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Complexification of proper hamiltonian *G*-spaces

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Complexification of Proper Hamiltonian G-Spaces

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Abstract. Let (M, τ) be a symplectic manifold and let G be a Lie group (with finitely many connected components) acting properly by symplectic diffeomorphisms on M. Then there is a proper Stein G-manifold X with a G-invariant Kähler form ω and a G-equivariant totally real embedding of maximal dimension $i : M \hookrightarrow X$ such that $i^*\omega = \tau$. Additionally, if τ possesses a moment map, this can be extended to a moment map of ω on X. The Kähler form and moment map are unique up to diffeomorphism around M fixing M pointwise.

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1. – Introduction

Let (M, τ) be a symplectic manifold and let G be a real Lie group acting properly by symplectic automorphisms on (M, τ). The goal of this paper is to complexify (M, τ, G). This is of interest since the symplectic reduction of a complex manifold is itself a complex space. This provides a method for analyzing the symplectic reduction of M via its embedding in the symplectic reduction of the complexification of M.

Historically, the starting point for complexifications is Whitney's classical theorem (see e.g. [Hir76]) stating that any smooth paracompact manifold M possesses a real analytic structure. Grauert [Gra58] proved that there is a Stein complexification X of M in the following sense.

There is a real analytic totally real embedding $i : M \hookrightarrow X$, and an anti-holomorphic involution $\sigma : X \to X$ with $\operatorname{Fix} \sigma = M$ such that the manifold X is Stein. In fact, there is a basis of Stein neighborhoods of M. Furthermore, X can be chosen so that M is a strong deformation retract of X. A Stein complexification satisfying all of the above conditions will be said to be a *Stein tube*.

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In this context, after an appropriate shrinking, closed 2-forms on M extend to Kähler forms on X:

THEOREM 1.1 ([HHL94]). *Let M be a manifold with a closed 2-form τ. Then there is a Stein tube $i : M \hookrightarrow X$ and a Kähler form ω on X, so that $i^*\omega = \tau$.*

Now let G act properly and smoothly on M. A *Stein G-tube* is a Stein tube X with a Lie group G acting properly on X by holomorphic transformations so that the embedding i, the involution σ and the strong deformation retract are G-equivariant and each G-stable neighborhood of M contains a G-stable Stein neighborhood.

THEOREM 1.2 ([Ku94], [HHK95], [He93]). *Each proper G-manifold M admits a Stein G-tube X.*

The main goal in this paper is to prove Theorem 1.1 under the presence of a proper G-action (Chapter 3):

THEOREM 1.3. *Let G be a (real) Lie group with finitely many components acting properly on a manifold M and let τ be a closed G-invariant 2-form on M. Then there is a Stein G-tube $i : M \hookrightarrow X$ and a G-invariant Kähler form ω on X with $i^*\omega = \tau$.*

If G is compact, then Theorem 1.3 is a consequence of Theorems 1.1 and 1.2 by using the averaging process. The case of a non-compact group requires substantially different techniques.

In Chapter 4 it is proved that even moment maps are extendable, i.e. if ν is a G-moment map of τ on M, then there is a G-moment map μ of ω on X with $i^*\mu = \nu$. In Chapter 5 it is shown that the construction is canonical up to local G-equivariant diffeomorphism around M.

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2. – Preliminaries

A smooth action of a Lie group G on a manifold or complex space M is said to be *proper* if the mapping $G \times M \to M \times M$, $(g, x) \mapsto (g \cdot x, x)$ is proper. This can be written in terms of sequences: if $x_n \to x$ and $g_n x_n \to y$, then there exists a subsequence $g_{n_k} \to g \in G$ with $gx = y$. Of course, compact groups always act properly. For proper actions, all isotropy groups $G_x = \{g \in G \mid g \cdot x = x\}$ $(x \in X)$ are compact subgroups of G, all orbits $G \cdot x = \{g \cdot x \mid g \in G\} \subset X$ are closed, and moreover the space of orbits X/G

is Hausdorff. Furthermore, there is a (local) slice S through each point x of a proper G-manifold M, i.e. a locally closed G_x-stable submanifold $S \ni x$ such that

$$G \times_{G_x} S \hookrightarrow M, \quad [g, s] \mapsto g \cdot s$$

is a G-equivariant open embedding [Pa61], where the G-manifold $G \times_{G_x} S$ is the associated bundle over G/G_x to the G_x-principal bundle $G \to G/G_x$. The slice S can be chosen G_x-equivariantly isomorphic to an open neighborhood of the origin in a G_x-representation space, where x is identified with the origin.

2.1. – The moment map

Let (M, ω) be a symplectic manifold, G a Lie group acting on M and assume that the symplectic form ω is G-invariant. Every $v \in \mathfrak{g} := \mathcal{L}ie(G)$ induces a fundamental vector field $\widetilde{v}$ on M and the contraction $\iota_{\widetilde{v}}\omega$ is a closed 1-form. Assume that $\iota_{\widetilde{v}}\omega$ is exact, i.e. there is a function μ^v on M with $\mathrm{d}\mu^v = \iota_{\widetilde{v}}\omega$. The functions μ^v define a map $\mu : M \to \mathfrak{g}^*$ by $\mu(x)(v) := \mu^v(x)$ for all $x \in M$ and $v \in \mathfrak{g}$.

Definition 2.1. Let M be a G-manifold, ω a closed (not necessarily non-degenerate) G-invariant 2-form on M and $\mu : M \to \mathfrak{g}^*$ a G-equivariant map satisfying $\iota_{\widetilde{v}}\omega = \mathrm{d}\mu^v$. Then μ is said to be an *(equivariant) moment map* with respect to ω and the G-action. If ω is symplectic, (M, G, ω, μ) is called a *Hamiltonian space* and the quotient $\mu^{-1}(0)/G$ its *symplectic reduction.*

If M has a complex structure and the form ω is a Kähler form arising from a G-invariant strictly plurisubharmonic function ρ, i.e. $\omega = 2i\partial\overline{\partial}\rho$, there is a moment map, namely

$$\mu^v(x) := \mathrm{d}\rho(J\widetilde{v}_x) \text{ for } v \in \mathfrak{g}.$$

Remark. From the point of view of classical mechanics the components of the moment map, i.e. the functions μ^v, are constants of motion with respect to any G-invariant Hamiltonian. So the flow of any such Hamiltonian stays on the simultaneous level set of these constants of motion, i.e. the μ-fibers. The observables on the level set are restrictions of global observables.

2.2. – Moment maps on Kähler manifolds

Let X be a proper Hamiltonian Kähler G-manifold with invariant Kähler form ω, moment map μ and $R := \mu^{-1}(0)$. Let J denote the almost complex structure of X. The induced vector field of $v \in \mathfrak{g}$ on X is denoted by $\widetilde{v}$.

Lemma 2.1. *The moment map has the following properties:*
(i) $\ker(\mathrm{d}\mu)_x = (T_x(G \cdot x))^{\perp_\omega}$ *for all* $x \in X$.
(ii) *For* $x \in R$ *the tangent space* $T_x(G \cdot x)$ *to the orbit in* x *is isotropic, i.e.* $\omega|_{T_x(G \cdot x)} \equiv 0$. *Furthermore* $T_x(G \cdot x) \cap \mathrm{J}T_x(G \cdot x) = \{0\}$.
(iii) *For* $v \in \mathfrak{g}$ *let* γ *be the flow curve of* $\mathrm{J}\widetilde{v}$ *with* $\gamma(0) = x$. *Then the curve* $\alpha = \mu^v \circ \gamma$ *is strictly increasing in a neighborhood of* 0 *or* $\widetilde{v}_x = 0$.

