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Nonlinear Parabolic Equations with Natural Growth Terms and Measure Initial Data

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Abstract. We investigate the existence and the stability of the solutions of a nonlinear evolution equation with a local quadratic term with respect to the gradient of the type $g(u)|Du|^2$ and for measure initial data. We extend the notion of renormalized solutions for this problem. Under a natural condition on the convergence of the initial data, we prove the compactness of the truncation of solutions in the energy space. Then we show that the integrability of g at infinity is a necessary and sufficient condition for the stability of the problem with respect to general measure data, as well as for the existence of renormalized solutions.

Mathematics Subject Classification (2000): 35K55 (primary), 35K60, 35R05 (secondary).

1. - Introduction

In this paper we investigate the problem of existence of solutions of the following initial-boundary value problem:

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,\nabla u)) + g(u)|\nabla u|^2 = f & \text{in } Q := \Omega \times (0,T)\,, \\ u = 0 & \text{on } \Sigma := \partial\Omega \times (0,T)\,, \\ u(0) = u_0 & \text{in } \Omega\,, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, $T > 0$, and we have set Q the cylinder $\Omega \times (0,T)$ and Σ its lateral surface. We assume that $a(x,t,\xi) : Q \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e. measurable with respect to (x,t) and continuous with respect to ξ) such that:

$$(a_1)\qquad a(x,t,\xi)\xi \geq \alpha|\xi|^2\,,$$

$$(a_2)\qquad |a(x,t,\xi)| \leq \beta|\xi|\,,$$

$$(a_3)\qquad (a(x,t,\xi) - a(x,t,\eta))(\xi - \eta) > 0\,,$$

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for every ξ, η ($\xi \neq \eta$) in $\mathbb{R}^N$ and almost every (x,t) in Q, where α, $\beta > 0$, and that the function $s \mapsto g(s)$ is continuous and satisfies the sign condition

$$(g_1) \qquad sg(s) \geq 0\,, \qquad \forall s \in \mathbb{R}\,.$$

The data are taken such that:

$$f \in L^1(Q)\,, \quad f \geq 0\,, \qquad u_0 \in \mathcal{M}_b^+(\Omega)\,,$$

where $\mathcal{M}_b^+(\Omega)$ is the space of positive Radon measures on Ω with bounded total mass (i.e. $u_0(\Omega) < +\infty$). In fact, while (g_1) is a structural assumption on equation (1.1) which plays a crucial role in our study, the assumption of positiveness on the data is made only to simplify some technical arguments.

In this context of nonlinear operators, if (g_1) holds true existence results for problem (1.1) have been proved in [LaMu] when f belongs to $L^2(0,T;H^{-1}(\Omega))$ and u_0 is in $L^2(\Omega)$. The case where f belongs to $L^1(Q)$ and $u_0 = 0$ is investigated in [DO] under few extra conditions on g that lead to a weak solution u of (1.1) in $L^2(0,T;H_0^1(\Omega))$. Finally in [P] problem (1.1) is studied under the assumptions adopted in the present paper and in the case when u_0 lies in $L^1(\Omega)$. As already pointed out in this last paper, the extension to general measure initial data seems to be not always possible. For instance, assume that

$$\exists\, \gamma\,,\, s_0 > 0: \quad g(s)\mathrm{sign(s)} \geq \gamma \qquad \forall\, \mathrm{s}\ : |\mathrm{s}| \geq \mathrm{s}_0\,.$$

Then looking for a weak solution u of (1.1) such that $g(u)|\nabla u|^2$ is in $L^1(Q)$ leads, in some sense, to a solution u in $L^2(0,T;H_0^1(\Omega))$. As a consequence of trace results, then u_0 must be in $L^1(\Omega)$.

Thus the aim of our work is to investigate the link between the behaviour of $g(s)$ at infinity and the measure u_0 which allows, or which is needed, to have solutions in some appropriate sense.

In fact, the main point in our study is the relationship between the possibility to find solutions of (1.1) and the stability properties of the equation, as they naturally arise when one tries to solve (1.1) by approximating the singular data f and u_0 with sequences of regular functions. For example, letting $\{f_\varepsilon\}$ and $u_{0\varepsilon}$ be a standard approximation of f and u_0 constructed by convolution, we consider the approximating problems:

$$(1.2) \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \mathrm{div}(a(x,t,\nabla u_\varepsilon)) + g(u_\varepsilon)|\nabla u_\varepsilon|^2 = f_\varepsilon & \text{in } Q\,, \\ u_\varepsilon = 0 & \text{on } \Sigma\,, \\ u_\varepsilon(0) = u_{0\varepsilon} & \text{in } \Omega\,, \end{cases}$$

and we study the possibility to find a solution of (1.1) as limit of a subsequence $\{u_\varepsilon\}$ of solutions of (1.2). We are going to prove that, regardless of any other assumption on $g(s)$ but for (g_1), a compactness result on the sequence $\{u_\varepsilon\}$

is always available. On the other hand, it may happen that the limit of u_ε is not a solution of (1.1). Precisely, we prove that if u_0 is not assumed to be in $L^1(\Omega)$, a necessary and sufficient condition to pass to the limit in (1.2) and get a solution of (1.1) is the integrability of $g(s)$ at infinity. In particular, if $f = 0$ and $u_0 \in \mathcal{M}_b^+(\Omega)$ is singular with respect to Lebesgue measure, then the assumption that $\int_0^{+\infty} g(s)ds = +\infty$ implies that the whole sequence u_ε converges to zero.

The main tool to obtain this result is the proof that, setting, for every $k > 0$, $T_k(s) = \max(-k, \min(s, k))$ the truncation function at levels $\pm k$, then $T_k(u_\varepsilon)$ is strongly compact in the energy space $L^2(0, T; H_0^1(\Omega))$. Let us recall that this kind of compactness results on the truncations of solutions of approximating problems, like (1.2), plays a crucial role in the existence theory for nonlinear equations with integrable or measure data. As for parabolic initial boundary value problems, the strong convergence in $L^2(0, T; H_0^1(\Omega))$ of truncations of solutions of approximating problems was proved, in case of L^1 data, in [Bl] (see also [BlMR]) if $g = 0$, and in [P] with the lower order term having natural growth. Adapting a technique recently introduced in [DMOP] for elliptic equations, here we extend these results to the case of measures as initial data, under the assumptions that the sequence $u_{0\varepsilon}$ converges to u_0 in what is called the narrow topology of measures (i.e. $\int_\Omega \varphi du_{0\varepsilon} \to \int_\Omega \varphi du_0$ for every φ in $C(\bar{\Omega})$) and satisfies a sort of compatibility condition with respect to the Lebesgue decomposition of u_0, loosely speaking that the L^1 part of u_0 is approximated in the strong topology of $L^1(\Omega)$. These requirements are satisfied, for instance, by the approximation of u_0 constructed through convolution. Moreover we prove, in Example 2.5, that this specific approximation of the initial data is actually necessary in some sense since the strong convergence of the truncations may be false if $u_{0\varepsilon}$ is only assumed to converge to u_0 in the narrow topology of measures.

As a consequence of these stability properties proved on the solutions of (1.2), we are led to the problem of finding a suitable definition of solution of (1.1) which may provide existence and stability at the same time, and this is why we choose to set our results in the framework of the so-called renormalized solutions. Let us recall that the definition of renormalized solutions was given first in [DL] in the context of hyperbolic equations of conservation laws and then adapted to second order elliptic problems in [BDGM], while in the theory of boundary value problems with L^1 data it has often been used recently in order to get uniqueness of solutions (see [LM], for the stationary case, [BlM], [BlMR], [CW] for evolution equations). Finally, in [DMOP] an extension of this framework to general measure data has been given for elliptic equations. We follow the approach of this last paper extending this notion to problem (1.1) when $u_0 \in \mathcal{M}_b^+(\Omega)$ and showing how the renormalized solutions emphasize the stability properties mentioned above by selecting only the stable solutions.

We are grateful to the referee for pointing out the work of [GV] in which a systematic study of blowing-up and extinction properties of the solutions of nonlinear parabolic problems is performed. In [GV], the authors give a

general construction of extended semigroups for possibly singular data which are measurable functions on Ω. Losely speaking the method uses monotone approximations of singular functions. This allows to define generalized solutions for such singular initial data and then to describe their blowing-up, extinction or singular properties. It is worth noting that such a construction can not be achieved (at least with the same technique) as far as measure initial data are concerned (since in this case no monotone approximations are available). Independently of this simple technical argument, we would like to emphasize that the existence (and stability) or nonexistence (and unstability) results obtained in the present paper could hardly be classified in the blowing-up (or extinction) theory for parabolic equations since they are valid on any time interval $[0, T]$. Our results are actually in the same spirit as those in the article [BF] which concern the semilinear equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u^p = 0 & \text{in } Q, \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

They prove that the absorption term is responsible for the lack of solutions in the case where p is large and u_0 is a singular measure. Similar convergence results with possibly boundary layer phenomena (the conclusion we obtain) are also observed in the semilinear case. Again this sort of effects seems to be closer to removable singularity type results rather than to blow-up or extinction properties.

The paper is planned in the following way; in Section 2 we will precise the notion and some basic properties of renormalized solutions and we will state the results that we obtain, whose proof, which is rather technical, is left to Section 3.

2. – Definition of renormalized solutions and statement of the results

Let us first fix some notations. Henceforward, we will consider, for every measure u_0, its Lebesgue decomposition by writing

$$u_0 = u_0^r + u_0^s, \quad u_0^r \in L^1(\Omega), \quad u_0^s = u_0 \llcorner E \text{ with } \operatorname{meas}(E) = 0,$$

where $u_0 \llcorner E$ is the restriction of u_0 to the set E, defined as a measure by setting $u_0 \llcorner E(B) = u_0(B \cap E)$ for every Borelian subset B in Ω. Moreover we will denote by $C_c^\infty([0, T) \times \Omega)$ the set of functions φ belonging to $C^\infty(\bar{Q})$ such that $\varphi = 0$ on $\Sigma \cup (\Omega \times \{T\})$.

The main idea of renormalized solutions is that, if S is a function in $W^{2,\infty}(\mathbb{R})$ such that S' has compact support, multiplying formally the equation

(1.1) by $S'(u)$ one gets:

$$
\frac{\partial S(u)}{\partial t} - \operatorname{div}(S'(u)a(x,t,\nabla u)) + S''(u)a(x,t,\nabla u)\nabla u + g(u)S'(u)|\nabla u|^2 = fS'(u)\,, \tag{2.1}
$$

so that the equation is in some sense restricted to the subset of Q where $|u| \leq L$, if L is such that $\mathrm{Supp}(S') \subset [-L, L]$, and u can be replaced by its truncation $T_L(u)$, which can be asked to belong to the energy space $L^2(0,T; H_0^1(\Omega))$. On the other hand, since the equation (2.1) only considers the properties of the truncations of u, the renormalized formulation usually needs to add an extra-condition to recover, in some sense, the behaviour of u at infinity. Moreover, for any such function $S(r)$ as in (2.1), it follows from the fact that $\mathrm{Supp}(S')$ is compact and from the renormalized equation that $S(u)$ belongs to $L^2(0,T; H_0^1(\Omega))$ and then $\frac{\partial S(u)}{\partial t}$ belongs to the space $L^1(Q) + L^2(0,T; H^{-1}(\Omega))$, which implies that $S(u)$ belongs to $C([0,T]; L^1(\Omega))$ (for a proof of this trace result see [P]). This means that (2.1) does not take into account the singular part in the Lebesgue decomposition of the measure u_0. This prompts to ask $S(u)$ to satisfy the initial condition $S(u)(t=0) = S(u_0^r)$ (as it is the case when dealing with L^1 data, see e.g. [BlM]). To understand how the extra condition at infinity appears and is linked to the singular part u_0^s of u_0, the following heuristic argument can be carried on.

Since we expect renormalized solutions to be weak solutions, let us try to recover from (2.1) the usual weak formulation, by taking a sequence of functions $\{S_n(r)\}$ converging to the identity function $I(r) = r$ and such that $S_n'(r)$ has compact support for every n. Since we will very often make use of these auxiliary functions linking the renormalized and the weak formulation, let us fix once for all the following notations.

Definition 2.1. Setting, for every $k > 0$, $T_k(s) = \max(-k, \min(s,k))$ the truncation function at levels $\pm k$, we define

$$
\theta_n(s) = T_1(s - T_n(s)), \quad h_n(s) = 1 - |\theta_n(s)|, \quad S_n(s) = \int_0^s h_n(r)dr\,, \quad \forall s \in \mathbb{R}. \tag{2.2}
$$
□

Note that $h_n(s)$ converges to 1 as n tends to infinity and has compact support, so that $S_n(s)$ is a sequence of $W^{2,\infty}(\mathbb{R})$ functions having a derivative with compact support and converging, as n tends to infinity, to the identity function $I(s) = s$. Let then now u satisfy the renormalized formulation (2.1), and for simplicity assume that $u_0 \geq 0$ and $u \geq 0$. Let us also assume that u satisfies some standard regularity, that is u belongs to $L^\infty(0,T; L^1(\Omega)) \cap L^1(0,T; W_0^{1,1}(\Omega))$, so that by (a_2) we also have $a(x,t,\nabla u)$ in $L^1(Q)^N$, and let $g(u)|\nabla u|^2$ belong to $L^1(Q)$ (we will prove later that every renormalized solution has this regularity). We can take $S = S_n$ in (2.1) in order to see whether u satisfies the weak formulation as n tends to infinity. Since we ask $S_n(u)(t=0) = S_n(u_0^r)$ we have for every φ in $C_c^\infty([0,T) \times \Omega)$:

$$\begin{aligned}
&-\int_\Omega S_n(u_0^r)\,\varphi(0)\,dx - \int_0^T \int_\Omega \frac{\partial \varphi}{\partial t} S_n(u)\,dx\,dt \\
(2.3)\qquad &+\int_Q a(x,t,\nabla u)\nabla\varphi\, h_n(u)\,dx\,dt - \int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt \\
&+\int_Q g(u)|\nabla u|^2 h_n(u)\varphi\,dx\,dt = \int_Q f h_n(u)\,\varphi\,dx\,dt\,.
\end{aligned}$$

Thanks to the regularity satisfied by u, we can pass to the limit, as n tends to infinity, to obtain:

$$\begin{aligned}
(2.4)\qquad &\lim_{n\to\infty}\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt = -\int_\Omega u_0^r\,\varphi(0)\,dx - \int_0^T\int_\Omega \frac{\partial\varphi}{\partial t}\,u\,dx\,dt \\
&+\int_Q a(x,t,\nabla u)\nabla\varphi\;dx\,dt + \int_Q g(u)|\nabla u|^2\varphi\,dx\,dt - \int_Q f\,\varphi\,dx\,dt\,.
\end{aligned}$$

Since we expect u to be also a weak solution it should also satisfy:

$$\begin{aligned}
&-\int_\Omega \varphi(0)\,du_0 - \int_0^T\int_\Omega \frac{\partial\varphi}{\partial t}\,u\,dx\,dt \\
&+\int_Q a(x,t,\nabla u)\nabla\varphi\;dx\,dt + \int_Q g(u)|\nabla u|^2\varphi\,dx\,dt = \int_Q f\,\varphi\,dx\,dt\,,
\end{aligned}$$

which compared with (2.4) gives (recall that $u_0 = u_0^r + u_0^s$):

$$\lim_{n\to\infty}\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt = \int_\Omega \varphi(0)\,du_0^s\,, \quad \forall\varphi\in C_c^\infty([0,T)\times\Omega)\,.$$

At least for positive data and solutions, this is the extra-condition we were looking for, except that we will ask it to hold true for φ in $C(\bar{Q})$, which is slightly stronger. We introduce now the definition of renormalized solutions for signed measures, later we prove that renormalized solutions corresponding to positive data are always positive. For any signed measure u_0, let $(u_0^s)^\pm$ denote the positive and negative parts of u_0^s, both being singular, $u_0^s = (u_0^s)^+ - (u_0^s)^-$.

