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Finiteness and Separation Theorems for Dolbeault Cohomologies with Support Conditions

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Abstract. We consider the following situation: Let Y be an n-dimensional compact complex space whose singular part is isolated and divided into two non-empty parts S_1 and S_2. Set $X = Y \setminus (S_1 \cup S_2)$ and denote by Ψ_j, $j = 1, 2$, the family of closed sets $C \subseteq X$ such that $S_j \cup (X \setminus C)$ is a neighborhood of S_j. Using integral formulas, then we prove that, for all p, r and any holomorphic vector bundle E over X, $H^{p,r}_{\Psi_1}(X, E)$ is Hausdorff and $\dim H^{p,r}_{\Psi_1}(X, E) = \dim H^{n-p,n-r}_{\Psi_2}(X, E^*)$. If $2 \leq r \leq n-2$, moreover $\dim H^{p,r}_{\Psi_1}(X, E) < \infty$. The reason for this is that all ends of X are 1-concave. We study also the case when the ends of X satisfy some other convexity conditions.

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1. – introduction

Let X be an n-dimensional complex manifold which is connected and non-compact. Further let E be a holomorphic vector bundle over X, and E^* the dual of E.

We use the following standard notations: $\mathcal{E}^{p,r}(X, E)$, $0 \leq p, r \leq n$, is the Fréchet space of E-valued C^∞-forms of bidegree (p, r) on X given the topology of uniform convergence of the forms and all their derivatives on compact sets; $\mathcal{E}^{p,-1}(X, E) := 0$. For each closed $C \subseteq X$, $\mathcal{D}^{p,r}_C(X, E)$ denotes the space of all $f \in \mathcal{E}^{p,r}(X, E)$ with $\operatorname{supp} f \subseteq C$. $\mathcal{D}^{p,r}_C(X, E)$ will be considered as a Fréchet space with the topology induced from $\mathcal{E}^{p,r}(X, E)$. $\mathcal{D}^{p,r}(X, E)$ is the space of all $f \in \mathcal{E}^{p,r}(X, E)$ with compact support endowed with the finest local convex topology such that, for each compact set $K \Subset X$, the embedding $\mathcal{D}^{p,r}_K(X, E) \to \mathcal{D}^{p,r}(X, E)$ is continuous (the Schwartz topology).

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A family Ψ of closed subsets of X will be called a *family of supports* (in the sense of Serre [Se]) if

(a) with each $C \in \Psi$ also all closed subsets of C belong to Ψ;

(b) for each $C \in \Psi$ there exists an open neighborhood U of C with $\overline{U} \in \Psi$,

(c) if $A, B \in \Psi$ then $A \cup B \in \Psi$.

If Ψ is a family of supports, then we denote by $\mathcal{D}_{\Psi}^{p,r}(X, E)$ the space of forms $f \in \mathcal{E}^{p,r}(X, E)$ with $\operatorname{supp} f \in \Psi$, given the finest local convex topology such that, for each $C \in \Psi$, the embedding $\mathcal{D}_{C}^{p,r}(X, E) \to \mathcal{D}_{\Psi}^{p,r}(X, E)$ is continuous. Further, then we consider the factor space

$$H_{\Psi}^{p,r}(X, E) = \frac{\mathcal{D}_{\Psi}^{p,r}(X, E) \cap \operatorname{Ker} \overline{\partial}}{\overline{\partial} \mathcal{D}_{\Psi}^{p,r-1}(X, E)}$$

as topological vector space endowed with the factor topology.

Definition 1.1. (i) A C^{∞} function $\rho: X \to \mathbb{R}$ will be called a *double exhausting function for* X if all critical points of ρ are non-degenerate, if ρ has no absolute minimum and no absolute maximum and if the sets $\{s \leq \rho \leq t\}$, $\inf \rho < s \leq t < \sup \rho$, are compact.

(ii) A double exhausting function ρ of X will be called *of type* $F_{\text{left}}(r)$, $r \in \{0, \ldots, n\}$, if there exists a number $s_0 \in]\inf \rho, \sup \rho[$ such that at least one of the following two conditions is fulfilled:

- $2 \leq r \leq n$ and the Levi form of ρ has at least $n-r+2$ positive eigenvalues on $\{\inf \rho < \rho \leq s_0\}$ (i.e. at the ends defined by the small values of ρ, X is $(r-1)$-concave).
- $0 \leq r \leq n-1$ and the Levi form of ρ has at least $r+1$ negative eigenvalues on $\{\inf \rho < \rho \leq s_0\}$ (i.e. at the ends defined by the small values of ρ, X is $(n-r)$-convex).

The number s_0 then will be called a *left r-exceptional value for* ρ.

(iii) A double exhausting function ρ of X will be called *of type* $F_{\text{right}}(r)$, $r \in \{0, \ldots, n\}$, if there exists a number $t_0 \in]\inf \rho, \sup \rho[$ such that at least one of the following two conditions is fulfilled:

- $1 \leq r \leq n$ and the Levi form of ρ has at least $n-r+1$ positive eigenvalues on $\{t_0 \leq \rho < \sup \rho\}$ (i.e. at the ends defined by the large values of ρ, X is r-convex).
- $0 \leq r \leq n-2$ and the Levi form of ρ has at least $r+2$ negative eigenvalues on $\{\rho = t_0\}$, and at least $r+1$ *negative* eigenvalues on $\{t_0 \leq \rho < \sup \rho\}$ (i.e. at the ends defined by the large values of ρ, X is $(n-r)$-concave where, moreover, the boundary of $\{\rho < t_0\}$ is even strictly $(n-r-1)$-concave and X is an *$(n-r)$-concave extension of* $\{\rho < t_0\}$).

The number t_0 then will be called a *right r-exceptional value for ρ.*

(iv) A double exhausting function ρ of X will be called *of type $F(r)$*, $r \in \{0, \dots, n\}$, if ρ is both of type $F_{\text{left}}(r)$ and of type $F_{\text{right}}(r)$. If s_0 is a left r-exceptional value for ρ, t_0 is a right r-exceptional value for ρ and moreover $s_0 < t_0$, then $[s_0, t_0]$ will be called an *r-exceptional interval for ρ.*

(v) With any double exhausting function ρ of X we associate the following two families of supports $\Phi(\rho)$ and $\Phi^*(\rho)$: $\Phi(\rho)$ consists of all closed sets $C \subseteq X$ such that, for some $t \in]\inf \rho, \sup \rho[$, $C \subseteq \{\rho \leq t\}$, and $\Phi^*(\rho)$ consists of all closed sets $C \subseteq X$ such that, for some $s \in]\inf \rho, \sup \rho[$, $C \subseteq \{\rho \geq s\}$. Note that both $\Phi(\rho)$ and $\Phi^*(\rho)$ are *cofinal* in the sense of Chirka and Stout [CS][(1)].

REMARK 1.2. If ρ is a double exhausting function for X which is of type $F_{\text{left}}(n)$, then we can always find a double exhausting function $\tilde{\rho}$ for X which is even of type $F(n)$ such that $\Phi(\tilde{\rho}) = \Phi(\rho)$ and $\Phi^*(\tilde{\rho}) = \Phi^*(\rho)$. This follows easily from the theorem of Green and Wu [GW] (see also [O]) that any connected non-compact complex manifold is n-convex.

Our main results are the following two theorems (the proofs will be given in Section 4):

THEOREM 1.3. *Let ρ be a double exhausting function for X, $\Phi = \Phi(\rho)$, $\Phi^* = \Phi^*(\rho)$, $0 \leq p \leq n$ and $1 \leq r \leq n$. If $1 \leq r \leq n-1$ and ρ is of type $F(r)$, or $r = n$ and ρ is of type $F_{\text{left}}(n)$, then:*

(i) $\dim H^{p,r}_{\Phi^*}(X, E) < \infty$

(ii) *$H^{p,r}_{\Phi^*}(X, E)$ is Hausdorff.*

(iii) *$H^{p,n-r+1}_{\Phi}(X, E)$ is Hausdorff.*

REMARK 1.4. If Φ^* is the family of compact subsets or the family of all closed sets, then finite dimensionality of $H^{p,r}_{\Phi^*}(X, E)$ always implies Hausdorffness of $H^{p,r}_{\Phi^*}(X, E)$, $1 \leq r \leq n$. This is not clear if $\Phi^* = \Phi^*(\rho)$ where ρ is a double exhausting function without additional conditions (cp. conjecture 1.6 in [LaLe2], where this problem is discussed). Part (i) of Theorem 1.3 will be proved in Section 2 in a direct way using well known local integral formulas whereas (ii) can be proved only in Section 4 using results on Serre duality obtained in [LaLe2].