For the proof confer [GuSt84].

One motivation for complexifying Hamiltonian spaces arises from the fact that the symplectic reduction of a complex proper Hamiltonian space with respect to a proper action is a complex space and henceforth carries a much richer structure. The goal is therefore to understand the structure of the symplectic reduction of the real object via its embedding into the complex one.

Let X be a complex proper G-manifold with G-invariant Kähler form ω with moment map μ. Define a sheaf on R by

$$\mathcal{O}_R(U) := \{f : U \to \mathbb{C} \mid \exists\, V \subset X \text{ open}, V \cap R = U \text{ and } \exists\, \tilde{f} \in \mathcal{O}(V), \quad \tilde{f}\,|_U = f\}$$

and the structure sheaf on R/G by

$$\mathcal{O}_{R/G}(U) := \mathcal{O}_R^G\left(\pi^{-1}(U)\right).$$

The exponent "G" denotes the G-invariant functions and $\pi : R \to R/G$ the projection.

THEOREM 2.2. *There is a complex structure on R/G making $(R/G, \mathcal{O}_{R/G})$ a complex space.*

A proof is given in [AHH98] (see [Amm97] for the case of semi-simple groups). In the present paper only the case of proper *free* actions on manifolds will be used:

PROPOSITION 2.3. *Let G act freely and properly on the Kähler manifold X by holomorphic Kähler isometries. Then the quotient $(R/G, \mathcal{O}_{R/G})$ is in a canonical way a complex manifold and the projection map is holomorphic.*

For the proof confer [HH00] or [OrigDiss].

This proposition provides the following

LEMMA 2.4. *Let X be a proper Hamiltonian complex G-manifold with G acting freely and $R := \mu^{-1}(0)$. Then to each holomorphic G-invariant map $\theta : X \to Y$ there is an induced holomorphic map*

$$\theta_{\text{ind}} : R/G \to Y.$$

It is important to observe that the zero moment level possesses a particular geometry. Restrictions of invariant Kähler forms to R induce Kähler forms on the quotient Y. For the case of an action of a compact group see e.g. [HHL94]. Using the local normal form for Hamiltonian manifolds there is an induced symplectic form on the quotient R/G. For this construction, known as the Marsden-Weinstein reduction, confer [GuSt84].

LEMMA 2.5. *Let ω be a G-invariant Kähler form on the proper complex G-manifold X with G acting freely, μ a moment map and $R := \mu^{-1}(0)$. Then there is a natural Kähler form ω_{red} on the symplectic reduction R/G.*

PROOF. Let $i_R : R \hookrightarrow X$ denote the embedding. Set $Q := TR \cap JTR$ and F the vector bundle spanned by the G-vector fields. The Kähler form ω respects the bundle splitting $TR = F \oplus^{\perp} Q$, i.e. for all $\eta_i \in F_x$ and $\kappa_i \in Q_x$ it follows

$$\omega_x(\eta_1 + \kappa_1, \eta_2 + \kappa_2) = \omega_x(\kappa_1, \kappa_2).$$

Since ω is G-invariant and Q is G-stable, the complex linear vector space isomorphism $(\pi_*)_x : Q_x \to T_{\pi(x)}(R/G)$ induces a positive $(1,1)$-form ω_{red} on R/G with $i_R^*\omega = \pi^*\omega_{\text{red}}$. Hence $d\pi^*\omega_{\text{red}} = 0$ and by the surjectivity of π the form ω_{red} is closed and therefore Kählerian. □

2.3. – Properties of Stein G-tubes

As the main object of interest we recall the definition of Stein G-tubes.

DEFINITION 2.2. Let G act properly on a (real) manifold M. A Stein manifold X with a proper G-action and a totally real G-equivariant embedding $i : M \hookrightarrow X$ is said to be a *Stein G-tube* if

(i) there is an anti-holomorphic involution $\sigma : X \to X$ with $M = \operatorname{Fix}\sigma$.
(ii) M is a strong deformation retract of X
(iii) Each G-stable neighborhood of M can be shrunk to a G-stable Stein open set in X which fulfills conditions 1 and 2 as well. (Shrinking Principle)

As mentioned in the introduction (Theorem 1.2), it is of fundamental importance for our considerations that every proper G-manifold possesses a Stein G-tube (see [He93], [Kut94], [HHK95]).

Stein G-tubes possess the following fundamental property.

PROPOSITION 2.6. *Let M be a real proper G-manifold with Stein G-tube X. Furthermore let Z be a complex G-manifold and $f : M \to Z$ a G-equivariant real analytic map. Then after shrinking of X the map f extends to a G-equivariant holomorphic map $\tilde{f} : X \to Z$.*

PROOF. Identify G-equivariantly a G-stable neighborhood of M with a neighborhood V of the zero section in the normal bundle of M with convex fibers. Then after shrinking of V the real analytic function f extends uniquely to a holomorphic function $\tilde{f}$ on V. Since $g^{-1} \circ \tilde{f} \circ g$ is an extension as well, by uniqueness it is equal to $\tilde{f}$ which is therefore G-equivariant. □

2.3.1. – Embedding of the real symplectic reduction

Let M be a manifold with proper free G-action. Let $i : M \hookrightarrow X$ be a Stein G-tube of M with G-invariant Kähler form ω and associated moment map μ so that $i^*\omega = 0$ and $i^*\mu = 0$. Denote by σ the anti-holomorphic involution on X.

LEMMA 2.7. *There is a G-invariant Kähler form $\overline{\omega}$ on X with an associated moment map $\bar{\mu}$ with $i^*\overline{\omega} = 0$ and $i^*\bar{\mu} = 0$ such that the embedding $i : M \hookrightarrow X$ induces a totally real embedding $i_{\text{ind}} : M/G \hookrightarrow \bar{\mu}^{-1}(0)/G$ of maximal dimension. The set $\bar{\mu}^{-1}(0)/G$ can be shrunk to a Stein tube.*

PROOF. The involution σ is G-equivariant and fixes M pointwise. Thus the form $\overline{\omega} := \omega - \sigma^*\omega$ is a Kähler form with $i^*\overline{\omega} = 0$ and $\bar{\mu} := \mu - \sigma^*\mu$ is an associated moment map with $i^*\bar{\mu} = 0$. Set $R := \bar{\mu}^{-1}(0)$ and notice that σ stabilizes R with $\mathrm{Fix}(\sigma|_R : R \to R) = M$. Hence there is an induced anti-holomorphic involution $\sigma_{\mathrm{ind}} : R/G \to R/G$ whose fixed point set is exactly the image of the induced embedding $i_{\mathrm{ind}} : M/G \hookrightarrow R/G$. A calculation of the dimensions

$$\dim_{\mathbb{R}} R/G = \dim_{\mathbb{R}} X - 2\dim G = 2(\dim M - \dim G)$$

shows that M/G is of half real dimension of R/G, hence totally real of maximal dimension.

In order to see that R/G can be shrunk to a Stein tube we use the fact that M/G possesses a Stein tube Y since M/G is a real manifold. Shrinking Y sufficiently, the embedding $i_{\mathrm{ind}} : M/G \hookrightarrow R/G$ extends to a holomorphic map $j : Y \to R/G$. This map j is biholomorphic in a neighborhood of M/G onto its image. Shrinking this neighborhood to a Stein neighborhood, the image is a Stein tube of M/G embedded in R/G. □

3. – Proof of the main theorem

Let M be a real proper G-manifold with a G-invariant closed 2-form τ.