DEFINITION 2.2. A measurable function u, almost everywhere finite, is said to be a renormalized solution of (1.1) if

- $T_k(u) \in L^2(0,T;H_0^1(\Omega))$ for every $k > 0$,

- we have:

$$
(2.5)\qquad
\begin{aligned}
&\lim_{n\to\infty}\int_{\{(x,t):n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt = \int_\Omega \varphi(0)\,d(u_0^s)^+\,,\\
&\hspace{22em}\forall\varphi\in C(\bar Q)\,,\\
&\lim_{n\to\infty}\int_{\{(x,t):-n-1\le u\le -n\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt = \int_\Omega \varphi(0)\,d(u_0^s)^-\,,\\
&\hspace{22em}\forall\varphi\in C(\bar Q)\,,
\end{aligned}
$$

- for every $S\in W^{2,\infty}(\mathbb{R})$ such that S' has compact support u satisfies in the sense of distributions in Q:

$$
(2.6)\qquad
\begin{aligned}
&\frac{\partial S(u)}{\partial t} - \operatorname{div}(a(x,t,\nabla u)S'(u)) + a(x,t,\nabla u)\nabla u S''(u)\\
&\qquad + g(u)|\nabla u|^2\,S'(u) = S'(u)f\,,
\end{aligned}
$$

and
- $S(u)(0) = S(u_0^r)$ in $L^1(\Omega)$. □

Firstly, let us see that requiring condition (2.5) to hold true for φ in $C(\bar Q)$ allows to prove some standard regularity properties on renormalized solutions, and moreover, if f and u_0 are positive, every renormalized solution is positive too. Let us point out that in the following we will make use of the integration by parts formula applied to functions which belong to $L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$ and have a time derivative in the space $L^2(0,T;H^{-1}(\Omega))+L^1(Q)$, like, for example, the function $S(u)$ appearing in the renormalized formulation (2.6). Such generalizations of the classical integration by parts formula can be found, for instance, in [BMP2], or in [CW].

Proposition 2.3. *Let u be a renormalized solution of* (1.1) *in the sense of Definition* 2.2. *Then u belongs to $L^\infty(0,T;L^1(\Omega))\cap L^q(0,T;W_0^{1,q}(\Omega))$ for every $q<\frac{N+2}{N+1}$, $g(u)|\nabla u|^2$ is in $L^1(Q)$ and the following estimates hold true:*

$$
\begin{aligned}
&\|T_k(u)\|^2_{L^2(0,T;H_0^1(\Omega))} \le k\,C_0 && \forall k>0\,,\\
&\int_{\{k\le |u|\le k+1\}} a(x,t,\nabla u)\nabla u\,dx\,dt \le C_0 && \forall k>0\,,\\
&\|g(u)|\nabla u|^2\|_{L^1(Q)} + \|u\|_{L^\infty(0,T;L^1(\Omega))} + \|u\|_{L^q(0,T;W_0^{1,q}(\Omega))} \le C_0\,,
\end{aligned}
$$

where C_0 is a positive constant depending on $|u_0|(\Omega)$ and $\|f\|_{L^1(Q)}$. Moreover, if $f\ge 0$ and $u_0\ge 0$ (in the sense of measures), then $u\ge 0$.

Proof. Let us choose $T_k(u)$ as test function in (2.6) with $S=S_n(r)$, where S_n is defined in (2.2). Then for n large enough we have that $T_k(u)=T_k(S_n(u))$, so that defining $\Theta_k(r) = \int_0^r T_k(t)\,dt$ we have:

$$\begin{aligned}\int_\Omega \Theta_k(S_n(u(\tau))\,dx &+ \int_0^\tau \int_{\{|u|\le k\}} a(x,t,\nabla u)\nabla u\,dx\,dt\\ &+ \int_0^\tau \int_\Omega g(u)S_n'(u)|\nabla u|^2\, T_k(u)\,dx\,dt\\ &\le \int_Q f\,S_n'(u)\,T_k(u)\,dx\,dt + \int_\Omega \Theta_k(S_n(u_0^r))\,dx\\ &+ \int_{\{n\le\, |u|\le n+1\}} a(x,t,\nabla u)\nabla u\,T_k(u)\,dx\,dt\,,\end{aligned}$$

for almost every τ in $(0,T)$. Using (a_1) and the sign condition (g_1) we obtain:

$$\begin{aligned}\int_\Omega \Theta_k(S_n(u(\tau))dx+\alpha\int_0^\tau\int_\Omega |\nabla T_k(u)|^2\,dx\,dt + k\int_{\{|u|\ge k\}} |g(u)||\nabla u|^2\,S_n'(u)\,dx\,dt\\ \le k\,\left(\|f\|_{L^1(Q)}+\|u_0^r\|_{L^1(\Omega)}\right) + k\int_{\{n\le\,|u|\le n+1\}} a(x,t,\nabla u)\nabla u\,dx\,dt\,,\end{aligned}$$

which yields, as n tends to infinity, thanks to (2.5) and Fatou's lemma,

$$\begin{aligned}\int_\Omega \Theta_k(u(\tau))\,dx + \alpha\int_0^\tau\int_\Omega |\nabla T_k(u)|^2\,dx\,dt + k\int_{\{|u|\ge k\}} |g(u)|\,|\nabla u|^2\,dx\,dt\\ \le k\left(\|f\|_{L^1(Q)} + |u_0|(\Omega)\right)\,,\end{aligned}\tag{2.7}$$

for almost every τ in $(0,T)$. As a consequence of (2.7) we obtain:

$$\int_\Omega \Theta_1(\tau)\,dx \le \|f\|_{L^1(Q)} + |u_0|(\Omega) \qquad \text{a.e. } \tau\in(0,T)\,,$$

$$\alpha\int_Q |\nabla T_k(u)|^2\,dx\,dt \le k\,\left(\|f\|_{L^1(Q)}+|u_0|(\Omega)\right)\,,$$

$$\begin{aligned}\int_Q |g(u)||\nabla u|^2\,dx\,dt &\le \max_{[-1,1]}|g(s)|\int_Q |\nabla T_1(u)|^2\,dx\,dt+\int_{\{|u|\ge 1\}} |g(u)||\nabla u|^2\,dx\,dt\\ &\le \left(1+\max_{[-1,1]}|g(s)|\right)\left(\|f\|_{L^1(Q)}+|u_0|(\Omega)\right).\end{aligned}$$

Thus $g(u)|\nabla u|^2$ is in $L^1(Q)$, and u is in $L^\infty(0,T;L^1(\Omega))$. Moreover the estimate on $T_k(u)$ also implies that u belongs to $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q<\frac{N+2}{N+1}$, according to the results in [ST] (see also [BDGO]). In fact, in the previous paper these regularity results are proved under the assumption that $N\ge 2$, but the case $N=1$ can be dealt with exactly as the case $N=2$, and u can still be proved to belong to $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q<\frac{N+2}{N+1}$ if $N=1$. Last estimate of Proposition 2.3 is obtained through similar arguments using $T_{k+1}(u)-T_k(u)$ as test function in (2.6).

Let now f and u_0 be positive. To see that $u \geq 0$, it is enough to take $T_k(u^-)$ as test function in (2.6) with $S = S_n$ and to use (g_1). Setting $\Theta_k^-(r) = \int_0^r T_k(s^-)\, ds$, we have:

$$\begin{aligned}\alpha \int_Q |\nabla T_k(u^-)|^2\, dx\, dt &\leq -\int_Q f\, T_k(u^-) dx\, dt + \int_{\{u\leq 0\}} g(u)|\nabla u|^2\, T_k(u^-)\, dx\, dt \\ &\quad - \int_\Omega \Theta_k^-(u_0^r)\, dx + \int_Q S_n''(u) a(x,t,\nabla u)\nabla u T_k(u^-) dx\, dt \\ &\leq k \int_{\{(x,t):\, -n-1\leq u\leq -n\}} a(x,t,\nabla u)\nabla u\, dx\, dt\,.\end{aligned}$$

Since u is a renormalized solution it satisfies (2.5), so that, letting n tend to infinity and using that $(u_0^s)^- = 0$, we get that u is positive. □

Note that Proposition 2.3 ensures that renormalized solutions have the regularity we asked in the above argument used to explain condition (2.5). Thus, using Proposition 2.3 and (2.5), the same reasoning now proves that a renormalized solution is also a weak solution. It is also easy to prove that the two concepts are in fact equivalent if f and u_0 belong respectively to $L^2(0,T; H^{-1}(\Omega))$ and to $L^2(\Omega)$.

PROPOSITION 2.4. *Every renormalized solution is a weak solution, the converse being true if f belongs to $L^2(0,T; H^{-1}(\Omega))$ and u_0 belongs to $L^2(\Omega)$.*

PROOF. We already proved that a renormalized solution is a weak solution. On the other hand, if f belongs to $L^2(0,T; H^{-1}(\Omega))$, say $f = -\mathrm{div}(F)$ with F in $L^2(Q)^N$, and u_0 is in $L^2(\Omega)$, then a weak solution u satisfies the initial condition $u(0) = u_0$ and the variational formulation:

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \int_Q a(x,t,\nabla u)\nabla v\, dx\, dt + \int_Q g(u)|\nabla u|^2\, v\, dx\, dt = \int_Q F\nabla v\, dx\, dt\,,$$

for every v in $L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q)$, where $\langle \cdot, \cdot \rangle$ denotes the duality between $H_0^1(\Omega) \cap L^\infty(\Omega)$ and $H^{-1}(\Omega) + L^1(\Omega)$. Taking $v = S'(u)\varphi$, with $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support and φ is in $C_c^\infty([0,T) \times \Omega)$ we easily see that u satisfies (2.6). Condition (2.5) is obtained choosing $v = \theta_n(u)$, where θ_n is defined in (2.2). Indeed we have, using (g_1) and integrating by parts:

$$\int_{\{n\leq |u|\leq n+1\}} a(x,t,\nabla u)\nabla u\, dx\, dt \leq \int_{\{|u_0|>n\}} |u_0| + \int_{\{n\leq |u|\leq n+1\}} |F|\,|\nabla u|\, dx\, dt\,,$$

which yields, by (a_1) and Young's inequality,

$$\frac{1}{2}\int_{\{n\leq |u|\leq n+1\}} a(x,t,\nabla u)\nabla u\, dx\, dt \leq \int_{\{|u_0|>n\}} |u_0| + c_0 \int_{\{n\leq |u|\leq n+1\}} |F|^2\, dx\, dt\,,$$

and (2.5) follows letting n go to infinity, since in this case $u_0^s = 0$. □

We are first concerned with the stability properties of renormalized solutions, which also include, as a consequence of Proposition 2.4, the study of the behaviour, as ε tends to zero, of the approximating sequence $\{u_\varepsilon\}$ of solutions of (1.2), where f_ε converges weakly to f in $L^1(Q)$ and $u_{0\varepsilon}$ converges to u_0 in what is called the narrow topology of measures, that is

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi\, du_{0\varepsilon} = \int_\Omega \varphi\, du_0\,, \qquad \forall\,\varphi \in C(\bar{\Omega})\,. \tag{2.8}$$

We will sometimes refer to (2.8) saying that $u_{0\varepsilon}$ converges tightly to u_0, which also implies the important estimate that $|u_{0\varepsilon}|(\Omega)$ is uniformly bounded with respect to ε. Under the assumptions (a_1), (a_2), (a_3), the stability properties of the renormalized solutions with respect to the data $(u_{0\varepsilon}, f_\varepsilon)$ are strongly related to the compactness of the sequence $\{T_k(u_\varepsilon)\}$ (for any fixed $k>0$) in the strong topology of the energy space $L^2(0,T;H_0^1(\Omega))$. By contrast with the case where $u_{0\varepsilon}$ is assumed to converge to u_0 strongly in $L^1(\Omega)$, the strong convergence of the truncations may be false under the only assumption (2.8). The following example, in the simplest case of the linear heat equation, shows that, even for smooth functions u_0 and $u_{0\varepsilon}$ satisfying (2.8) the strong convergence of the truncations is violated if the sequence $u_{0\varepsilon}$ does not converge to u_0 in measure on Ω.

EXAMPLE 2.5. Assume that u_0 is a positive smooth bounded function and consider an approximation of positive functions $u_{0\varepsilon}$ such that $\{u_{0\varepsilon}\}$ converges to u_0 in the sense of (2.8) but $u_{0\varepsilon}$ does not converge to u_0 in measure. It is quite easy to construct a similar example, for instance the sequence $u_{0\varepsilon} = \sum_{j=1}^{\frac{1}{\varepsilon}} \frac{1}{\varepsilon^2}\chi_{(\varepsilon j-\varepsilon^3,\varepsilon j)}$, where $\varepsilon=\varepsilon_n=\frac{1}{n}$, converges tightly to 1 but it converges to zero in measure.

Let us consider the solutions u_ε of the heat equation:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = 0 & \text{in } Q\,,\\ u_\varepsilon = 0 & \text{on } \Sigma\,,\\ u_\varepsilon(0) = u_{0\varepsilon} & \text{in } \Omega\,. \end{cases}$$

By standard linear theory (see also [BG]) we have that u_ε strongly converges in $L^1(Q)$ to the unique solution u of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{in } Q\,,\\ u = 0 & \text{on } \Sigma\,,\\ u(0) = u_0 & \text{in } \Omega\,, \end{cases}$$

which is a bounded smooth function on Q. Suppose now that the sequence $T_k(u_\varepsilon)$ strongly converges to $T_k(u)$ in $L^2(0,T;H_0^1(\Omega))$ for every $k>0$, and take an auxiliary function of real variable $\mathcal{T}_k(s)$ which is a smooth truncation, in the sense that $\mathcal{T}_k(s)$ is a C^2 odd function such that $\mathcal{T}_k(s) = s$ if $|s| \le \frac{k}{2}$, and $\mathcal{T}_k(s) = k$ if $s > k$. Then the function $\mathcal{T}_k(u_\varepsilon)$ solves the equation:

$$\begin{cases} \dfrac{\partial \mathcal{T}_k(u_\varepsilon)}{\partial t} - \Delta \mathcal{T}_k(u_\varepsilon) + \mathcal{T}_k''(u_\varepsilon)|\nabla u_\varepsilon|^2 = 0 & \text{in } Q, \\ \mathcal{T}_k(u_\varepsilon) = 0 & \text{on } \Sigma, \\ \mathcal{T}_k(u_\varepsilon)(0) = \mathcal{T}_k(u_{0\varepsilon}) & \text{in } \Omega, \end{cases}$$

and by the strong convergence of truncations we deduce (recall that $\mathcal{T}_k''$ has compact support) that $\{\mathcal{T}_k(u_\varepsilon)\}$ is strongly convergent in $L^2(0,T;\, H_0^1(\Omega))$ and $\frac{\partial \mathcal{T}_k(u_\varepsilon)}{\partial t}$ is strongly convergent in the space $L^1(Q) + L^2(0,T;\, H^{-1}(\Omega))$. But the space $W = \{u \in L^2(0,T;\, H_0^1(\Omega)),\, \frac{\partial u}{\partial t} \in L^1(Q) + L^2(0,T;\, H^{-1}(\Omega))\}$ with its natural norm continuously injects into $C([0,T];\, L^1(\Omega))$ (see [P]), which implies that the sequence $\mathcal{T}_k(u_{0\varepsilon})$ strongly converges in $L^1(\Omega)$ to $\mathcal{T}_k(u_0)$. Since for k large we have $\mathcal{T}_k(u_0) = u_0$, then for any $\sigma > 0$

$$\{x \,:\, |u_{0\varepsilon} - u_0| > \sigma\} \subset \{x \,:\, u_{0\varepsilon} > k\} \cup \{x \,:\, |\mathcal{T}_k(u_{0\varepsilon}) - \mathcal{T}_k(u_0)| > \sigma\},$$

which yields

$$\operatorname{meas}\{x \,:\, |u_{0\varepsilon} - u_0| > \sigma\} \le \frac{1}{k}\|u_{0\varepsilon}\|_{L^1(\Omega)} + \operatorname{meas}\{x \,:\, |\mathcal{T}_k(u_{0\varepsilon}) - \mathcal{T}_k(u_0)| > \sigma\}.$$