THEOREM 1.5. *Let ρ be a double exhausting function for X, $\Phi = \Phi(\rho)$, $\Phi^* = \Phi^*(\rho)$, $0 \leq p \leq n$ and $1 \leq r \leq n-1$. Suppose one of the following two conditions is fulfilled: $1 \leq r \leq n-2$ and ρ is both of type $F(r)$ and $F(r+1)$, or $r = n-1$ and ρ is both of type $F(n-1)$ and $F_{\text{left}}(n)$. Then*

$$\dim H^{p,n-r}_{\Phi}(X, E) = \dim H^{n-p,r}_{\Phi^*}(X, E^*) < \infty,$$

where E^* is the dual bundle of E.

(1) *Cofinal* means that there exists a sequence C_j in the family such that all other sets of the family are contained in some C_j.

Note the following

COROLLARY 1.6. *Suppose X is 1-concave, i.e. there exists a C^∞-function $\varphi : X \to \mathbb{R}$ without absolute minimum such that the sets $\{\varphi \geq t\}$, $t > \inf\varphi$, are compact and, for certain $t_0 > \inf\varphi$, φ is strictly plurisubharmonic on $\{\varphi < t_0\}$.*

Moreover we assume that the ends of X are divided into two parts, i.e., for certain $t_1 > \inf\varphi$, $\{\varphi < t_1\} = U_1 \cup U_2$, where U_1 and U_2 are non-empty open sets with $U_1 \cap U_2 = \emptyset$, and we denote by Ψ_j, $j = 1, 2$, the family of all closed sets $C \subseteq X$ such that $C \cap \overline{U}_j$ is compact.Then, for all p:

(i) $H^{p,r}_{\Psi_j}(X, E)$ *is Hausdorff for $j = 1, 2$ and all r.*

(ii) *If $\{j, j^*\} = \{1, 2\}$, then* $\dim H^{p,r}_{\Psi_j}(X, E) = \dim H^{n-p,n-r}_{\Psi_{j^*}}(X, E^*)$ *for all r.*

(iii) $\dim H^{p,r}_{\Psi_j}(X, E) < \infty$ *for $j = 1, 2$ if $2 \leq r \leq n - 2$.*

(iv) $H^{p,0}_{\Psi_j}(X, E) = 0$ *and* $H^{p,n}_{\Psi_j}(X, E) = 0$ *for $j = 1, 2$.*

REMARK 1.7. An example for the situation considered in this corollary can be obtained as follows: Let Y be a compact complex space whose singular part is isolated and divided into two non-empty parts S_1 and S_2. Set $X = Y \setminus (S_1 \cup S_2)$ and denote by Ψ_j, $j = 1, 2$, the family of closed sets $C \subseteq X$ such that $S_j \cup (X \setminus C)$ is a neighborhood of S_j.

This example was first studied by K. Miyazawa [Mi]: Using Ohsawa's L^2-methods he proved that $H^{p,n-1}_{\Psi_j}(X, E)$ is Hausdorff provided the bundle $K_X^{-1} \otimes E$ is extendable to S_j as a holomorphic vector bundle.

PROOF OF COROLLARY 1.6. Since X is connected and, for all $C \in \Psi_1 \cup \Psi_2$, $X \setminus C$ is open and non-empty, by uniqueness of holomorphic functions, we have

$$H^{p,0}_{\Psi_j}(X, E) = 0 \quad \text{for } j = 1, 2\,. \tag{1.1}$$

To prove the other statements, we assume that the critical points of φ are non-degenerate (as always possible, cp., e.g., [GP]). Then it is easy to construct two double exhausting functions ρ_1 and ρ_2 for X with $\rho_j|_{U_j} = \varphi|_{U_j}$ for $j = 1, 2$ and $\rho_j|_{U_{j^*}} = -\varphi|_{U_{j^*}}$ if $\{j, j^*\} = \{1, 2\}$. Then

$$\Psi_1 = \Phi^*(\rho_1) \quad \text{and} \quad \Psi_2 = \Phi^*(\rho_2)$$

and both ρ_1 and ρ_2 are of type $F_{\text{left}}(n)$ and of type $F(r)$ for $2 \leq r \leq n - 2$. By part (i) of Theorem 1.3 this implies statement (iii) of the corollary.

Moreover this implies that $H^{p,r}_{\Psi_j}(X, E)$ is Hausdorff if $r \in \{2, \dots, n - 2, n\}$ (by part (ii) of Theorem 1.3) or $r \in \{1, 3, \dots, n - 1\}$ (by part (iii) of Theorem 1.3). As $\{2, \dots, n - 2, n\} \cup \{1, 3, \dots, n - 1\} = \{1, 2, 3, \dots, n\}$. Together with (1.1) this implies statement (i) of the corollary.

(ii) follows from (i) by well known arguments from the theory of Serre duality (cp. Lemma 3.1 (iii) below).

(iv) follows from (1.1) and (ii). □

2. – Shrinking of the support and a first finiteness result

In this section, X is an n-dimensional complex manifold with a double exhausting function ρ, and $E \to X$ is a holomorphic vector bundle over X. If $D \subseteq X$ is open and $K \subseteq \overline{D}$ is closed, then we denote by $\mathcal{C}_K^{p,r}(\overline{D}, E)$, $0 \le p, r \le n$, the space of all continuous E-valued (p,r)-forms f on $\overline{D}$ with $\operatorname{supp} f \subseteq K$, and by $\mathcal{H}_K^{p,r}(\overline{D}, E)$ we denote the space of all forms in $\mathcal{C}_K^{p,r}(\overline{D}, E)$ which are locally Hölder continuous with exponent $1/2$. If $K = \overline{D}$ then we write $\mathcal{C}^{p,r}(\overline{D}, E)$ and $\mathcal{H}^{p,r}(\overline{D}, E)$ instead of $\mathcal{C}_{\overline{D}}^{p,r}(\overline{D}, E)$ and $\mathcal{H}_{\overline{D}}^{p,r}(\overline{D}, E)$. If K is compact, then we consider $\mathcal{C}_K^{p,r}(\overline{D}, E)$ and $\mathcal{H}_K^{p,r}(\overline{D}, E)$ as Banach spaces given, respectively, the maximum norm and the Hölder norm with exponent $1/2$. We write

$$\mathcal{C}_{\Phi^*(\rho)}^{p,r}(\overline{D}, E) = \bigcup_{C \in \Phi^*(\rho)} \mathcal{C}_{C \cap \overline{D}}^{p,r}(\overline{D}, E) \text{ and } \mathcal{H}_{\Phi^*(\rho)}^{p,r}(\overline{D}, E) = \bigcup_{C \in \Phi^*(\rho)} \mathcal{H}_{C \cap \overline{D}}^{p,r}(\overline{D}, E).$$

For each interval $I \subseteq \mathbb{R}$, we set

$$DI = D(I) = \{z \in X \mid \rho(z) \in I\}.$$

LEMMA 2.1. *Let $p \in \{0, 1, \dots, n\}$ and $r \in \{1, \dots, n\}$. Suppose ρ is of type $F_{\text{left}}(r)$ and $s_0 \in]\inf \rho, \sup \rho[$ is a left r-exceptional value for ρ. Then:*

(i) *If $2 \le r \le n$ and the Levi form of ρ has at least $n - r + 2$ positive eigenvalues on $D]\inf \rho, s_0]$, then, for all s, ε with $\inf \rho < s - \varepsilon < s < s_0$, there exists a continuous linear operator*

$$A : \mathcal{C}_{D[s,s_0]}^{p,r}(D]\inf \rho, s_0], E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}_{D[s-\varepsilon,s_0]}^{p,r-1}(D]\inf \rho, s_0], E)$$

such that $\overline{\partial} A f = f$ for all $f \in \mathcal{C}_{D[s,s_0]}^{p,r}(D]\inf \rho, s_0], E) \cap \operatorname{Ker} \overline{\partial}$.

