\varphi_n$ satisfies (5.31). Now we show that for every fixed $\varepsilon > \varepsilon'$ and for λ close enough to 1, $\lambda\varphi_n$ cannot have an ordinary oscillation region. Indeed, let us suppose that, given $\varepsilon > 0$, for every $n \in \mathbb{N}$ there exists $\widetilde{B}_n$ such that $|\widetilde{B}_n| \geq (\frac{1}{2}+\varepsilon)|B|$ and $\operatorname{osc}_{\widetilde{B}_n} \lambda\varphi_n \leq \varepsilon^{-1}R^{\frac{1}{2}}$. Now, since $\varphi_n \to u'$ in $L^1_{\mathrm{loc}}(B \setminus K)$ and $B \setminus K$ is a bounded open set, then φ_n converges in measure to u' so, by fixing a convenient value of λ and by setting for every n

$$X_n = \left\{ x \in B \setminus K \mid |\varphi_n(x) - u'(x)| > \frac{1}{2}\left(\frac{1}{\varepsilon'} - \frac{\lambda^{-1}}{\varepsilon}\right) R^{\frac{1}{2}} \right\},$$

for n large enough we get $|X_n| < (\varepsilon - \varepsilon')|B|$. For such a value of n we take $\widetilde{B} = \widetilde{B}_n \setminus X_n$, then

$$\begin{aligned} |\widetilde{B}| &\geq |\widetilde{B}_n| - |X_n| \\ &\geq \left(\frac{1}{2} + \varepsilon\right)|B| - (\varepsilon - \varepsilon')|B| \\ &= \left(\frac{1}{2} + \varepsilon'\right)|B|. \end{aligned}$$

Moreover, for every $x, y \in \widetilde{B}$ we have $x, y \notin X_n$ and so

$$|\varphi_n(x) - u'(x)| + |\varphi_n(y) - u'(y)| \leq \left(\frac{1}{\varepsilon'} - \frac{\lambda^{-1}}{\varepsilon}\right) R^{\frac{1}{2}}.$$

Then

$$\begin{aligned} |u'(x) - u'(y)| &\leq |\varphi_n(x) - \varphi_n(y)| + \left(\frac{1}{\varepsilon'} - \frac{\lambda^{-1}}{\varepsilon}\right) R^{\frac{1}{2}} \\ &\leq \frac{\lambda^{-1}}{\varepsilon} R^{\frac{1}{2}} + \left(\frac{1}{\varepsilon'} - \frac{\lambda^{-1}}{\varepsilon}\right) R^{\frac{1}{2}} \\ &= \frac{1}{\varepsilon'} R^{\frac{1}{2}}, \end{aligned}$$

which is a contradiction since u' splits B. Therefore, for n large enough $\lambda\varphi_n$ realizes the function u as required in the thesis. □

The previous result enables us to take, after a slight modification of ε, the function u splitting B with the smoothness needed to apply Coarea Formula which will be employed in the next lemma.

LEMMA 5.2. *For every $\varepsilon' < \varepsilon$ and for every $K \in \mathcal{S}_{\varepsilon'}(B)$ there exists a closed set H such that $\mathcal{H}^{N-1}(H) \leq 4\varepsilon R^{N-1}$ and $K \cup H \in \mathcal{D}_{\varepsilon}(B)$.*

PROOF. Let u be as in Lemma 5.1, for every $l \in \mathbb{R}$ we introduce the following notation.

$$\begin{aligned} X_l &= \{x \in B \setminus K \mid u(x) = l\}, \\ S_l &= \{x \in B \setminus K \mid u(x) < l\}, \\ S^l &= \{x \in B \setminus K \mid u(x) > l\}. \end{aligned}$$

We set

$$\bar{l} = \sup\left\{l \in \mathbb{R} \mid |S_l| < \frac{1}{2}|B|\right\}$$

and we see that $\bar{l} \in \mathbb{R}$. Then we take two open contiguous intervals $I_1, I_2 \in \mathbb{R}$ each one of measure $\frac{1}{2\varepsilon}R^{\frac{1}{2}}$ and with $\bar{l}$ as common extremum. By applying Coarea Formula, from (5.31) we get the following estimate for $i = 1, 2$.

$$\fint_{I_i} \mathcal{H}^{N-1}(X_l)dl \le 2\varepsilon R^{-\frac{1}{2}} \int_B |\nabla u| \le 2\varepsilon R^{N-1}.$$

Therefore there exist $l_1 \in I_1$ and $l_2 \in I_2$ such that $\mathcal{H}^{N-1}(X_{l_i}) \le 2\varepsilon R^{N-1}$ for $i = 1, 2$. Let $H = X_{l_1} \cup X_{l_2}$, then

$$\mathcal{H}^{N-1}(H) \le 4\varepsilon R^{N-1}.$$

By observing that each connected component C of $B \setminus (K \cup H)$ is contained in one of the sets S_{l_1}, S^{l_2} and $S^{l_1} \cap S_{l_2}$, we get one of the following estimates on its measure:

$$\begin{aligned} |C| &\le |S_{l_1}| \le \frac{1}{2}|B| \quad (\text{since } l_1 < \bar{l}), \\ |C| &\le |S^{l_2}| \le \frac{1}{2}|B| \quad (\text{since } l_2 > \bar{l}) \end{aligned}$$

or

$$|C| \le |S^{l_1} \cap S_{l_2}| \le \left(\frac{1}{2} + \varepsilon\right)|B| \quad \left(\text{since } \operatorname{osc}_{S^{l_1} \cap S_{l_2}} u \le \frac{1}{\varepsilon}R^{\frac{1}{2}} \text{ and } u \text{ splits } B\right).$$

So we can conclude that $K \cup H \in \mathcal{D}_\varepsilon(B)$. □

The following assertions are simple consequences of the above result and of Theorem 4.1.

COROLLARY 5.1. *For every $\varepsilon > \varepsilon' > 0$*

$$s(\varepsilon') \ge \sigma(\varepsilon) - 4\varepsilon.$$

COROLLARY 5.2.

$$s_0 = \sigma_0 = b_{N-1}.$$

PROOF OF THEOREM 1.1. Given $\varepsilon > 0$, we take $\delta > 0$ such that

$$s(\delta) \ge (1 - \varepsilon)s_0 = (1 - \varepsilon)b_{N-1}$$

and so the thesis is proved. □

Proof of Theorem 1.2. Let $\varepsilon > 0$ be given. We fix δ as in Theorem 4.1 and we fix $K \in \mathcal{S}_{\frac{\delta}{5}}(B)$ such that $\mathcal{H}^{N-1}(K) \leq (\sigma_0 + \frac{\delta}{5})R^{N-1}$. By Lemma 5.2 we take a closed set H such that $\mathcal{H}^{N-1}(H) \leq \frac{4}{5}\delta$ and $K \cup H \in \mathcal{D}_\delta(B)$. Obviously

$$\mathcal{H}^{N-1}(K \cup H) \leq (b_{N-1} + \delta)R^{N-1}.$$

Because of the ε-closeness of K to a disk D, guaranteed by Theorem 4.1, we have that

$$\mathbf{A}_K(B) \leq (\varepsilon^2 + \varepsilon)R^{1-N}\mathcal{H}^{N-1}(K) \leq (\varepsilon^2 + \varepsilon)(b_{N-1} + \delta).$$

We can always take D such that the orthogonal projection of the irregular part of K has null $(N-1)$ measure. Then from

$$\mathcal{H}^{N-1}(K) \leq (b_{N-1} + \delta)R^{N-1} \leq \mathcal{H}^{N-1}(p(K)) + (\varepsilon + \delta)R^{N-1},$$

the smallness of $\mathbf{T}_K(B)$ follows. By the arbitrariness of ε and δ we deduce the thesis. □

6. - Discrete geometric tools

Now we are going to introduce a tool which will turn out to be useful in the subsequent developments, it will allow a sort of discretization of the main problems discussed in this paper. Roughly speaking, we will enclose the singular set $K \subset B$ in a finite set made of small balls $B_i \subset B$. This set, which will be named *punching*, has, for some aims, nicer properties than the set K because the measure properties of the last one can be controlled by the analogous one of the punchings through a similarity argument. Let $B \subset \mathbb{R}^N$ be a given ball with radius R, we take a finite set $\mathcal{B}$, made of balls B_i with radius r_i, and define the *measure* of $\mathcal{B}$ as

$$\mu(\mathcal{B}) = \sum_i r_i^{N-1}. \tag{6.32}$$

Definition 6.1. For every $\varepsilon > 0$, $\mathcal{B}$ will be named ε-punching, or punching with a thickness less than or equal to ε, if it is made of balls B_i with radius r_i such that $\forall i$, $r_i \leq \varepsilon R$ and such that all the connected components of $B \setminus \bigcup_{B_i \in \mathcal{B}} B_i$ have a measure less than or equal to $(\frac{1}{2} + \varepsilon)|B|$.

We shall denote by $\mathcal{P}_\varepsilon(B)$ the set of all the ε-*punchings* of B, which is, roughly speaking, the discrete counterpart of the set $\mathcal{D}_\varepsilon$ introduced above. Analogously, beside σ_ε, we introduce the following notation.

$$m_\varepsilon(B) = \inf_{\mathcal{B} \in \mathcal{P}_\varepsilon(B)} \mu(\mathcal{B}),$$

and, by similarity,

$$m_\varepsilon(B) = m(\varepsilon)R^{N-1},$$

where we have set

$$m(\varepsilon) = m_\varepsilon(B_1).$$

Finally, we note that $m(\varepsilon)$ is a monotone decreasing function; we set

$$m_0 = \lim_{\varepsilon\to 0} m(\varepsilon) = \sup m(\varepsilon) \le 1, \tag{6.33}$$

as one can easily check by covering a disk with suitably small balls. We finally give the suitable version of Definition 2.1 whose notation is kept.