For the reader's convenience, we sum up the main steps of the proof. We start with the case where M is the acting group G itself, realize the G-equivariant complexification of the space and construct an invariant Kähler form on this complexification. The next case treated is to suppose that M is a product $G \times S$ with G acting by multiplication on the first factor and S is an arbitrary real manifold. Here we split the given 2-form τ into a part τ_G arising from a 2-form on G, a part τ_S arising from a 2-form on S and the rest, namely τ_M, containing the "mixed terms". Then we construct the corresponding Kähler forms separately. For this, the form τ_M has to be split again. Finally for the general case, we use the fact that M can be realized as a G-equivariant quotient $G \times_K S$ of the product $G \times S$ by a compact subgroup K of G. The situation is lifted to $G \times S$ where the previous case solves the problem. Averaging over K and Kähler reduction of the complexification of $G \times S$ due to a moment map with respect to the K-action are the essential tools in the last step in order to push down the solution on the complexification of $G \times_K S$.

NOTATION. Let J be the almost complex structure of a complex manifold X and η a k-form on X. Define the k-form $\mathrm{J}\eta$ by $\mathrm{J}\eta(v_1, \dots, v_k) := \eta(\mathrm{J}v_1, \dots, \mathrm{J}v_k)$ for all vector fields $v_1, \dots, v_k$ and for a 0-form f, i.e. a function, $\mathrm{J}f := f$. Furthermore, $\mathrm{d}^c\eta := i(\partial - \overline{\partial})\eta$.

3.1. – The group case

In the first step let M be the group G itself and let the G-action be defined by left multiplication.

NOTATION. Throughout this section we will let e denote both the neutral element in the group G and its image in an associated Stein G-tube $G^\star$.

PROPOSITION 3.1 [Wi93]. *Let G be a real Lie group. Then there is a Stein G-tube $i : G \hookrightarrow G^\star$ admitting a submanifold Σ with $e \in \Sigma$ so that*

(i) $T_e\Sigma = \mathrm{J}T_eG$
(ii) *the map $G \times \Sigma \to G^\star$, $(g, s) \mapsto g \cdot s$ is a G-equivariant diffeomorphism.*

LEMMA 3.2. *There is a Stein G-tube $i : G \hookrightarrow G^\star$ and a G-invariant strictly plurisubharmonic function $\rho_+ : G^\star \to \mathbb{R}^{\geq 0}$ with $\{\rho_+ = 0\} = G$ and $i^*\mathrm{d}^c\rho_+ = 0$.*

PROOF. Let Σ be the slice cited in Proposition 3.1. For sufficiently small Σ an open neighborhood of $0 \in T_e\Sigma$ can be identified with a neighborhood of $e \in \Sigma$ with 0 corresponding to e. Consider the square of the norm function on $T_e\Sigma$ pulled back to Σ via this identification. Extend this function G-invariantly to $G^\star \cong G \times \Sigma$ and denote it ρ_+. Shrinking Σ and thereby $G^\star$ again, ρ_+ is strictly plurisubharmonic and $i^*\mathrm{d}^c\rho_+ \equiv 0$. □

3.1.1. – The 2-form τ is "G-exact"

The following lemma will be used in the case in which the G-invariant 2-form τ on G is equal to $\mathrm{d}\alpha$ for some *G-invariant* 1-form α on the group. (τ is "G-exact".)

LEMMA 3.3. *Let α be a G-invariant 1-form on G. Then there is a Stein G-tube $i : G \hookrightarrow G^\star$ and a G-invariant function $\rho : G^\star \to \mathbb{R}$ with $i^*\mathrm{d}^c\rho = \alpha$.*

PROOF. The slice Σ used in Proposition 3.1 satisfies $T_e\Sigma = \mathrm{J}T_eG$. So there is a function ρ on Σ regarded as being G-invariantly extended to $G \times \Sigma \cong G^\star$ and which satisfies

$$\alpha_e(\zeta) = (\mathrm{d}^c\rho)_e(\zeta) = (\mathrm{d}\rho)_e(\mathrm{J}\zeta) \qquad \forall \zeta \in \mathfrak{g}.$$

By the G-invariance of both $\mathrm{d}^c\rho$ and α we obtain

$$i^*\mathrm{d}^c\rho = \alpha .$$ □

COROLLARY 3.4. *Let α be a G-invariant 1-form on G. Then there is a Stein G-tube $i : G \hookrightarrow G^\star$ and a G-invariant strictly plurisubharmonic function $\rho : G^\star \to \mathbb{R}$ so that*

$$i^*\mathrm{dd}^c\rho = \mathrm{d}\alpha .$$

PROOF. By Lemma 3.3 there is a G-invariant function ρ_0 on some Stein G-tube $G^\star$ so that $i^*\mathrm{d}^c\rho_0 = \alpha$. Choosing $G^\star$ sufficiently small, there is a G-invariant strictly plurisubharmonic function ρ_+ with $i^*\mathrm{d}^c\rho_+ = 0$. Scaling ρ_+ with a sufficiently large factor $\lambda \in \mathbb{R}^{>0}$ the bilinear form

$$(\mathrm{dd}^c\rho_0 + \mathrm{dd}^c(\lambda \cdot \rho_+))_e \in (\wedge^2\mathfrak{g})^*$$

is non degenerate. Hence by G-invariance there is a G-stable neighborhood of $G \subset G^\star$, so that $\rho := \rho_0 + \lambda \cdot \rho_+$ is strictly plurisubharmonic and still holds $i^* d^c \rho = \alpha$. The proof is completed by shrinking this set to a G-stable Stein neighborhood of G. □

3.1.2. – The 2-form τ is arbitrary on the group

The next step is to consider an arbitrary closed G-invariant 2-form τ on G.

First assume G to be connected and simply connected. Then τ defines a central Lie algebra extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ by defining the Lie bracket on $\hat{\mathfrak{g}} \cong \mathbb{R} \times \mathfrak{g}$ as $[(s, \xi), (t, \zeta)] := (\tau(\xi, \zeta), [\xi, \zeta])$. Let $\widehat{G}$ be the unique, connected, simply connected Lie group associated to $\hat{\mathfrak{g}}$. Associated to the natural projection $\hat{\mathfrak{g}} \to \mathfrak{g}$ there are a surjective Lie group morphism $\pi_G : \widehat{G} \to G$ with kernel $\mathbb{R}$ and the induced $\widehat{G}$-equivariant holomorphic map $\pi_G^\star : \widehat{G}^\star \to G^\star$ on some Stein $\widehat{G}$- and G-tubes. Here we regard the G-action on $G^\star$ pulled back via π_G to a $\widehat{G}$-action. Note that $\mathbb{R}$ acts on $\widehat{G}^\star$ as a subgroup of $\widehat{G}$. Define $\hat{\tau} := \pi_G^\star \tau$. The functional

$$\alpha_e : \hat{\mathfrak{g}} \to \mathbb{R}, \quad (s, \zeta) \mapsto (-2s)$$

defines a $\widehat{G}$-invariant 1-form α on $\widehat{G}$. For $(s, \zeta), (t, \xi) \in \hat{\mathfrak{g}}$ it follows that

$$\begin{aligned}(d\alpha)_e((s, \zeta), (t, \xi)) &= -\frac{1}{2} \alpha_e(\tau_e(\zeta, \xi), [\zeta, \xi]) \\ &= \tau_e(\zeta, \xi) = \hat{\tau}_e((s, \zeta), (t, \xi)).\end{aligned}$$

Hence, by the $\widehat{G}$-invariance of both sides, $d\alpha = \hat{\tau}$.