(ii) *If $0 \le r \le n-1$ and the Levi form of ρ has at least $r+1$ negative eigenvalues on $D]\inf \rho, s_0]$, then, for all $s \in \mathbb{R}$ and all $\varepsilon > 0$ with $\inf \rho < s - \varepsilon < s < s_0 - \varepsilon$, there exists a continuous linear operator*

$$A : \mathcal{C}_{D[s,s_0]}^{p,r}(D]\inf \rho, s_0], E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}_{D[s-\varepsilon,s_0-\varepsilon]}^{p,r-1}(D]\inf \rho, s_0 - \varepsilon], E)$$

such that $\overline{\partial} A f = f|_{D]\inf \rho, s_0 - \varepsilon]}$ for all $f \in \mathcal{C}_{D[s,s_0]}^{p,r}(D]\inf \rho, s_0], E) \cap \operatorname{Ker} \overline{\partial}$.

REMARK 2.2. We believe that part (ii) of this lemma can be stated in the following stronger form (similar to part (i)):

(ii′) *If $0 \le r \le n-1$ and the Levi form of ρ has at least $r+1$ negative eigenvalues on $D]\inf \rho, s_0]$, then, for all $s \in \mathbb{R}$ and all $\varepsilon > 0$ with $\inf \rho < s - \varepsilon < s < s_0$, there exists a continuous linear operator*

$$A : \mathcal{C}_{D[s,s_0]}^{p,r}(D]\inf \rho, s_0], E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}_{D[s-\varepsilon,s_0]}^{p,r-1}(D]\inf \rho, s_0], E)$$

such that $\overline{\partial} A f = f$ for all $f \in \mathcal{C}_{D[s,s_0]}^{p,r}(D]\inf \rho, s_0], E) \cap \operatorname{Ker} \overline{\partial}$.

A hint that (ii′) should be true is Theorem 3.1 in [LaLe1] which claims: *If* $0 \le r \le n-1$ *and the Levi form of* ρ *has at least* $r+1$ *negative eigenvalues on* $D]\inf\rho, s_0]$*, and the level set* $\{\rho = s_0\}$ *is smooth, then, for all* $s \in]\inf\rho, s_0[$ *and each* $f \in \mathcal{C}^{p,r}_{D[s,s_0]}(D]\inf\rho, s_0], E) \cap \operatorname{Ker}\overline{\partial}$*, there exists* $u \in \mathcal{H}^{p,r-1}(D]\inf\rho, s_0], E)$ *with* $\operatorname{supp} u \in \Phi^*(\rho)$ *and* $\overline{\partial}u = f$. But it seems that a complete proof of (ii′) requires certain technical effort, which we want to avoid, because for the purpose of the present paper statement (ii) is sufficient.

PROOF OF LEMMA 2.1. Essentially, this lemma is contained already in [Ho], [FiLi], [Li], [HeLe], althought not explicitely stated. Using the terminology of [HeLe], in the following we show how to obtain it from those assertions which are explicitly stated and proved in [HeLe].

PROOF OF PART (i). That the Levi form of ρ has at least $n-r+2$ positive eigenvalues on $D]\inf\rho, s_0]$ means, in the sense of Definitions 4.3 and 12.1 of [HeLe], that ρ is $(n-r+2)$-convex(2) on $D]\inf\rho, s_0]$, and that $D]\inf\rho, s_0]$ is a strictly $(n-r+1)$-convex extension of $D]\inf\rho, s]$. Therefore, by Lemma 12.3 in [HeLe], there exists a finite number of open sets $W_j \subseteq X$ $(0 \le j \le N)$ and $V_j \Subset X$ $(1 \le j \le N)$ with

$$D]\inf\rho, s[= W_0 \subseteq W_1 \subseteq \ldots \subseteq W_N = D]\inf\rho, s_0[$$

such that, for each $j \in \{1, \ldots, N\}$, we have one of the following cases:

CASE 1. V_j is C^∞ smooth and strictly convex in the real linear sense (with respect to some local holomorphic coordinates in a neighborhood of $\overline{V_j}$) such that $\overline{W_{j-1}} \cap \overline{V_j} = \emptyset$ and $W_j = W_{j-1} \cup V_j$.

CASE 2. $[W_{j-1}, W_j, V_j]$ is an $(n-r+1)$-convex extension element in X (in the sense of Definition 12.2 of [HeLe]).

For $j = 0, 1, \ldots, N$, we consider the following statement $S(j)$: *There exists a continuous linear operator*

$$A_j : \mathcal{C}^{p,r}_{\overline{W_j} \cap D[s,s_0]}(\overline{W_j}, E) \cap \operatorname{Ker}\overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}_{\overline{W_j} \cap D[s-\varepsilon,s_0]}(\overline{W_j}, E)$$

such that $\overline{\partial}A_j f = f$ *for all* $f \in \mathcal{C}^{p,r}_{\overline{W_j} \cap D[s,s_0]}(\overline{W_j}, E) \cap \operatorname{Ker}\overline{\partial}$. We have to prove that $S(N)$ is true. It is trivial that $S(0)$ is true (set $A_0 = 0$).

Assume now that, for some $j \in \{1, \ldots, N\}$, $S(j-1)$ is true and prove that this implies $S(j)$.

First consider Case 1. Then we take the Henkin operator T for V_j (cp. Theorem 2.12. in [HeLe] for the definition of T, and Theorem 9.1 in [HeLe] for the estimates). T is a continuous linear operator

$$T : \mathcal{C}^{p,r}(\overline{V_j}, E) \cap \operatorname{Ker}\overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}(\overline{V_j}, E)$$

(2)Note that in [HeLe], *q-convexity in the sense of Andreotti-Grauert* is called $(n-q)$*-convexity.*

with $\overline{\partial} Tf = f$ for all $f \in \mathcal{C}^{p,r}(\overline{V_j}, E) \cap \operatorname{Ker} \overline{\partial}$. For $f \in \mathcal{C}^{p,r}_{\overline{W_j \cap D[s,t]}}(\overline{W_j}, E) \cap \operatorname{Ker} \overline{\partial}$ now it remains to set $A_j f = A_{j-1} f$ on $\overline{W_{j-1}}$ and $A_j f = Tf$ on $\overline{V_j}$.

Now we consider Case 2. Then we take open sets $U \Subset U' \Subset V_j$ such that also $[W_{j-1}, W_j, U]$ and $[W_{j-1}, W_j, U']$ are $(n-r+1)$-convex extension elements. Moreover we take a real C^∞ function χ on X such that $\chi \equiv 1$ in a neighborhood of $\overline{W_j} \setminus W_{j-1}$ and $\operatorname{supp} \chi \Subset U \cap D]s - \varepsilon, \sup \rho[$. By Theorems 7.8 and 9.1 in [HeLe], we have continuous linear operators

$$T_j : \mathcal{C}^{p,r}(\overline{W_j \cap V_j}, E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}(\overline{W_j \cap U'}, E)$$

and

$$T_{j-1} : \mathcal{C}^{p,r-1}(\overline{W_{j-1} \cap U'}, E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}^{p,r-2}(\overline{W_{j-1} \cap U}, E)$$

such that $\overline{\partial} T_j f = f|_{\overline{W_j \cap U'}}$ for all $f \in \mathcal{C}^{p,r}(\overline{W_j \cap V_j}, E) \cap \operatorname{Ker} \overline{\partial}$ and $\overline{\partial} T_{j-1} f = f|_{\overline{W_{j-1} \cap U}}$ for all $f \in \mathcal{C}^{p,r-1}(\overline{W_{j-1} \cap U'}, E) \cap \operatorname{Ker} \overline{\partial}$. It remains to set

$$A_j = (1-\chi) A_{j-1} + \chi T_j - \overline{\partial} \chi \wedge T_{j-1}(A_{j-1} - T_j). \qquad \square$$