DEFINITION 6.2. The punching $\mathcal{B}$ is said to be ε-close to D if

$$\mu(\mathcal{B} \setminus P(\varepsilon R)) \le \varepsilon\mu(\mathcal{B}), \tag{6.34}$$

$$\mathcal{H}^{N-1}\left(D \setminus p\left(\bigcup_{B'\in\mathcal{B}} B'\right)\right) \le \varepsilon R^{N-1}. \tag{6.35}$$

If $r_1 < r_2$, we shall denote by $A_{r_1}^{r_2}$ the annulus of B enclosed between the radii r_1 and r_2 and, if $\mathcal{B}$ is any collection of balls, we shall use the notation

$$\mu_{r_1}^{r_2}(\mathcal{B}) = \mu(\{B' \in \mathcal{B} \mid B' \cap A_{r_1}^{r_2} \ne \emptyset\}).$$

We shall make use of two main operations on the punchings, namely restrictions and extensions which we are going to define. If $B' \subset B$ is any ball and $\mathcal{B} \in \mathcal{P}_\delta(B)$, we define the *restriction* $\mathcal{B}'$ of $\mathcal{B}$ to B' as

$$\mathcal{B}' = \{B_i \in \mathcal{B} \mid B_i \cap B' \ne \emptyset\}.$$

A *δ-extension* $\mathcal{B}''$ of a given $\mathcal{B} \in \mathcal{P}_\delta(B)$ to a ball $B'' \supset B$ with radius $R'' > R$ is defined as a set of balls obtained by adding new balls B_i with a radius r_i such that $\sup r_i \le \delta R''$ and such that two different connected components of $B \setminus \bigcup_{B_i\in\mathcal{B}} B_i$ are not merged in the largest ball and so they are contained in two different connected components of $B'' \setminus \bigcup_{B_i\in\mathcal{B}''} B_i$. We point out that, as happens in the continuous case, that, in general, restrictions and extensions of punchings are not still punchings.

A particular extension, which will be employed in Section 3, is the one which we shall call *extension by reflection* which allows to define a $\lambda\delta$ extension of a δ-punching $\mathcal{B}$ from B to λB ($\lambda > 1$), by using the balls which intersect $A_{\lambda^{-1}R}^{R}$. More precisely, let $B_i = B(x_i, r_i)$ the balls such that $B_i \cap A_{\lambda^{-1}R}^{R} \ne \emptyset$, the extension by reflection $\mathcal{B}''$ of $\mathcal{B}$ is defined by adding to $\mathcal{B}$ the balls $B(y_i, \lambda^2 r_i)$, with $y_i = \frac{x_i R^2}{|x_i|^2}$. The measure of the new set of balls is given by $\mu(\mathcal{B}) + \lambda^{2(N-1)}\mu_{\lambda^{-1}R}^{R}(\mathcal{B})$. We remark that such a construction prevents merging of connected components. Such an extension can be iterated up to $\lambda^2 R$, but in the next step, in order to prevent merging, it is not necessary to multiply once more by λ^2 the radii, it suffices to add balls with center $\lambda^2 x_i$ and radius $\lambda^2 r_i$ as in the previous extension. Thus we get a δ-extension of $\mathcal{B}$. With this strategy we can iterate up to the ball $\lambda^n B$ and the contribution of the new balls inherent to $A^{\lambda^i R}_{\lambda^{i-1} R}$ is estimated by $\lambda^{(i+1)(N-1)} \mu^R_{\lambda^{-1} R}(\mathcal{B})$, (the term $+1$ being necessary only if i is odd). By iterating the construction, we get a $\lambda^2\delta$-punching of any ball $B'' \supset B$, with the same center of B. In the following, we shall refer to such an extension as to the *radial extension* of $\mathcal{B}$ to B'' *from the annulus* $A^R_{\lambda^{-1}R}$. This is at most a $\lambda^2\delta$-extension of $\mathcal{B}$ and is actually a δ-extension when $R'' = \lambda^n R$, with n even.

The following definition states, roughly speaking, that a set K is well *packed* in the balls of a punching $\mathcal{B}$ if $\mathcal{B}$ has a small measure and if the part of K left out of the balls of $\mathcal{B}$ or in a few of them is small.

DEFINITION 6.3. We shall say that a ε-punching $\mathcal{B}$ of B with radius R is a ε-packing of a closed subset $K \subset B$, if

$$\mu(\mathcal{B}) \leq (1+\varepsilon) R^{N-1} \tag{6.36}$$

and, if $\mathcal{B}'$ is any subset of $\mathcal{B}$ such that $\mu(\mathcal{B}) < 2\mu(\mathcal{B}')$, then

$$\mathcal{H}^{N-1}\left(K \setminus \bigcup_{B\in\mathcal{B}'} B\right) < \left(\varepsilon + R^{1-N}(\mu(\mathcal{B}) - \mu(\mathcal{B}'))\right)\mathcal{H}^{N-1}(K). \tag{6.37}$$

7. – Discrete bisection

This section contains the discrete version of the arguments in Section 3 which has been fully developed for the reader's convenience. Indeed, even if the arguments are constructed in the same way, some extra difficulties due to the discrete formulation must be taken into account.

THEOREM 7.1. *For every $\varepsilon > 0$ there exists δ such that, if B is a ball of radius R and $\mathcal{B} \in \mathcal{P}_\delta(B)$ with $\mu(\mathcal{B}) \leq (m_0+\delta)R^{N-1}$, then there exists a connected component C of $B \setminus \bigcup_{B\in\mathcal{B}} B$, such that $|C| > (\frac{1}{2} - \varepsilon)|B|$.*

The same remarks in the beginning of Section 3 can be made also here. The above property is obviously scale invariant, so we can work on a fixed ball B. In order to prove Theorem 7.1 by contradiction, we assume that a constant $\gamma < \frac{1}{2}$ can be fixed in such a way that, for δ arbitrarily small, there exists a δ-punching $\mathcal{B}$ with $\mu(\mathcal{B}) < (m_0+\delta)R^{N-1}$ such that, for every C connected component of $B \setminus \bigcup_{B\in\mathcal{B}} B$, we have $|C| < \gamma|B|$. Therefore, by continuity, we can fix R' and R'' depending on γ (but constant with respect to δ), with $R' < R < R''$, such that, if B' and B'' are any pair of balls, $B' \subset B \subset B''$, of radius respectively greater or equal to $R' - 2\delta R$ and less than or equal to R'', $\mathcal{B}'$ is the restriction of $\mathcal{B}$ to B' and $\mathcal{B}''$ is any 2δ-extension of $\mathcal{B}'$ from B' to B'', then $\mathcal{B}'$ and $\mathcal{B}''$ are 2δ-punchings respectively of B' and B''. Now we are going to introduce some functions which will be employed in the following. Let $\rho > 0$, we put $S_\rho = A^\rho_{\rho(1+\sqrt{\delta})^{-1}}$, moreover for every $B_i \in \mathcal{B}$ we define

$$c_i(\rho) = \begin{cases} 1 & \text{if } B_i \cap S_\rho \neq \emptyset \\ 0 & \text{if } B_i \cap S_\rho = \emptyset . \end{cases}$$

We will denote by $f(\rho)$ the measure $\mu^\rho_{\rho(1+\sqrt{\delta})^{-1}}(\mathcal{B})$, normalized by $\sqrt{\delta}$, namely, if $n = \text{card}(\mathcal{B})$,

$$f(\rho) = \frac{1}{\sqrt{\delta}} \mu^\rho_{\rho(1+\sqrt{\delta})^{-1}}(\mathcal{B}) = \frac{1}{\sqrt{\delta}} \sum_{i=1}^n c_i(\rho) r_i^{N-1} , \tag{7.38}$$

so we can define the following density function

$$d(\rho) = \frac{f(\rho)}{\rho^{N-1}} . \tag{7.39}$$

Throughout this section we will use the same notation $h(\delta)$ as in Section 3 to denote infinitesimal quantities as δ tends to zero. The analogous of Lemma 3.1 is the following result which also takes into account that $d(\rho)$ needs now to be estimated also for $\rho \geq R$.

LEMMA 7.1. *There exists a positive constant $\overline{d}$, such that*

$$\int_{R'}^R |d(\rho) - \overline{d}| \rho^{N-2} d\rho + \int_R^\infty d(\rho) \rho^{N-2} d\rho = h(\delta) . \tag{7.40}$$

We shall give the proof of Lemma 7.1 after some preliminary steps which include the analogous ones of those in Section 3 and some more ones due to the necessity of justifying some arguments which become less obvious in the discrete formulation. This is the case of the two following lemmas.

