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A Waiting Time Phenomenon for Thin Film Equations

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Abstract. We prove the occurrence of a waiting time phenomenon for solutions to fourth order degenerate parabolic differential equations which model the evolution of thin films of viscous fluids. In space dimension less or equal to three, we identify a general criterion on the growth of initial data near the free boundary which guarantees that for sufficiently small times the support of strong solutions locally does not increase. It turns out that this condition only depends on the smoothness of the diffusion coefficient in its point of degeneracy. Our approach combines a new version of Stampacchia's iteration lemma with weighted energy or entropy estimates. On account of numerical experiments, we conjecture that the aforementioned growth criterion is optimal.

Mathematics Subject Classification (2000): 35K25 (primary), 35K55, 35K65, 35K99, 76D08, 76D27 (secondary).

1. – Introduction

In this paper we prove a waiting time phenomenon for nonnegative, mass conserving solutions to the fourth order degenerate parabolic equation

$$u_t + \operatorname{div}(|u|^n \nabla \Delta u) = 0 \quad \text{in} \quad \mathbb{R}^+ \times \Omega \tag{1.1}$$

in space dimension $N \in \{1, 2, 3\}$. Here, either Ω is a open bounded domain of class $C^{1,1}$ or $\Omega \equiv \mathbb{R}^N$. In the former case, we assume the normal derivatives of both u and Δu to vanish on $\partial\Omega$. Further, we suppose the initial datum $u_0 \in H^1(\Omega)$ to be nonnegative and to have compact support.

Equation (1.1) serves as a model problem for a class of higher order diffusion equations arising in materials science and fluid dynamics (cf. [13], [24] and the references therein). In the form presented above, it describes the surface tension driven evolution of the height u of a thin film of viscous fluid that spreads on a horizontal surface.

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Physically, the diffusion growth exponent n is determined by the flow condition at the liquid-solid interface: Navier-type slip conditions infer values of $n \in (0, 3)$, whereas a no-slip condition entails $n = 3$ (cf. [24] and [4]). Note that for spreading droplets a no-slip condition implies infinite energy dissipation at the triple junction liquid-solid-gas (for details, cf. [15]). Mathematically, this is reflected by the fact that selfsimilar source-type solutions with compact support do exist if and only if $n < 3$ (see [8], [16]).

It is an important peculiarity of the aforementioned evolution equation that it allows for the construction of nonnegativity preserving solutions (cf. [7], [18], [9], [3] and [13]). This is remarkable since solutions to nondegenerate fourth order problems in general may become locally negative even if initial data are strictly positive.

Another crucial feature is that (1.1) implicitly defines a free boundary problem for which the free boundary is given by $\partial[\text{supp}(u(t,\cdot))]$. As we are dealing with a fourth order equation, we do not expect well-posedness without imposing a third condition at the free boundary in addition to the natural ones

$$u|_{\partial[\text{supp}(u(t,\cdot))]} = u^n \nabla \Delta u|_{\partial[\text{supp}(u(t,\cdot))]} = 0 .$$

Besides the work of Otto [25] — who constructs in space dimension $N = 1$ for $n = 1$ solutions with constant nonzero contact angle — the analytical study of (1.1) concentrates on a distinguished class of solutions, hereafter referred to as *strong solutions*, which satisfy an additional integral estimate, the so called *entropy estimate*. For solutions with compactly supported initial data, such an estimate holds true provided $0 < n < 3$. It implies in particular that the corresponding solutions exhibit a zero contact angle at the free boundary. Hence, strong solutions are expected to be unique.

An important feature of strong solutions is that their support has the property of finite speed of propagation, i.e.:

$$\text{supp}(u_0) \subset B(0, R) \implies \text{supp}(u(t)) \subset B(0, R + c \cdot t^{\frac{1}{nN+4}}), \tag{1.2}$$

with c depending only on n, N and on $\|u_0\|_1$. In one space dimension, this was proven for $0 < n < 3$ in [5], [6], [21]; in higher space dimensions it was obtained for $\frac{1}{8} < n < 2$ in [11]. We observe that estimate (1.2) is optimal, since the speed of propagation coincides with that of source-type solutions [8], [16]. In addition, it is known that for $n < 3$ the support of strong solutions covers any bounded subdomain of Ω as t tends to infinity ([13], see also [5] and [12] for sharp estimates of its measure). Nevertheless, more refined results on the motion of the free boundary are expected to hold provided additional informations are given on the local behaviour of u_0. In particular, if the initial datum is sufficiently flat at one point of the interface, then a *waiting time* should exist, during which — locally at that point — the support does not expand.