Proof of part (ii). That the Levi form of ρ has at least $r+1$ negative eigenvalues on $D]\inf \rho, s_0]$ means, in the sense of Definitions 5.1 and 15.1 in [HeLe], that ρ is $(r+1)$-concave on $D]\inf \rho, s_0]$, and that $D]\inf \rho, s_0]$ is a strictly r-concave extension of $D]\inf \rho, s]$. Then it follows from Lemma 15.5 in [HeLe](3) that there exists a finite number of open sets $W_j \subseteq X$ $(0 \le j \le N)$ and $V_j \Subset D]s - \varepsilon, s_0[$ $(1 \le j \le N)$ with

$$D]\inf \rho, s] = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_N = D]\inf \rho, s_0 - \varepsilon]$$

such that, for each $j \in \{1, \ldots, N\}$, $[W_{j-1}, W_j, V_j]$ is an r-concave extension element in X (in the sense of Definition 15.4 in [HeLe]), and the bundle E is holomorphically trivial over a neighborhood of $\overline{V_j}$. For $j = 0, 1, \ldots, N$, we consider the following statement $S(j)$: *There exists a continuous linear operator*

$$A_j : \mathcal{C}^{p,r}_{D[s,s_0]}(D]\inf \rho, s_0], E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}_{\overline{W_j \cap D[s-\varepsilon, s_0-\varepsilon]}}(\overline{W_j}, E)$$

such that $\overline{\partial} A_j f = f|_{\overline{W_j}}$ *for all* $f \in \mathcal{C}^{p,r}_{D[s,s_0]}(D]\inf \rho, s_0], E)$. We have to prove that $S(N)$ is true. It is trivial that $S(0)$ is true (set $A_0 = 0$).

Assume now that, for some $j \in \{1, \ldots, N\}$, $S(j-1)$ is true. Then we take open sets $U \Subset U' \Subset V_j$ such that also $[W_{j-1}, W_j, U]$ and $[W_{j-1}, W_j, U']$ are r-concave extension elements. Since $U' \Subset V_j$ and, by Lemma 13.5 (i) in [HeLe], V_j is biholomorphically equivalent to some pseudoconvex domain in $\mathbb{C}^n$, it is clear that there exists a contiunuous linear operator

$$T_j : \mathcal{C}^{p,r}(\overline{V_j}, E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}(\overline{U'}, E)$$

(3) There is a misprint in this lemma, *q-convex* must be replaced by *q-concave*.

such that $\overline{\partial} T_j f = f|_{\overline{U'}}$ for all $f \in \mathcal{C}^{p,1}(\overline{V_j}, E) \cap \operatorname{Ker} \overline{\partial}$. Then the sections $A_{j-1}f - T_j f$, $f \in \mathcal{C}^{p,r}_{D[s,s_0]}(D]\inf \rho, s_0], E)$, belong to $\mathcal{C}^{p,r}(\overline{U' \cap W_{j-1}}, E) \cap \operatorname{Ker} \overline{\partial}$.

Now we distinguish the cases $r = 1$ and $r > 1$. First let $r = 1$. Then the sections $A_{j-1}f - T_j f$, $f \in \mathcal{C}^{p,1}_{D[s,s_0]}(D]\inf \rho, s_0], E)$, are holomorphic on $U' \cap W_{j-1}$. By Theorem 13.8 in [HeLe] such sections extend holomorphically to U'. Moreover, from the proof of this theorem it is clear that this extensions are given by a continuous linear operator

$$S : \mathcal{C}^{p,0}(\overline{U' \cap W_{j-1}}, E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}^{p,0}(\overline{U'}, E) \cap \operatorname{Ker} \overline{\partial}$$

($S = L_2^v + L_{23}^v$ with the notation from [HeLe]). It remains to observe that $\overline{W_j} \subseteq \overline{W_{j-1}} \cup \overline{U'}$ and to set

$$A_j f = \begin{cases} A_{j-1} f & \text{on } \overline{W_{j-1}} \\ T_j f + S(A_{j-1} f - T_j f) & \text{on } \overline{U'} \end{cases}$$

for $f \in \mathcal{C}^{p,r}_{D[s,s_0]}(D]\inf \rho, s_0], E) \cap \operatorname{Ker} \overline{\partial}$.

Now let $2 \leq r \leq n-1$. Then, by Theorem 14.2 in [HeLe], we have a continuous linear operator

$$T_{j-1} : \mathcal{C}^{p,r-1}(\overline{W_{j-1} \cap U'}, E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}^{p,r-2}(\overline{W_{j-1} \cap U}, E)$$

such that $\overline{\partial} T f_{j-1} = f|_{\overline{W_{j-1} \cap U}}$ for all $f \in \mathcal{C}^{p,r-1}(\overline{W_{j-1} \cap U'}, E) \cap \operatorname{Ker} \overline{\partial}$. Take a real C^∞ function χ on X such that $\chi \equiv 1$ in a neighborhood of $\overline{W_j} \setminus W_{j-1}$ and $\operatorname{supp} \chi \Subset U \cap D]s - \varepsilon, s_0[$. It remains to set

$$A_j = (1 - \chi) A_{j-1} + \chi T_j - \overline{\partial} \chi \wedge T_{j-1}(A_{j-1} - T_j) . \qquad \square$$

Corollary 2.3. *Let $p \in \{0, 1, \ldots, n\}$ and $r \in \{1, \ldots, n\}$. Suppose ρ is of type $F_{\mathrm{left}}(r)$ and s_0 is a left r-exceptional value for ρ. Then, for each $t \in]s_0, \infty]$, the natural map*

$$\frac{\mathcal{C}^{p,r}_{D[s_0,t]}(D]\inf \rho, t], E) \cap \operatorname{Ker} \overline{\partial}}{\mathcal{C}^{p,r}_{D[s_0,t]}(D]\inf \rho, t], E) \cap \overline{\partial}\big(\mathcal{C}^{p,r-1}_{\Phi^*(\rho)}(D]\inf \rho, t], E)\big)}$$

$$\longrightarrow \frac{\mathcal{C}^{p,r}_{\Phi^*(\rho)}(D]\inf \rho, t], E) \cap \operatorname{Ker} \overline{\partial}}{\mathcal{C}^{p,r}_{\Phi^*(\rho)}(D]\inf \rho, t], E) \cap \overline{\partial}\big(\mathcal{C}^{p,r-1}_{\Phi^*(\rho)}(D]\inf \rho, t], E)\big)}$$

is an isomorphism.

Proof. The injectivity of this map is trivial. To prove the surjectivity, consider a form $f \in \mathcal{C}^{p,r}_{\Phi^*(\rho)}(D]\inf \rho, t], E) \cap \operatorname{Ker} \overline{\partial}$. We have to find a form $u \in \mathcal{C}^{p,r-1}_{\Phi^*(\rho)}(D]\inf \rho, t], E)$ such that $\operatorname{supp}(f - \overline{\partial} u) \subseteq D[s_0, t]$.

Take $s \in]\inf \rho, s_0[$ with $\operatorname{supp} f \subseteq D[s, t]$, and let $\varepsilon > 0$ be so small that $s - \varepsilon > \inf \rho$ and $s_0 + 2\varepsilon$ is also a left r-exceptional value of ρ. By lemma 2.1 then there exists $u_0 \in \mathcal{C}^{p,r-1}_{D[s-\varepsilon, s_0+\varepsilon]}(D]\inf \rho, s_0 + \varepsilon], E)$ with $\overline{\partial} u_0 = f|_{D]\inf \rho, s_0+\varepsilon]}$. Take a C^∞-function χ on X such that $\chi \equiv 1$ on $D]\inf \rho, s_0]$ and $\chi \equiv 0$ in a neighborhood of $D[s_0 + \varepsilon, \sup \rho[$. Then, after extending by zero, the form $u := \chi u_0$ has the required property. $\square$

LEMMA 2.4. *Let $p \in \{0, 1, \ldots, n\}$ and $r \in \{1, \ldots, n\}$. Suppose ρ is of type $F(r)$ and $[s_0, t_0]$ is an r-exceptional interval for ρ. Then, for all s, ε with $\inf\rho < s - \varepsilon < s \leq s_0$, there exist continuous linear operators*

$$A : \mathcal{C}^{p,r}_{D[s,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}_{D[s-\varepsilon,t_0]}(D]\inf\rho, t_0], E)$$

and

$$K : \mathcal{C}^{p,r}_{D[s,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial} \longrightarrow \mathcal{H}^{p,r}_{D[s,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}$$

such that $\overline{\partial}Af = f + Kf$ for all $f \in \mathcal{C}^{p,r}_{D[s,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}$.