LEMMA 7.2. *Let $\delta^{\frac{1}{3}} R \leq R_0 \leq R$ be given and let B_0 be the ball with the same center of B and radius R_0. If i is such that $B_i \not\subset B_0$, then*

$$\frac{1}{\sqrt{\delta}} \int_{R_0}^\infty c_i(\rho) \frac{d\rho}{\rho} = 1 + h(\delta) . \tag{7.41}$$

PROOF. Let $\overline{\rho}_1$ and $\overline{\rho}_2$ be respectively the smallest and the greatest values of $|x|$, for $x \in \overline{B}_i$. By setting $\rho_1 = \max(\overline{\rho}_1, R_0)$ and $\rho_2 = \overline{\rho}_2(1+\sqrt{\delta})$, we have

$$\begin{aligned} \int_{R_0}^\infty c_i(\rho) \frac{d\rho}{\rho} &= \log \rho_2 - \log \rho_1 = \log\left(\frac{\overline{\rho}_2}{\overline{\rho}_1}\right) - \log\left(\frac{\rho_1}{\overline{\rho}_1}\right) + \log(1+\sqrt{\delta}) \\ &= \log\left(\frac{\overline{\rho}_2}{\overline{\rho}_1}\right) - \log\left(\frac{\rho_1}{\overline{\rho}_1}\right) + \sqrt{\delta} + o(\sqrt{\delta}) . \end{aligned}$$

Moreover we have

$$1 \leq \frac{\rho_1}{\overline{\rho}_1} \leq \frac{\overline{\rho}_2}{\overline{\rho}_1} \leq \frac{1}{1-2\delta\delta^{\frac{-1}{3}}} = 1 + O(\delta^{\frac{2}{3}})$$

and therefore

$$\log\left(\frac{\overline{\rho}_2}{\overline{\rho}_1}\right) = O(\delta^{\frac{2}{3}}), \qquad \log\left(\frac{\rho_1}{\overline{\rho}_1}\right) = O(\delta^{\frac{2}{3}}),$$

then the thesis is proved. □

LEMMA 7.3. *For every $r \in [R', R]$ the following equality holds.*

$$\int_r^\infty d(\rho)\rho^{N-2}d\rho = (1+h(\delta))\mu_r^R(\mathcal{B}) = \mu_r^R(\mathcal{B}) + h(\delta)\,. \tag{7.42}$$

PROOF. We note that, by (7.39), (7.38) and (7.41),

$$\begin{aligned}\int_r^\infty d(\rho)\rho^{N-2}d\rho &= \int_r^\infty \frac{f(\rho)}{\rho}d\rho = \sum\nolimits_i \frac{1}{\sqrt{\delta}}\left(\int_r^\infty \frac{c_i(\rho)}{\rho}d\rho\right) r_i^{N-1} \\ &= \sum_i (1+h(\delta))\, r_i^{N-1} \\ &= (1+h(\delta))\,\mu_r^R(\mathcal{B})\,,\end{aligned}$$

where the sum is extended to the balls $B_i \in \mathcal{B}$ which intersect A_r^R, since for all the other indexes $c_i(\rho) = 0$ for $\rho \geq r$. □

An upper bound on the mean value of $d(\rho)$ can now be easily deduced by estimating $\mu_r^R(\mathcal{B})$, leading to the analogous of Lemma 3.2.

LEMMA 7.4. *For every $r \in [R', R]$ the following inequality holds.*

$$\mu_r^R(\mathcal{B}) \leq m_0(R^{N-1} - r^{N-1}) + h(\delta)\,. \tag{7.43}$$

PROOF. Let B' be the ball with the same center of B and radius equal to $r - 2\delta R \geq R' - 2\delta R$ and $\mathcal{B}'$ the restriction of $\mathcal{B}$ to B'. As we have already noted, $\mathcal{B}'$ is a 2δ*-punching* of B'. Since $\mu(\mathcal{B}') \geq \mu(\mathcal{B}) - \mu_r^R(\mathcal{B})$, we have by definition

$$\mu(\mathcal{B}) - \mu_r^R(\mathcal{B}) \geq m(2\delta)(r-2\delta R)^{N-1} = (m_0 + h(\delta))r^{N-1}\,. \tag{7.44}$$

Moreover, by hypothesis we know that

$$\mu(\mathcal{B}) \leq (m_0+\delta)R^{N-1} = (m_0 + h(\delta))R^{N-1}\,. \tag{7.45}$$

By combining (7.44) and (7.45), we get (7.43). □

By (7.42) and (7.43) we obtain a first integral estimate on $d(\rho)$, indeed for every $r \in [R', R]$ the following inequality holds (see (3.17)).

$$\int_r^{\infty} d(\rho)\rho^{N-2}d\rho \leq m_0(R^{N-1} - r^{N-1}) + h(\delta)\,. \tag{7.46}$$

Now we want to get a uniform lower estimate on $d(\rho)$ which matches (7.46) and, to this aim, we are going to consider for a fixed $r \in [R', R]$ the restriction $\mathcal{B}'$ of $\mathcal{B}$ to the concentric ball to B of radius r and then the radial extension $\mathcal{B}''$ of $\mathcal{B}'$ to the ball B'' of radius R'' from the annulus S_r. The aim of the next lemma is to prove an estimate on the measure of this new set of balls $\mathcal{B}''$.

Lemma 7.5. *For a given $r \in [R', R]$, let $\mathcal{B}''$ the above introduced radial extension of $\mathcal{B}'$ to the ball B''. Then, the following estimate holds*

$$\mu_r^{R''}(\mathcal{B}'') \leq d(r)\left(\int_r^{R''} \rho^{N-2}d\rho + h(\delta)\right)\,. \tag{7.47}$$

Proof. Since $\mathcal{B}''$ is defined as the radial extension of $\mathcal{B}'$ from S_r, its measure can be conveniently evaluated by adding the contributions related to a generic annulus $A^{r(1+\sqrt{\delta})^i}_{r(1+\sqrt{\delta})^{i-1}}$ which have been computed in the end of Section 6. We will denote by μ_i^* the measure of the balls competing the i-th annulus. For $i = 1$ we have

$$\begin{aligned}\mu_1^* &= (1+\sqrt{\delta})^{2(N-1)}\mu^r_{r(1+\sqrt{\delta})^{-1}}(\mathcal{B}) = (1+\sqrt{\delta})^{2(N-1)}\sqrt{\delta}f(r)\\ &= (1+\sqrt{\delta})^{2(N-1)}\sqrt{\delta}d(r)r^{N-1} = (1+\sqrt{\delta})^{2(N-1)}d(r)\int_r^{r(1+\sqrt{\delta})} r^{N-2}d\rho\,.\end{aligned} \tag{7.48}$$

In the second annulus, by scaling the balls which intersect S_r we find

$$\begin{aligned}\mu_2^* &\leq (1+\sqrt{\delta})^{2(N-1)}\sqrt{\delta}f(r) = \sqrt{\delta}d(r)((1+\sqrt{\delta})^2 r)^{N-1}\\ &\leq (1+\sqrt{\delta})^{N-1}d(r)\int_{r(1+\sqrt{\delta})}^{r(1+\sqrt{\delta})^2} \rho^{N-2}d\rho\,.\end{aligned} \tag{7.49}$$

By iterating the argument, for every i we get

$$\mu_i^* \leq (1+\sqrt{\delta})^{2(N-1)}d(r)\int_{r(1+\sqrt{\delta})^{i-1}}^{r(1+\sqrt{\delta})^i} \rho^{N-2}d\rho\,, \tag{7.50}$$

where the term 2 appearing in the first exponent can be omitted when i is even. Summing up with respect to i, we get (7.47). □

In the next lemma we shall prove the desired uniform lower estimate on the density function $d(\rho)$. To this aim we set

$$\overline{d} = \inf_{\rho \in [R', R]} d(\rho)$$

and we take $\overline{\rho} \in [R', R]$ such that

$$d(\overline{\rho}) \leq (1+\delta)\overline{d}\,. \tag{7.51}$$

We can now state the corresponding case of Lemma 3.3.

Lemma 7.6. *The following lower bound holds*

$$\overline{d} \geq m_0(N-1) + h(\delta)\,. \tag{7.52}$$

Proof. By taking $\mathcal{B}''$ as in the previous lemma for $r = \overline{\rho}$, we get

$$\mu(\mathcal{B}'') \leq \mu(\mathcal{B}) - \mu_{\overline{\rho}}^{R}(\mathcal{B}) + \mu_{\overline{\rho}}^{R''}(\mathcal{B}'')\,. \tag{7.53}$$

Since by (7.42)

$$\mu_{\overline{\rho}}^{R}(\mathcal{B}) = \int_{\overline{\rho}}^{\infty} d(\rho)\rho^{N-2}d\rho + h(\delta)\,,$$

by (7.47), for $r = \overline{\rho}$, and by (7.51), we deduce

$$\begin{aligned} \mu(\mathcal{B}'') &\leq \mu(\mathcal{B}) - \int_{\overline{\rho}}^{\infty} d(\rho)\rho^{N-2}d\rho + d(\overline{\rho})\int_{\overline{\rho}}^{R''} \rho^{N-2}d\rho + h(\delta) \\ &\leq (m_0+\delta)R^{N-1} - \int_{\overline{\rho}}^{R} \overline{d}\rho^{N-2}d\rho + (1+\delta)\overline{d}\int_{\overline{\rho}}^{R''} \rho^{N-2}d\rho + h(\delta) \\ &= (m_0+\delta)R^{N-1} + \overline{d}\int_{R}^{R''} \rho^{N-2}d\rho + h(\delta)\,. \end{aligned} \tag{7.54}$$

Since $\mathcal{B}''$ is a 2δ-punching of B'' and therefore

$$\mu(\mathcal{B}'') \geq m(2\delta)(R'')^{N-1} = (m_0 + h(\delta))(R'')^{N-1}\,, \tag{7.55}$$

by (3.21) and (3.22) we finally get

$$\begin{aligned} \overline{d} &\geq \left(\int_{R}^{R''} \rho^{N-2}d\rho\right)^{-1} \Big((m_0+h(\delta))(R'')^{N-1} - (m_0+\delta)R^{N-1}\Big) + h(\delta) \\ &= m_0(N-1) + h(\delta)\,. \end{aligned} \tag{7.56}$$

□

Proof of Lemma 7.1. By integrating (7.52) we get

$$\int_{R'}^{R} \overline{d}\rho^{N-2}d\rho \geq m_0(R^{N-1} - (R')^{N-1}) + h(\delta)\,,$$

so, by using (7.46) for $r = R'$, since $d(\rho) - \overline{d} \geq 0$, we finally have (7.40). □