Since $u_{0\varepsilon}$ is bounded in $L^1(\Omega)$ and $\mathcal{T}_k(u_{0\varepsilon})$ converges to $\mathcal{T}_k(u_0)$ in measure, letting first ε tend to zero, then k go to infinity, we conclude that $u_{0\varepsilon}$ converges to u_0 in measure, getting a contradiction, so the sequence of truncations can not converge strongly in $L^2(0,T;\, H_0^1(\Omega))$. □

The previous example shows that the strong convergence of truncations can not in general be expected for every choice of the approximating sequence $u_{0\varepsilon}$ tightly converging to u_0, but in some sense we need an approximation of u_0 which is consistent with its Lebesgue decomposition into u_0^r and u_0^s, since the L^1 part should be approximated in the strong topology of $L^1(\Omega)$. Moreover to simplify a few technical arguments we will restrict the whole analysis to the case of positive data f_ε and $u_{0\varepsilon}$, which as a consequence of Proposition 2.3 implies that we deal with positive solutions u_ε of (1.1). Thus we consider a sequence $\{u_{0\varepsilon}\} \subset \mathcal{M}_b^+(\Omega)$ such that:

$$\begin{aligned} & u_{0\varepsilon} = u_{0\varepsilon}^r + u_{0\varepsilon}^s = \mu_\varepsilon^1 + \mu_\varepsilon^2 + u_{0\varepsilon}^s, \\ & u_{0\varepsilon}^r \in L^1(\Omega), \quad u_{0\varepsilon}^s = u_{0\varepsilon} \llcorner E_\varepsilon \text{ with } \operatorname{meas}(E_\varepsilon) = 0, \\ \mu_\varepsilon^1 \ge 0, \quad & \mu_\varepsilon^1 \to u_0^r \quad \text{strongly in } L^1(\Omega), \\ \mu_\varepsilon^2 \ge 0, \quad & \mu_\varepsilon^2 + u_{0\varepsilon}^s \to u_0^s \quad \text{in the sense of (2.8).} \end{aligned} \tag{2.9}$$

We also take a sequence of positive functions $\{f_\varepsilon\} \subset L^1(Q)$ weakly convergent to f in $L^1(Q)$, and we study the stability of the sequence $\{u_\varepsilon\}$ of renormalized solutions of the Cauchy-Dirichlet problems:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(a(x,t,\nabla u_\varepsilon)) + g(u_\varepsilon)|\nabla u_\varepsilon|^2 = f_\varepsilon & \text{in } Q\,, \\ u_\varepsilon = 0 & \text{on } \Sigma\,, \\ u_\varepsilon(0) = u_{0\varepsilon} & \text{in } \Omega\,. \end{cases} \tag{2.10}$$

If f_ε and $u_{0\varepsilon}$ are, for instance, bounded functions, we recover from next theorem the study of (1.2); in fact we allow f_ε to belong only to $L^1(Q)$ and $u_{0\varepsilon}$ to be a general measure also containing a nonzero singular part $u_{0\varepsilon}^s$ in its Lebesgue decomposition, so that we consider the cases that u_0^s is approximated both with singular measures and with L^1 functions. Our main stability result is the following, which is new even for $g \equiv 0$, and which shows how the integrability of $g(s)$ plays a decisive role in the perturbed equation.

THEOREM 2.6. *Let $\{f_\varepsilon\} \subset L^1(Q)$ be a sequence of positive functions weakly converging to f in $L^1(Q)$ and let $\{u_{0\varepsilon}\} \subset \mathcal{M}_b^+(\Omega)$ tightly converge to u_0 in the sense of* (2.8) *and satisfying* (2.9). *Let u_ε be renormalized solutions of* (2.10) *in the sense of Definition* 2.2. *Then there exist a measurable function u, and a subsequence still indexed by ε, such that*

$$T_k(u_\varepsilon) \to T_k(u) \quad \textit{strongly in } L^2(0,T;H_0^1(\Omega)) \textit{ for every } k > 0\,.$$

Moreover we have:

(i) *if $\int_0^{+\infty} g(s)ds < +\infty$ then:*

$$g(u_\varepsilon)|\nabla u_\varepsilon|^2 \to g(u)|\nabla u|^2 \quad \textit{strongly in } L^1(Q)$$

$$\lim_{n\to\infty} \int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt = \int_\Omega \varphi(0)\,du_0^s \qquad \forall \varphi \in C(\bar{Q})\,,$$

and u is a renormalized solution of (1.1).

(ii) *if $\int_0^{+\infty} g(s)ds = +\infty$ then:*

$$\lim_{\varepsilon\to 0} \int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,dx\,dt = \int_Q g(u)|\nabla u|^2\,\varphi\,dx\,dt + \int_\Omega \varphi(0)\,du_0^s \qquad \forall \varphi \in C(\bar{Q})\,,$$

$$\lim_{n\to\infty} \int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\;dx\,dt = 0\,,$$

and u is a renormalized solution of problem (1.1) *with initial datum u_0^r.* □

REMARK 2.7. Last conclusion of Theorem 2.6 says that, if $u_0 \in \mathcal{M}_b^+(\Omega)$ is concentrated on a set of zero Lebesgue measure (like, for instance, the Dirac mass), then every sequence of renormalized solutions u_ε of (2.10) admits a subsequence converging to a renormalized solution u of (1.1) with zero initial datum, since in this case $u_0^r = 0$. If also $f = 0$, it can be easily proved that $u = 0$ is the only possible limit function u, since it is the only renormalized solution of (1.1) with zero data (this also follows from Proposition 2.3, since u should be positive and negative at the same time). Thus in the case that $f = 0$ and u_0 is singular with respect to the Lebesgue measure we deduce, under the assumption (ii) of Theorem 2.6, that every sequence of renormalized solutions of (2.10) converges to zero. In particular this applies to every sequence of solutions of the regular problems (1.2) defined through convolution of the data. □

The proof of Theorem 2.6 is rather technical and will be achieved in several steps in Section 3. In the case $\int_0^{+\infty} g(s)ds < +\infty$, Theorem 2.6 leads to our main result, which extends those proved in [P] with L^1 data, and which is new even for $g \equiv 0$ as far as renormalized solutions are concerned.

THEOREM 2.8. *Let assumptions $(a_1) - (a_3)$ and (g_1) be satisfied, and moreover assume that $\int_0^{+\infty} g(s)ds < +\infty$. Let $f \in L^1(Q)$, $f \geq 0$, and let $u_0 \in \mathcal{M}_b^+(\Omega)$. Then there exists a renormalized solution of* (1.1).

PROOF. Choosing two sequences $\{f_\varepsilon\}$ and $\{u_{0\varepsilon}\}$ of bounded functions satisfying the assumptions of Theorem 2.6 as ε tends to zero (for instance, it is enough to take a standard convolution of f in Q and u_0 in Ω), then by the results in [BMP2] there exist weak solutions u_ε of (2.10), which are also renormalized solutions thanks to Proposition 2.4. Then it is enough to apply Theorem 2.6, (i), to conclude. □

In the case where $\int_0^{+\infty} g(s)ds = +\infty$, the conclusion of Theorem 2.6 shows that there is no stability with respect to initial data except for $L^1(\Omega)$ data. This result is emphasized, in some sense, by the following theorem.

THEOREM 2.9. *Assume that (a_1)-(a_3) and (g_1) hold true, and that $\int_0^{+\infty} g(s)ds = +\infty$. Let $f \in L^1(Q)$, $f \geq 0$, and let $u_0 \in \mathcal{M}_b^+(\Omega)$. Then there exists a renormalized solution of* (1.1) *if and only if u_0 is in $L^1(\Omega)$.* □

REMARK 2.10. Note that Theorem 2.9 does not say anything on the nonexistence of weak solutions under the assumption that $\int_0^{+\infty} g(s)ds = +\infty$, so that this problem is still open. Nevertheless, as a consequence of Theorem 2.6, under this condition on g, weak solutions, even if they exist, are certainly not stable with respect to the convergence of the data as considered above. □

REMARK 2.11. With the same techniques and very close proofs, similar results can be obtained for the general problem

$$
(2.11) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{in } Q\,, \\ u = 0 & \text{on } \Sigma\,, \\ u(0) = u_0 & \text{in } \Omega\,, \end{cases}
$$

where $a(x,t,s,\xi)$ satisfies, for every s in $\mathbb{R}$, every ξ, η ($\xi \neq \eta$) in $\mathbb{R}^N$ and almost every (x,t) in Q:

$$\begin{aligned}
&a(x,t,s,\xi)\cdot\xi \geq \alpha|\xi|^2, \quad \alpha > 0,\\
&|a(x,t,s,\xi)| \leq \beta(k(x,t)+|\xi|), \quad \beta > 0 \quad k \in L^2(Q),\\
&(a(x,t,s,\xi)-a(x,t,s,\eta))\cdot(\xi-\eta) > 0,
\end{aligned}$$

and $H(x,t,s,\xi)$ satisfies, for every s in $\mathbb{R}$, every ξ in $\mathbb{R}^N$ and almost every (x,t) in Q:

$$H(x,t,s,\xi)s \geq 0,$$

and

$$|H(x,t,s,\xi)| \leq h(x,t)+g(s)|\xi|^2 \quad h(x,t) \in L^1(Q),\ g : \mathbb{R} \to \mathbb{R}^+ \text{ continuous}.$$

In particular, under the assumption that $\int_0^{+\infty} g(s)ds < +\infty$, for every $f \in L^1(Q)$, $f \geq 0$, and for every $u_0 \in \mathcal{M}_b^+(\Omega)$ it is possible to find a renormalized solution of problem (2.11). On the other hand, the results of Theorem 2.6 in the case (ii), and those of Theorem 2.9, can be generalized to (2.11) if $a(x,t,s,\xi)$ satisfies

$$|a(x,t,s,\xi)| \leq \beta|\xi|, \quad \beta > 0 \quad \text{for every } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N \text{ and a.e. } (x,t) \in Q,$$

and $H(x,t,s,\xi)$ satisfies, for every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ and almost every $(x,t) \in Q$,

$$\exists\, L > 0 : \quad H(x,t,s,\xi)\mathrm{sign(s)} \geq \mathrm{g(s)}|\xi|^2 \quad \text{for every } s \in \mathbb{R}\text{: } |s| \geq L,$$

with $\int_L^{+\infty} g(s)ds = +\infty$. □

3. – Proof of the results

We start the proof of Theorem 2.6 with the following lemma on the *a priori* estimates obtained on the renormalized solutions.

Lemma 3.1. *Let u_ε be renormalized solutions of* (2.10), *with $\{f_\varepsilon\}$ and $\{u_{0\varepsilon}\}$ satisfying the assumptions of Theorem* 2.6. *Then there exists a positive constant C, not depending on ε, and a positive constant C_k, which depends on k but not on ε, such that the following estimates hold true:*

$$\begin{aligned}
&\|T_k(u_\varepsilon)\|^2_{L^2(0,T;H^1_0(\Omega))} \leq kC && \forall k > 0,\\
&\|g(u_\varepsilon)|\nabla u_\varepsilon|^2\|_{L^1(Q)}+\|u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}+\|u_\varepsilon\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C && \forall q < \frac{N+2}{N+1},\\
&\|a(x,t,\nabla u_\varepsilon)\|_{L^q(\Omega)^N} \leq C && \forall q < \frac{N+2}{N+1},\\
&\|a(x,t,\nabla T_k(u_\varepsilon))\|_{L^2(Q)^N} \leq C_k && \forall k > 0.
\end{aligned}$$

Moreover there exist a subsequence, still indexed by ε, and a measurable function u belonging to $L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega))$ for every $q < \frac{N+2}{N+1}$ such that $T_k(u)$ belongs to $L^2(0,T;H_0^1(\Omega))$ for every $k > 0$ and:

$$
\begin{aligned}
&u_\varepsilon \to u \quad \text{weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \text{ , strongly in } L^1(Q) \text{ and a.e. in } Q,\\
&T_k(u_\varepsilon) \to T_k(u) \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)) \text{ and a.e. in } Q,\\
&a(x,t,\nabla T_k(u_\varepsilon)) \to \sigma_k \quad \text{weakly in } L^2(Q)^N \text{ for every } k>0\,,\\
&a(x,t,\nabla u_\varepsilon)) \to \sigma \quad \text{weakly in } L^q(\Omega)^N \text{ for every } q < \frac{N+2}{N+1}\,,
\end{aligned}
\tag{3.1}
$$

where σ_k belongs to $L^2(Q)^N$ and σ belongs to $L^q(\Omega)^N$ for every $q < \frac{N+2}{N+1}$.