PROOF. Take $\delta > 0$ so small that $[s + 2\delta, t_0]$ is also r-exceptional for ρ. Set $U_0 = D]\inf\rho, s + \delta[$. By Lemma 2.1 then there exists a continuous linear operator

$$A_0 : \mathcal{C}^{p,r}_{D[s,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}_{D[s-\varepsilon,s+\delta]}(\overline{U_0}, E)$$

such that $\overline{\partial}A_0 f = f|_{\overline{U_0}}$ for all $f \in \mathcal{C}^{p,r}_{D[s,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}$. Since, in a neighborhood of $\{\rho = t_0\}$, ρ has at least $n - r + 1$ positive, or at least $r + 2$ negative eigenvalues, it follows from Theorems 7.8, 9.1, 14.2 in [HeLe] that we can find a finite number of open sets $U_1, \ldots, U_N \Subset D]s, \sup\rho[$ with

$$D]\inf\rho, t_0] \subseteq U_0 \cup U_1 \cup \ldots \cup U_N$$

such that, for each $j \in \{1, \ldots, N\}$, there exists a continuous linear operator

$$A_j : \mathcal{C}^{p,r}_{D[s,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}(\overline{U_j \cap D]\inf\rho, t_0]}, E)$$

such that $\overline{\partial}A_j f = f|_{\overline{U_j \cap D]\inf\rho,t_0]}}$ for all $f \in \mathcal{C}^{p,r}_{D[s,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}$. Take C^∞ functions χ_j, $j = 0, 1, \ldots, N$, with $\chi_j \equiv 0$ in a neighborhood of $X \setminus U_j$ and $\sum_{j=0}^N \chi_j \equiv 1$ in a neihgborhood of $D]\inf\rho, t_0]$. It remains to set

$$A = \sum_{j=1}^N \chi_j A_j \quad \text{and} \quad K = \sum_{j=0}^N \overline{\partial}\chi_j \wedge A_j\,.$$

COROLLARY 2.5. *Let $p \in \{0, 1, \ldots, n\}$ and $r \in \{1, \ldots, n\}$. Suppose ρ is of type $F(r)$ and $[s_0, t_0]$ is an r-exceptional interval for ρ. Then, for all $\varepsilon > 0$ with $\inf\rho < s_0 - \varepsilon$, the space*

$$\mathcal{C}^{p,r}_{D[s_0,t_0]}(D]\inf\rho, t_0], E) \cap \overline{\partial}\Big(\mathcal{C}^{p,r-1}_{D[s_0-\varepsilon,t_0]}(D]\inf\rho, t_0], E)\Big)$$

is topologically closed (whith respect to the maximum norm) and of finite codimension in $C^{p,r}_{D[s_0,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}$. *In particular,*

$$\dim\left(\frac{C^{p,r}_{D[s_0,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}}{C^{p,r}_{D[s_0,t_0]}(D]\inf\rho, t_0], E) \cap \overline{\partial}\left(C^{p,r-1}_{\Phi^*(\rho)}(D]\inf\rho, t_0], E)\right)}\right) < \infty. \tag{2.2}$$

PROOF. If A is the operator from Lemma 2.4, then, by this lemma and by Ascoli's theorem, $\overline{\partial}A$ is a Fredholm operator in the Banach space $C^{p,r}_{D[s_0,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}$. Hence the image of this operator is of finite codimension and topologically closed in $C^{p,r}_{D[s_0,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}$. As (2.1) is a subspace of $C^{p,r}_{D[s_0,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}$ which contains the image of $\overline{\partial}A$, (2.1) is also topologically closed. □

LEMMA 2.6. *Let* $p \in \{0, 1, \ldots, n\}$, $r \in \{1, \ldots, n\}$. *Suppose* ρ *is of type* $F(r)$, $[s_0, t_0]$ *is an* r*-exceptional interval for* ρ, *and* s, ε *are numbers with* $\inf\rho < s - \varepsilon < s \le s_0$. *Then, for each* $f \in C^{p,r}(X, E) \cap \operatorname{Ker}\overline{\partial}$ *with* $\operatorname{supp} f \subseteq D[s, \sup\rho[$, *the following holds:*

If there exists $u_0 \in C^{p,r-1}(D]\inf\rho, t_0], E)$ *with* $\operatorname{supp} u_0 \subseteq D[s-\varepsilon, t_0]$ *such that* $f|_{D]\inf\rho,t_0]} = \overline{\partial}u_0$, *then there exists* $u \in C^{p,r-1}(X, E)$ *with* $\operatorname{supp} u \subseteq D[s-\varepsilon, \sup\rho[$ *such that* $f = \overline{\partial}u$.

PROOF. Since ρ is of type $F(r)$ and $[s_0, t_0]$ is r-exceptional for ρ, at least one of the following two conditions is satisfied:

(I) $1 \le r \le n-2$ *and the Levi form of* ρ *has at least* $r+1$ *negative eigenvalues on* $D[t_0, \sup\rho[$[(4)]

(II) *The Levi form of* ρ *has at least* $n-r+1$ *positive eigenvalues on* $D[t_0, \sup\rho[$.

First consider the case when condition (I) is fulfilled. Then, in the sense of Definition 15.1 in [HeLe], X is an r-concave extension of $D]\inf\rho, t_0[$. Therefore, by Theorem 16.1 in [HeLe], we can find $u \in C^{p,r-1}(X, E)$ such that $f = \overline{\partial}u$ on X, where, moreover, we can achieve that, for any given $\delta > 0$, $u = u_0$ on $D]\inf\rho, t_0-\delta]$. It remains to choose $\delta > 0$ so small that $t_0-\delta \ge s-\varepsilon$.

Now we consider the case when condition (II) is fulfilled. Then, in the sense of Definition 12.1 in [HeLe], X is an $(n-r)$-convex extension of $D]\inf\rho, t_0[$ and, by means of the proof of Lemma 13.3 in [HeLe], we can find a sequence $(\rho_j)_{j\in\mathbb{N}}$ of double exhausting functions of X as well as sequences $(W_j)_{j\in\mathbb{N}}$ and $(V_j)_{j\in\mathbb{N}^*}$ of open sets in X such that the following holds:

- For each j, the Levi form of ρ_j has at least $n-r+1$ positive eigenvalues on $D[t_0, \sup\rho[$.

(4) In the present proof we do not use explicitely the hypothesis that ρ has at least $r+2$ negative eigenvalues on $\{\rho = t_0\}$, but we use Lemma 2.4 and we do not know whether the assertion of that lemma remains true without this hypothesis.