PROOF OF THEOREM 7.1. We assume by contradiction that the thesis is false, so we can use all the construction made in this section. We set $R^* = \frac{R+R'}{2}$ and for every $y \in \partial B$ we denote by B^y the ball with radius R^* contained in B and tangent in y to ∂B, whereas B^* is the ball with the same center of B and radius R^*. For $\lambda \geq 0$, let B^y_λ be the ball with the same center of B^y and radius $R^* + \lambda R$. For $x \in B$ let us introduce the function

$$p_\lambda(x) = \fint_{\partial B_R} \mathbf{1}_{B^y_\lambda}(x) d\mathcal{H}^{N-1}(y) \leq 1 . \tag{7.57}$$

Because of the radial symmetry of the mappings $\mathbf{1}_{B^*}$ and p_λ, we can deal with $\mathbf{1}_{B^*}(\rho)$ and $p_\lambda(\rho)$. We can claim that

$$\int_{\mathbb{R}^N} p_0(x)dx = |B^*| \tag{7.58}$$

and therefore

$$\int_{\mathbb{R}^N} (\mathbf{1}_{B^*} - p_0)dx = 0 , \text{ namely } \int_0^\infty (\mathbf{1}_{B^*} - p_0)(\rho)\rho^{N-1}d\rho = 0 . \tag{7.59}$$

We note that, since

$$(\mathbf{1}_{B^*} - p_0)(\rho)\begin{cases} = 0 & \text{if } 0 < \rho < R' \\ > 0 & \text{if } R' < \rho < R^* \\ < 0 & \text{if } R^* < \rho < R , \end{cases}$$

then we can find a positive constant $\overline{c}$ such that

$$\int_0^\infty (\mathbf{1}_{B^*} - p_0)(\rho)\rho^{N-2}d\rho > 2\overline{c} \tag{7.60}$$

and, subsequently, for λ sufficiently small

$$\int_0^\infty (\mathbf{1}_{B^*} - p_\lambda)(\rho)\rho^{N-2}d\rho \geq \overline{c} \tag{7.61}$$

and so, since $\mathbf{1}_{B^*} = p_\lambda$ for $\rho \leq R'$,

$$\int_{R'}^\infty p_\lambda(\rho)\rho^{N-2}d\rho \leq \int_{R'}^{R^*} \rho^{N-2}d\rho - \overline{c} . \tag{7.62}$$

We fix $\lambda > 0$ in such a way to have (7.62) and we shall take a δ-punching $\mathcal{B}$ with $\sqrt{\delta} < \frac{\lambda}{3}$. By(7.40) and (7.62) we get

$$\begin{aligned} \int_{R'}^\infty d(\rho)\rho^{N-2}p_\lambda(\rho)d\rho &= \overline{d}\int_{R'}^\infty \rho^{N-2}p_\lambda(\rho)d\rho + h(\delta) \\ &\leq \overline{d}\int_{R'}^{R^*} \rho^{N-2}d\rho - \overline{c}\,\overline{d} + h(\delta) . \end{aligned} \tag{7.63}$$

For every $y \in \partial B$ let $\mathcal{B}^y \subset \mathcal{B}$ be the restriction of the punching to the ball B^y and $\mathcal{B}^* \subset \mathcal{B}$ be the restriction to B^*. Now our aim is to estimate

$$\fint_{\partial B} \mu(\mathcal{B}^y) d\mathcal{H}^{N-1}$$

and we shall begin by computing the mean value of $\mu_{R'}^R(\mathcal{B}^y)$, since the balls which do not intersect $A_{R'}^R$ belong to all the punchings $\mathcal{B}^y$. So we shall take into account in the following sums only indexes i such that $B_i \not\subset B'$, where B' is the ball with the same center of B and radius R'. For each i we set

$$c_i^*(y) = \begin{cases} 1 & \text{if } B_i \in \mathcal{B}^y \\ 0 & \text{if } B_i \notin \mathcal{B}^y , \end{cases}$$

then

$$\fint_{\partial B} \mu_{R'}^R(\mathcal{B}^y) d\mathcal{H}^{N-1} = \sum_i \left(\fint_{\partial B} c_i^*(y) d\mathcal{H}^{N-1} \right) r_i^{N-1} . \tag{7.64}$$

Let $B_i(\sqrt{\delta}R)$ be the closed $\sqrt{\delta}R$-neighborhood of B_i. We denote by z_i a point of $B_i(\sqrt{\delta}R)$ having maximum modulus and we put $\rho_i = |z_i|$. We see that, if $c_i^*(y) \neq 0$, then, for every $x \in B_i(\sqrt{\delta}R)$, x belongs to the $(2r_i + \sqrt{\delta}R)$-neighborhood of B^y, with $2r_i + \sqrt{\delta}R \leq \lambda R$, so $\mathbf{1}_{B_\lambda^y}(x) = 1$. Thus, we can claim that

$$\fint_{\partial B} c_i^*(y) d\mathcal{H}^{N-1} \leq \fint_{\partial B} \mathbf{1}_{B_\lambda^y}(z_i) d\mathcal{H}^{N-1} = p_\lambda(\rho_i) .$$

Since when $c_i(\rho) \neq 0$ we have $\rho \leq \rho_i$ and consequently $p_\lambda(\rho) \geq p_\lambda(\rho_i)$, then by (7.41) we have

$$\begin{aligned} \fint_{\partial B} c_i^*(y) d\mathcal{H}^{N-1} &\leq p_\lambda(\rho_i) \leq \frac{1}{\sqrt{\delta}} \int_{R'}^\infty p_\lambda(\rho_i) c_i(\rho) \frac{d\rho}{\rho} + p_\lambda(\rho_i) h(\delta) \\ &\leq \frac{1}{\sqrt{\delta}} \int_{R'}^\infty p_\lambda(\rho) c_i(\rho) \frac{d\rho}{\rho} + h(\delta) . \end{aligned} \tag{7.65}$$

Therefore, by combining (7.64) and (7.65) and by using (7.38), (7.39), (7.40) and (7.63), we can establish the following estimate

$$\begin{aligned} \fint_{\partial B} \mu_{R'}^R(\mathcal{B}^y) d\mathcal{H}^{N-1} &\leq \sum_i \left(\frac{1}{\sqrt{\delta}} \int_{R'}^\infty p_\lambda(\rho) c_i(\rho) \frac{d\rho}{\rho} + h(\delta) \right) r_i^{N-1} \\ &\leq \int_{R'}^\infty p_\lambda(\rho) f(\rho) \frac{d\rho}{\rho} + h(\delta) \\ &= \int_{R'}^\infty d(\rho) \rho^{N-2} p_\lambda(\rho) f(\rho) \frac{d\rho}{\rho} + h(\delta) \\ &\leq \int_{R'}^{R^*} d(\rho) \rho^{N-2} d\rho - \bar{c}\bar{d} + h(\delta). \end{aligned} \tag{7.66}$$

For every $y \in \partial B$, we can pass from $\mu^R_{R'}(\mathcal{B}^y)$ to $\mu(\mathcal{B}^y)$ by adding the measure of the set of balls of $\mathcal{B}$ contained in B', namely $\mu(\mathcal{B})-\mu^R_{R'}(\mathcal{B})$. So from (7.66) and by Lemma 7.3 we get

$$\begin{aligned}
\fint_{\partial B} \mu(\mathcal{B}^y) d\mathcal{H}^{N-1} &\leq \mu(\mathcal{B}) - \mu^R_{R'}(\mathcal{B}) + \int_{R'}^{R^*} d(\rho)\rho^{N-2} d\rho - \overline{c}\,\overline{d} + h(\delta) \\
&\leq (m_0+\delta)R^{N-1} - \int_{R^*}^{\infty} d(\rho)\rho^{N-2} d\rho - \overline{c}\,\overline{d} + h(\delta)\,.
\end{aligned}$$

Finally, by (7.40) and Lemma 7.6 we conclude

$$\begin{aligned}
\fint_{\partial B} \mu(\mathcal{B}^y) d\mathcal{H}^{N-1} &\leq (m_0+\delta)R^{N-1} - m_0(R^{N-1} - (R^*)^{N-1}) - \overline{c}\,\overline{d} + h(\delta) \\
&\leq m_0(R^*)^{N-1} - \overline{c}\,\overline{d} + h(\delta) < m(2\delta)(R^*)^{N-1}\,,
\end{aligned}$$

for a small δ. Then we can find $y \in \partial B$ such that $\mu(\mathcal{B}^y) < m(2\delta)(R^*)^{N-1}$, which leads to a contradiction, since $\mathcal{B}^y$ is a 2δ-punching of B^y. □

8. – Flatness of minimal punchings

The analogous of Theorem 4.1 holds for minimal punchings. More precisely we shall prove the following theorem.