LEMMA 3.5. *Let τ be a closed, G-invariant 2-form on G. Then there is a Stein G-tube $i : G \hookrightarrow G^\star$ and a G-invariant Kähler form ω on $G^\star$ with $i^* \omega = \tau$.*

PROOF. Suppose first that G is connected and simply connected. By Lemma 3.3 and Corollary 3.4 there is an exact $\widehat{G}$-invariant Kähler form $\widehat{\omega}$ on some Stein $\widehat{G}$-tube $\hat{\imath} : \widehat{G} \hookrightarrow \widehat{G}^\star$ with $\hat{\imath}^* \widehat{\omega} = \hat{\tau}$.

The next step will be to push down $\widehat{\omega}$ to a Kähler form ω on a Stein G-tube $G^\star$. Let Z denote the vector field induced by the central $\mathbb{R}$-action on $\widehat{G}^\star$. The 1-form $\iota_Z \widehat{\omega}$ is closed and $\mathbb{R}$-invariant. Since $\widehat{G}^\star$ can be retracted to the simply connected Lie group $\widehat{G}$, there is a moment map $\mu : \widehat{G}^\star \to \mathbb{R} \cong \mathcal{L}ie(\mathbb{R})^*$ defined by

$$\mu(x) := \int_e^x \iota_Z \widehat{\omega}.$$

Set $R := \mu^{-1}(0)$ and note that $\widehat{G} \cdot e \subset R$. By Lemma 2.5 there is a Kähler form ω on a Stein G-tube of G and ω is even G-invariant due to the $\widehat{G}$-invariance of $\widehat{\omega}$. Here we identify $R/\mathbb{R}$ with $G^\star$ using the universal property introduced in Lemma 2.4.

Set $i_R : R \hookrightarrow \widehat{G}^\star$. The form ω fulfills $i_R^*(\pi_G^\star)^*\omega = i_R^*\widehat{\omega}$. Since $\widehat{G} \cdot e \subset R$, it also follows that $\hat{\imath}^*(\pi_G^\star)^*\omega = \hat{\imath}^*\widehat{\omega}$. Thus

$$\pi_G^* i^*\omega = \hat{\imath}^*(\pi_G^\star)^*\omega = \hat{\imath}^*\widehat{\omega} = \hat{\tau} = \pi_G^*\tau$$

and, by the surjectivity of π_G,

$$i^*\omega = \tau .$$

Now let G be arbitrary. There is a Lie group morphism p of the identity component H of the universal covering to G. This induces a locally biholomorphic H-equivariant map $p^\star : H^\star \to G^\star$ with H acting on $G^\star$ via p.

Thus there is an H-invariant Kähler form $\widetilde{\omega}$ on $H^\star$ such that if $\tilde{\imath} : H \hookrightarrow H^\star$ is the canonical embedding, then $\tilde{\imath}^*\widetilde{\omega} = p^*\tau$.

Let U and V be open neighborhoods of $e \in G^\star$ and $e \in H^\star$ respectively so that $p^\star|_V : V \to U$ is biholomorphic. We may assume that the intersection of every G-orbit with U is connected. Since $p^\star$ is H-equivariant, $\omega|_U := ((p^\star)^{-1})^*\widetilde{\omega}$ defines a Kähler form on U satisfying

$$\mathcal{L}_{\tilde{\xi}}\omega = 0 \text{ for all } \xi \in h = \mathcal{L}ie(H) \cong \mathrm{g} .$$

So ω can be extended G-equivariantly on $G \cdot U$. Finally $p^\star \circ \tilde{\imath} = i \circ p$ implies that on $\hat{\imath}^{-1}(V)$

$$p^* i^*\omega = \tilde{\imath}^*(p^\star)^*\omega = \tilde{\imath}^*\widetilde{\omega} = p^*\tau .$$

By $\widehat{G}$-invariance this holds globally and, by the surjectivity of p,

$$i^*\omega = \tau . \qquad \square$$

3.1.3. – A basic property for closed G-invariant 1-forms

The following lemma will be necessary for a construction in the product case section.

LEMMA 3.6. *Let λ be a closed G-invariant 1-form on G. Then there is a Stein G-tube $i : G \hookrightarrow G^\star$ and a pluriharmonic, G-invariant function $\theta : G^\star \to \mathbb{R}$ with $i^*\mathrm{d}^c\theta = \lambda$ and $\theta|_G \equiv 0$.*

PROOF. Let $\xi_1, \ldots, \xi_n$ be a basis of $\mathrm{g} := \mathcal{L}ie(G)$ and $\tilde{\xi}_i$ denote the induced vector field of $\xi_i, i = 1, \ldots, n$. Since λ is both G-invariant and closed, $\lambda(\tilde{\xi}_i)$ is constant. For $\lambda = 0$ there is nothing to prove, so we can assume that $\lambda(\tilde{\xi}_i) = \delta_{1i}$. Now we choose a Stein G-tube $G^\star$ so that $\tilde{\xi}_1(x), \ldots, \tilde{\xi}_n(x), J\tilde{\xi}_1(x), \ldots, J\tilde{\xi}_n(x)$ form a basis of $T_x(G^\star)$ for all $x \in G^\star$. The structure constants c_{ij}^k of the Lie algebra g with respect to the fixed basis are defined by

$$[\xi_i, \xi_j] = \sum_k c_{ij}^k \xi_k .$$

The closedness of λ shows that $c_{ij}^1 = 0$ for all $i, j = 1, \dots, n$ since

$$0 = (\mathrm{d}\lambda)_e(\xi_i, \xi_j) = -\frac{1}{2}\lambda_e([\xi_i, \xi_j]) = -\frac{1}{2}c_{ij}^1 \quad \text{for all } i, j = 1, \dots, n\,.$$

Now the pointwise dual $\beta(x) := (J\tilde{\xi}_1(x))^*$ defines a (smooth) 1-form β on $G^\star$. We will see that β is closed, d^c-closed and G^0-invariant where G^0 denotes the component of G containing e.

Let $\zeta_1, \zeta_2 \in \{\tilde{\xi}_1, \dots, \tilde{\xi}_n, J\tilde{\xi}_1, \dots, J\tilde{\xi}_n\}$ and calculate

$$\mathrm{d}\beta(\zeta_1, \zeta_2) = \frac{1}{2}\zeta_1(\beta(\zeta_2)) - \frac{1}{2}\zeta_2(\beta(\zeta_1)) - \frac{1}{2}\beta([\zeta_1, \zeta_2])\,.$$

The first terms vanish, since $\beta(\zeta_1)$, $\beta(\zeta_2)$ are constant. Furthermore the term $\beta([\zeta_1, \zeta_2])$ vanishes, because $[\zeta_1, \zeta_2]$ is a linear combination of the vector fields $\tilde{\xi}_i, \mathrm{J}\tilde{\xi}_j$ for $i, j = 2, \dots, n$, i.e. $i, j \neq 1$ since the constants c_{ij}^1 vanish. Thus $\mathrm{d}\beta = 0$.

Analogously $\mathrm{d}^c\beta = 0$:

$$\begin{aligned}\mathrm{d}^c\beta(\zeta_1, \zeta_2) &= -\mathrm{J}\mathrm{d}\mathrm{J}\beta(\zeta_1, \zeta_2) = -\mathrm{d}\mathrm{J}\beta(\mathrm{J}\zeta_1, \mathrm{J}\zeta_2)\\ &= -\frac{1}{2}\mathrm{J}\zeta_1(\beta(\zeta_2)) + \frac{1}{2}\mathrm{J}\zeta_2(\beta(\zeta_1)) - \frac{1}{2}\beta(\mathrm{J}[\zeta_1, \zeta_2])\,.\end{aligned}$$

The individual terms vanish for the same reason as above, because $\beta(\zeta_i)$ is constant and $\mathrm{J}[\zeta_1, \zeta_2] = [\mathrm{J}\zeta_1, \zeta_2]$. Finally,

$$\mathcal{L}_{\tilde{\xi}_i}\beta = \mathrm{d}(\beta(\tilde{\xi}_i)) = 0\,,$$

since $\beta(\tilde{\xi}_i)$ is constant. Thus β is G^0-invariant.