PROOF. Since $\|f_\varepsilon\|_{L^1(Q)}$ and $\|u_{0\varepsilon}\|_{L^1(\Omega)}$ are uniformly bounded on ε, the estimates follow from Proposition 2.3. and from the growth assumption (a_2). The almost everywhere and the strong convergence of u_ε in $L^1(Q)$ are standard results (see [BG], [BlM] for example). □

In order to show that the sequence of truncations $\{T_k(u_\varepsilon)\}$ is in fact strongly compact in $L^2(0,T;H_0^1(\Omega))$, we make use of different techniques already employed in similar contexts (see [BlMR], [P]). First of all, we introduce the auxiliary function of real variable $\varphi_\rho(s) = se^{\rho s^2}$, which was first used in [BMP] to deal with hamiltonian terms having quadratic growth with respect to the gradient. Indeed, an essential role will be played by the following property enjoyed by $\varphi_\rho(s)$ (the proof is trivial):

$$
a\varphi_\rho'(s) - b|\varphi_\rho(s)| \ge \frac{a}{2}\,, \qquad \forall a,b>0\,, \qquad \forall \rho : \rho \ge \frac{b^2}{4a^2}\,. \tag{3.2}
$$

We also need to recall the following definition of a time-regularization of $T_k(u)$, which was first introduced in [La], then used in several papers afterwards (see [DO], [BDGO], [P], [BlMR]). Let z_ν be a sequence of functions such that:

$$
\begin{aligned}
&z_\nu \in H_0^1(\Omega)\cap L^\infty(\Omega)\,, && \|z_\nu\|_{L^\infty(\Omega)} \le k\,,\\
&z_\nu \to T_k(u_0^r) && \text{a.e. in } \Omega \text{ as } \nu \text{ tends to infinity}\,,\\
&\frac{1}{\nu}\|z_\nu\|^2_{H_0^1(\Omega)} \to 0 && \text{as } \nu \text{ tends to infinity}\,.
\end{aligned}
\tag{3.3}
$$

Existence of such a sequence z_ν is easy to establish. Then, for fixed $k>0$, and $\nu>0$, we denote by $T_k(u)_\nu$ the unique solution of the problem

$$
\begin{cases}
\dfrac{\partial T_k(u)_\nu}{\partial t} = \nu(T_k(u) - T_k(u)_\nu) & \text{in the sense of distributions}\,,\\
T_k(u)_\nu(0) = z_\nu & \text{in } \Omega\,.
\end{cases}
\tag{3.4}
$$

Then we have that $T_k(u)_\nu$ belongs to $L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$ and $\frac{\partial T_k(u)_\nu}{\partial t}$ belongs to $L^2(0,T;H_0^1(\Omega))$, and it can be easily proved (see also [La]) that

$$
(3.5)\qquad \begin{array}{ll} T_k(u)_\nu \to T_k(u) & \text{strongly in } L^2(0,T;H_0^1(\Omega)), \text{ a.e. in } Q, \\ & \text{and } *\text{-weakly in } L^\infty(Q) \\ \|T_k(u)_\nu\|_{L^\infty(Q)} \le k & \forall \nu > 0\,. \end{array}
$$

In order to deal with the singular term u_0^s in the initial datum, we adapt an idea of [DMOP]. Namely, we consider a sequence of compact subsets $K_\delta \subset E$ (recall that $u_0^s = u_0 \llcorner E$) such that

$$
(3.6)\qquad u_0^s(E\setminus K_\delta) < \delta\,,
$$

and then a sequence of functions ψ_δ belonging to $C_c^\infty(\Omega)$ and satisfying:

$$
(3.7)\qquad \begin{array}{c} 0 \le \psi_\delta \le 1\,, \quad \psi_\delta \equiv 1 \quad \text{on} \quad K_\delta\,, \\ \psi_\delta \to 0 \quad \text{a.e. in } \Omega, \text{ as } \delta \text{ tends to zero}, \end{array}
$$

which also implies that ψ_δ converges to zero weakly-$*$ in $L^\infty(\Omega)$. Then we define v_δ as the solution of the heat equation:

$$
(3.8)\qquad \begin{cases} \dfrac{\partial v_\delta}{\partial t} - \Delta v_\delta = 0 & \text{in } Q\,, \\ v_\delta = 0 & \text{on } \Sigma\,, \\ v_\delta(0) = \psi_\delta & \text{in } \Omega\,. \end{cases}
$$

It can be easily seen that, for fixed $\delta > 0$, v_δ is a smooth function on $\bar{Q}$ such that $0 \le v_\delta \le 1$, and moreover, as δ tends to zero,

$$
(3.9)\qquad \begin{array}{ll} v_\delta \to 0 & \text{strongly in } L^2(0,T;H_0^1(\Omega))\,, \\ \dfrac{\partial v_\delta}{\partial t} \to 0 & \text{strongly in } L^2(0,T;H^{-1}(\Omega))\,. \end{array}
$$

We are now ready to prove our result.

PROOF OF THEOREM 2.6.

1. *Proof of the strong convergence of truncations.* Throughout the proof, we will always make use of the renormalized formulation of (2.10) written with $S = S_n$ as defined in (2.2), that is

$$
(3.10)\qquad \begin{aligned} \frac{\partial S_n(u_\varepsilon)}{\partial t} &- \operatorname{div}(a(x,t,\nabla u_\varepsilon)S_n'(u_\varepsilon)) + S_n''(u_\varepsilon)a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon \\ &+ g(u_\varepsilon)|\nabla u_\varepsilon|^2\, S_n'(u_\varepsilon) = f_\varepsilon\, S_n'(u_\varepsilon)\,. \end{aligned}
$$

Our goal will be to prove the asymptotic estimate:

$$
(3.11)\qquad \lim_{\varepsilon\to 0}\int_Q \{a(x,t,\nabla T_k(u_\varepsilon)) - a(x,t,\nabla T_k(u))\}\,\nabla\,(T_k(u_\varepsilon) - T_k(u))\,dx\,dt = 0\,,
$$

for every $k > 0$, since then the strict monotonicity assumption (a_3) implies that $T_k(u_\varepsilon)$ strongly converges to $T_k(u)$ in $L^2(0,T;H_0^1(\Omega))$ (see for instance Lemma 5 in [BMP]). In order to get (3.11), we proceed in several steps.

STEP 1. Let v_δ and $\varphi_\rho(s)$ be defined as above. We take $(T-t)\varphi_\rho((k-u_\varepsilon)^+)v_\delta$ as test function in (3.10). Since $\varphi_\rho((k-u_\varepsilon)^+) \equiv 0$ if $u_\varepsilon > k$, choosing n large leads to $\varphi_\rho((k-u_\varepsilon)^+) = \varphi_\rho((k-S_n(u_\varepsilon))^+) = \varphi_\rho((k-u_\varepsilon)^+)S_n'(u_\varepsilon)$ and $S_n''(u_\varepsilon)\varphi_\rho((k-u_\varepsilon)^+) \equiv 0$. Then, setting $\Phi_k(s) = \int_0^s \varphi_\rho((k-t)^+)\,dt$ and integrating by parts we get:

$$
\begin{aligned}
&-T\int_\Omega \Phi_k(u_{0\varepsilon}^r)\psi_\delta\,dx + \int_Q \Phi_k(u_\varepsilon)v_\delta\,dx\,dt - \int_Q (T-t)\Phi_k(u_\varepsilon)\frac{\partial v_\delta}{\partial t}\,dx\,dt \\
&+\int_Q (T-t)a(x,t,\nabla u_\varepsilon)\nabla v_\delta\,\varphi_\rho((k-u_\varepsilon)^+)\,dx\,dt \\
&-\int_Q (T-t)a(x,t,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon)\,\varphi_\rho'((k-u_\varepsilon)^+)\,v_\delta\,dx\,dt \\
&+\int_Q (T-t)g(u_\varepsilon)|\nabla u_\varepsilon|^2\,\varphi_\rho((k-u_\varepsilon)^+)v_\delta\,dx\,dt \\
=&\int_Q (T-t)f_\varepsilon\,\varphi_\rho((k-u_\varepsilon)^+)v_\delta\,dx\,dt\,.
\end{aligned}
$$

Using the fact that $\varphi_\rho((k-u_\varepsilon)^+) = 0$ if $u_\varepsilon > k$, we can replace u_ε with $T_k(u_\varepsilon)$ in all the integrals above; setting $M_k = \max_{[0,k]} g(s)$ we get, using also (a_1):

$$
\begin{aligned}
&T\int_\Omega \Phi_k(u_{0\varepsilon}^r)\psi_\delta\,dx + \alpha\int_Q (T-t)|\nabla T_k(u_\varepsilon)|^2\,\varphi_\rho'((k-u_\varepsilon)^+)\,v_\delta\,dx\,dt \\
&\le M_k\int_Q (T-t)|\nabla T_k(u_\varepsilon)|^2\,\varphi_\rho((k-u_\varepsilon)^+)v_\delta\,dx\,dt \\
&\quad+\int_Q \Phi_k(u_\varepsilon)v_\delta\,dx\,dt - \int_Q (T-t)\Phi_k(u_\varepsilon)\frac{\partial v_\delta}{\partial t}\,dx\,dt \\
&\quad-\int_Q (T-t)f_\varepsilon\,\varphi_\rho((k-u_\varepsilon)^+)v_\delta\,dx\,dt \\
&\quad+\int_Q (T-t)a(x,t,\nabla T_k(u_\varepsilon))\nabla v_\delta\,\varphi_\rho((k-u_\varepsilon)^+)\,dx\,dt\,,
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\int_Q (T-t)|\nabla T_k(u_\varepsilon)|^2\,v_\delta[\alpha\varphi_\rho'((k-u_\varepsilon)^+) - M_k\varphi_\rho((k-u_\varepsilon)^+)]\,dx\,dt \\
&\quad+T\int_\Omega \Phi_k(u_{0\varepsilon}^r)\psi_\delta\,dx \le \int_Q \Phi_k(u_\varepsilon)v_\delta\,dx\,dt - \int_Q (T-t)\Phi_k(u_\varepsilon)\frac{\partial v_\delta}{\partial t}\,dx\,dt \\
&\quad+\int_Q (T-t)a(x,t,\nabla T_k(u_\varepsilon))\nabla v_\delta\,\varphi_\rho((k-u_\varepsilon)^+)\,dx\,dt \\
&\quad-\int_Q (T-t)f_\varepsilon\,\varphi_\rho((k-u_\varepsilon)^+)v_\delta\,dx\,dt\,.
\end{aligned}
$$

Now we use property (3.2) with $a = \alpha$ and $b = M_k$, so that for ρ sufficiently large we obtain:

$$
\begin{aligned}
&T \int_\Omega \Phi_k(u_{0\varepsilon}^r)\psi_\delta \, dx + \frac{\alpha}{2} \int_Q (T-t)|\nabla T_k(u_\varepsilon)|^2 \, v_\delta \, dx\, dt \\
&\leq \int_Q \Phi_k(u_\varepsilon) v_\delta \, dx\, dt - \int_Q (T-t)\Phi_k(u_\varepsilon)\frac{\partial v_\delta}{\partial t}\, dx\, dt \\
&\quad + \int_Q (T-t) a(x,t,\nabla T_k(u_\varepsilon))\nabla v_\delta \, \varphi_\rho((k-u_\varepsilon)^+)\, dx\, dt \\
&\quad - \int_Q (T-t) f_\varepsilon \, \varphi_\rho((k-u_\varepsilon)^+) v_\delta \, dx\, dt .
\end{aligned} \tag{3.12}
$$

Since by (3.1) $T_k(u_\varepsilon)$ weakly converges in $L^2(0,T; H_0^1(\Omega))$ and $\Phi_k'(s) = \varphi_\rho((k-s)^+)$, which implies that $\Phi_k(s)$ is a bounded function with a compactly supported derivative, and $\Phi_k(s) \geq 0$ if $s \geq 0$, we have that

$$
\begin{aligned}
\Phi_k(u_\varepsilon) \to \Phi_k(u) \ \text{weakly in } L^2(0,T; H_0^1(\Omega)),\\
\text{weakly-}* \text{ in } L^\infty(Q) \text{ and a.e. in } Q .
\end{aligned}
$$

Using also the weak convergence in $L^2(Q)$ of the sequence $a(x,t,\nabla T_k(u_\varepsilon))$ allows then to pass to the limit as ε tends to zero in (3.12) to obtain:

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \frac{\alpha}{2} \int_Q (T-t)|\nabla T_k(u_\varepsilon)|^2 \, v_\delta \, dx\, dt \\
&\leq \int_Q \Phi_k(u) v_\delta \, dx\, dt - \int_Q (T-t)\Phi_k(u)\frac{\partial v_\delta}{\partial t}\, dx\, dt \\
&\quad + \int_Q (T-t)\sigma_k \nabla v_\delta \, \varphi_\rho((k-u)^+)\, dx\, dt \\
&\quad - \int_Q (T-t) f \, \varphi_\rho((k-u)^+) v_\delta \, dx\, dt .
\end{aligned} \tag{3.13}
$$

Due to the convergences properties of v_δ in (3.9) and to the fact that $\Phi_k(u)$ belongs to $L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q)$ and σ_k belongs to $L^2(Q)^N$, we conclude that:

$$
\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \int_Q (T-t)|\nabla T_k(u_\varepsilon)|^2 \, v_\delta \, dx\, dt = 0 . \tag{3.14}
$$

On the other hand, recall that v_δ solves (3.8), so by the properties of the heat equation we have $\|v_\delta(t+t_0)\|_{L^\infty(\Omega)} \leq c_0 t_0^{-\frac{N}{2}} \|\psi_\delta\|_{L^1(\Omega)}$ for every t_0 in $(0,T)$ and every t in $(0, T-t_0)$, so that we have:

$$
\begin{aligned}
\int_Q |\nabla T_k(u_\varepsilon)|^2 \, v_\delta \, dx\, dt &\leq \frac{2}{T} \int_0^{\frac{T}{2}} \int_\Omega (T-t)|\nabla T_k(u_\varepsilon)|^2 \, v_\delta \, dx\, dt \\
&\quad + c_0 \left(\frac{T}{2}\right)^{-\frac{N}{2}} \|\psi_\delta\|_{L^1(\Omega)} \int_Q |\nabla T_k(u_\varepsilon)|^2 \, dx\, dt ,
\end{aligned}
$$

which yields, by Lemma 3.1,

$$\int_Q |\nabla T_k(u_\varepsilon)|^2 v_\delta \, dx\, dt \le \frac{2}{T}\int_Q (T-t)|\nabla T_k(u_\varepsilon)|^2 v_\delta \, dx\, dt + k\left(\frac{T}{2}\right)^{-\frac{N}{2}} c_0 \|\psi_\delta\|_{L^1(\Omega)}.$$

Then by (3.7) and (3.14) we obtain:

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \int_Q |\nabla T_k(u_\varepsilon)|^2 \, v_\delta \, dx\, dt = 0. \tag{3.15}$$

STEP 2. Hereafter, we study the behaviour of the sequence $\{\nabla T_k(u_\varepsilon)\}$ in that part of the cylinder Q which is far from the support of the singular measure u_0^s. This is done, following the idea of [DMOP], by localizing in some sense equation (3.10) through multiplication of each test function by $1 - v_\delta$. Apart from this localization procedure, we take advantage of some methods already used to get strong convergence of truncations, see [P] and [BlMR], in particular we prove the estimate (3.11) separately reasoning on the positive and negative parts of $T_k(u_\varepsilon) - T_k(u)$. To perform this task in Step 3 and Step 4, the present step is devoted to establish the preliminary essential estimate:

$$\lim_{\delta\to 0} \limsup_{h\to\infty} \limsup_{\varepsilon\to 0} \int_{\{h\le u_\varepsilon\le h+1\}} a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon (1 - v_\delta) \, dx\, dt = 0. \tag{3.16}$$

In order to obtain (3.16), we take $\theta_h(S_n(u_\varepsilon))(1 - v_\delta)$ as test function in (3.10), where θ_h is defined in (2.2), and we get, integrating by parts,

$$\begin{aligned}
&\int_{\{h\le S_n(u_\varepsilon)\le h+1\}} a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon S_n'(u_\varepsilon)(1 - v_\delta)\, dx\, dt \\
&\quad + \int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2 \theta_h(S_n(u_\varepsilon))S_n'(u_\varepsilon)\,(1 - v_\delta)\, dx\, dt \\
&\le \int_Q f_\varepsilon\, \theta_h(S_n(u_\varepsilon))(1 - v_\delta)S_n'(u_\varepsilon)\, dx\, dt - \int_Q \frac{\partial v_\delta}{\partial t}\int_0^{S_n(u_\varepsilon)} \theta_h(r)\, dr\, dx\, dt \\
&\quad + \int_Q S_n'(u_\varepsilon)\theta_h(S_n(u_\varepsilon))a(x,t,\nabla u_\varepsilon)\nabla v_\delta\, dx\, dt \\
&\quad + \int_\Omega \int_0^{S_n(u_{0\varepsilon}^r)} \theta_h(r)dr\,(1 - \psi_\delta)\, dx \\
&\quad + \int_{\{n\le u_\varepsilon\le n+1\}} a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon\, \theta_h(S_n(u_\varepsilon))(1 - v_\delta)\, dx\, dt.
\end{aligned}$$