THEOREM 8.1. $m_0 = 1$ *and for every* $\varepsilon > 0$ *there exists* δ *such that every* δ*-punching of* B*, with measure less than or equal to* $(m_0+\delta)R^{N-1} = (1+\delta)R^{N-1}$*, turns out to be* ε*-close to some disk through the center of* B*.*

The discrete formulation does not allow a proof as obvious as in Section 4. Indeed, before proving the main theorem, we set some preliminary results. Let B be a given ball and let $(\mathcal{B}_n)_{n\in\mathbb{N}}$ be a sequence of punchings of B with infinitesimal thickness such that, for every $n \in \mathbb{N}$, $\mu_n(B) \leq (m_0 + \frac{1}{n})R^{N-1} \leq (1+\frac{1}{n})R^{N-1}$ and let C_n be the connected component of B, relative to $\mathcal{B}_n$, having maximal measure. By Theorem 3.1 we get $|C_n| \to \frac{1}{2}|B|$ as $n \to \infty$, furthermore we note that $(C_n)_{n\in\mathbb{N}}$ is a sequence of sets with finite perimeter in B (Caccioppoli sets), that is $\mathbf{1}_{C_n} \in \mathrm{BV}(B)$ for every n, $\mathrm{BV}(B)$ denoting the space of functions of bounded variation defined on B (see [14], [15]). Indeed $\partial C_n \cap B \subset \bigcup_{B_i\in\mathcal{B}_n} \partial B_i$. Then for every $n \in \mathbb{N}$

$$P(C_n, B) \leq N b_N \left(1+\frac{1}{n}\right) R^{N-1}\,, \tag{8.67}$$

where $P(A,B) = |D\mathbf{1}_A|(B)$ denotes the perimeter of A in B. Therefore $(\mathbf{1}_{C_n})_{n\in\mathbb{N}}$ is a bounded sequence in $\mathrm{BV}(B)$ and, by the compactness theorem on

BV functions ([15, Theorem 1.19]), we know that there exists a subsequence $(\mathbf{1}_{C_{n_k}})_{k\in\mathbb{N}}$, by which we shall replace the whole $(\mathbf{1}_{C_n})_{n\in\mathbb{N}}$, converging in $L^1(B)$ to a function which can be identified as the characteristic function of a measurable set $\overline{C}$. Then, as in Section 4, we have (4.29) and (4.30). We are now going to see that $B \cap \partial\overline{C}$ is a disk. This will show, in particular, that, for any given $\varepsilon > 0$, $\mathcal{B}_n$ satisfies (6.35) for n large and for $D = B \cap \partial\overline{C}$. Indeed, if $x \in D$ and $x \notin \bigcup_{B\in\mathcal{B}_n} p(B)$, then the chord c_x through x and orthogonal to D is such that $c_x \subset C_n$ or $c_x \subset B \setminus C_n$. Therefore $|\bigcup_x c_x|$ tends to zero, so $\mathcal{H}^{N-1}(D \setminus \bigcup_{B\in\mathcal{B}} p(B))$ tends to zero.

LEMMA 8.1. *Let $\overline{C}$ be as above, then*

$$P(\overline{C}, B) \le m_0 b_{N-1} R^{N-1} \le b_{N-1} R^{N-1}. \tag{8.68}$$

PROOF. By the definition of *perimeter* we have

$$P(\overline{C}, B) = \int_B |D\mathbf{1}_{\overline{C}}| = \sup_{\varphi} \int_{\overline{C}} \operatorname{div}\varphi dx,$$

where the sup is taken on the set of the vector valued functions $\varphi \in C_0^1(B, \mathbb{R}^N)$ such that $|\varphi(x)| \le 1$ for a.e. x. So, let φ be any function as above, to prove the thesis we need to estimate $\int_{\overline{C}} \operatorname{div}\varphi dx$. To this aim, we begin by taking $\varepsilon > 0$ and we note that, since $\partial C_n \subset \bigcup_i \partial B_i$, then there exists a family of mutually disjoint borel measurable sets $A_i \subset \partial B_i$ such that $\partial C_n = \bigcup_i A_i$. Now, for every $n \in \mathbb{N}$, let φ_i be a constant vector corresponding to a value assumed by φ on ∂B_i and let ν be the normal field to ∂B_i, we introduce the sets $A_i^+ = \{x \in \partial B_i : \varphi_i \cdot \nu \ge 0\}$ which is a hemisphere and the corresponding disk D_i. Thus, for every n such that $\operatorname{osc}_X \varphi \le \varepsilon$ when $\operatorname{diam}(X) \le \frac{2}{n}$, we can proceed to the following estimate in which we will use the divergence theorem.

$$\begin{aligned} \int_{\partial C_n} \varphi \cdot \nu d\mathcal{H}^{N-1} &= \sum_i \int_{A_i} \varphi \cdot \nu d\mathcal{H}^{N-1} \\ &\le \sum_i \left(\int_{A_i} \varphi_i \cdot \nu d\mathcal{H}^{N-1} + \int_{A_i} |\varphi - \varphi_i|\, |d\nu| \right) \\ &\le \sum_i \int_{A_i^+} \varphi_i \cdot \nu d\mathcal{H}^{N-1} + \sum_i \int_{A_i} \varepsilon |d\nu| \\ &\le \sum_i \int_{D_i^+} d\mathcal{H}^{N-1} + \varepsilon N b_N \sum_i r_i^{N-1} \\ &\le (b_{N-1} + \varepsilon N b_N) \sum_i r_i^{N-1}. \end{aligned} \tag{8.69}$$

Then, since for every ε (8.69) is definitively true, we get

$$\begin{aligned} \int_{\overline{C}} \operatorname{div}\varphi dx &= \lim_n \int_{\partial C_n} \varphi \cdot \nu d\mathcal{H}^{N-1} \\ &\le \lim_n (b_{N-1} + \varepsilon N b_N) \left(m_0 + \frac{1}{n} \right) R^{N-1} \end{aligned}$$

and, by the arbitrariness of ε, we get the thesis. □

We know that C satisfies the inequality (4.30) and that, if the equality holds, that the reduced boundary $\partial^*\overline{C}$ can be identified with a disk of B. The reverse inequality has been just proved in the previous lemma which enables us to conclude that $\partial^*\overline{C}$ is a disk and that $m_0 = 1$. Now we are in a position to prove the main theorem of this section.

Proof of Theorem 8.1. Let us suppose that there exists $\overline{\varepsilon} > 0$ such that for every n there exists a punching $\mathcal{B}_n$ with measure less than or equal to $(1+\frac{1}{n})R^{N-1}$ such that, for every hyperplane P through the center of B, $\mathcal{B}_n$ is not *$\overline{\varepsilon}$-close* to $D = P \cap B$. Now, since $(\mathcal{B}_n)_{n\in\mathbb{N}}$ has infinitesimal thickness, the conclusion reached above allows us to claim that the boundary in B of $\overline{C}$ coincides with a disk $D = P \cap B$. Since $\mathcal{B}_n$ is not ε-close to D and (6.35) has already been checked, then there exists a finite set $\mathcal{B}'_n$ of balls $B_i \in \mathcal{B}_n$, with a measure greater than $\overline{\varepsilon}\mu(\mathcal{B}) \geq \overline{\varepsilon}(1-\overline{\varepsilon}b_{N-1}^{-1})R^{N-1}$, such that $\bigcup_{B_i\in\mathcal{B}'_n} B_i \cap D(\overline{\varepsilon}R) = \emptyset$. Thus, we can repeat the same computations of (8.69), by taking $\operatorname{spt}\varphi \subset \overline{D}(\overline{\varepsilon}R)$. This leads to the conclusion $P(\overline{C}, B) \leq (b_{N-1}-\overline{\varepsilon})R^{N-1}$, which is a contradiction. □

9. – Similarity arguments

The main results of this section are summarized in the following theorem.

Theorem 9.1. *For every $\varepsilon > 0$ and $\delta_1 > 0$ there exist ε' (defined in Section 1 and depending on ε) and δ_2 such that, if B is δ_2-split by the set K and condition* (M_{δ_1}) *(depending on ε') holds, then there exists a ε-packing of K.*

This will allow us, in the next section, to take advantage of the results established so far and in particular Theorem 8.1teor:alp. The proof of Theorem 9.1 will follow from some results which we are going to establish in the following lemmas. If $\mathcal{B}$ is any set of balls, we will use the notation $\mu'_{\mathcal{B}} = \mathcal{H}^{N-1}(K \setminus \bigcup_{B\in\mathcal{B}} B)$.

Lemma 9.1. *Let $\varepsilon' > 0$ be fixed (and so t and n are also fixed). Let B be a ball with radius R such that* (M_{δ_1}) *holds. If $\widetilde{\mathcal{B}}$ is any set of disjoint balls contained in B which are δ_1-split by K and have a radius bigger or equal to $t^n r$, then*

$$\mu(\widetilde{\mathcal{B}}) \leq R^{N-1}\left(1+\frac{\varepsilon'}{2}\right)\left(1-\frac{\mu'_{\widetilde{\mathcal{B}}}}{\mathcal{H}^{N-1}(K)}\right). \tag{9.70}$$

Proof. For every $B' \in \widetilde{\mathcal{B}}$, with radius r, we obtain by (M_{δ_1})

$$r^{N-1} \leq \left(1+\frac{\varepsilon'}{2}\right)\frac{\mathcal{H}^{N-1}(K\cap B')}{\mathcal{H}^{N-1}(K)}\,R^{N-1}. \tag{9.71}$$

Since the balls of $\widetilde{\mathcal{B}}$ are disjoint, we get

$$\mu(\widetilde{\mathcal{B}}) = \sum r^{N-1} \leq R^{N-1}\left(1+\frac{\varepsilon'}{2}\right)\left(1-\frac{\mu'_{\widetilde{\mathcal{B}}}}{\mathcal{H}^{N-1}(K)}\right). \qquad \square$$

The proof of Theorem 9.1 can be easily deduced from the following Lemma which will be proved in the sequel.