Now there is a contractible open neighborhood U of $e \in G^\star$ which intersects each G-orbit in a connected set. Define $\theta : U \to \mathbb{R}$ by

$$\theta(x) = \int_e^x \beta\,.$$

We can consider θ to be extended G-invariantly on $G \cdot U$ since

$$\mathcal{L}_{\tilde{\xi}_i}\theta = \mathrm{d}\theta(\tilde{\xi}_i) = \beta(\tilde{\xi}_i) = 0\,.$$

Furthermore, due to the G-invariance of both $i^*\mathrm{d}^c\theta$ and λ and

$$(\mathrm{d}^c\theta)_e(\tilde{\xi}_i) = (\mathrm{d}\theta)_e(\mathrm{J}\tilde{\xi}_i) = \beta_e(\mathrm{J}\tilde{\xi}_i) = \delta_{1i} = \lambda_e(\tilde{\xi}_i)\,,$$

it follows that $i^*\mathrm{d}^c\theta = \lambda$. Finally, $\theta|_G \equiv 0$ follows from $i^*\beta = 0$. □

3.2. – The product case

Now we turn to the product case, i.e. $M = G \times S$ and G acts on M by left multiplication on the first factor.

For any Stein tube $S^\star$ let $i_S : S \hookrightarrow S^\star$ be the totally real embedding and analogously for any Stein G-tube $G^\star$ set $i_G : G \hookrightarrow G^\star$. Let $\pi_G : G \times S \to G$ and $\pi_S : G \times S \to S$ denote the projections and $\pi_G^\star$ and $\pi_S^\star$ their holomorphic extensions to $G^\star \times S^\star$ respectively.

Let us first consider a 2-form τ of a special type. Given a closed 1-form η' on S and a G-invariant closed 1-form λ' on G set $\eta := \pi_S^* \eta'$ and $\lambda := \pi_G^* \lambda'$ and let $\tau := \lambda \otimes \eta$ be the associated closed G-invariant 2-form on $G \times S$ seen as a section in the bundle $\pi_G^* T^* G \otimes \pi_S^* T^* S$.

3.2.1. – Extension of $\lambda \otimes \eta$ for λ and η closed

LEMMA 3.7. *Let $\tau = \lambda \otimes \eta$ be as above. Then there is a Stein G-tube $i = i_G \times i_S : G \times S \hookrightarrow G^\star \times S^\star = X$ and a closed, G-invariant $(1, 1)$-form ω on X with $i^*\omega = \tau$.*

PROOF. Fix a closed 1-form $\widetilde{\eta}'$ on $S^\star$ with $i_S^* \widetilde{\eta}' = \eta'$. Set $\widetilde{\eta} := (\pi_S^\star)^* \widetilde{\eta}'$. By Lemma 3.6 there is a G-invariant pluriharmonic function $\theta' : G^\star \to \mathbb{R}$ on some Stein G-tube $i_G : G \hookrightarrow G^\star$ with $i_G^* \mathrm{d}^c \theta' = \lambda'$ and $\theta'|_G \equiv 0$ and set $\theta := \theta' \circ \pi_G^\star$. We define the G-invariant 2-form

$$\omega := -\mathrm{d}\theta \otimes \mathrm{J}\widetilde{\eta} + \theta \mathrm{d}^c \widetilde{\eta} + \mathrm{d}^c \theta \otimes \widetilde{\eta} .$$

Locally there is a function b, so that

$$\mathrm{d}b \overset{\text{locally}}{=} \widetilde{\eta} .$$

Thus

$$\omega \overset{\text{locally}}{=} -\mathrm{d}\mathrm{d}^c(\theta \cdot b) ,$$

since

$$\begin{aligned} -\mathrm{d}\mathrm{d}^c(\theta \cdot b) &= -\mathrm{d}(\theta \mathrm{d}^c b + b \mathrm{d}^c \theta) \\ &= -\mathrm{d}\theta \otimes \mathrm{d}^c b - \theta \mathrm{d}\mathrm{d}^c b + \mathrm{d}^c \theta \otimes \mathrm{d}b \\ &= -\mathrm{d}\theta \otimes \mathrm{J}\widetilde{\eta} + \theta \mathrm{d}^c \widetilde{\eta} + \mathrm{d}^c \theta \otimes \widetilde{\eta} . \end{aligned}$$

So ω is closed, G-invariant and of type $(1, 1)$. To show that $i^*\omega = \tau$ note that, since $i^*\theta = 0$, it follows that $i^* \mathrm{d}\theta = 0$. By definition $i^* \mathrm{d}^c \theta = \lambda$ and $i^* \widetilde{\eta} = \eta$. Thus

$$i^*\omega = -i^* \mathrm{d}\theta \otimes i^* \mathrm{J}\widetilde{\eta} + i^*\theta \cdot i^* \mathrm{d}^c \widetilde{\eta} + i^* \mathrm{d}^c \theta \otimes i^* \widetilde{\eta} = \lambda \otimes \eta = \tau . \qquad \square$$

3.2.2. – The main lemma

Fix an arbitrary point $s_0 \in S$ and the embeddings $i_e : S \hookrightarrow G \times S, s \mapsto (e, s)$ and $i_{s_0} : G \hookrightarrow G \times S, g \mapsto (g, s_0)$.

Now the main Lemma can be formulated.

LEMMA 3.8. *Let τ be a closed, G-invariant 2-form on $G \times S$ with $i_e^*\tau = 0$ and $i_{s_0}^*\tau = 0$. Then there is a closed G-invariant (1, 1)-form ω on $G^\star \times S^\star$ with $i^*\omega = \tau$.*

PROOF. Fix a basis $\lambda_1', \dots, \lambda_n'$ of the vector space of G-invariant 1-forms on G, so that the subsystem $\lambda_r', \dots, \lambda_n'$ forms a basis of the closed invariant forms. Set $\lambda_i := \pi_G^*\lambda_i'$. The general form of τ is

$$\tau = \sum_k \eta_k \otimes \lambda_k + \sum_{i,j} f_{ij}\lambda_i \wedge \lambda_j ,$$

where $f_{ij} = \pi_S^* f_{ij}'$ and $\eta_k = \pi_S^*\eta_k'$ with f_{ij}' functions and η_k' 1-forms on S. Note that $f_{ij}(s_0) = 0$, since $i_e^*\tau = 0$. Now we decompose τ:

$$\tau_c := \sum_{k=r}^{n} \eta_k \otimes \lambda_k, \qquad \tau_r := \tau - \tau_c = \sum_{k=1}^{r-1} \eta_k \otimes \lambda_k + \sum_{i,j}^{n} f_{ij}\lambda_i \wedge \lambda_j .$$

The bundle $\bigwedge^3 T^*(G \times S)$ splits canonically into

$$\bigwedge\nolimits^3 T^*S \bigoplus \left(T^*G \otimes \bigwedge\nolimits^2 T^*S\right) \bigoplus \left(\bigwedge\nolimits^2 T^*G \otimes T^*S\right) \bigoplus \bigwedge\nolimits^3 T^*G .$$

Since the 3-form $\mathrm{d}\tau$ vanishes, its $(T^*G \otimes \bigwedge^2 T^*S)$-component vanishes and thus $\sum_{k=1}^n \mathrm{d}\eta_k \otimes \lambda_k = 0$. Hence the forms $\eta_k, k = 1, \dots, n$ and τ_c are closed. Lemma 3.7 solves the problem for τ_c, i.e. there is G-invariant closed (1, 1)-form ω_c on $G^\star \times S^\star$ with $i^*\omega_c = \tau_c$.