In this paper we present conditions on the initial data which guarantee for strong solutions the existence of such a waiting time. To put it concisely, if

$0 < n < 2$ and $u_0(x)$ grows at most like $|x - x_0|^{\frac{4}{n}}$ in a neighbourhood of a point $x_0 \in \partial[\mathrm{supp}(u_0)]$, then a waiting time phenomenon occurs at x_0. If $2 \le n < 3$ and the space dimension is one, the same result holds provided $u_{0x}(x)$ grows at most like $|x - x_0|^{\frac{4}{n}-1}$. To the best of our knowledge these are the first mathematically rigorous results on a waiting time phenomenon for fourth order degenerate parabolic equations of the form (1.1) (for preliminary numerical results, cf. [20]; for related formal considerations, see also [26]).

Let us sketch the outline of our approach. The general strategy is to use certain integral estimates to obtain a recursive inequality for suitable space-time L^p-norms of the solution. To take full advantage of that inequality, we use an extension of Stampacchia's lemma which to our knowledge is new and which might be of independent interest. We notice that, in the analysis of thin film equations, the combination of weighted integral estimates and iterative procedures has been developed by Hulshof and Shishkov [21] to obtain the estimate (1.2).

For $n < 2$, our method is based on local entropy estimates (cf. (2.1)). In space dimension $N = 1$, we assume that:

$$u_0(x) = 0 \quad \forall x \in [x_0 - 2r_0, x_0] \quad \text{for some } r_0 > 0\,, \tag{1.3}$$

and that initial data satisfy:

$$\limsup_{r\to 0} r^{-\frac{4}{n}(\alpha+1)} \fint_{B(x_0,r)} u_0^{\alpha+1}(x)\,dx < \infty \quad \text{for some } \alpha \in I_n\,, \tag{1.4}$$

where

$$I_n := \left(\max\left\{0, \frac{1}{2} - n\right\}, 2 - n\right)\,.$$

Under these conditions we prove (Theorem 4.1) that a positive time $T^\star$ exists such that

$$u(t,\cdot) = 0 \text{ in } [x_0 - r_0, x_0] \quad \forall\, t \in [0, T^\star)\,. \tag{1.5}$$

In higher space dimension, we reformulate (1.3) requiring an "external cone property" for $\partial[\mathrm{supp}(u_0)]$ at x_0: we assume that there exists a cone $C(x_0, 2\theta)$ with vertex in x_0 and opening angle 2θ such that

$$u_0 = 0 \text{ a.e. in } C(x_0, 2\theta) \cap B(x_0, 2r_0) \text{ for some } r_0 > 0\,. \tag{1.6}$$

If in addition (1.4) holds, then (Theorem 5.1) for $\frac{1}{8} < n < 2$ a positive time $T^\star$ exists such that

$$u(t,\cdot) = 0 \text{ a.e. in } C(x_0,\theta) \cap B(x_0, r_0) \quad \forall\, t \in [0, T^\star)\,.$$

If $2 \le n < 3$, the entropy estimate cannot be exploited any longer and we have to base our method on a weighted energy estimate (cf. (2.3)) proved by Bernis [6]. To the best of our knowledge, this estimate is only known to hold globally in space, in one space dimension and for strong solutions to the Cauchy problem. For this reason we restrict ourselves to $N = 1$, replace the local condition (1.3) by a global one, namely

$$u_0(x) = 0 \quad \forall x < x_0 , \tag{1.7}$$

and introduce a "flatness" assumption which involves the L^2-norm of the gradient of initial data:

$$\limsup_{r\to 0} r^{-2(\frac{4}{n}-1)} \fint_{B(x_0,r)} u_{0x}^2(x)\, dx < \infty . \tag{1.8}$$

Under assumptions (1.7) and (1.8) we prove (Theorem 6.1) that a positive time $T^\star$ exists such that

$$u(t,\cdot) = 0 \;\text{ in } (-\infty, x_0] \qquad \forall t \in [0, T^\star) .$$

Though condition (1.8) is stronger, it is nevertheless consistent with (1.4): again, it yields $4/n$ as critical exponent for the existence of a waiting time.