LEMMA 9.2. *For every $\varepsilon' > 0$ and $\delta_1 > 0$ there exists δ_2 such that, if B is any δ_2-split ball satisfying* (M_{δ_1}), *then there exists $\mathcal{B}$, ε'-punching of B such that, if $\mathcal{B}'$ is any subset of $\mathcal{B}$ such that $\mu(\mathcal{B}') \geq \frac{1}{2}\mu(\mathcal{B})$, then*

$$\mu(\mathcal{B}') \leq (1+\varepsilon')R^{N-1}\left(1-\frac{\mu'_{\mathcal{B}'}}{\mathcal{H}^{N-1}(K)}\right). \tag{9.72}$$

PROOF OF THEOREM 9.1. Let $\varepsilon > 0$ and $\delta_1 > 0$ be given. We fix $\varepsilon' \leq \varepsilon$ such that

$$1-\frac{m(\varepsilon')}{1+\varepsilon'} < \varepsilon \tag{9.73}$$

holds, as we can do since the left hand side tends to zero as ε' tend to zero. We can also take δ_2 small enough as in Lemma 9.2 so that, if B is δ_2-split and enjoys (M_{δ_1}), we can find a ε'-punching (and so ε-punching, since $\varepsilon' \leq \varepsilon$) $\mathcal{B}$ which satisfies (9.72). Then (6.36) trivially follows from (9.72) for $\mathcal{B}' = \mathcal{B}$. In order to prove (6.37), we observe that, since, by definition, $\mu(\mathcal{B}) \geq m(\varepsilon')R^{N-1}$ and $2\mu(\mathcal{B}') \geq \mu(\mathcal{B})$, then (9.72) and (9.73) give

$$\frac{\mu'_{\mathcal{B}'}}{\mathcal{H}^{N-1}(K)} \leq 1-\frac{\mu(\mathcal{B}')}{(1+\varepsilon')R^{N-1}} \leq 1-\frac{m(\varepsilon')}{1+\varepsilon'}+\frac{\mu(\mathcal{B})-\mu(\mathcal{B}')}{(1+\varepsilon')R^{N-1}} < \varepsilon+\frac{\mu(\mathcal{B})-\mu(\mathcal{B}')}{R^{N-1}}.$$

By the definition of $\mu'_{\mathcal{B}'}$, we get (6.37). $\square$

In order to prove Lemma 9.2, we need to construct $\mathcal{B}$. To this aim, let ε' (and consequently t, n) and δ_1 be fixed and B be any ball which satisfies (M_{δ_1}). Let $\mathcal{B}_1$ be a set, maximal by inclusion, of pairwise disjoint δ_1-*split* balls contained in B and radius tR. In the same way, we define $\mathcal{B}_2$ by taking disjoint δ_1-*split* balls having radius t^2R and disjoint from the former ones, thus recursively, for $k = 1 \cdots n$, we define $\mathcal{B}_k$ as a maximal set of disjoint balls of radius t^kR . In order to simplify the notation, we put $\overline{\mu}_k = \mu(\mathcal{B}_k) = \mathrm{card}(\mathcal{B}_k)(t^kR)^{N-1}$, $\overline{\mu} = \sum_k \overline{\mu}_k$, $\overline{\mu}' = \mu'_{\bigcup \mathcal{B}_k}$. Lemma 9.1 applies in particular to all the subsets $\widetilde{\mathcal{B}}$ of $\bigcup_k \mathcal{B}_k$. By (9.70) for $\widetilde{\mathcal{B}} = \bigcup_k \mathcal{B}_k$, we deduce that $\overline{\mu} \leq R^{N-1}(1+\frac{\varepsilon'}{2})$ and so there exists an index i which singles out a scale to which, roughly speaking, we find few balls, namely their total measure is such that

$$\overline{\mu}_i = \mu(\mathcal{B}_i) \leq \frac{\overline{\mu}}{n} \leq \frac{R^{N-1}\left(1+\frac{\varepsilon'}{2}\right)}{n}.$$

By fixing i as above, we introduce the following sets

$$\mathcal{B}^1 = \bigcup_{k \neq i} : \bigcup_{B' \in \mathcal{B}_k} \{(1+3t)B'\}, \qquad \mathcal{B}^2 = \bigcup_{B' \in \mathcal{B}_i} \{4B'\},$$
$$\mathcal{B} = \mathcal{B}^1 \cup \mathcal{B}^2, \qquad U = \bigcup_{B' \in \mathcal{B}} B'.$$

So, every ball $B_j \in \mathcal{B}$ is obtained, by definition, by suitably increasing the radius of a ball $B' \in \bigcup_k \mathcal{B}_k$. We shall refer to B' as to the *ball corresponding* to B_j.

LEMMA 9.3. *Let i be fixed as above and $B' \subset B$ be a ball of radius $r = t^i R$, such that $2B' \not\subset U$. Then B' cannot be δ_1- split.*

PROOF. Fix any ball B'' in $\bigcup_{k=1}^{i} \mathcal{B}_k$. By construction, the $3t^i R$-neighborhood of B'' is contained in a ball $B''' \in \mathcal{B}$. By assumption $2B' \not\subset B'''$, so there exists $z \in 2B'$ such that $d(z, B'') \geq 3t^i R$. So, if $\overline{x}$ is the center of B', by the triangular inequality we deduce $d(\overline{x}, B'') \geq t^i R$. This implies $B' \cap B'' = \emptyset$ and by the maximality of $\mathcal{B}_i$ we get the conclusion. □

Now, we can show that $\mathcal{B}$ is as required in Lemma 9.2.

LEMMA 9.4. *Let $\mathcal{B}$ be the above defined set and let $\mathcal{B}' \subset \mathcal{B}$ be any subset such that $\mu(\mathcal{B}') \geq \frac{1}{2}\mu(\mathcal{B})$, then* (9.72) *holds.*

PROOF. Let $\widetilde{\mathcal{B}} \subset \bigcup_k \mathcal{B}_k$ be the set of the balls in $\bigcup_k \mathcal{B}_k$ corresponding to those belonging to $\mathcal{B}'$. We set $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_1 \cup \widetilde{\mathcal{B}}_2$, with $\widetilde{\mathcal{B}}_1 = \widetilde{\mathcal{B}} \setminus \mathcal{B}_i$ and $\widetilde{\mathcal{B}}_2 = \widetilde{\mathcal{B}} \cap \mathcal{B}_i$. We shall denote by r the radius of the generic ball B' belonging to $\widetilde{\mathcal{B}}$. Firstly, we remark that

$$\sum_{B' \in \widetilde{\mathcal{B}}_2} r^{N-1} \leq \frac{2\mu(\widetilde{\mathcal{B}})}{n},$$

because, otherwise, from $2\mu(\mathcal{B}') \geq \mu(\mathcal{B})$ we get $2\mu(\widetilde{\mathcal{B}}) \geq \overline{\mu}$ and therefore

$$\sum_{B' \in \widetilde{\mathcal{B}}_2} r^{N-1} \leq \overline{\mu}_i \leq \frac{\overline{\mu}}{n} \leq \frac{2\mu(\widetilde{\mathcal{B}})}{n}.$$

Thus, from (1.4) and (9.70), we can establish the following inequalities

$$\begin{aligned} \mu(\mathcal{B}') &= \sum_{B' \in \widetilde{\mathcal{B}}_1} ((1+3t)r)^{N-1} + \sum_{B \in \widetilde{\mathcal{B}}_2} (4r)^{N-1} \\ &\leq \left((1+3t)^{N-1} + \frac{2^{2N-1}}{n}\right)\mu(\widetilde{\mathcal{B}}) \leq \frac{1+\varepsilon'}{1+\frac{\varepsilon'}{2}}\mu(\widetilde{\mathcal{B}}) \qquad (9.74) \\ &\leq (1+\varepsilon')R^{N-1}\left(1 - \frac{\mu'_{\mathcal{B}'}}{\mathcal{H}^{N-1}(K)}\right), \end{aligned}$$

since $\mu'_{\mathcal{B}'} \leq \mu'_{\widetilde{\mathcal{B}}}$, by construction. □

What we still have to show for ending the proof of Lemma 9.2 is that, if B is furthermore δ_2-split, with a suitable value of δ_2, the above introduced set $\mathcal{B}$ actually is a ε'-punching of B. To this aim we need to estimate the measure of the connected components of $B \setminus U$. Firstly, we note that there exists $k \in \mathbb{N}$, with $k = k(\delta_1)$, such that, for every pair of balls which are not δ_1-*split*, with radius r and with the centers at a distance less than $\frac{r}{k}$, then each ordinary oscillation region of the first one has a nonempty intersection with each ordinary oscillation region of the other one. Now, put $\eta = \frac{t^i R}{2k}$, we take an η-*net* $\mathcal{N}$ of the ball B^1 with the same center as B and radius $(1 - t^i)R$, such that $\omega = \text{card}(\mathcal{N})$ is of the same order as $(\frac{k}{t^i})^N$. Thus ω is a function of (δ_1, t, n) and so $\omega = \omega(\delta_1, \varepsilon')$. We define π as the minimal distance projection from B to $\overline{B}^1$. Let C be any connected component of $B \setminus U$, by introducing the set

$$\mathcal{N}' = \{x \in \mathcal{N} \colon | \, d(x, \pi(C)) \leq \eta\},$$

we see that $\pi(C) \subset \bigcup_{x \in \mathcal{N}'} B(x, t^i R)$. Let S be the union of all the δ_1-*split* balls contained in B, with radius $r \leq t^i R$ and $C' = C \setminus (S \cup K)$.