Therefore, as n tends to infinity, using condition (2.5) for renormalized solutions, Lebesgue's theorem and Lemma 3.1 we obtain:

$$
\begin{aligned}
&\int_{\{h\le u_\varepsilon\le h+1\}} a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon(1-v_\delta)\,dx\,dt \\
&\quad+\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\,\theta_h(u_\varepsilon)\,(1-v_\delta)\,dx\,dt \\
&\le \int_Q f_\varepsilon\theta_h(u_\varepsilon)(1-v_\delta)dx\,dt-\int_Q\frac{\partial v_\delta}{\partial t}\int_0^{u_\varepsilon}\theta_h(r)dr\,dx\,dt \\
&\quad+\int_Q a(x,t,\nabla u_\varepsilon)\nabla v_\delta\theta_h(u_\varepsilon)dx\,dt \\
&\quad+\int_\Omega\int_0^{u_{0\varepsilon}^r}\theta_h(r)dr(1-\psi_\delta)dx+\int_\Omega(1-\psi_\delta)du_{0\varepsilon}^s\,.
\end{aligned}\tag{3.17}
$$

Now since $\mu_\varepsilon^2\ge 0$, so that $u_{0\varepsilon}^r\ge\mu_\varepsilon^1$, we have:

$$
\begin{aligned}
\int_\Omega\int_0^{u_{0\varepsilon}^r}\theta_h(r)dr(1-\psi_\delta)dx&\le\int_\Omega\int_0^{\mu_\varepsilon^1}\theta_h(r)dr(1-\psi_\delta)dx+\int_\Omega\int_{\mu_\varepsilon^1}^{u_{0\varepsilon}^r}\theta_h(r)dr(1-\psi_\delta)dx \\
&\le\int_\Omega\mu_\varepsilon^1\chi_{\{\mu_\varepsilon^1>h\}}\,dx+\int_\Omega(u_{0\varepsilon}^r-\mu_\varepsilon^1)(1-\psi_\delta)\,dx\,,
\end{aligned}
$$

It follows then from (3.17), using also (g_1):

$$
\begin{aligned}
&\int_{\{h\le u_\varepsilon\le h+1\}} a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon(1-v_\delta)dx\,dt \\
&\quad+\int_{\{u_\varepsilon>h+1\}} g(u_\varepsilon)|\nabla u_\varepsilon|^2(1-v_\delta)dx\,dt \\
&\le \int_Q f_\varepsilon\theta_h(u_\varepsilon)(1-v_\delta)dx\,dt-\int_Q\frac{\partial v_\delta}{\partial t}\int_0^{u_\varepsilon}\theta_h(r)dr\,dx\,dt \\
&\quad+\int_\Omega(1-\psi_\delta)du_{0\varepsilon}^s+\int_\Omega\mu_\varepsilon^1\chi_{\{\mu_\varepsilon^1>h\}}dx+\int_\Omega(1-\psi_\delta)d\mu_\varepsilon^2 \\
&\quad+\int_Q a(x,t,\nabla u_\varepsilon)\nabla v_\delta\,\theta_h(u_\varepsilon)dx\,dt\,.
\end{aligned}\tag{3.18}
$$

Since

$$
\begin{aligned}
&\int_Q f_\varepsilon\theta_h(u_\varepsilon)(1-v_\delta)dx\,dt-\int_Q\frac{\partial v_\delta}{\partial t}\int_0^{u_\varepsilon}\theta_h(r)dr\,dx\,dt \\
&\quad\le\int_{\{u_\varepsilon\ge h\}}\left(|f_\varepsilon|+\left|\frac{\partial v_\delta}{\partial t}\right|\,|u_\varepsilon|\right)dx\,dt\,,
\end{aligned}
$$

using the fact that u_ε strongly converges in $L^1(Q)$ and that $\frac{\partial v_\delta}{\partial t}$ is bounded in Q (for fixed δ) we obtain:

$$
\lim_{h\to\infty}\limsup_{\varepsilon\to 0}\left[\int_Q f_\varepsilon\,\theta_h(u_\varepsilon)(1-v_\delta)dx\,dt-\int_Q\frac{\partial v_\delta}{\partial t}\int_0^{u_\varepsilon}\theta_h(r)dr\,dx\,dt\right]=0\,.
$$

Moreover, due to the weak convergence of $a(x,t,\nabla u_\varepsilon)$ in (3.1) and to the smoothness of v_δ, we have:

$$\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u_\varepsilon)\nabla v_\delta\,\theta_h(u_\varepsilon)dx\,dt = \int_Q \sigma\nabla v_\delta\,\theta_h(u)dx\,dt\,,$$

so that, since σ belongs to $L^q(Q)^N$ for every $q<\frac{N+2}{N+1}$ and $\theta_h(u)$ tends to zero in the weak-$*$ topology of $L^\infty(Q)$, letting h tend to infinity yields

$$\lim_{h\to\infty}\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u_\varepsilon)\nabla v_\delta\,\theta_h(u_\varepsilon)dx\,dt = 0\,.$$

The above results, together with the fact that μ_ε^1 strongly converges in $L^1(\Omega)$ while $\mu_\varepsilon^2+u_{0\varepsilon}^s$ tightly converges to u_0^s, allow to obtain from (3.18):

$$\limsup_{h\to\infty}\limsup_{\varepsilon\to 0}\int_{\{h\le u_\varepsilon\le h+1\}} a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon(1-v_\delta)dx\,dt \le \int_\Omega (1-\psi_\delta)du_0^s \le \delta\,,$$

in view of (3.6) and (3.7). This concludes the proof of (3.16). Moreover, note that from (3.18) we have also obtained:

$$\lim_{\delta\to 0}\limsup_{h\to\infty}\limsup_{\varepsilon\to 0}\int_{\{u_\varepsilon>h+1\}} g(u_\varepsilon)|\nabla u_\varepsilon|^2(1-v_\delta)dx\,dt = 0\,.$$

STEP 3. Let us take the time-regularization $T_k(u)_\nu$ defined in (3.4), which strongly converges to $T_k(u)$ in $L^2(0,T;H_0^1(\Omega))$, and choose $\varphi_\rho((u_\varepsilon-T_k(u)_\nu)^-)$ $(1-v_\delta)$ as a test function in (3.10). Since $\varphi_\rho(0)=0$, and $T_k(u)_\nu\le k$, if $u_\varepsilon>k$ we have $\varphi_\rho((u_\varepsilon-T_k(u)_\nu)^-)=0$, so that $\varphi_\rho((u_\varepsilon-T_k(u)_\nu)^-)=\varphi_\rho((T_k(u_\varepsilon)-T_k(u)_\nu)^-)$, and all the integrals appearing in (3.10) are in fact taken only on the subset $\{(x,t)\,:\,u_\varepsilon\le k\}$. Finally since $S_n''(u_\varepsilon)\varphi_\rho((u_\varepsilon-T_k(u)_\nu)^-)\equiv 0$ if $n>k$, we obtain, for n large enough:

$$\begin{aligned}
&\int_0^T\left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t},\varphi_\rho((u_\varepsilon-T_k(u)_\nu)^-)(1-v_\delta)\right\rangle dt\\
&\quad-\int_{\{u_\varepsilon\le T_k(u)_\nu\}} a(x,t,\nabla T_k(u_\varepsilon))\nabla[T_k(u_\varepsilon)-T_k(u)_\nu]\varphi_\rho'((u_\varepsilon-T_k(u)_\nu)^-)(1-v_\delta)dx\,dt\\
&\quad+\int_Q g(u_\varepsilon)|\nabla T_k(u_\varepsilon)|^2\varphi_\rho((u_\varepsilon-T_k(u)_\nu)^-)(1-v_\delta)dx\,dt\\
&=\int_Q f_\varepsilon\,\varphi_\rho((u_\varepsilon-T_k(u)_\nu)^-)(1-v_\delta)dx\,dt\\
&\quad+\int_Q a(x,t,\nabla T_k(u_\varepsilon))\nabla v_\delta\,\varphi_\rho((u_\varepsilon-T_k(u)_\nu)^-)dx\,dt
\end{aligned}$$

which yields, using (a_1) and recalling that $M_k = \max_{[0,k]} g(s)$ (for the sake of shortness the explicit dependence on x and t is omitted in the derivations below),

$$\begin{aligned}
&\int_{\{u_\varepsilon \le T_k(u)_\nu\}} a(\nabla T_k(u_\varepsilon))\nabla[T_k(u_\varepsilon) - T_k(u)_\nu]\,\varphi'_\rho((u_\varepsilon - T_k(u)_\nu)^-)\,(1 - v_\delta) \\
&\le \frac{M_k}{\alpha}\int_Q a(\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1 - v_\delta)dx\,dt \\
&\quad + \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, \varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1 - v_\delta)\right\rangle dt \\
&\quad - \int_Q a(\nabla T_k(u_\varepsilon))\nabla v_\delta\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)dx\,dt \\
&\quad - \int_Q f_\varepsilon\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1 - v_\delta)dx\,dt\,.
\end{aligned} \tag{3.19}$$

Let us denote, henceforth, by $\omega(\varepsilon, \nu, \delta)$ any quantity depending on ε, ν, δ such that

$$\limsup_{\delta\to 0}\limsup_{\nu\to\infty}\limsup_{\varepsilon\to 0}\ \omega(\varepsilon, \nu, \delta) = 0\,,$$

the order in which the limit on each parameter is taken being essential in what follows.

Due to (3.1) and since $\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)$ converges to $\varphi_\rho((u - T_k(u)_\nu)^-)$ almost everywhere in Q as ε tends to zero and is bounded by $\varphi_\rho(2k)$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}&\int_Q a(\nabla T_k(u_\varepsilon))\nabla v_\delta\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)dx\,dt \\
&= \int_Q \sigma_k \nabla v_\delta\,\varphi_\rho((u - T_k(u)_\nu)^-)dx\,dt\,.
\end{aligned}$$

Now, as ν tends to infinity, $\varphi_\rho((u - T_k(u)_\nu)^-)$ converges almost everywhere (and then weakly-$*$ in $L^\infty(Q)$) to $\varphi_\rho((u - T_k(u))^-) \equiv 0$, so that

$$\lim_{\nu\to\infty}\lim_{\varepsilon\to 0}\int_Q a(\nabla T_k(u_\varepsilon))\nabla v_\delta\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)dx\,dt = 0\,.$$

Using (3.1), the weak convergence of f_ε to f in $L^1(Q)$ and the properties of $T_k(u)_\nu$, it is easy to prove that last term in (3.19) converges to zero as ε tends to zero and then ν tends to infinity. As a consequence of the above estimates, (3.19) yields:

$$\begin{aligned}
&\int_{\{u_\varepsilon \le T_k(u)_\nu\}} a(\nabla T_k(u_\varepsilon))\nabla[T_k(u_\varepsilon) - T_k(u)_\nu]\,\varphi'_\rho((u_\varepsilon - T_k(u)_\nu)^-)\,(1 - v_\delta) \\
&\le \frac{M_k}{\alpha}\int_Q a(\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1 - v_\delta)dx\,dt \\
&\quad + \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, \varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1 - v_\delta)\right\rangle dt + \omega(\varepsilon, \nu, \delta)\,.
\end{aligned} \tag{3.20}$$

Now we estimate the first term in the right hand side of (3.20) as follows:

$$
\begin{aligned}
&\int_Q a(\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta) \\
(3.21)\qquad &= \int_{\{u_\varepsilon \le T_k(u)_\nu\}} a(\nabla T_k(u_\varepsilon))\nabla[T_k(u_\varepsilon) - T_k(u)_\nu]\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta) \\
&\quad + \int_Q a(\nabla T_k(u_\varepsilon))\nabla T_k(u)_\nu\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)dx\,dt\,.
\end{aligned}
$$

We claim that last term of the above equality goes to zero as first ε tends to zero and then ν tends to infinity. Indeed, we have, as before,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_Q a(\nabla T_k(u_\varepsilon))\nabla T_k(u)_\nu\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)dx\,dt \\
&\qquad = \int_Q \sigma_k \nabla T_k(u)_\nu\,\varphi_\rho((u - T_k(u)_\nu)^-)(1-v_\delta)dx\,dt\,,
\end{aligned}
$$

and then the strong convergence of $T_k(u)_\nu$ in $L^2(0,T;H_0^1(\Omega))$ as ν tends to infinity implies:

$$
\lim_{\nu\to\infty}\lim_{\varepsilon\to 0}\int_Q a(\nabla T_k(u_\varepsilon))\nabla T_k(u)_\nu\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)dx\,dt = 0\,,
$$

because $\varphi_\rho((u - T_k(u))^-) \equiv 0$. We then deduce from (3.21):

$$
\begin{aligned}
&\int_Q a(\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)dx\,dt = \omega(\varepsilon,\nu,\delta) \\
&+ \int_{\{u_\varepsilon \le T_k(u)_\nu\}} a(\nabla T_k(u_\varepsilon))\nabla[T_k(u_\varepsilon) - T_k(u)_\nu]\,\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)dx\,dt.
\end{aligned}
$$

Therefore (3.20) now implies:

$$
\begin{aligned}
&\int_{\{u_\varepsilon \le T_k(u)_\nu\}} \{a(\nabla T_k(u_\varepsilon)) - a(\nabla T_k(u)_\nu)\}\nabla[T_k(u_\varepsilon) - T_k(u)_\nu]\left(\varphi_\rho' - \frac{M_k}{\alpha}\varphi_\rho\right)(1-v_\delta)dx\,dt \\
&\quad \le \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, \varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)\right\rangle dt \\
&\quad - \int_{\{u_\varepsilon \le T_k(u)_\nu\}} a(\nabla T_k(u)_\nu)\nabla[T_k(u_\varepsilon) - T_k(u)_\nu]\left(\varphi_\rho' - \frac{M_k}{\alpha}\varphi_\rho\right)(1-v_\delta)dx\,dt \\
&\quad + \omega(\varepsilon,\nu,\delta)\,,
\end{aligned}
$$

where $(\varphi_\rho' - \frac{M_k}{\alpha}\varphi_\rho)$ denotes $(\varphi_\rho' - \frac{M_k}{\alpha}\varphi_\rho)((u_\varepsilon - T_k(u)_\nu)^-(x,t))$.