So it remains to construct an extension ω_r of τ_r. We calculate

$$0 = \mathrm{d}\tau_r = \sum_{k=1}^{r-1} \mathrm{d}\lambda_k \otimes \eta_k + \sum_{i,j} \lambda_i \wedge \lambda_j \otimes \mathrm{d}f_{ij} .$$

In order to see that η_k is exact for $k = 1, \dots, r-1$ notice that $\Lambda_1 := \mathrm{d}\lambda_1, \dots, \Lambda_{r-1} := \mathrm{d}\lambda_{r-1}$ are linearly independent in the vector space of G-invariant 2-forms independent of S. We complete them to a basis $\Lambda_1, \dots, \Lambda_m$. Now we can apply the dual basis vector Λ_k^* to the upper equation and obtain the exactness of η_k:

$$\eta_k = -\sum_{i,j} \Lambda_k^*(\lambda_i \wedge \lambda_j)\mathrm{d}f_{ij} .$$

Let $(\frac{\partial}{\partial\lambda_i})_{i=1,\dots,n}$ be the G-vector fields dual to $(\lambda_i)_{i=1,\dots,n}$, i.e. $\lambda_i(\frac{\partial}{\partial\lambda_j}) = \delta_{ij}$. Since

$$\mathrm{d}\lambda_k\left(\frac{\partial}{\partial\lambda_i}, \frac{\partial}{\partial\lambda_j}\right) = -\frac{1}{2}c_{ij}^k ,$$

where c_{ij}^k denote the according Lie algebra structure constants, it follows that

$$(*) \qquad \frac{1}{2}\sum_k \eta_k c_{ij}^k = \mathrm{d}f_{ij} .$$

For simplicity we define G-invariant functions $b_k : G \times S \to \mathbb{R}$ by $\mathrm{d}b_k = \eta_k$ and $b_k(s_0) = 0$. Due to $f_{ij}(s_0) = 0$, the equation $(*)$ transforms to

$$\frac{1}{2}\sum_k b_k c_{ij}^k = f_{ij}\,.$$

We calculate

$$\begin{aligned}\mathrm{d}\left(-\sum_k b_k\lambda_k\right) &= \sum_k \lambda_k \otimes \eta_k + \frac{1}{2}\sum_{k,i,j} b_k c_{ij}^k \lambda_i \wedge \lambda_j\\ &= \sum_k \lambda_k \otimes \eta_k + \sum_{i,j} f_{ij}\lambda_i \wedge \lambda_j = \tau_r\,.\end{aligned}$$

By Lemma 3.3 there are G-invariant functions ρ'_k on $G^\star$ with $i^*\mathrm{d}^{\mathrm{c}}\rho'_k = \lambda'_k$ and $\rho'_k|_G \equiv 0$; set $\rho_k := \rho'_k \circ \pi_G$. Extend the b_k to functions on $G^\star \times S^\star$ independent of $G^\star$, denote these extensions b_k as well, and define

$$\rho := -\sum_k b_k\,\rho_k\,.$$

It follows that $\omega_r := \mathrm{dd^c}\rho$ is an exact, G-invariant $(1,1)$-form with

$$\begin{aligned}i^*\omega_r &= -\mathrm{d}i^*\sum_k(\rho_k \mathrm{d^c} b_k + b_k \mathrm{d^c}\rho_k)\\ &= -\mathrm{d}\left(\sum_k b_k\lambda_k\right) = \tau_r\,,\end{aligned}$$

since $i^*\rho_k \equiv 0$. □

3.2.3. – Extension of an arbitrary 2-form τ

The general extension is obtained by decomposing τ into relevant pieces.

LEMMA 3.9. *Let τ be a closed, G-invariant 2-form on $G \times S$. Then there is a Stein G-tube $i : G \times S \hookrightarrow G^\star \times S^\star$ and a closed, G-invariant $(1,1)$-form ω on $G^\star \times S^\star$ with $i^*\omega = \tau$.*

PROOF. We decompose τ into three parts $\tau = \tau_G + \tau_M + \tau_S$, so that each part is still closed and G-invariant; τ_G and τ_S will be 2-forms arising from 2-forms on G and S respectively while τ_M contains the "mixed terms". In order to obtain the decomposition define the G-invariant closed 2-form $\tau'_G := i_{s_0}^*\tau$ on G. By Lemma 3.5 there is a G-invariant Kähler form ω'_G on $G^\star$ with $i_G^*\omega'_G = \tau'_G$. We set $\omega_G := (\pi_G^\star)^*\omega'_G$ and obtain $i^*\omega_G = \tau_G$. Analogously, for the closed 2-form $\tau'_S := i_e^*\tau$, by Theorem 1.1 there is a closed $(1,1)$-form ω'_S with the desired properties on $S^\star$ and set $\omega_S := (\pi_S^\star)^*\omega'_S$.

The difference $\tau_M := \tau - \tau_G - \tau_S$ is a G-invariant closed 2-form containing the "mixed terms". This can be extended to a G-invariant closed $(1,1)$-form ω_M on $G^\star \times S^\star$ by the main Lemma (Lemma 3.8).

Thus by adding the constructed components, i.e. setting $\omega := \omega_G + \omega_M + \omega_S$, a G-invariant closed $(1,1)$-form ω is obtained with $i^*\omega = \tau$. □

3.2.4. – Extension as a Kähler form

Finally, it is an elementary matter to adjust the above extension to obtain a Kähler form.

LEMMA 3.10. *Let τ be a closed G-invariant 2-form on $G \times S$. Then there is a Stein G-tube $i : G \times S \hookrightarrow X \subset G^\star \times S^\star$ and a G-invariant Kähler form ω on X with $i^*\omega = \tau$.*

PROOF. By Lemma 3.9 there is a closed G-invariant $(1,1)$-form ω_0 on some Stein G-tube X with $i^*\omega_0 = \tau$. For X sufficiently small there is a G-invariant strictly plurisubharmonic function $\rho_+ : X \to \mathbb{R}^{\geq 0}$ with

$$\{\rho_+ = 0\} = \{\mathrm{d}\rho_+ = 0\} = M = G \times S \subset X\,.$$

Fix a G-invariant partition of unity $\{\chi_\alpha\}$ so that $(\operatorname{supp}\chi_\alpha)/G \subset X/G$ is compact and the interiors of $\operatorname{supp}\chi_\alpha$ form a locally finite cover of G-stable open sets in X. Choose $\varepsilon_\alpha > 0$ so that $V_\alpha := \{x \mid \chi_\alpha(x) > \varepsilon_\alpha\}$ is a cover as well. The conditions $\{\rho_+ = 0\} = M$ and $\{\mathrm{d}\rho_+ = 0\} = M$ imply

$$\mathrm{dd^c}(\chi_\alpha\rho_+)|_M = (\chi_\alpha \mathrm{dd^c}\rho_+)|_M\,,$$

since the terms $\mathrm{d}\chi_\alpha \wedge \mathrm{d^c}\rho_+$, $\mathrm{d^c}\chi_\alpha \wedge \mathrm{d}\rho_+$ and $\rho_+\mathrm{dd^c}\chi_\alpha$ vanish on M. The sets V_α/G are relatively compact, so that there are constants $c_\alpha > 0$ such that

$$\omega_0 - c_\alpha \mathrm{dd^c}(\chi_\alpha\rho_+)$$

is a Kähler form on a G-stable neighborhood of $M \cap V_\alpha$ in X since the form $\mathrm{dd^c}(\chi_\alpha\rho_+)$ is a Kähler form in some open neighborhood of $M \cap V_\alpha$. Set $\rho := \sum c_\alpha\chi_\alpha\rho_+$ and note that the sum is locally finite. Thus the form

$$\omega := \omega_0 - \mathrm{dd^c}\rho$$

is a G-invariant real $(1,1)$-form on a G-stable neighborhood of M which is positive on M. Thus there is a possibly smaller Stein G-tube $G^\star \times S^\star$, again denoted by X, such that ω is a G-invariant Kähler form. The fact $\mathrm{d}\rho_+|_M = 0$ yields $i^*\mathrm{d}\rho_+ = i^*\mathrm{d^c}\rho_+ = 0$, hence $i^*\mathrm{dd^c}\rho = 0$ which implies $i^*\omega = \tau$. □