A natural question is whether the growth exponent $4/n$ is optimal. In Section 7 we will present numerical experiments which strongly support the conjecture that no waiting time phenomenon occurs if $u_0(x) \geq c|x - x_0|^\gamma$ for some $\gamma < 4/n$ at a Lipschitz-regular boundary point $x_0 \in \partial\{u_0 > 0\}$. If $n < 2$, we actually expect instantaneous spreading under the weaker condition

$$\limsup_{r\to 0} r^{-\frac{4}{n}} \fint_{B(x_0,r)} u_0(x)\, dx = \infty . \tag{1.9}$$

If on the other hand $n \geq 2$, then the question is more delicate, mainly due to a less strong regularizing effect of the operator [12], and oscillations of initial data near the interface could become important. It is worthwhile to observe that, for $n < 2$, conditions (1.4) and (1.9) remind of the condition

$$\limsup_{r\to 0} r^{-\frac{2}{m}} \fint_{B(x_0,r)} u_0(x)\, dx < \infty$$

for the porous medium equation $u_t = \operatorname{div}(u^m \nabla u)$ (cf. Section 7 for a more detailed comparison). Nevertheless, we underline that the two operators have a deeply different structure — for instance, the role of the critical exponents $n = 2$, $n = 3$ does not have a counterpart in the second order case.

The paper is organized as follows: In Section 2 we introduce the concept of strong solutions, and we present weighted energy and entropy estimates which we are going to use in the sequel; in Section 3 we formulate and prove the extension of Stampacchia's lemma; Sections 4 to 6 are devoted to the proofs of a waiting time phenomenon in the cases $N = 1$ and $0 < n < 2$, $N \in \{2, 3\}$ and $\frac{1}{8} < n < 2$, $N = 1$ and $2 \leq n < 3$, respectively; finally, the question of optimality is discussed in Section 7.

Throughout the whole paper, we use the standard notation for Sobolev spaces, denoting by $W^{k,p}(\Omega)$ the space of k-times weakly differentiable functions with weak derivatives in $L^p(\Omega)$; we abbreviate $W^{k,2}(\Omega)$ by $H^k(\Omega)$. Given $p > 0$ and a measurable function u in Ω, we let $\|u\|_p = (\int_\Omega |u|^p)^{1/p}$, which coincides with the Lebesgue norm for $p \geq 1$. χ_A denotes the characteristic function of a set A, and $\{u > 0\}$ is a short form for $\{(t,x) \in (0,T) \times \Omega : u(t,x) > 0\}$. Finally, $C^{\alpha,\beta}((0,T)\times\Omega)$ denotes the subset of those elements of $C((0,T)\times\Omega)$ which are of class C^α (or C^β) with respect to time (or with respect to the spatial variables), respectively.

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2. – Definition and properties of strong solutions

Let Ω be an open domain of $\mathbb{R}^N$ of class $C^{1,1}$. By a weak solution of (1.1) we mean the following.

DEFINITION 2.1. A nonnegative function $u \in L^\infty(\mathbb{R}^+; H^1(\Omega) \cap L^1(\Omega))$ is called a weak solution of (1.1) if:

(i) $u \in H^1_{\text{loc}}(\mathbb{R}^+_0; (H^{1,q}(\Omega))')$ for all $q > \frac{4N}{2N+(2-N)n}$;

(ii) if $N = 1$: $u \in C^{1,4}(\{u > 0\})$, $u^{\frac{n}{2}} u_{xxx} \in L^2(\{u > 0\})$, and u solves the equation in the sense that

$$\int_0^\infty \int_\Omega u\zeta_t + \int_0^\infty \int_{\{u(t)>0\}} u^n u_{xxx}\zeta_x = 0$$

for any $\zeta \in C^1_0(\mathbb{R}^+ \times \overline{\Omega})$;

(iii) if $N > 1$: $\chi_{\{u>0\}}u^{n-2}|\nabla u|^3$, $\chi_{\{u>0\}}u^{n-1}|\nabla u|^2$ and $u^n|\nabla u|$ belong to L^1_{loc} $(\mathbb{R}^+_0; L^1(\Omega))$, and u solves the equation in the sense that

$$\int_0^T \langle u_t(t), \zeta(t)\rangle = \frac{n(n-1)}{2}\int_0^T\int_{\{u(t)>0\}} u^{n-2}|\nabla u|^2 \nabla u \nabla\zeta + \frac{n}{2}\int_0^T\int_{\{u(t)>0\}} u^{n-1}|\nabla u|^2 \Delta\zeta$$
$$+ \frac{n}{2}\int_0^T\int_{\{u(t)>0\}} u^{n-1} D^2\zeta \nabla u \nabla u + \int_0^T\int_\Omega u^n \nabla u \nabla \Delta\zeta$$

for any $T > 0$ and any $\zeta \in C^3(\overline{\Omega}_T)$ such that $\nabla\zeta \cdot \underline{\nu} = 0$ on S_T.