- $\rho_0 = \rho$ and, for each $j \geq 1$, there are numbers α_j, β_j with $s_0 < \alpha_j < \beta_j < \sup \rho$ such that $\rho_j = \rho$ on $X \setminus D[\alpha_j, \beta_j]$.
- For each j, there is a number t_j with $W_j := \{\inf \rho < \rho_j < t_j\}$.
- $W_0 = D]\inf \rho, t_0]$, $W_j \subseteq W_{j+1}$ for all j, and $\bigcup_{j \in \mathbb{N}} W_j = X$.
- $V_j \Subset D]s_0, \sup \rho[$ for all j.
- For each $j \geq 1$, one of the following conditions is satisfied: (i) *V_j is C^∞ smooth and strictly convex in the real linear sense (with respect to some local holomorphic coordinates in a neighborhood of $\overline{V_j}$) such that $\overline{W_{j-1}} \cap \overline{V_j} = \emptyset$ and $W_j = W_{j-1} \cup V_j$.* (ii) *$[W_{j-1}, W_j, V_j]$ is an $(n-r)$-convex extension element in X (in the sense of Definition* 12.2 *of* [HeLe]*).*

LEMMA A. *For each j there exist continuous linear operators*

$$A_j : C^{p,r}_{\overline{W_j} \cap D[s, \sup \rho[}(\overline{W_j}, E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}_{\overline{W_j} \cap D[s-\varepsilon, \sup \rho[}(\overline{W_j}, E)$$

and

$$K_j : C^{p,r}_{\overline{W_j} \cap D[s, \sup \rho[}(\overline{W_j}, E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}^{p,r}_{\overline{W_j} \cap D[s, \sup \rho[}(\overline{W_j}, E) \cap \operatorname{Ker} \overline{\partial}$$

such that $\overline{\partial} A_j f = f + K_j f$ for all $f \in C^{p,r}_{\overline{W_j} \cap D[s, \sup \rho[}(\overline{W_j}, E)$.

PROOF OF LEMMA A. As $W_0 = D]\inf \rho, t_0]$, for $j = 0$ the assertion holds by Lemma 2.4. Let $j \geq 1$ and assume that the assertion holds for $j-1$. If condition (i) is satisfied, then we take the Henkin operator T for V_j (cp. the proof of Lemma 2.1) and set $A_j f = A_{j-1} f$, $K_j f = K_{j-1} f$ on $\overline{W_{j-1}}$ and $A_j f = Tf$, $K_j f = 0$ on $\overline{V_j}$ for $f \in C^{p,r}_{\overline{W_j} \cap D[s, \sup \rho[}(\overline{W_j}, E) \cap \operatorname{Ker} \overline{\partial}$.

Now let condition (ii) be satisfied. Then we take an open set $U \Subset V_j$ such that also $[W_{j-1}, W_j, U]$ is an $(n-r)$-convex extension element. By Theorems 7.8 and 9.1 in [HeLe], there is a continuous linear operator

$$T : C^{p,r}(\overline{W_j \cap V_j}, E) \cap \operatorname{Ker} \overline{\partial} \longrightarrow \mathcal{H}^{p,r-1}(\overline{W_j \cap U}, E)$$

such that $\overline{\partial} Tf = f|_{\overline{W_j \cap U}}$ for all $f \in C^{p,r}(\overline{W_j \cap V_j}, E) \cap \operatorname{Ker} \overline{\partial}$. Further let χ be a C^∞ function with $\operatorname{supp} \chi \Subset U$ and $\chi \equiv 0$ in a neighborhood of $\overline{W_j} \setminus W_{j-1}$. Setting $A_j = (1-\chi) A_{j-1} + \chi T$ and $K_j = \overline{\partial} \chi \wedge (T - A_{j-1})$ we complete the proof of lemma A.

For the proof of Lemma 2.6 now it sufficient to prove the following two lemmas:

LEMMA B. *For all j, there exists $u_j \in C^{p,r-1}_{\overline{W_j} \cap D[s-\varepsilon, \sup \rho[}(\overline{W_j}, E)$ with $f|_{\overline{W_j}} = \overline{\partial} u_j$.*

LEMMA C. *For all $j \geq 1$, any $v \in C^{p,r-1}_{\overline{W_{j-1}} \cap D[s-\varepsilon, \sup \rho[}(\overline{W_{j-1}}, E) \cap \operatorname{Ker} \overline{\partial}$ can be approximated uniformly on $\overline{W_{j-1}}$ by forms from $C^{p,r-1}_{\overline{W_j} \cap D[s-\varepsilon, \sup \rho[}(\overline{W_j}, E) \cap \operatorname{Ker} \overline{\partial}$.*

Indeed, by Lemma B, then we can find a sequence $u_j \in \mathcal{C}^{p,r-1}_{\overline{W_j \cap D[s-\varepsilon, \sup \rho]}}(\overline{W_j}, E)$, $j \in \mathbb{N}$, with $f|\overline{W_j} = \overline{\partial} u_j$, and, by lemma C, we can modify this sequence so that it converges uniformly on each compact set to some $u \in \mathcal{C}^{p,r-1}(X, E)$, which then has the required properties.

Proof of Lemma B. For $j = 0$ the assertion holds by hypothethis, because $\overline{W_0} = D]\inf\rho, t_0]$. Assume that $j \geq 1$ and the assertion is already proved for $j - 1$.

To prove that then the assertion holds also for j, we first assume that condition (i) is satisfied. Then we set $u_j = u_{j-1}$ on $\overline{W_{j-1}}$ and $u_j = Tf$ on $\overline{V_j}$ where T is the Henkin operator for V_j (cp. the proof of Lemma 2.1).

Now we assume that condition (ii) is satisfied. We have to prove that $f|_{\overline{W_j}}$ belongs to the space

$$\mathcal{C}^{p,r}_{\overline{W_j} \cap D[s, \sup \rho[}(D]\inf\rho, t_0], E) \cap \overline{\partial}\big(\mathcal{C}^{p,r-1}_{\overline{W_j} \cap D[s-\varepsilon, \sup \rho[}(\overline{W_j}, E)\big) . \tag{2.3}$$

Take open sets $U \Subset U' \Subset V_j$ such that also $[W_{j-1}, W_j, U]$ and $[W_{j-1}, W_j, U']$ are $(n-r)$-convex extension elements in X. By Theorem 7.8 in [HeLe] there exists $v \in \mathcal{C}^{p,r-1}(\overline{W_j} \cap U', E)$ such that $f|_{\overline{W_j} \cap U'} = \overline{\partial} v$. By Theorem 10.1 in [HeLe], there is a sequence $w_\nu \in \mathcal{C}^{p,r-1}(\overline{U'}, E) \cap \operatorname{Ker} \overline{\partial}$, $\nu \in \mathbb{N}$, which converges to $v - u_{j-1}$, uniformly on $\overline{U \cap W_{j-1}}$. Take a C^∞ function χ with $\operatorname{supp} \chi \Subset U$ and $\chi \equiv 1$ in a neighborhood of $\overline{W_j} \setminus W_{j-1}$, and set $\varphi_\nu = (1-\chi)u_{j-1} + \chi(v - w_\nu)$ on $\overline{W_j}$. Then the sequence $\overline{\partial}\varphi_\nu = f + \overline{\partial}\chi \wedge (v - u_{j-1} - w_\nu)$ belongs to the space (2.3) and converges to f, uniformly on $\overline{W_j}$. This completes the proof of Lemma B, because it follows from lemma A (in the same way as Corollary 2.5 follows from Lemma 2.4) that the space (2.3) is topologically closed with respect to the maximum norm.

Proof of Lemma C. Take an open set $U \Subset V_j$ such that also $[W_{j-1}, W_j, U]$ is an $(n-r)$-convex extension element in X. By Theorem 10.1 in [HeLe], there is a sequence $w_\nu \in \mathcal{C}^{p,r-1}(V_j, E) \cap \operatorname{Ker} \overline{\partial}$, $\nu \in \mathbb{N}$, which converges to v, uniformly on $\overline{U \cap W_{j-1}}$. Take a C^∞ function χ with $\operatorname{supp} \chi \Subset U$ and $\chi \equiv 1$ in a neighborhood of $\overline{W_j} \setminus W_{j-1}$, and set $\varphi_\nu = (1-\chi)v + \chi(w_\nu)$ on $\overline{W_j}$. Then the sequence $\overline{\partial}\varphi_\nu = \overline{\partial}\chi \wedge (w_\nu - v)$ belongs to the space (2.3) and converges to zero, uniformly on $\overline{W_j}$. Since the space (2.3) is topologically closed (cp. the proof of Lemma B), it follows from Banach's open mapping theorem that there exists a sequence $\psi_\nu \in \mathcal{C}^{p,r-1}_{\overline{W_j} \cap D[s-\varepsilon, \sup \rho[}(\overline{W_j}, E)$ which also converges to zero, uniformly on $\overline{W_j}$, such that $\overline{\partial}\psi_\nu = \overline{\partial}\varphi_\nu$. Then the sequence $v_\nu := \varphi_\nu - \psi_\nu$ belongs to $\mathcal{C}^{p,r-1}_{\overline{W_j} \cap D[s-\varepsilon, \sup \rho[}(\overline{W_j}, E) \cap \operatorname{Ker} \overline{\partial}$ and converges to v, uniformly on $\overline{W_{j-1}}$. □