Lemma 9.5. *Let* x, y *be in* $\mathcal{N}'$, *then there exists a finite chain* $\{x_1, x_2, \dots, x_m\} \subset \mathcal{N}'$ *such that* $x_1 = x$, $x_m = y$ *and, for every* $i < m$, $d(x_i, x_{i+1}) < 2\eta$ *(obviously, we can take* $m \leq \omega(\delta_1, \varepsilon')$*).*

Proof. Fix $x \in \mathcal{N}'$ and let $\mathcal{N}''$ be the set of $y \in \mathcal{N}'$ such that the thesis holds. We note that, if $z \in \mathcal{N}'$ and $d(z, \mathcal{N}'') \leq 2\eta$, then $z \in \mathcal{N}''$. Moreover $\mathcal{N}'' \neq \emptyset$, because $x \in \mathcal{N}''$. We are going to show that the hypothesis $\mathcal{N}' \setminus \mathcal{N}'' \neq \emptyset$ leads to claim that $d(\mathcal{N}'', \mathcal{N}' \setminus \mathcal{N}'') \leq 2\eta$ and this is a contradiction to what we have stated above. The equality $\mathcal{N}' = \mathcal{N}''$ will follow and the thesis will be proved. Let us introduce the sets

$$A' = \{x \in \pi(C) \colon | \, d(x, \mathcal{N}'') \leq \eta\}$$

and, analogously,

$$A'' = \{x \in \pi(C) \colon | \, d(x, \mathcal{N}' \setminus \mathcal{N}'') \leq \eta\}.$$

Now, by the definition of $\mathcal{N}'$, we get that $A' \cup A'' = \pi(C)$, while the condition $\mathcal{N}'' \neq \emptyset$ implies $A' \neq \emptyset$ and the condition $\mathcal{N}' \setminus \mathcal{N}'' \neq \emptyset$ implies $A'' \neq \emptyset$. The sets A' and A'' are closed. Since $\pi(C)$ is connected, then there exists $z \in A' \cap A''$. Therefore, we have that $d(z, \mathcal{N}') \leq \eta$ and $d(z, \mathcal{N}'') \leq \eta$, then the claim. □

Lemma 9.6. *There exists* $c = c(\delta_1, \varepsilon')$ *such that*

$$\operatorname{osc}_{C'} u \leq c\sqrt{R}.$$

Proof. Fix $\overline{x}$ and $\overline{y}$ in C'. Let $\lambda < 1$ be suitably chosen and let $(B_h)_{h \in \mathbb{N}}$ be a monotone decreasing sequence of balls, contained in B, with radius $\lambda^h t^i R$, such that $B_0 = B(\pi(\overline{x}), t^i R)$ and for every $h \in \mathbb{N}$, $\overline{x} \in B_h$, i.e. $\{\overline{x}\} = \bigcap_h B_h$. Since $\overline{x} \notin S$ and the radii are less than or equal to $t^i R$, by the definition of S these balls cannot be δ_1-*split*, then each one of them has an ordinary oscillation region $\widetilde{B}_h$. Thus, by choosing λ suitably close to 1, we can claim that, for every $h \in \mathbb{N}$, $\widetilde{B}_h \cap \widetilde{B}_{h+1} \neq \emptyset$. Obviously $\lambda = \lambda(\delta_1)$. We fix an arbitrary point $z_0 \in \widetilde{B}_0$ and for every $h \in \mathbb{N}$, $h \geq 1$, we take $z_h \in \widetilde{B}_{h-1} \cap \widetilde{B}_h$; since z_h and z_{h+1} both belong to $\widetilde{B}_h$, we get the following estimate.

$$\begin{aligned} |u(z_h) - u(z_{h+1})| &\leq \operatorname{osc}_{\widetilde{B}_h} u \\ &\leq \delta_1^{-1}\sqrt{\lambda^h t^i R}\,. \end{aligned} \tag{9.75}$$

Since $z_h \to \overline{x}$ and $\overline{x} \notin K$, then we have

$$\begin{aligned} |u(z_0) - u(\overline{x})| &\leq \sum_{h=0}^{\infty} |u(z_h) - u(z_{h+1})| \\ &\leq \delta_1^{-1}\sqrt{t^i R}\sum_{h=0}^{\infty} \sqrt{\lambda^h}\,. \end{aligned} \tag{9.76}$$

By repeating the above argument, we can find w_0 belonging to the ordinary oscillation region of $B(\pi(\overline{y}), t^i R)$, such that

$$|u(w_0) - u(\overline{y})| \leq \delta_1^{-1}\sqrt{t^i R}\sum_{h=0}^{\infty} \sqrt{\lambda^h}\,. \tag{9.77}$$

By definition of $\mathcal{N}'$, we can find $x, y \in \mathcal{N}'$ such that both $d(x, \pi(\overline{x}))$ and $d(y, \pi(\overline{y}))$ are less than or equal to η. Furthermore, we can find a finite chain $\{x_1, x_2, \ldots, x_m\} \subset \mathcal{N}'$ such that $x_1 = x$, $x_m = y$, as in Lemma 9.5, also we put $x_0 = \pi(\overline{x})$, $x_{m+1} = \pi(\overline{y})$. We claim that for every $j = 0, \ldots, m+1$, the ball $B(x_j, t^i R)$ is not δ_1-*split*. Indeed $B(x_j, t^i R) \cap \pi(C) \supset B(x_j, \eta) \cap \pi(C) \neq \emptyset$. Therefore $B(x_j, 2t^i R) \cap C \neq \emptyset$ and so $B(x_j, 2t^i R) \not\subset U$ and the claim follows from Lemma 9.3. Thus let $\tilde{A}_j$ be an ordinary oscillation region of $B(x_j, t^i R)$. Since, for every j, $d(x_j, x_{j+1}) \leq 2\eta$, then we have that, for $j = 0, \ldots, m$, $\tilde{A}_j \cap \tilde{A}_{j+1} \neq \emptyset$. Let us fix $v_j \in \tilde{A}_j \cap \tilde{A}_{j+1}$. Obviously, $\tilde{A}_0 = \widetilde{B}_0$ and we can take $z_0 = v_0$. Likewise, we can take $v_{m+1} = w_0$. Therefore, for every $j = 1, \ldots, m+1$, v_j and v_{j-1} are both by construction in $\tilde{A}_j$. Thus

$$\begin{aligned} |u(v_j) - u(v_{j-1})| &\leq \operatorname{osc}_{\tilde{A}_j} u \\ &\leq \delta_1^{-1}\sqrt{t^i R}\,. \end{aligned} \tag{9.78}$$

By the triangular inequality, we get

$$\begin{aligned} |u(z_0) - u(w_0)| &= \left|\sum_{j=1}^{m+1} u(v_j) - u(v_{j-1})\right| \leq (m+1)\delta_1^{-1}\sqrt{t^i R} \\ &\leq (\omega(\delta_1, \varepsilon') + 1)\delta_1^{-1}\sqrt{t^i R}\,. \end{aligned} \tag{9.79}$$

Finally, by combining the above inequalities, with (9.76)) and (9.77), we can easily deduce the following estimate which concludes the proof.

$$
\begin{aligned}
|u(\overline{x}) - u(\overline{y})| &\leq |u(\overline{x}) - u(z_0)| + |u(z_0) - u(w_0)| + |u(w_0) - u(\overline{y})| \\
&\leq \delta_1^{-1}\sqrt{t^i}\left(\omega(\delta_1, \varepsilon') + 1 + 2\sum_{h=0}^{\infty}\sqrt{\lambda^h}\right)\sqrt{R} = c(\delta_1, \varepsilon')\sqrt{R},
\end{aligned} \tag{9.80}
$$

provided we take $c(\delta_1, \varepsilon') = \delta_1^{-1}\sqrt{t^i}\left(\omega(\delta_1, \varepsilon') + 1 + 2\sum_{h=0}^{\infty}\sqrt{\lambda^h}\right)$. □

If we fix $\delta_2 < c(\delta_1, \varepsilon')^{-1}$ and we take B δ_2-split, then we get from the above lemma

$$
|C'| < \left(\frac{1}{2} + \delta_2\right)|B|. \tag{9.81}
$$

Proof of Lemma 9.2. Let B and K, closed subset of B, be given, we fix δ_2 as above and we assume that B is δ_2-split by K and satisfies (M_{δ_1}). Since $\mathcal{B}$ has a thickness less than or equal to $4tR$ and so, by (1.3), less than or equal to $\varepsilon' R$, in order to prove that $\mathcal{B}$ is a ε'-punching of B, it remains to show that the measure of any connected component C of $B \setminus \bigcup_{B' \in \mathcal{B}} B'$ is less than or equal to $(\frac{1}{2} + \varepsilon')|B|$. To this aim, in view of (9.81), we just need to estimate $|S|$. By a Vitali covering type argument, we can say that

$$
S \subset \bigcup_{j \in J} 5B_j ,
$$

where J is a set of indexes and the sets B_j are mutually disjoint δ_1-*split* balls with radius $r_j \geq t^n r$, by construction. By Lemma 9.1 we find

$$
\sum_{j \in J} r_j^{N-1} \leq \left(1 + \frac{\varepsilon'}{2}\right) R^{N-1} \leq 2R^{N-1}. \tag{9.82}
$$

Therefore

$$
|S| \leq 5^N \sum_{j \in J} |B_j| \leq 5^N b_N \sup_{j \in J} r_j \sum_{j \in J} r_j^{N-1} \leq 2 \cdot 5^N t b_N R^N. \tag{9.83}
$$

Then, by (9.81) and (9.83), we deduce that

$$
|C| \leq \left(\frac{1}{2} + \delta_2\right)|B| + |S| \leq \left(\frac{1}{2} + \delta_2\right)|B| + 2 \cdot 5^N t |B|. \tag{9.84}
$$

By also taking, as allowed by (1.3),

$$
\delta_2 < \varepsilon' - 2 \cdot 5^N t ,
$$

we get $|C| \leq (\frac{1}{2} + \varepsilon')|B|$, which, combined with (9.74) shows that $\mathcal{B}$ is an ε'-punching of B. Finally, (9.72) follows from Lemma 9.4. □

10. – Proof of Theorem 1.3

We begin this section by showing how Theorem 8.1 combines with Theorem 9.1.