Since we have, because of the weak convergence of $T_k(u_\varepsilon)$ and of the strong convergence of $T_k(u)_\nu$ in $L^2(0,T;H_0^1(\Omega))$,

$$\begin{aligned}
&\int_{\{u_\varepsilon \le T_k(u)_\nu\}} a(\nabla T_k(u)_\nu)\nabla[T_k(u_\varepsilon)-T_k(u)_\nu]\left(\varphi_\rho' - \frac{M_k}{\alpha}\varphi_\rho\right)(1-v_\delta)dx\,dt\\
&= -\int_Q a(\nabla T_k(u)_\nu)\nabla\left(\varphi_\rho((T_k(u_\varepsilon)-T_k(u)_\nu)^-)\right)(1-v_\delta)dx\,dt\\
&\quad -\frac{M_k}{\alpha}\int_Q a(\nabla T_k(u)_\nu)\nabla[T_k(u_\varepsilon)-T_k(u)_\nu]\varphi_\rho((T_k(u_\varepsilon)-T_k(u)_\nu)^-)(1-v_\delta)dx\,dt\\
&= \omega(\varepsilon,\nu,\delta)
\end{aligned}$$

we obtain, choosing by (3.2) ρ sufficiently large:

$$\begin{aligned}
&\frac{1}{2}\int_{\{u_\varepsilon \le T_k(u)_\nu\}} \{a(\nabla T_k(u_\varepsilon)) - a(\nabla T_k(u)_\nu)\}\nabla[T_k(u_\varepsilon)-T_k(u)_\nu](1-v_\delta)dx\,dt\\
&\le \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, \varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)\right\rangle dt + \omega(\varepsilon,\nu,\delta).
\end{aligned} \tag{3.22}$$

Now we investigate the behaviour of the time-derivative term. Since $\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-) = \varphi_\rho((S_n(u_\varepsilon) - T_k(u)_\nu)^-)$ for n sufficiently large, we have, thanks to (3.4):

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, \varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)\right\rangle dt\\
&= \int_0^T \left\langle \frac{\partial (S_n(u_\varepsilon) - T_k(u)_\nu)}{\partial t}, \varphi_\rho((S_n(u_\varepsilon) - T_k(u)_\nu)^-)(1-v_\delta)\right\rangle dt\\
&\quad + \nu\int_Q (T_k(u) - T_k(u)_\nu)\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)dx\,dt.
\end{aligned}$$

Let us set $\Phi_\rho^-(s) = \int_0^s \varphi_\rho(t^-)\,dt$; then, since $\Phi_\rho^-(s) \le 0$ and $0 \le v_\delta \le 1$, integrating by parts we obtain, for n large enough:

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, \varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)\right\rangle dt\\
&\le -\int_\Omega \Phi_\rho^-(u_{0\varepsilon}^r - z_\nu)(1-\psi_\delta)dx\\
&\quad + \nu\int_Q (T_k(u) - T_k(u)_\nu)\varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta)dx\,dt\\
&\quad + \int_Q \frac{\partial v_\delta}{\partial t}\Phi_\rho^-(u_\varepsilon - T_k(u)_\nu)dx\,dt.
\end{aligned}$$

Now we can pass to the limit as ε tends to zero by means of the Lebesgue theorem, using that $u_{0\varepsilon}^r$ converges to u_0^r in measure (in fact we have that μ_ε^2

converges to zero in measure) and that $\Phi_\rho^-(u_{0\varepsilon}^r - z_\nu)$ is uniformly bounded in ε. Then, since we also have that $\varphi_\rho(t^-)t \le 0$, we get:

$$\limsup_{\varepsilon\to 0} \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, \varphi_\rho(u_\varepsilon - T_k(u)_\nu)^-(1-v_\delta) \right\rangle dt$$
$$\le -\int_\Omega \Phi_\rho^-(u_0^r - z_\nu)(1-\psi_\delta)dx + \int_Q \frac{\partial v_\delta}{\partial t}\Phi_\rho^-(u - T_k(u)_\nu)dx\,dt\,,$$

which yields, letting ν tend to infinity and using (3.3),

$$\limsup_{\nu\to\infty}\limsup_{\varepsilon\to 0} \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, \varphi_\rho((u_\varepsilon - T_k(u)_\nu)^-)(1-v_\delta) \right\rangle dt$$
$$\le -\int_\Omega \Phi_\rho^-(u_0^r - T_k(u_0^r))(1-\psi_\delta)\,dx + \int_Q \frac{\partial v_\delta}{\partial t}\Phi_\rho^-(u - T_k(u))dx\,dt\,.$$

Since $\Phi_\rho^-(t - T_k(t)) = 0$ for any t, we obtain:

$$\text{(3.23)}\qquad \limsup_{\nu\to\infty}\limsup_{\varepsilon\to 0} \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, \varphi_\rho(u_\varepsilon - T_k(u)_\nu)^-(1-v_\delta) \right\rangle dt \le 0.$$

Thanks to (3.23) we get, from (3.22),

$$\int_{\{u_\varepsilon\le T_k(u)_\nu\}} \{a(x,t,\nabla T_k(u_\varepsilon)) - a(x,t,\nabla T_k(u)_\nu)\}\nabla[T_k(u_\varepsilon) - T_k(u)_\nu](1-v_\delta)dx\,dt$$
$$\le \omega(\varepsilon,\nu,\delta)\,,$$

which in turn implies that

$$\text{(3.24)}\quad \int_{\{u_\varepsilon\le T_k(u)_\nu\}} a(x,t,\nabla T_k(u_\varepsilon))\nabla[T_k(u_\varepsilon) - T_k(u)_\nu](1-v_\delta)dx\,dt \le \omega(\varepsilon,\nu,\delta).$$

STEP 4. This step consists in taking $(T_k(u_\varepsilon) - T_k(u)_\nu)^+(1-v_\delta)$ as a test function in (3.10). Since u_ε is positive and by assumption (g_1) this leads to:

$$\int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, (T_k(u_\varepsilon) - T_k(u)_\nu)^+(1-v_\delta) \right\rangle dt$$
$$+ \int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon) - T_k(u)_\nu)^+]S_n'(u_\varepsilon)(1-v_\delta)dx\,dt$$
$$\text{(3.25)}\qquad \le \int_Q a(x,t,\nabla u_\varepsilon)\nabla v_\delta\, S_n'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)_\nu)^+dx\,dt$$
$$- \int_Q a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon\, S_n''(u_\varepsilon)\,(T_k(u_\varepsilon) - T_k(u)_\nu)^+(1-v_\delta)dx\,dt$$
$$+ \int_Q f_\varepsilon S_n'(u_\varepsilon)\,(T_k(u_\varepsilon) - T_k(u)_\nu)^+(1-v_\delta)dx\,dt\,.$$

We now investigate the behaviour of each term in the right hand side of (3.25). Due to the definition of S_n, we have:

$$\begin{aligned}&\int_Q a(x,t,\nabla u_\varepsilon)\nabla v_\delta\, S_n'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u)_\nu)^+dx\,dt\\ &=\int_Q a(x,t,\nabla T_{n+1}(u_\varepsilon))\nabla v_\delta\, S_n'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u)_\nu)^+dx\,dt\,,\end{aligned}$$

so that (3.1) implies

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u_\varepsilon)\nabla v_\delta\, S_n'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u)_\nu)^+dx\,dt\\ &=\int_Q \sigma_{n+1}\nabla v_\delta S_n'(u)(T_k(u)-T_k(u)_\nu)^+dx\,dt\,.\end{aligned}$$

Thanks to (3.5), letting ν tend to infinity then yields:

$$\lim_{\substack{\nu\to\infty\\ \varepsilon\to 0}}\int_Q a(x,t,\nabla u_\varepsilon)\nabla v_\delta\, S_n'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u)_\nu)^+dx\,dt=0\,, \tag{3.26}$$

where $\lim\limits_{\substack{\nu\to\infty\\ \varepsilon\to 0}}$ stays for $\lim\limits_{\nu\to\infty}\lim\limits_{\varepsilon\to 0}$. Similarly we will write below $\limsup\limits_{\substack{\delta\to 0\\ n\to\infty\\ \nu\to\infty\\ \varepsilon\to 0}}$ to denote the lim sup taken subsequently on the different parameters, first ε, then ν, n, δ.

In the same spirit, we have (recall that $S_n''(u_\varepsilon)=-\chi_{\{n\le u_\varepsilon\le n+1\}}$):

$$\begin{aligned}0&\le-\int_Q a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon\, S_n''(u_\varepsilon)\,(T_k(u_\varepsilon)-T_k(u)_\nu)^+(1-v_\delta)dx\,dt\\ &\le 2k\int_{\{n\le u_\varepsilon<n+1\}} a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon\,(1-v_\delta)dx\,dt\,,\end{aligned}$$

hence by (3.16) we get:

$$\lim_{\delta\to 0}\limsup_{\substack{n\to\infty\\ \nu\to\infty\\ \varepsilon\to 0}}-\int_Q a(x,t,\nabla u_\varepsilon)\nabla u_\varepsilon\, S_n''(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u)_\nu)^+(1-v_\delta)dx\,dt=0. \tag{3.27}$$

Similarly we have, using the weak convergence of f_ε in $L^1(Q)$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_Q f_\varepsilon S_n'(u_\varepsilon)\,(T_k(u_\varepsilon)-T_k(u)_\nu)^+(1-v_\delta)dx\,dt\\ &=\int_Q f S_n'(u)\,(T_k(u)-T_k(u)_\nu)^+(1-v_\delta)dx\,dt\,,\end{aligned}$$

and then, as ν tends to infinity, we get:

$$\lim_{\substack{\nu\to\infty\\ \varepsilon\to 0}} \int_Q f_\varepsilon S_n'(u_\varepsilon)\,(T_k(u_\varepsilon) - T_k(u)_\nu)^+(1 - v_\delta)dx\,dt = 0\,. \tag{3.28}$$

Gathering (3.26), (3.27) and (3.28), we have that (3.25) implies:

$$\begin{aligned}&\int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, (T_k(u_\varepsilon)-T_k(u)_\nu)^+(1-v_\delta)\right\rangle dt\\ &+\int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon)-T_k(u)_\nu)^+]S_n'(u_\varepsilon)(1-v_\delta)dx\,dt \le \omega(\varepsilon,\nu,n,\delta),\end{aligned} \tag{3.29}$$

where $\omega(\varepsilon, \nu, n, \delta)$ is such that:

$$\lim_{\delta\to 0} \limsup_{\substack{n\to\infty\\ \nu\to\infty\\ \varepsilon\to 0}} \omega(\varepsilon, \nu, n, \delta) = 0\,.$$

The time derivative term can be dealt with by using the following lemma, which extends similar results of [BlMR], [DO].

LEMMA 3.2. *Let $k > 0$ be fixed and let $T_k(u)_\nu$ be the time-regularization of $T_k(u)$ defined in* (3.4). *Then we have, for $n > k$:*

$$\liminf_{\substack{\nu\to\infty\\ \varepsilon\to 0}} \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, (T_k(u_\varepsilon) - T_k(u)_\nu)^+(1 - v_\delta)\right\rangle dt \ge 0\,.$$

PROOF. First note that for $n > k$ we have $T_k(S_n(u_\varepsilon)) = T_k(u_\varepsilon)$ and $T_k(S_n(u)) = T_k(u)$, hence $T_k(S_n(u))_\nu = T_k(u)_\nu$; moreover it can be easily seen that $(T_k(s) - T_k(u)_\nu)^+ = (s - T_k(u)_\nu)^+ - (k - s)^-$ for every $s \ge 0$, so that we have:

$$\begin{aligned}&\int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, (T_k(u_\varepsilon) - T_k(u)_\nu)^+(1 - v_\delta)\right\rangle dt\\ &= \int_0^T \left\langle \frac{\partial (S_n(u_\varepsilon) - T_k(S_n(u))_\nu)}{\partial t}, (S_n(u_\varepsilon) - T_k(S_n(u))_\nu)^+(1 - v_\delta)\right\rangle dt\\ &+ \int_0^T \left\langle \frac{\partial T_k(S_n(u))_\nu}{\partial t}, (S_n(u_\varepsilon) - T_k(S_n(u))_\nu)^+(1 - v_\delta)\right\rangle dt\\ &- \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, (k - S_n(u_\varepsilon))^-\,(1 - v_\delta)\right\rangle dt\,.\end{aligned}$$

Integrating by parts, using the definition of $T_k(u)_\nu$ we obtain, upon setting $\Psi_k(r) = \int_0^r (k-s)^- \, ds$:

$$
\begin{aligned}
&\int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, (T_k(u_\varepsilon) - T_k(u)_\nu)^+(1-v_\delta) \right\rangle dt \\
&= -\frac{1}{2}\int_\Omega |(S_n(u_{0\varepsilon}^r) - z_\nu)^+|^2(1-\psi_\delta)dx - \int_\Omega \Psi_k(S_n(u_\varepsilon)(T))(1-v_\delta(T))dx \\
&\quad + \frac{1}{2}\int_\Omega |(S_n(u_\varepsilon) - T_k(u)_\nu)^+|^2(T)(1-v_\delta(T))dx \\
&\quad + \int_\Omega \Psi_k(S_n(u_{0\varepsilon}^r))\,(1-\psi_\delta)dx \\
&\quad + \nu \int_Q (T_k(S_n(u)) - T_k(S_n(u))_\nu)(S_n(u_\varepsilon) - T_k(S_n(u))_\nu)^+(1-v_\delta)dx\,dt \\
&\quad + \int_Q \frac{\partial v_\delta}{\partial t}\left(\frac{|(S_n(u_\varepsilon) - T_k(S_n(u))_\nu)^+|^2}{2} - \Psi_k(S_n(u_\varepsilon))\right)dx\,dt\,.
\end{aligned}
\tag{3.30}
$$

By definition of $\Psi_k(r)$, it can be easily seen that for every fixed $z \in \mathbb{R}$ such that $|z| \le k$ the function $s \mapsto \frac{|(s-z)^+|^2}{2} - \Psi_k(s)$ is positive for every s in $\mathbb{R}^+$, so that we deduce from (3.30):

$$
\begin{aligned}
&\int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, (T_k(u_\varepsilon) - T_k(u)_\nu)^+(1-v_\delta) \right\rangle dt \\
&\ge -\frac{1}{2}\int_\Omega |(S_n(u_{0\varepsilon}^r) - z_\nu)^+|^2\,(1-\psi_\delta)dx \\
&\quad + \int_\Omega \Psi_k(S_n(u_{0\varepsilon}^r))(1-\psi_\delta)dx \\
&\quad + \int_Q \frac{\partial v_\delta}{\partial t}\left(\frac{|(S_n(u_\varepsilon) - T_k(S_n(u))_\nu)^+|^2}{2} - \Psi_k(S_n(u_\varepsilon))\right)dx\,dt \\
&\quad + \nu \int_Q (T_k(S_n(u)) - T_k(S_n(u))_\nu)(S_n(u_\varepsilon) - T_k(S_n(u))_\nu)^+(1-v_\delta)dx\,dt\,.
\end{aligned}
\tag{3.31}
$$

Recalling that $u_{0\varepsilon}^r$ converges in measure to u_0^r in Ω, and that S_n is bounded by $n+1$, we can pass to the limit as ε tends to zero in (3.31) by means of Lebesgue's theorem and we obtain:

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, (T_k(u_\varepsilon) - T_k(u)_\nu)^+(1-v_\delta) \right\rangle dt \\
&\ge -\frac{1}{2}\int_\Omega |(S_n(u_0^r) - z_\nu)^+|^2\,(1-\psi_\delta)\,dx + \int_\Omega \Psi_k(S_n(u_0^r))\,(1-\psi_\delta)dx \\
&\quad + \nu \int_Q (T_k(S_n(u)) - T_k(S_n(u))_\nu)(S_n(u) - T_k(S_n(u))_\nu)^+(1-v_\delta)dx\,dt \\
&\quad + \int_Q \frac{\partial v_\delta}{\partial t}\left(\frac{|(S_n(u) - T_k(S_n(u))_\nu)^+|^2}{2} - \Psi_k(S_n(u))\right)dx\,dt\,,
\end{aligned}
$$

which yields, since $(T_k(S_n(u)) - T_k(S_n(u))_\nu)(S_n(u) - T_k(S_n(u))_\nu)^+ \geq 0$ almost everywhere in Q,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, (T_k(u_\varepsilon) - T_k(u)_\nu)^+(1-v_\delta) \right\rangle dt \\
&\geq -\frac{1}{2}\int_\Omega |(S_n(u_0^r) - z_\nu)^+|^2\,(1-\psi_\delta)dx + \int_\Omega \Psi_k(S_n(u_0^r))\,(1-\psi_\delta)dx \\
&\quad + \int_Q \frac{\partial v_\delta}{\partial t}\left[\frac{|(S_n(u) - T_k(S_n(u))_\nu)^+|^2}{2} - \Psi_k(S_n(u))\right] dx\,dt\,.
\end{aligned}$$

Due to (3.3), (3.5), to conclude the proof it is enough to use again Lebesgue's theorem as ν tends to infinity together with the fact that $\frac{|(s-T_k(s))^+|^2}{2} - \Psi_k(s) = 0$ for every s in $\mathbb{R}$. □