3.3. – The general case via Abels' theorem

The main Theorem will be proved via a real analytic version of Abels' global Slice Theorem. It is known that for any proper G-action on a $\mathcal{C}^\infty$-manifold M there is a compatible real analytic structure on M making the action real analytic ([Il93]). In fact this structure is unique ([Ku96]).

The following theorem is valid for Lie groups G which admit a maximal compact subgroup K unique up to conjugation. Therefore let us restrict in the sequel to the case where G possesses only finitely many components where such a maximal compact subgroup K exists in general.

THEOREM 3.11 [HHK96]. *Let G act properly (and real analytically) on a manifold M and let K be a maximal compact subgroup. Then there is a K-stable real analytic submanifold $S \subset M$ so that the map*

$$G \times_K S \xrightarrow{\sim} M$$
$$[g, s] \mapsto g \cdot s$$

is a G-equivariant real analytic bijection with real analytic inverse.

REMARK. The theorem is based on Abels' theorem ([Ab74]) that proves the same statement in the category of smooth manifolds.

The Stein G-tube of an Abels representation $M = G \times_K S$ is constructed concretely as the categorical quotient $(G^\star \times S^\star)//K$ ([HHK96]), i.e. the quotient with respect to the K-invariant holomorphic functions. The categorical quotient of a Stein manifold with respect to a compact group is a Stein space ([He91]). This allows us to construct the Kähler extension by pushing down an extension from $G^\star \times S^\star$ to $(G^\star \times S^\star)//K$.

PROOF OF THE MAIN THEOREM. Let $M = G \times_K S$ and τ be a closed G-invariant 2-form on M. As mentioned above $i : M \hookrightarrow (G^\star \times S^\star)//K$ is a Stein G-tube. We lift the situation to $G \times S$ via the projection $p : G \times S \to G \times_K S$ which extends to a holomorphic projection $p^\star : G^\star \times S^\star \to (G^\star \times S^\star)//K$. The inclusion $\hat{\imath} : G \times S \hookrightarrow G^\star \times S^\star$ is a Stein G-tube as well. Of course, $p^\star \circ \hat{\imath} = i \circ p$. Note that $G \times S$ and $G^\star \times S^\star$ are endowed with $(G \times K)$-actions making $\hat{\imath}$ equivariant. The 2-form $\hat{\tau} := p^*\tau$ is $(G \times K)$-invariant. By Lemma 3.10 there is a G-invariant Kähler form $\widehat{\omega}$ on $G^\star \times S^\star$ with $\hat{\imath}^*\widehat{\omega} = \hat{\tau}$. By the averaging process $\widehat{\omega}$ can be assumed K-invariant as well. For $v \in \mathfrak{k}$ let $\widetilde{v}_K$ denote the associated K-vector field on $G \times S$ and $G^\star \times S^\star$ respectively. The 1-form $\iota_{\widetilde{v}_K} \hat{\tau}$ vanishes for all $v \in \mathfrak{k}$. Fix an arbitrary point $x_0 \in G \times S$ and define

$$\mu^v(x) = \int_{x_0}^{x} \iota_{\widetilde{v}_K} \widehat{\omega}$$

on the K-stable Stein G-tube $X = G^\star \times S^\star$. Note that the associated map $\mu : X \to \mathfrak{k}^*$ vanishes identically on $G \times S$. Furthermore $k^*\mu^v - \mu^{\mathrm{Ad}(k)v}$ is constant and vanishes on $G \times S$, hence vanishes identically. So μ is a K-moment map with $G \times S \subset R := \mu^{-1}(0)$. By Lemma 2.5 there is an induced Kähler form ω on R/K satisfying $i_R^*(p^\star)^*\omega = i_R^*\widehat{\omega}$ and hence $\hat{\imath}^*(p^\star)^*\omega = \hat{\imath}^*\widehat{\omega}$. Since TR and $\mathrm{J}TR$ span $TX|_R$, the image of the map $R \to X//K$ induced by the embedding of R contains a G-stable open neighborhood V of $G \times_K S \subset X//K$. Shrinking X to a G-stable Stein neighborhood of $G \times S$ in the $p^\star$-preimage of V makes the induced G-equivariant map $R/K \to X//K$ biholomorphic such that we can identitfy these spaces. Due to the G-equivariance of the projection the form ω is even G-invariant. In order to show $i^*\omega = \tau$ calculate

$$p^*i^*\omega = \hat{\imath}^*(p^\star)^*\omega = \hat{\imath}^*\widehat{\omega} = \hat{\tau} = p^*\tau$$

and by surjectivity of p we obtain finally

$$i^*\omega = \tau\,. \qquad \square$$

Note that the above proof only requires the existence of an Abels representation. Thus, even if G has infinitely many components, the main theorem holds for $M = G \times_K S$ of this type. In particular, we have the following local version.

THEOREM 3.12. *Let M be a manifold with proper G-action and τ a closed G-invariant 2-form. For each $x_0 \in M$ there is a G-stable neighborhood U of x_0 in the Stein G-tube X and a G-invariant Kähler form ω on U with $(i|_{i^{-1}(U)})^*\omega = \tau$.*

4. – Extension of the moment map

Next it will be shown that if the totally real manifold possesses a moment map, then this is extendable to a moment map with respect to the Kähler form on the complexification.

THEOREM 4.1. *Let $\nu : M \to \mathfrak{g}^*$ be a moment map on M with respect to a closed G-invariant 2-form τ and ω a closed G-invariant 2-form on some Stein G-tube X with $i^*\omega = \tau$. Then there is a moment map $\mu : X \to \mathfrak{g}^*$ with respect to ω with $i^*\mu = \nu$.*

PROOF. Let $v \in \mathfrak{g}$ and $\widetilde{v}$ denote the induced vector field on M and X respectively. The 1-form $\iota_{\widetilde{v}}\tau$ on M is exact by assumption and M is a strong deformation retract of X. Thus, fixing $x_0 \in M$,

$$\mu^v(x) = \int_{x_0}^{x} \iota_{\widetilde{v}}\omega + \nu^v(x_0)$$

is well-defined on X and fulfills $i^*\mu = \nu$. Note that the map $\mathfrak{g} \to C^\infty(X)$, $v \mapsto \mu^v$, is linear. The associated map $\mu : X \to \mathfrak{g}^*$ satisfies the moment map condition $\iota_{\widetilde{v}}\omega = \mathrm{d}\mu^v$. Thus we must only prove the G-equivariance of μ, i.e.

$$\mu^v(g \cdot x) = \mu^w(x) \text{ for all } x \in X$$

with $w = \mathrm{Ad}(g)v$. Note that $\widetilde{w} = g_*\widetilde{v}$ and thus

$$\begin{aligned} \mathrm{d}(\mu^v(g \cdot x) - \mu^w(x)) &= g^*\iota_{\widetilde{v}}\omega - \iota_{\widetilde{w}}\omega \\ &= \iota_{g_*\widetilde{v}}g^*\omega - \iota_{\widetilde{w}}\omega \\ &= \iota_{\widetilde{w}}\omega - \iota_{\widetilde{w}}\omega = 0. \end{aligned}$$

So $g^*\mu^v - \mu^w \in \mathfrak{g}^*$ is constant. But for any $x \in M$

$$g^*\mu^v(x) - \mu^w(x) = \nu^v(g \cdot x) - \nu^w(x) = 0$$

by the G-equivariance of ν. □

5. – Construction is canonical

Stein G-tubes can be considered as "germs", i.e. two Stein G-tubes of a proper G-manifold M are G-equivariant biholomorphic after sufficient shrinking of both. The following theorem shows that any two G-invariant Kähler extensions of a G-invariant 2-form on X are likewise equivalent.