LEMMA 10.1. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that, if K has a δ-packing $\mathcal{B}$, then there exists a hyperplane P through the center of B such that* (2.5) *holds.*

PROOF. The assertion trivially follows from Theorem 8.1 (whose use is allowed by (6.36)) and from (6.37). □

Let $B = B(x, R)$ and let D be a disk through the center of B. Given $\varepsilon > 0$, we take the two disks of B, D_1 and D_2, obtained by cutting B with two hyperplanes parallel to D and distant ε from D. Thus, the ball B turns out to be divided in three regions, namely the set enclosed between D_1 and D_2, which will be denoted by F_ε, and the remaining two regions E_1 and E_2, symmetric with respect to D. Let S be a subset of $E_1 \cup E_2$ and let p denote the orthogonal projection of B on the hyperplane containing D_1. Let $K \subset B$ be a closed subset, $u : B \to \mathbb{R}$ satisfying condition (WS) on $B \setminus K$ with $\| u \|_{2N}^{*} = 1$. We introduce the notation $A = D_1 \setminus p(K \cup S)$ and $a = R^{1-N}\mathcal{H}^{N-1}(A)$, and we claim the following statement.

LEMMA 10.2. *If $a \neq 0$ there exists a chord C of B, orthogonal to D, such that $C \cap (S \cup K) = \emptyset$, on which the following estimate holds*

$$\int_{C \cap F_\varepsilon} | \nabla u | \leq c a^{-1} \sqrt{\varepsilon R}\,, \tag{10.85}$$

where c denotes a positive constant only depending on the dimension N.

PROOF. Let $\mathcal{N}_{\varepsilon R}$ be an εR-*net* of D, then there exists a positive constant c, only depending on N, such that $\operatorname{card}(\mathcal{N}_{\varepsilon R}) \leq c\varepsilon^{1-N}$. Then, by observing that

$$F_\varepsilon \subset \bigcup_{y \in \mathcal{N}_{\varepsilon R}} B(y, 2\varepsilon R)$$

and by using (WS), we can establish the following estimate

$$\begin{aligned} \int_{F_\varepsilon} | \nabla u | &\leq \sum_{y \in \mathcal{N}_{\varepsilon R}} \int_{B(y, 2\varepsilon R)} | \nabla u | \leq \operatorname{card}(\mathcal{N}_{\varepsilon R})(2\varepsilon R)^{N-\frac{1}{2}} \\ &\leq c\varepsilon^{1-N}(2\varepsilon R)^{N-\frac{1}{2}} \leq c\sqrt{\varepsilon R}\, R^{N-1}\,. \end{aligned} \tag{10.86}$$

Thus, by Fubini theorem and (10.86), we can easily deduce the assertion. □

LEMMA 10.3. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that, if B is δ-split by K, K has a δ-packing and P is as in Lemma 10.1, the following estimate holds.*

$$\mathcal{H}^{N-1}(p(K)) > (1-\varepsilon)b_{N-1}R^{N-1}. \tag{10.87}$$

PROOF. Given $\varepsilon > 0$, by Lemma 10.1, if δ is suitably chosen and if K has a δ-packing, we can find a hyperplane P such that, with the notation introduced in Lemma 10.2,

$$\mathcal{H}^{N-1}(K \setminus F_\varepsilon) < \varepsilon R^{N-1}.$$

Thus we assume the above construction with $D = P \cap B$ and we take a function u which satisfies (WS) with $\| u \|_{2N}^{*} = 1$ on $B \setminus K$ and which splits B. By (WS) and by [22, Theorem 6.1] we know that there exists a neighborhood S of $K \setminus F_\varepsilon$ and a positive constant c such that u is Hölder continuous with index $\frac{1}{2}$ on each of the two sets $E_1 \setminus S$ and $E_2 \setminus S$ with a given Hölder norm c_H and such that

$$\mathcal{H}^{N-1}(\partial S) \le c\mathcal{H}^{N-1}(K \setminus F_\varepsilon) \le c\varepsilon R^{N-1}. \tag{10.88}$$

Let σ be the *jump* of u between E_1 and E_2, defined by $\sigma = R^{-\frac{1}{2}}(\inf_{E_2\setminus S} u - \sup_{E_1\setminus S} u)$, under a suitable choice of the indexes. By taking the chord C found in Lemma 10.2, by (10.85) we can state

$$\sigma \le R^{-\frac{1}{2}} \int_{C\cap F_\varepsilon} |\nabla u| \le ca^{-1}\sqrt{\varepsilon}.$$

Since B is δ-split, by the Hölder continuity of u on the sets E_i we get $\sigma \ge \frac{1}{2}\delta^{-1}$ and therefore the above inequality gives

$$a \le \frac{c\sqrt{\varepsilon}}{\sigma} \le c\sqrt{\varepsilon}\delta. \tag{10.89}$$

By recalling the definition of A and by combining (10.88) with (10.89), we get

$$\begin{aligned}\mathcal{H}^{N-1}(p(K)) &\ge \mathcal{H}^{N-1}(D_1) - \mathcal{H}^{N-1}(p(S)) - \mathcal{H}^{N-1}(A)\\ &= \mathcal{H}^{N-1}(D_1) - \mathcal{H}^{N-1}(\partial S) - aR^{N-1}\\ &\ge \mathcal{H}^{N-1}(D_1) - c\varepsilon R^{N-1} - c\sqrt{\varepsilon}\delta R^{N-1}.\end{aligned} \tag{10.90}$$

Now, since $\mathcal{H}^{N-1}(D \setminus D_1) \le c\varepsilon^2 R^{N-1}$, by the arbitrariness of ε, from (10.90) we get the thesis. □

COROLLARY 10.1. *For every $\varepsilon > 0$ there exists δ such that, if K has a δ-packing and B is δ-split, then there exists a hyperplane P through the center of B such that K is ε-close to P.*

PROOF OF THEOREM 1.3. We fix ε and consequently δ as in Corollary 10.1. Then we apply Theorem 9.1 with ε given by such a δ and so we find δ_2 such that, if B is δ_2-split by K and condition (M_{δ_1}) holds, there exists a δ-packing of K. We take $\delta_2 \leq \delta$ and so we have the hypotheses of Corollary 10.1. □

PROOF OF CROLLARY 1.4. Given ε, c_0 and δ_1, we fix δ_2 as in Theorem 3.1 As in the proof of Theorem 1.2, we find $\mathbf{A}_K(B) \leq (\varepsilon^2 + \varepsilon)R^{1-N}\mathcal{H}^{N-1}(K) \leq (\varepsilon^2 + \varepsilon)c_0$. By the arbitrariness of ε we get the thesis. □

11. – Proof of the main results through similarity arguments

Theorem 1.3 allows a more elementary proof of Theorem 1.1 and Theorem 1.2 (of course the results in Section 5 will not be considered to be known in this last section).

LEMMA 11.1. *For every $\varepsilon > 0$ there exists δ such that, for every $K \in \mathcal{S}_\delta(B)$ satisfying* (1.2), *K is ε-close to some hyperplane P through the center of B.*

PROOF. Given $\varepsilon > 0$, we fix ε' as given by Theorem 1.3 (ε' just depends on ε). Subsequently, we fix δ_1 such that

$$\frac{s_0 + \delta_1}{s(\delta_1)} < 1 + \frac{\varepsilon'}{2},$$

as we can do since $\frac{s_0+\delta_1}{s(\delta_1)} \to 1$ as $\delta_1 \to 0$. Then we fix δ_2 as in Theorem 1.3, and we take $\delta \leq \min\{\delta_1, \delta_2\}$. We observe that (1.2) implies (M_{δ_1}). Indeed, if a ball B' with radius $r \leq R$ is δ_1-split, we have

$$\begin{aligned} R^{1-N}\mathcal{H}^{N-1}(K) &\leq s_0 + \delta \leq s_0 + \delta_1 \\ &\leq (s_0 + \delta_1)\frac{\mathcal{H}^{N-1}(K \cap B')}{s(\delta_1)r^{N-1}} \\ &< \left(1 + \frac{\varepsilon'}{2}\right) r^{1-N}\mathcal{H}^{N-1}(K \cap B'). \end{aligned}$$

So, if K is as in the statement, from Theorem 1.3 we deduce the thesis. □

COROLLARY 11.1.

$$s_0 = b_{N-1}. \tag{11.91}$$

PROOF. Fix any $\varepsilon > 0$ and then a constant δ as given by the previous lemma. We can always take K such that (1.2) holds and then we have from (2.5), which is guaranteed by the previous lemma,

$$(b_{N-1} - \varepsilon)R^{N-1} \leq \mathcal{H}^{N-1}(p(K)) \leq \mathcal{H}^{N-1}(K) \leq (s_0 + \delta)R^{N-1}.$$

Since we can take δ and ε arbitrarily small in the above inequality, we get $s_0 \geq b_{N-1}$. Since the reverse inequality of (11.91) has already been observed, the thesis follows. □

PROOF OF THEOREM 1.1. The theorem is just a restatement of the last corollary (see Section 5). □

PROOF OF THEOREM 1.2. Given $\varepsilon > 0$, by Lemma 11.1 we have $\delta > 0$ such that, for every $K \in \mathcal{S}_\delta(B)$ satisfying (1.2), K is ε-close to some hyperplane P through the center of B. Therefore, the estimates on $\mathbf{A}_K(B)$ and $\mathbf{T}_K(B)$ follow as in Section 5. □

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Dipartimento Interuniversitario di Matematica
Via E. Orabona, 4
70125 Bari, Italy
solimini@dm.uniba.it