Corollary 2.7. *Let $p \in \{0, 1, \ldots, n\}$ and $r \in \{1, \ldots, n\}$. Suppose ρ is of type $F(r)$ and $[s_0, t_0]$ is an r-exceptional interval for ρ. Then the natural map*

$$R_{t_0} : H^{p,r}_{\Phi^*(\rho)}(X, E) \longrightarrow \frac{\mathcal{C}^{p,r}_{\Phi^*(\rho)}(D]\inf\rho, t_0], E) \cap \operatorname{Ker} \overline{\partial}}{\mathcal{C}^{p,r-1}_{\Phi^*(\rho)}(D]\inf\rho, t_0], E) \cap \overline{\partial}\big(\mathcal{C}^{p,r-1}_{\Phi^*(\rho)}(D]\inf\rho, t_0], E)\big)}$$

is injective.

Proof. Let $F \in H^{p,r}_{\Phi^*(\rho)}(X, E)$ with $R_{t_0}(F) = 0$ be given. Take $f \in \mathcal{D}^{p,r}_{\Phi^*(\rho)}(X, E)$ which defines F. Then $f|_{D]\inf\rho,t_0]} = \overline{\partial}u_0$ for some $u_0 \in \mathcal{C}^{p,r-1}_{\Phi^*(\rho)}(D]\inf\rho, t_0], E)\big)$. Take s, ε such that $\inf\rho < s-2\varepsilon < s \leq s_0$ and $\operatorname{supp} u_0 \subseteq D[s, t_0]$. Then, by Lemma 2.6, we can find $u \in \mathcal{C}^{p,r-1}(X, E)$ with $\operatorname{supp} u \subseteq D[s-\varepsilon, \sup\rho[$ and $f = \overline{\partial}u$. By regularity of $\overline{\partial}$ (cp., e.g., Corollary 2.15 (ii) in [HeLe]) this implies the existence of $u_\infty \in \mathcal{E}^{p,r-1}(X, E)$ with $\operatorname{supp} u_\infty \subseteq D[s-2\varepsilon, \sup\rho[$ and $f = \overline{\partial}u_\infty$. As $D[s-2\varepsilon, \sup\rho[$ belongs to $\Phi^*(\rho)$, this means $F = 0$. $\square$

Lemma 2.8. *Let $0 \leq p \leq n$ and $1 \leq r \leq n$. Suppose one of the following two conditions is fulfilled: $1 \leq r \leq n-1$ and ρ is of type $F(r)$, or $r = n$ and ρ is of type $F_{\text{left}}(n)$. Then*

$$\dim H^{p,r}_{\Phi^*(\rho)}(X, E) < \infty .$$

Proof. By Remark 1.2 we may assume that ρ is of type $F(r)$ also for $r = n$. Take an r-exceptional interval $[s_0, t_0]$ for ρ. Then, by Corollaries 2.7, 2.3 and 2.5,

$$\begin{aligned}\dim H^{p,r}_{\Phi^*(\rho)}(X) &\leq \dim \frac{\mathcal{C}^{p,r}_{\Phi^*(\rho)}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}}{\mathcal{C}^{p,r-1}_{\Phi^*(\rho)}(D]\inf\rho, t_0], E) \cap \overline{\partial}\big(\mathcal{C}^{p,r-1}_{\Phi^*(\rho)}(D]\inf\rho, t_0], E)\big)} \\ &= \dim \frac{\mathcal{C}^{p,r}_{D[s_0,t_0]}(D]\inf\rho, t_0], E) \cap \operatorname{Ker}\overline{\partial}}{\mathcal{C}^{p,r}_{D[s_0,t_0]}(D]\inf\rho, t_0], E) \cap \overline{\partial}\big(\mathcal{C}^{p,r-1}_{\Phi^*(\rho)}(D]\inf\rho, t_0], E)\big)} < \infty . \quad \square\end{aligned}$$

3. – Serre duality

Here we first recall some well known facts on Serre duality, and then we give the formulation of the main result of [LaLe2] which we need for the proof of Theorems 1.3 and 1.5. Let X be an n-dimensional complex manifold with a double exhausting function ρ, let $\Phi = \Phi(\rho)$, $\Phi^* = \Phi^*(\rho)$ (cf. Definition 1.1), and let $E \to X$ be a holomorphic vector bundle over X.

Denote by $\big(H^{p,q}_{\Phi}(X, E)\big)'$ the space of all continuous linear functionals on $H^{p,q}_{\Phi}(X, E)$, $0 \leq p, q \leq n$. Since, for $C \in \Phi$ and $C^* \in \Phi^*$, the intersection $C \cap C^*$ is compact, for each $\varphi \in \mathcal{D}^{n-p,n-q}_{\Phi^*}(X, E^*) \cap \operatorname{Ker}\overline{\partial}$, by setting

$$\varphi'(f) := \int_X \varphi \wedge f \qquad \text{for} \quad f \in \mathcal{D}^{p,q}_{\Phi}(X, E) \cap \operatorname{Ker}\overline{\partial} ,$$

we obtain a continuous linear functional φ' on $\mathcal{D}^{p,q}_{\Phi}(X, E) \cap \operatorname{Ker}\overline{\partial}$, where, by Stokes' theorem, $\varphi'(f) = 0$ if $\varphi \in \overline{\partial}\mathcal{D}^{n-p,n-q-1}_{\Phi^*}(X, E^*)$ or $f \in \overline{\partial}\mathcal{D}^{p,q-1}_{\Phi}(X, E)$. Hence in this way a natural linear map

$$S_{p,q} : H^{n-p,n-q}_{\Phi^*}(X, E^*) \longrightarrow \big(H^{p,q}_{\Phi}(X, E)\big)' \tag{3.1}$$

is defined, $0 \leq p, q \leq n$.

LEMMA 3.1. *For all integers p, q with $0 \leq p, q \leq n$ we have:*

(i) *$S_{p,q}$ is always surjective.*

(ii) *If $H_{\Phi^*}^{n-p,n-q}(X, E^*)$ is Hausdorff, then $S_{p,q}$ is an isomorphism.*

(iii) *If both $H_{\Phi^*}^{n-p,n-q}(X, E^*)$ and $H_{\Phi}^{p,q}(X, E)$ are Hausdorff, then moreover*

$$\dim H_{\Phi^*}^{n-p,n-q}(X, E^*) = \dim H_{\Phi}^{p,q}(X, E)\,.$$

PROOF. PART (i). Let $F \in \big(H_{\Phi}^{p,q}(X, E)\big)'$ be given and let $p : \mathcal{D}_{\Phi}^{p,q}(X, E) \cap \operatorname{Ker}\overline{\partial} \longrightarrow H_{\Phi}^{p,q}(X, E)$ be the canonical projection. We have to find a form $\varphi \in \mathcal{D}_{\Phi^*}^{n-p,n-q}(X, E^*)$ with

$$\int_X \varphi \wedge f = (F \circ p)(f) \qquad \text{for all} \quad f \in \mathcal{D}_{\Phi}^{p,q}(X, E) \cap \operatorname{Ker}\overline{\partial}\,. \tag{3.2}$$

By the Hahn-Banach theorem, we can find a continuous linear functional $F' : \mathcal{D}_{\Phi}^{p,q}(X, E) \to \mathbb{C}$ with $F' = F \circ p$ on $\mathcal{D}_{\Phi}^{p,q}(X, E) \cap \operatorname{Ker}\overline{\partial}$. Since F' is continuous on $\mathcal{D}_{\Phi}^{p,q}(X, E)$, it follows (cp., e.g. Lemma 2.4 in [LaLe2]) that, $\operatorname{supp} F' \in \Phi^*$. Denote by T the $(n-p, n-q)$-current defined by F' on X. Since F' vanishes on $\overline{\partial}\mathcal{D}^{p,q-1}(X, E)$, then $\overline{\partial}T = 0$. Moreover, $\operatorname{supp} T = \operatorname{supp} F' \in \Phi^*$. By regularity of $\overline{\partial}$ (cp., e.g., Lemma 2.15 in [HeLe]), now we can find a form $\varphi \in \mathcal{D}_{\Phi^*}^{n-p,n-q}(X, E^*)$ such that $\varphi - T = \overline{\partial}S$ for certain E^*-valued current S with $\operatorname{supp} S \in \Phi^*$. We shall see that (3.2) holds for this φ, indeed, since $\operatorname{supp} S \in \Phi^*$, it is clear that $\overline{\partial}S(f) = 0$ and therefore $\int_X \varphi \wedge f = F'(f) = (F \circ p)(f)$ for all $f \in \mathcal{D}_{\Phi}^{p,q}(X, E)$.