THEOREM 5.1. *Let M be a proper G-manifold and $i : M \hookrightarrow X$ an associated Stein G-tube. For a closed G-invariant 2-form τ on M suppose that ω_0 and ω_1 are G-invariant Kähler forms on X with $i^*\omega_0 = i^*\omega_1 = \tau$. Then there are G-stable neighborhoods U_0, U_1 of M and a G-equivariant diffeomorphism $\varphi : U_0 \to U_1$ with $\varphi|_M = \mathrm{id}_M$ so that*

$$\varphi^*\omega_1 = \omega_0 .$$

PROOF. Using a G-invariant Riemannian metric on X, the exponential map on JTM identifies G-equivariantly a G-stable neighborhood V of the zero section with a G-stable neighborhood U of $M \subset X$. We can assume the set $V_x := V \cap JT_xM$ to be convex for all $x \in M$, so that, via the identification, the G-equivariant map $(t, v) \mapsto (1-t)\cdot v$ can be regarded as a smooth G-equivariant homotopy on U, i.e. a smooth map

$$\psi : [0,1] \times U \to U$$

defining $\psi_t := \psi(t, .) : U \to U$ with $\psi_0 = \mathrm{id}_U$, $\psi_1(U) = M$, $\psi_t|_M = \mathrm{id}_M$ and ψ_t is G-equivariant.

Define the sections $\sigma_s : X \to [0,1] \times X, x \mapsto (s, x)$ and note that for any k-form η on $[0,1] \times X$

$$\frac{\partial}{\partial t}(\sigma_t^*\eta) = \sigma_t^*\mathcal{L}_{\frac{\partial}{\partial t}}\eta .$$

Now consider the closed 2-form $\omega := \omega_1 - \omega_0$. It follows that $\psi_0^*\omega = 0$, since $i^*\omega = 0$. Furthermore $\psi_1^*\omega = \omega$. In order to establish the existence of a G-invariant 1-form β_0 with $\omega = \mathrm{d}\beta_0$, we will use a slightly modified version of a calculation in [GuSt84].

$$\begin{aligned}\omega = \psi_1^*\omega - \psi_0^*\omega &= \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} [\psi_t^*\omega]\mathrm{d}s = \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} [\sigma_t^*\psi^*\omega]\mathrm{d}s \\ &= \int_0^1 [\sigma_s^*\mathcal{L}_{\frac{\partial}{\partial t}}\psi^*\omega]\mathrm{d}s = \int_0^1 [\sigma_s^*\mathrm{d}\iota_{\frac{\partial}{\partial t}}\psi^*\omega]\mathrm{d}s = \mathrm{d}\left(\int_0^1 [\sigma_s^*\iota_{\frac{\partial}{\partial t}}\psi^*\omega]\mathrm{d}s\right)\end{aligned}$$

For simplicity, set $\beta_0 := \int_0^1 [\sigma_s^*\iota_{\frac{\partial}{\partial t}}\psi^*\omega]\mathrm{d}s$ and notice that β_0 is G-invariant and $i^*\beta_0 = 0$. Consider β_0 as a function on JTM and pull it back via the exponential map to a G-invariant function $f : U \to \mathbb{R}$. This function satisfies $f|_M \equiv 0$ and $\mathrm{d}f|_M = \beta_0|_M$. Thus $\beta := \beta_0 - \mathrm{d}f$ is a G-invariant 1-form with $\beta|_M \equiv 0$ and $\mathrm{d}\beta = \omega_1 - \omega_0$.

Thus we can apply Moser's method to the curve $\omega_t := (1-t)\cdot\omega_0 + t\cdot\omega_1$ of G-invariant Kähler forms on U. For this, define the G-invariant time-dependent vector field ξ_t by

$$\imath_{\xi_t}\omega_t = -\beta\,.$$

Since $\xi_t|_M \equiv 0$ there is a G-stable neighborhood U_0 so that the flow

$$\varphi_t : U_0 \to X$$

is defined for all $t \in [0,1]$ satisfying $\varphi_t|_M = \mathrm{id}_M$. The general formula on time-dependent forms ([MDSa95], p. 92) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\omega_t &= \varphi_t^*\frac{\partial\omega_t}{\partial t} + \varphi_t^*(\imath_{\xi_t}\mathrm{d}\omega_t) + \varphi_t^*(\mathrm{d}\imath_{\xi_t}\omega_t)\\ &= \varphi_t^*\mathrm{d}\beta + 0 + \varphi_t^*(-\mathrm{d}\beta) = 0\,.\end{aligned}$$

Thus, from $\varphi_0^*\omega_0 = \omega_0$ we obtain

$$\varphi_t^*\omega_t = \omega_0\,.$$

The map $\varphi := \varphi_1 : U_0 \to \varphi_1(U_0)$ is a G-equivariant diffeomorphism with $\varphi|_M = \mathrm{id}_M$ and $\varphi^*\omega_1 = \omega_0$. □

COROLLARY 5.2. *In addition to the assumptions of Theorem* 5.1, *let a moment map* $\nu : M \to \mathfrak{g}^*$ *on* M *be given with respect to* τ *and* μ_0 *and* μ_1 *be moment maps with respect to* ω_0 *and* ω_1 *and assume that* $i^*\mu_0 = i^*\mu_1 = \nu$, *where* ν *is a moment map with respect to* τ. *Then the constructed diffeomorphism* φ *satisfies*

$$\varphi^*\mu_1 = \mu_0\,.$$

PROOF. For $v \in \mathfrak{g}$ the map φ stabilizes the induced vector field $\widetilde{v}$, i.e. $\varphi_*\widetilde{v} = \widetilde{v}$ and hence

$$\begin{aligned}\mathrm{d}(\varphi^*\mu_1^v - \mu_0^v) &= \varphi^*\mathrm{d}\mu_1^v - \mathrm{d}\mu_0^v\\ &= \varphi^*\imath_{\widetilde{v}}\omega_1 - \imath_{\widetilde{v}}\omega_0\\ &= \imath_{\widetilde{v}}\varphi^*\omega_1 - \imath_{\widetilde{v}}\omega_0\\ &= \imath_{\widetilde{v}}\omega_1 - \imath_{\widetilde{v}}\omega_0 = 0\,.\end{aligned}$$

Thus $\varphi^*\mu_1 - \mu_0 \in \mathfrak{g}^*$ is constant. But $\varphi(x) = x$ for any $x \in M$ and hence

$$\varphi^*\mu_1(x) - \mu_0(x) = \nu(x) - \nu(x) = 0\,.$$

Therefore

$$\varphi^*\mu_1 = \mu_0\,.$$ □

In summary the G-invariant Kähler extension (with moment map) is unique as germ up to diffeomorphisms which are the identity on M.

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