Thanks to the previous lemma we can deduce, from (3.29):

$$\limsup_{\substack{\delta\to 0\\ n\to\infty\\ \nu\to\infty\\ \varepsilon\to 0}} \int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon) - T_k(u)_\nu)^+]S_n'(u_\varepsilon)\,(1-v_\delta)dx\,dt \leq 0\,. \tag{3.32}$$

STEP 5. In this step we establish the asymptotic estimate (3.11) through the use of (3.15), (3.24) and (3.32). To this end we write:

$$\begin{aligned}
&\int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon) - T_k(u)_\nu)]S_n'(u_\varepsilon)dx\,dt \\
&= \int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon) - T_k(u)_\nu)]S_n'(u_\varepsilon)v_\delta\,dx\,dt \\
&\quad + \int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon) - T_k(u)_\nu)]S_n'(u_\varepsilon)\,(1-v_\delta)dx\,dt\,.
\end{aligned} \tag{3.33}$$

As soon as $n > k$, since $\mathrm{Supp}(S_n') \subset [-(n+1), n+1]$ and thanks to (3.1), the first term of the right hand side of (3.33) is estimated as follows:

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon) - T_k(u)_\nu)]S_n'(u_\varepsilon)v_\delta\,dx\,dt \\
&= \limsup_{\varepsilon\to 0} \int_Q a(x,t,\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)\,v_\delta\,dx\,dt \\
&\quad - \int_Q \sigma_{n+1}\nabla T_k(u)_\nu S_n'(u)\,v_\delta\,dx\,dt\,.
\end{aligned}$$

Using assumption (a_2) and the strong convergence of $T_k(u)_\nu$ to $T_k(u)$ in $L^2(0,T;H_0^1(\Omega))$, we deduce from the above equality that (recall the definition of S_n):

$$\begin{aligned}&\limsup_{\substack{\nu\to\infty\\ \varepsilon\to 0}}\int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon)-T_k(u)_\nu)]S_n'(u_\varepsilon)\,v_\delta\,dx\,dt\\ &\quad\le \beta\limsup_{\varepsilon\to 0}\int_Q |\nabla T_k(u_\varepsilon)|^2\,v_\delta\,dx\,dt-\int_Q \sigma_{n+1}\nabla T_k(u)\,v_\delta\,dx\,dt\,.\end{aligned}$$

Now the definition of σ_{n+1} in (3.1) and the pointwise convergence of u_ε to u in Q imply that

$$\sigma_{n+1}\chi_{\{u<k\}}=\sigma_k\chi_{\{u<k\}}\qquad\text{a.e. in } Q \text{ for every } n>k\,. \tag{3.34}$$

It follows that:

$$\int_Q \sigma_{n+1}\nabla T_k(u)\,v_\delta\,dx\,dt=\int_Q \sigma_k\nabla T_k(u)\,v_\delta\,dx\,dt\,,$$

and as a consequence of (3.9) and (3.15) we obtain:

$$\limsup_{\substack{\delta\to 0\\ n\to\infty\\ \nu\to\infty\\ \varepsilon\to 0}}\int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon)-T_k(u)_\nu)]S_n'(u_\varepsilon)\,v_\delta\,dx\,dt\le 0\,. \tag{3.35}$$

We now estimate the second term in the right hand side of (3.33).

We have:

$$\begin{aligned}&\int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon)-T_k(u)_\nu)]S_n'(u_\varepsilon)\,(1-v_\delta)dx\,dt\\ &=\int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon)-T_k(u)_\nu)^+]S_n'(u_\varepsilon)\,(1-v_\delta)dx\,dt\\ &\quad+\int_{\{u_\varepsilon\le T_k(u)_\nu\}} a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon)-T_k(u)_\nu)]S_n'(u_\varepsilon)\,(1-v_\delta)dx\,dt\,,\end{aligned}$$

since $(u_\varepsilon-T_k(u)_\nu)^-=(T_k(u_\varepsilon)-T_k(u)_\nu)^-$.

Due to the estimate (3.32), and using the fact that $T_k(u)_\nu\le k$ almost everywhere in Q and $S_n'(u_\varepsilon)=1$ if $u_\varepsilon\le n$, we then deduce that:

$$\begin{aligned}&\limsup_{\substack{\delta\to 0\\ n\to\infty\\ \nu\to\infty\\ \varepsilon\to 0}}\int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon)-T_k(u)_\nu)]S_n'(u_\varepsilon)\,(1-v_\delta)dx\,dt\\ &\le\lim_{\delta\to 0}\limsup_{\substack{\nu\to\infty\\ \varepsilon\to 0}}\int_{\{u_\varepsilon\le T_k(u)_\nu\}} a(x,t,\nabla T_k(u_\varepsilon))\nabla[(T_k(u_\varepsilon)-T_k(u)_\nu)]\,(1-v_\delta)dx\,dt\,.\end{aligned}$$

In view of estimate (3.24), it follows that:

$$\limsup_{\substack{\delta\to 0\\ n\to\infty\\ \nu\to\infty\\ \varepsilon\to 0}} \int_Q a(x,t,\nabla u_\varepsilon)\nabla[(T_k(u_\varepsilon)-T_k(u)_\nu)]S_n'(u_\varepsilon)\,(1-v_\delta)dx\,dt \le 0\,. \tag{3.36}$$

Gathering (3.33), (3.35), (3.36), we obtain, using again the properties of S_n:

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_Q a(x,t,\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)\,dx\,dt\\
&\le \limsup_{\substack{\nu\to\infty\\ \varepsilon\to 0}}\int_Q a(x,t,\nabla u_\varepsilon)\nabla T_k(u)_\nu S_n'(u_\varepsilon)dx\,dt\\
&\le \limsup_{\nu\to\infty}\int_Q \sigma_{n+1}\nabla T_k(u)_\nu\, S_n'(u)dx\,dt = \int_Q \sigma_k\nabla T_k(u)\,dx\,dt\,,
\end{aligned}$$

because of (3.34).

The monotone character of $a(x,t,\xi)$ together with the definition of σ_k and Lemma 3.1 allow to conclude through the usual monotonicity argument that (3.11) holds true.

From (3.11) and due to the strict monotonicity of $a(x,t,\xi)$ in (a_3), it follows (see for instance Lemma 5 in [BMP]) that:

$$T_k(u_\varepsilon)\to T_k(u) \qquad \text{strongly in } L^2(0,T;H_0^1(\Omega)) \text{ for every } k>0. \tag{3.37}$$

For a subsequence, still indexed by ε, (3.37) implies:

$$\nabla T_k(u_\varepsilon)\to\nabla T_k(u) \qquad \text{almost everywhere and in measure on } Q.$$

Let us also note that since we have, for every $\sigma>0$:

$$\begin{aligned}
\operatorname{meas}\{(x,t)\,:\,|\nabla u_\varepsilon-\nabla u|>\sigma\} &\le \operatorname{meas}\{(x,t)\,:\,|u_\varepsilon|>k\}\\
&+\operatorname{meas}\{(x,t)\,:\,|u|>k\}+\operatorname{meas}\{(x,t)\,:\,|\nabla(T_k(u_\varepsilon))-\nabla(T_k(u))|>\sigma\}\,,
\end{aligned}$$

we can deduce, using also the $L^1(Q)$ estimate on u_ε, that ∇u_ε converges to ∇u in measure, and therefore, always up to subsequences,

$$\nabla u_\varepsilon\to\nabla u \qquad \text{a.e. in } Q\,.$$

REMARK 3.3. If $a(x,t,\xi)$ is only assumed to be monotone, but not strictly monotone, in (a_3), then (3.11) still holds true while (3.37) may be false. However (3.11) in this case implies that

$$a(x,t,\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)\to a(x,t,\nabla T_k(u))\nabla T_k(u) \qquad \text{weakly in } L^1(Q)\,. \ \square$$

2. *Proof of* (i) *and* (ii). First of all, note that the strong convergence of truncations allows to deduce, without any further assumption on g, that u satisfies the renormalized equation (2.6). Indeed, for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support, with $\mathrm{Supp}(S') \subset [-L, L]$, we have from the renormalized formulation of (2.10):

$$\frac{\partial S(u_\varepsilon)}{\partial t} - \mathrm{div}(a(x,t,\nabla T_L(u_\varepsilon))S'(T_L(u_\varepsilon))) + S''(u_\varepsilon)a(x,t,\nabla T_L(u_\varepsilon))\nabla T_L(u_\varepsilon) \tag{3.38}$$
$$+ g(u_\varepsilon)|\nabla T_L(u_\varepsilon)|^2 S'(T_L(u_\varepsilon)) = f_\varepsilon S'(T_L(u_\varepsilon)).$$

Using the growth assumption (a_2) and (3.37) it is easy to see that $a(x, t, \nabla T_L(u_\varepsilon))$ $\nabla T_L(u_\varepsilon)$ is strongly convergent in $L^1(Q)$ to $a(x, t, \nabla T_L(u))\nabla T_L(u)$. Since $g(u_\varepsilon)S'(T_L(u_\varepsilon))$ is bounded in $L^\infty(Q)$, similarly the term $g(u_\varepsilon)|\nabla T_L(u_\varepsilon)|^2$ $S'(T_L(u_\varepsilon))$ strongly converges in $L^1(Q)$. Moreover, since S is a bounded function, $S(u_\varepsilon)$ converges to $S(u)$ in $L^1(Q)$ as ε tends to zero, so that $\frac{\partial S(u_\varepsilon)}{\partial t}$ converges to $\frac{\partial S(u)}{\partial t}$ in distributional sense. Thus from (3.38) we deduce, as ε tends to zero, that u satisfies the renormalized formulation:

$$\frac{\partial S(u)}{\partial t} - \mathrm{div}(a(x, t, \nabla u)S'(u)) + S''(u)a(x, t, \nabla u)\nabla u \tag{3.39}$$
$$+ g(u)|\nabla u|^2 S'(u) = f S'(u).$$

Note also that, as a consequence of (3.38), $\frac{\partial S(u_\varepsilon)}{\partial t}$ can be split into two sequences such that one is strongly convergent in $L^2(0, T; H^{-1}(\Omega))$ and the other is weakly convergent in $L^1(Q)$; since we also know that $S(u_\varepsilon)$ strongly converges to $S(u)$ in $L^2(0, T; H_0^1(\Omega))$ by (3.37), we can deduce (see [P]) that $S(u_\varepsilon)$ strongly converges to $S(u)$ in $C([0, T]; L^1(\Omega))$. As a consequence of the initial condition $S(u_{0\varepsilon}) = S(u_{0\varepsilon}^r)$ and of (2.9) we deduce that $S(u)(0) = S(u_0^r)$. Thus what really distinguishes between the case in which g is integrable or not is the behaviour of the sequence $\{g(u_\varepsilon)|\nabla u_\varepsilon|^2\}$ and of the energy term in (2.5).

Case (i). Let us now assume that $\int_0^{+\infty} g(r)dr < +\infty$ in order to prove that $g(u_\varepsilon)|\nabla u_\varepsilon|^2$ is strongly convergent in $L^1(Q)$. To this aim, we define the function $G_k(r) = \int_0^r g(t)\chi_{\{t>k\}}\,dt$ and choose $G_k(S_n(u_\varepsilon))$ as a test function in (3.10). Using (2.5) for u_ε we have, setting $G_k(\infty) = \int_k^{+\infty} g(t)\,dt$:

$$\int_{\{n\le u_\varepsilon\le n+1\}} a(x, t, \nabla u_\varepsilon)\nabla u_\varepsilon\, G_k(S_n(u_\varepsilon))dx\,dt \le G_k(\infty)c_2.$$

Then we obtain, by (g_1) and (a_1):

$$\int_0^T \left\langle \frac{\partial S_n(u_\varepsilon)}{\partial t}, G_k(S_n(u_\varepsilon)) \right\rangle dt$$
$$+ \alpha \int_Q g(S_n(u_\varepsilon))|\nabla u_\varepsilon|^2 (S_n'(u_\varepsilon))^2 \chi_{\{S_n(u_\varepsilon)>k\}} dx\,dt$$
$$\le \int_Q f_\varepsilon S_n'(u_\varepsilon)\, G_k(u_\varepsilon) dx\,dt + c_2 G_k(\infty).$$

Since G_k is non-decreasing, we easily get, using the properties of f_ε and $u_{0\varepsilon}$ and integrating by parts:

$$\alpha \int_Q g(S_n(u_\varepsilon))|\nabla u_\varepsilon|^2 \chi_{\{S_n(u_\varepsilon)>k\}}(S_n'(u_\varepsilon))^2 dx\, dt$$
$$\leq G_k(\infty)\left(\|f_\varepsilon\|_{L^1(Q)} + \|u_{0\varepsilon}^r\|_{L^1(\Omega)} + c_2\right) \leq G_k(\infty)\, c_3 ,$$

and then, as n tends to infinity,

$$\alpha \int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2 \chi_{\{u_\varepsilon>k\}} dx\, dt \leq G_k(\infty) c_3 .$$

By Lebesgue's theorem, since g is integrable, we have that $\lim_{k\to+\infty} G_k(\infty)=0$, so that we conclude:

$$\lim_{k\to\infty} \sup_\varepsilon \int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2 \chi_{\{u_\varepsilon>k\}} dx\, dt = 0 . \tag{3.40}$$

Estimate (3.40) together with (3.37) allow to show that $g(u_\varepsilon)|\nabla u_\varepsilon|^2$ is equi-integrable in $L^1(Q)$; indeed, for any subset B of Q we have:

$$\int_B g(u_\varepsilon)|\nabla u_\varepsilon|^2 dx\, dt \leq \left(\max_{[0,k]} g(s)\right) \int_B |\nabla T_k(u_\varepsilon)|^2 dx\, dt$$
$$+ \int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2 \chi_{\{u_\varepsilon>k\}} dx\, dt ,$$

which yields:

$$\sup_\varepsilon \int_B g(u_\varepsilon)|\nabla u_\varepsilon|^2 dx\, dt \leq \left(\max_{[0,k]} g(s)\right) \sup_\varepsilon \int_B |\nabla T_k(u_\varepsilon)|^2 dx\, dt$$
$$+ \sup_\varepsilon \int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2 \chi_{\{u_\varepsilon>k\}} dx\, dt .$$

Since the sequence $\{|\nabla T_k(u_\varepsilon)|\}$ is equi-integrable in $L^2(Q)$ by (3.37), letting meas(B) tend to zero and then k go to infinity, using (3.40) we deduce that

$$\lim_{\text{meas}(B)\to 0} \sup_\varepsilon \int_B g(u_\varepsilon)|\nabla u_\varepsilon|^2 dx\, dt = 0 .$$

Note that we also have that $g(u_\varepsilon)|\nabla u_\varepsilon|^2$ almost everywhere converges to $g(u)|\nabla u|^2$, so that, by Vitali theorem,

$$g(u_\varepsilon)|\nabla u_\varepsilon|^2 \to g(u)|\nabla u|^2 \quad \text{strongly in } L^1(Q).$$

We are left with the proof of (2.5). To this goal, let us observe that (3.16) now implies, thanks to (3.37),

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,(1-v_\delta)dx\,dt = 0\,. \tag{3.41}$$

Moreover, choosing $S = S_n$ in (3.39) and $\varphi \in C_c^\infty([0,T)\times\Omega)$ as a test function, and letting n tend to infinity, we find, using the fact that $a(x,t,\nabla u)$ belongs to $L^q(Q)^N$ for $q < \frac{N+2}{N+1}$:

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt\\
&= \int_Q a(x,t,\nabla u)\nabla\varphi\,dx\,dt - \int_Q \frac{\partial\varphi}{\partial t}u\,dx\,dt\\
&\quad + \int_Q g(u)|\nabla u|^2\varphi\,dx\,dt - \int_Q f\,\varphi\,dx\,dt - \int_\Omega u_0^r\varphi(0)\,dx\\
&\qquad\qquad \forall\varphi\in C_c^\infty([0,T)\times\Omega)\,.
\end{aligned}\tag{3.42}$$

On the other hand, since renormalized solutions are weak solutions we have that u_ε satisfies:

$$\begin{aligned}
\int_\Omega \varphi(0)du_{0\varepsilon} &= \int_Q a(x,t,\nabla u_\varepsilon)\nabla\varphi\,dx\,dt + \int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,dx\,dt\\
&\quad - \int_Q f_\varepsilon\varphi\,dx\,dt - \int_Q \frac{\partial\varphi}{\partial t}u_\varepsilon\,dx\,dt\,.
\end{aligned}$$

Since $a(x,t,\nabla u_\varepsilon)$ strongly converges to $a(x,t,\nabla u)$ in $L^1(Q)$ and thanks to the strong convergence of $g(u_\varepsilon)|\nabla u_\varepsilon|^2$ in $L^1(Q)$, we obtain, as ε tends to zero:

$$\begin{aligned}
\int_\Omega \varphi(0)du_0 &= \int_Q a(x,t,\nabla u)\nabla\varphi\,dx\,dt\\
&\quad + \int_Q g(u)|\nabla u|^2\varphi\,dx\,dt - \int_Q f\varphi\,dx\,dt - \int_Q \frac{\partial\varphi}{\partial t}u\,dx\,dt\,,
\end{aligned}$$

for every φ in $C_c^\infty([0,T)\times\Omega)$. Comparing with (3.42) we obtain that

$$\begin{aligned}
\lim_{n\to\infty}\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt &= \int_\Omega \varphi(0)\,du_0^s\\
&\qquad \forall\varphi\in C_c^\infty([0,T)\times\Omega)\,.
\end{aligned}\tag{3.43}$$

Let us prove that (3.43) actually holds for φ in $C^\infty(\bar{Q})$; indeed, we have

$$\begin{aligned}
&T\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt\\
&= T\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,(1-v_\delta)dx\,dt\\
&\quad + \int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,(T-t)v_\delta\,dx\,dt\\
&\quad + \int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,t\,v_\delta dx\,dt\,,
\end{aligned}\tag{3.44}$$

where v_δ is defined in (3.8). But for every fixed $\eta > 0$ we have, since v_δ solves (3.8):

$$\left|\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\, t\, v_\delta dx\, dt\right| \le \eta\, \|\varphi\|_{L^\infty(Q)} \int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\, dx\, dt$$
$$+\eta^{-\frac{N}{2}} \|\psi_\delta\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(Q)}\, T \int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\, dx\, dt$$
$$\le c_4(\eta + \eta^{-\frac{N}{2}} \|\psi_\delta\|_{L^1(\Omega)}),$$

because of Proposition 2.3 and of the strong convergence of $T_k(u_\varepsilon)$ and since $|u_{0\varepsilon}(\Omega)|$ is bounded. Letting first δ and then η tend to zero in the above inequality we obtain:

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\left|\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\, t\, v_\delta\, dx\, dt\right| = 0.$$

Then from (3.44), using also (3.41) and (3.43) with $\varphi\, v_\delta(T-t)$ instead of φ (note that for every $\varphi \in C^\infty(\bar{Q})$ we have $\varphi\, v_\delta(T-t) \in C_c^\infty([0,T)\times\Omega)$) we conclude that

$$\lim_{n\to\infty}\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\, dx\, dt = \int_\Omega \varphi(0)\, du_0^s \qquad \forall \varphi \in C^\infty(\bar{Q}),$$

then by a density argument for every φ in $C(\bar{Q})$, so that u satisfies (2.5) and is a renormalized solution of (1.1). This concludes the proof of (i).

CASE (ii). In (3.39) written with $S = S_n$, let us take $\theta_h(G(u))\varphi$ as a test function, where θ_h is defined in Definition 2.1, $G(s) = \int_0^s g(r)dr$ and φ is in $C_c^\infty([0,T)\times\Omega)$, $\varphi \ge 0$. First of all observe that, if $n > \sup\{s : G(s) = h+1\}$, then $\theta_h(G(u)) = \theta_h(G(S_n(u)))$ since G is nondecreasing, so we have, integrating by parts, and since $u_0^r \ge 0$:

$$\int_0^T \left\langle \frac{\partial S_n(u)}{\partial t}, \theta_h(G(u))\varphi \right\rangle dt \le -\int_Q \frac{\partial\varphi}{\partial t}\int_0^{S_n(u)} \theta_h(G(r))\, dr\, dx\, dt.$$

Then we deduce, using also that $\theta_h(G(u)) = 1$ if $u > n > \sup\{s : G(s) = h+1\}$,

$$\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\, dx\, dt \le \int_Q a(x,t,\nabla u)\nabla\varphi\, S_n'(u)\theta_h(G(u)) dx\, dt$$
$$+ c_5\|\varphi\|_{L^\infty(Q)}\int_{\{G(u)>h\}} \{g(u)|\nabla u|^2 S_n'(u) + S_n'(u) a(x,t,\nabla u)\nabla u\, g(u)\}\, dx\, dt$$
$$-\int_Q f\, S_n'(u)\theta_h(G(u))\,\varphi\, dx\, dt - \int_Q \frac{\partial\varphi}{\partial t}\int_0^{S_n(u)} \theta_h(G(r))\, dr\, dx\, dt.$$

Due to (a_2) it follows that:

$$\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt \le c_5\|\varphi\|_{L^\infty(Q)}\int_{\{G(u)>h\}}\Big(g(u)|\nabla u|^2+|f|\Big)\,dx\,dt$$
$$+c_5\int_{\{G(u)>h\}}|\nabla u|\,|\nabla\varphi|dx\,dt-\int_Q\frac{\partial\varphi}{\partial t}\int_0^{S_n(u)}\theta_h(G(r))\,dr\,dx\,dt\,.$$

Then as first n and then h tend to infinity, since u is in $L^1(0,T;W_0^{1,1}(\Omega))$ and meas$\{G(u)>h\}$ tends to zero as h tends to infinity, we find

$$\lim_{n\to\infty}\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt=0\quad\forall\varphi\in C_c^\infty([0,T)\times\Omega),\varphi\ge0\,. \tag{3.45}$$

Choosing $\varphi=v_\delta(T-t)$ in (3.45), using (3.41) and reasoning as in the case (i) we conclude that

$$\lim_{n\to\infty}\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,dx\,dt=0\,. \tag{3.46}$$

Let us now recover the limit of the sequence $\{g(u_\varepsilon)|\nabla u_\varepsilon|^2\}$. Since u_ε is a weak solution we have:

$$\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,dx\,dt=\int_\Omega u_{0\varepsilon}^r\varphi(0)\,dx-\int_Q a(x,t,\nabla u_\varepsilon)\nabla\varphi\,dx\,dt+\int_Q\frac{\partial\varphi}{\partial t}u_\varepsilon\,dx\,dt$$
$$+\int_Q f_\varepsilon\varphi\,dx\,dt+\int_\Omega\varphi(0)du_{0\varepsilon}^s\qquad\forall\varphi\in C_c^\infty([0,T)\times\Omega),$$

so that, as ε tends to zero, we obtain:

$$\begin{aligned}\lim_{\varepsilon\to0}\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,dx\,dt&=\int_Q f\,\varphi\,dx\,dt+\int_\Omega\varphi(0)\,du_0\\&\quad-\int_Q a(x,t,\nabla u)\nabla\varphi\,dx\,dt+\int_Q\frac{\partial\varphi}{\partial t}u\,dx\,dt.\end{aligned} \tag{3.47}$$

But we can also take (3.39) with $S=S_n$ and pass to the limit as n tends to infinity, to get, by means of (3.46):

$$-\int_\Omega u_0^r\varphi(0)\,dx+\int_Q a(x,t,\nabla u)\nabla\varphi\,dx\,dt+\int_Q g(u)|\nabla u|^2\varphi\,dx\,dt$$
$$=\int_Q f\,\varphi\,dx\,dt+\int_Q\frac{\partial\varphi}{\partial t}u\,dx\,dt\,,$$

which compared with (3.47) gives that

$$\lim_{\varepsilon\to0}\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,dx\,dt=\int_Q g(u)|\nabla u|^2\,\varphi\,dx\,dt+\int_\Omega\varphi(0)\,du_0^s\,,\qquad\forall\varphi\in C_c^\infty([0,T)\times\Omega)\,. \tag{3.48}$$

The same type of arguments as in the proof of the case (i) allow to prove that (3.48) actually holds true for φ in $C(\bar{Q})$. Indeed, by density arguments, it is enough to establish (3.48) for φ in $C^\infty(\bar{Q})$; to this purpose, let us note that we proved, at the end of Step 2 in the proof of the strong convergence of truncations,

$$\lim_{\delta\to 0} \limsup_{\substack{h\to\infty \\ \varepsilon\to 0}} \int_{\{u_\varepsilon>h\}} g(u_\varepsilon)|\nabla u_\varepsilon|^2(1-v_\delta)\,dx\,dt = 0. \tag{3.49}$$

Then we have, for any φ in $C^\infty(\bar{Q})$,

$$T\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi(1-v_\delta)\,dx\,dt = T\int_{\{u_\varepsilon\le h\}} g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,(1-v_\delta)\,dx\,dt + T\int_{\{u_\varepsilon>h\}} g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi(1-v_\delta)\,dx\,dt.$$

Since by (3.37) and the fact that $g(u)|\nabla u|^2$ belongs to $L^1(Q)$ we have, taking the limit in ε and then in h,

$$\lim_{\substack{h\to\infty \\ \varepsilon\to 0}} \int_{\{u_\varepsilon\le h\}} g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,(1-v_\delta)\,dx\,dt = \int_Q g(u)|\nabla u|^2\varphi\,(1-v_\delta)\,dx\,dt,$$

using also (3.49) and the fact that v_δ tends to zero almost everywhere in Q as δ tends to zero we deduce that:

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\quad & T\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi(1-v_\delta)\,dx\,dt \\ &= T\int_Q g(u)|\nabla u|^2\varphi\,dx\,dt.\end{aligned} \tag{3.50}$$

Remark that (3.50) still holds true if the limit-sup as ε tends to zero is replaced by the limit-inf as ε tends to zero (because of the positive character of the integrand in (3.49)). Now we have

$$\begin{aligned} T\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,dx\,dt = T&\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi(1-v_\delta)\,dx\,dt \\ &+\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,(T-t)v_\delta\,dx\,dt \\ &+\int_Q g(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,v_\delta\,t\,dx\,dt.\end{aligned} \tag{3.51}$$

Since $g(u_\varepsilon)|\nabla u_\varepsilon|^2$ is bounded in $L^1(Q)$ (see Lemma 3.1), we can repeat the technique that we used to control the last term of (3.44) to show that the last term of (3.51) tends to zero when taking the limit-sup as ε tends to zero and then the limit as δ tends to zero. Taking the limit-sup and the limit-inf as ε tends to zero in (3.51), and then the limit as δ tends to zero in the resulting inequalities, using (3.48) with $(T-t)\varphi\,v_\delta$ instead of φ, (3.50) and the properties of v_δ, we obtain (3.48) with φ in $C(\bar{Q})$. Together with (3.46), this proves the assertion in (ii), and the fact that u is a renormalized solution with datum u_0^r then follows straightforwardly. □

This strange phenomenon which happens in the case that g is not integrable at infinity is definitely hidden in the renormalized formulation, in the sense specified by Theorem 2.9. Due to the conclusion of the case (ii) of Theorem 2.6, all we have to prove is that if u is a renormalized solution of (1.1) and if $\int_0^{+\infty} g(s)\,ds = +\infty$, then $u_0^s = 0$. The proof is similar to that of (3.45).

Proof of Theorem 2.9. The existence is proved through the stability result of Theorem 2.6, (ii). Assume now that u is a renormalized solution, hence it satisfies (2.6). Choosing $S = S_n$ in (2.6) and the test function $\theta_h(G(u))\varphi$, where $G(s) = \int_0^s g(r)dr$ and φ is in $C_c^\infty([0,T)\times\Omega)$, $\varphi \ge 0$, we have:

$$\begin{aligned}
&\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\theta_h(G(u))\varphi\,dx\,dt\\
&\le \|\varphi\|_{L^\infty(Q)}\int_{\{G(u)>h\}}\Big(a(x,t,\nabla u)\nabla u\,g(u)+g(u)|\nabla u|^2+|f|\Big)\,dx\,dt\\
&\quad+\int_{\{G(u)>h\}}|a(x,t,\nabla u)|\,|\nabla\varphi|\,dx\,dt+\int_0^T\left\langle\frac{\partial S_n(u)}{\partial t},\theta_h(G(u))\right\rangle\varphi\,dt\,.
\end{aligned}\tag{3.52}$$

Since $G(s)$ is a positive, unbounded nondecreasing function, choosing $n > \sup\{s : G(s) = h+1\}$ we have that $\theta_h(G(u)) = \theta_h(G(S_n(u)))$ and moreover $\theta_h(G(u)) = 1$ if $u \ge n$, so that (3.52) implies, using (a_2):

$$\begin{aligned}
\int_{\{n\le u\le n+1\}} a(x,t,\nabla u)\nabla u\,\varphi\,dx\,dt \le{}& c_0\|\varphi\|_{L^\infty(Q)}\int_{\{G(u)>h\}}\Big(g(u)|\nabla u|^2+|f|\Big)\,dx\,dt\\
&+\int_{\{G(u)>h\}}|a(x,t,\nabla u)|\,|\nabla\varphi|\,dx\,dt\\
&-\int_Q\frac{\partial\varphi}{\partial t}\int_0^{S_n(u)}\theta_h(G(r))dr\,dx\,dt\,.
\end{aligned}$$

Since meas$\{G(u) > h\}$ tends to zero as h tends to infinity, letting first n and then h tend to infinity, we deduce from the fact that $g(u)|\nabla u|^2$ is in $L^1(Q)$ and from (2.5):

$$\int_\Omega \varphi(0)\,du_0^s \le 0 \qquad \forall\varphi\in C_c^\infty([0,T)\times\Omega)\,,\ \varphi\ge 0\,,$$

which implies, choosing $\varphi = (T-t)\psi$, where $\psi \in C_c^\infty(\Omega)$, that u_0^s must be zero, so that the proof of Theorem 2.9 is complete. □

Remark 3.4. If $a(x,t,\xi)$ is only assumed to be monotone, but not strictly monotone, we can not use the result of strong convergence of truncations (3.37) but from (3.11), which can still be obtained, we have (see Remark 3.3) that $a(x,t,\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)$ weakly converges to $a(x,t,\nabla T_k(u))\nabla T_k(u)$ in $L^1(Q)$. This is not enough, in general, to pass to the limit in the lower order term

$g(u_\varepsilon)|\nabla u_\varepsilon|^2 S'(u_\varepsilon)$ of the renormalized approximating equation. Nevertheless, if instead of (1.1) we consider the Cauchy-Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,\nabla u)) + g(u)a(x,t,\nabla u)\nabla u = f & \text{in } Q\,, \\ u = 0 & \text{on } \Sigma\,, \\ u(0) = u_0 & \text{in } \Omega\,, \end{cases}$$

under the assumptions on g, f and u_0 considered in the present paper similar results to those of Theorem 2.6 can be proved for non strictly monotone operators, that is for operators satisfying standard coercivity and growth conditions and only a large monotonicity assumption. □

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