PART (ii). By part (i) we only have to prove that $S_{p,q}$ is injective if $H_{\Phi^*}^{n-p,n-q}(X, E^*)$ is Hausdorff. But this follows from the fact that, by the Hahn-Banach theorem and regularity of $\overline{\partial}$ (see, e.g., Lemma 2.5 in [LaLe2] for the details), the kernel of $S_{p,q}$ is always equal to

$$\overline{\overline{\partial}\mathcal{D}_{\Phi^*}^{n-p,n-q-1}(X, E^*)} \Big/ \overline{\partial}\mathcal{D}_{\Phi^*}^{n-p,n-q-1}(X, E^*)\,.$$

PART (iii). If $H_{\Phi}^{p,q}(X, E)$ is Hausdorff, then, by the Hahn-Banach theorem,

$$\dim\big(\big(H_{\Phi}^{p,q}(X, E)\big)'\big) = \dim H_{\Phi}^{p,q}(X, E)\,.$$

Together with part (ii) this yields the assertion. $\square$

DEFINITION 3.2. Let $0 \le p \le n$ and $1 \le q \le n$. For $\lambda \in]\inf\rho, \sup\rho[$ we set[(5)]

$$X_\lambda = \{z \in X \mid \rho(z) > \lambda\},$$
$$\Phi_\lambda = \{C \cap X_\lambda \mid C \in \Phi\} \quad \text{and} \quad \Phi^*_\lambda = \{C \mid C \in \Phi^* \text{ and } C \subseteq X_\lambda\}.$$

We say $H^{p,q}_\Phi(X, E)$ is *α-Hausdorff* if, for each $C \in \Phi$, the space $\mathcal{D}^{p,q}_C(X, E) \cap \bar\partial \mathcal{D}^{p,q-1}_\Phi(X, E)$ is topologically closed in $\mathcal{D}^{p,q}_C(X, E)$ (with respect to the Fréchet topology induced by $\mathcal{E}^{p,q}(X, E)$).

We say $H^{p,q}_\Phi(X, E)$ is *β-Hausdorff* if it is α-Hausdorff and moreover, for each $s \in]\inf\rho, \sup\rho[$, there exists $\lambda \in]\inf\rho, s[$ such that $H^{p,q}_{\Phi_\lambda}(X_\lambda, E)$ is also α-Hausdorff.[(6)]

In [LaLe2] the following duality result is obtained which is basic for our proof of Theorems 1.3 and 1.5:

THEOREM 3.3. *Let $0 \le p \le n$ and $1 \le q \le n$. If $H^{p,q}_\Phi(X, E)$ is β-Hausdorff then $H^{n-p,n-q+1}_{\Phi^*}(X, E^*)$ is Hausdorff.*

4. – Proof of Theorems 1.3 and 1.5

PROOF OF THEOREM 1.3. Part (i) is already proved (Lemma 2.8). To prove (ii) and (iii), take a left r-exceptional value s_0 of ρ as well as a number $t_0 \in]s_0, \sup\rho[$ such that if $1 \le r \le n-1$, then t_0 is a right r-exceptional value of ρ. Let $\rho_{s,t}$ be the restriction of ρ to $D]s, t[$. Then, for all $s \in [\inf\rho, s_0]$ and $t \in [t_0, \sup\rho]$, $\rho_{s,t}$ is a double exhausting function for the manifold $D]s, t[$ which is of type $F(r)$ if $1 \le r \le n-1$ and of type $F_{\text{left}}(n)$ if $r = n$. Hence, by Lemma 2.8,

$$\dim H^{n-p,r}_{\Phi^*(\rho_{s,t})}(D]s, t[, E^*) < \infty \quad \text{for all } s \in [\inf\rho, s_0] \text{ and } t \in [t_0, \sup\rho].$$

Then, in particular, for all $s \in [\inf\rho, s_0]$ and $t \in [t_0, \sup\rho]$ and for each $C^* \in \Phi^*(\rho_{s,t})$, the space

$$\mathcal{D}^{n-p,r}_{C^*}(D]s, t[, E^*) \cap \bar\partial \mathcal{D}^{n-p,r-1}_{\Phi^*(\rho_{s,t})}(D]s, t[, E^*)$$

(5) With the notations of section 2, $X_\lambda = D]\lambda, \sup\rho[$, $\Phi_\lambda = \Phi(\rho|_{X_\lambda})$ and $\Phi^*_\lambda = \Phi^*(\rho|_{X_\lambda})$.

(6) We do not know, whether β-Hausdorffness is really stronger than α-Hausdorffnes. In all examples, where we are able to prove α-Hausdorffness, it is immediately clear that the same proof gives even β-Hausdorffness, because the reason for the possiblity to prove α-Hausdorffness is in all these examples that the double exhausting function ρ has some properties *outside some compact set*, which does not change if we replace X by $\{z \in X \mid s < \rho(z) < t\}$ when $s \ge \inf\rho$ and $t \le \sup\rho$ are sufficiently close to $\inf\rho$, resp. $\sup\rho$.

is of finite codimension in $\mathcal{D}_{C*}^{n-p,r}(D]s,t[,E^*)$ and hence, by Lemma 2.6 in [LaLe2], topological closed in $\mathcal{D}_{C*}^{n-p,r}(D]s,t[,E^*)$, i.e. $H_{\Phi*(\rho_{s,t})}^{n-p,r}(D]s,t[,E^*)$ is α-Hausdorff for all $s \in [\inf \rho, s_0]$ and $t \in [t_0, \sup \rho]$. Hence, for each $s \in [\inf \rho, s_0]$, $H_{\Phi*(\rho_{s,\sup \rho})}^{n-p,r}(D]s,\sup \rho[,E^*)$ is β-Hausdorf. Therefore we may apply the duality Theorem 3.3 and obtain that $H_{\Phi(\rho_{s,\sup\rho})}^{p,n-r+1}(D]s,\sup \rho[,E)$ is Hausdorff for all $s \in [\inf \rho, s_0]$ and $t \in [t_0, \sup \rho]$.

In particular (for $s = \inf \rho$), this means that part (iii) is true.

Moreover this implies that $H_\Phi^{p,n-r+1}(X,E)$ is also β-Hausdorff. Hence, again by the duality Theorem 3.3, it follows that $H_{\Phi*}^{n-p,r}(X,E^*)$ is Hausdorff. Replacing in the beginning of the proof E by E^* and p by $n-p$, we obtain part (ii). □

PROOF OF THEOREM 1.5. By Theorem 1.3 (iii), $H_\Phi^{p,n-r}(X,E)$ is Hausdorff. Moreover we obtain from Theorem 1.3 (ii) (replacing E by E^* and p by $n-p$) that $H_{\Phi*}^{n-p,r}(X,E^*)$ is Hausdorff. Therefore, by Lemma 3.1, $\dim H_\Phi^{p,n-r}(X,E) = \dim H_{\Phi*}^{n-p,r}(X,E^*)$. The inequality $\dim H_{\Phi*}^{n-p,r}(X,E^*) < \infty$ follows from Theorem 1.3 (i) (replacing again E by E^* and p by $n-p$). □

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