In the sequel, we will concentrate on a particular class of solutions — strong solutions — which satisfy an additional integral estimate — the entropy estimate. In its global version, this estimate provides in particular $L^2((0,T);H^2(\Omega))$-regularity of certain powers of the solution u. On account of corresponding nonexistence results for selfsimilar source-type solutions if $n \geq 3$, compactly supported solutions with such regularity are only expected to exist in the parameter range $n \in (0,3)$. Hence, we confine ourselves in the subsequent definition to those values of n:

DEFINITION 2.2. Let the initial datum u_0 be of class $H^1(\Omega;\mathbb{R}_0^+) \cap L^1(\Omega)$. A function $u \in L^\infty(\mathbb{R}_0^+; H^1(\Omega) \cap L^1(\Omega))$ is called *strong solution* of (1.1) with initial datum u_0 if:

(i) u is a weak solution of (1.1);
(ii) for arbitrary $\alpha \in (\max\{-1, \frac{1}{2}-n\}, 2-n) \setminus \{0\}$, u satisfies the *α-entropy estimate*; i.e. a positive constant $C = C(\alpha, n, N)$ exists such that for arbitrary $t > 0$ and arbitrary $\zeta \in C^2(\Omega)$ with $\operatorname{supp}(\nabla\zeta) \subset\subset \Omega$:

$$\begin{aligned}
&\frac{1}{\alpha(\alpha+1)}\int_\Omega \zeta^4 u^{\alpha+1}(t) + C^{-1}\int_0^t\int_\Omega \zeta^4\left[|\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |D^2 u^{\frac{\alpha+n+1}{2}}|^2\right] \\
&\leq \frac{1}{\alpha(\alpha+1)}\int_\Omega \zeta^4 u_0^{\alpha+1} + C\int_0^t\int_{\{\zeta>0\}} u^{\alpha+n+1}\left(|\nabla\zeta|^4 + \zeta^2|\Delta\zeta|^2.\right)
\end{aligned} \tag{2.1}$$

(iii) u attains its initial data in the sense that $u(t) \to u_0$ in $L^1(\Omega)$ as $t \searrow 0$.

REMARK 2.1. The existence of strong solutions in the sense of Definition 2.2 has been proved in [3] and [10] for space dimension $N = 1$. In higher space dimensions, corresponding results up to now only could be established for $N \in \{2,3\}$ and $n > 1/8$ (cf. [13] and [11]). It is worth mentioning that, for initial data which are positive almost everywhere, solutions satisfying an entropy estimate also exist for higher values of n.

REMARK 2.2. In one space dimension, a strong solution is also a solution in the sense of the concept presented in Definition 2.1 (iii), provided $n > 1/8$.

A crucial property of strong solutions is that of finite speed of propagation. For $n < 2$, the results presented in [5], [11] can be summarized as follows:

(FSP) *if $u(t_0) = 0$ a.e. in $B(x_0, r_0) \subset \Omega$, there exists a positive constant $T_0 = T_0(n, N, r_0, \|u_0\|_1)$ and a nondecreasing function $r \in C(0, T_0; \mathbb{R}_0^+)$ such that $r(0) = 0$ and $u(t) = 0$ a.e. in $B(x_0, r_0 - r(t))$ $\forall t \in (t_0, t_0 + T_0)$.*

This result is based on estimate (2.1); in particular, the assumption $n < 2$ allows to choose α positive, hence to take full advantage of the sign properties in the entropy estimate. By piling up strong solutions in bounded domains, (FSP) in turn permits to construct a strong solution on the whole of $\mathbb{R}^N$, provided

$$\operatorname{supp}(u_0) \text{ is compact in } \mathbb{R}^N. \tag{2.2}$$

If $n \geq 2$, the analytical methods are different — mainly due to a change in the structure of the operator. In this case, the energy estimate has to be exploited, but up to now the results are restricted to space dimension $N = 1$. In [6], Bernis proves for $2 \leq n < 3$ that strong solutions on $\Omega = \mathbb{R}$ have the property of finite speed of propagation in the following sense:

(FSP′) *there exists a nondecreasing function* $r \in C(\mathbb{R}_0^+; \mathbb{R}_0^+)$, $r(0) = 0$, *such that if* $\operatorname{supp}(u_0) \subset B(x_0, r_0)$, *then* $\operatorname{supp}(u(t)) \subset B(x_0, r_0 + r(t))$ $\forall t > 0$.

His proof is based on the following weighted energy estimate:

$$\begin{aligned} &\int_{-\infty}^{r} (r-x)^6 u_x^2(t) + \int_0^t \int_{-\infty}^{r} (r-x)^6 (u^{\frac{n+2}{2}})_{xxx}^2 \\ &\qquad \leq \int_{-\infty}^{r} (r-x)^6 u_{0x}^2 + C(n) \int_0^t \int_{-\infty}^{r} u^{n+2}. \end{aligned} \tag{2.3}$$

Let us now derive from the entropy estimate (2.1) and from the energy estimate (2.3) those integral estimates which will be used as main ingredients in our procedure. For $n < 2$, we find for any $\alpha \in I_n$ a positive constant C such that we have for arbitrary $t \in (0, T)$:

$$\begin{aligned} &\frac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 u^{\alpha+1}(t) + C^{-1} \int_0^t \int_\Omega \zeta^4 \left[|\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |D^2 u^{\frac{\alpha+n+1}{2}}|^2 \right] \\ &\qquad \leq \frac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 u_0^{\alpha+1} + C \int_0^T \int_{\{\zeta>0\}} u^{\alpha+n+1} \left(|\nabla \zeta|^4 + \zeta^2 |\Delta \zeta|^2 \right), \end{aligned}$$

which by density holds for any $\zeta \in W_0^{1,\infty}(\Omega)$ such that $\zeta \Delta \zeta \in L^\infty(\Omega)$. We may take the supremum in $(0, T)$ on the left hand side. As a consequence:

$$\begin{aligned} &\sup_{t \in (0,T)} \int_\Omega \zeta^4 u^{\alpha+1}(t) + C^{-1} \int_0^T \int_\Omega \zeta^4 \left[|\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |D^2 u^{\frac{\alpha+n+1}{2}}|^2 \right] \\ &\qquad \leq \int_\Omega \zeta^4 u_0^{\alpha+1} + C \int_0^T \int_{\{\zeta>0\}} u^{\alpha+n+1} \left(|\nabla \zeta|^4 + \zeta^2 |\Delta \zeta|^2 \right) \end{aligned} \tag{2.4}$$

for any $\alpha \in I_n$ and any $\zeta \in W_0^{1,\infty}(\Omega)$ such that $\zeta \Delta \zeta \in L^\infty(\Omega)$.

If $2 \leq n < 3$ and $N = 1$ we proceed by the same arguments, and we obtain from (2.3) that

$$\begin{aligned} &\sup_{t \in (0,T)} \int_{-\infty}^{r} (r-x)^6 u_x^2(t) + \int_0^T \int_{-\infty}^{r} (r-x)^6 (u^{\frac{n+2}{2}})_{xxx}^2 \\ &\qquad \leq \int_{-\infty}^{r} (r-x)^6 u_{0x}^2 + C \int_0^t \int_{-\infty}^{r} u^{n+2}. \end{aligned} \tag{2.5}$$

We shall frequently use Gagliardo-Nirenberg's inequality [17], [23]. In the formulation we provide below, some of the summability powers are allowed to be less than one; a proof of this extension may be found in [14]. The additional observation on relevant constants follows immediately from a rescaling argument.

THEOREM 2.3 (Gagliardo-Nirenberg). *Let* $0 < q < p$, $1 \le r \le \infty$, $m \in \mathbb{N}$, $m > 0$. *Let* $\Omega \subset \mathbb{R}^N$ *be open bounded with* $\partial\Omega$ *piecewise smooth. Suppose* u *belongs to* $L^q(\Omega)$ *and its derivatives of order* m *belong to* $L^r(\Omega)$. *Then the following inequalities hold* (*with constants* C_1, C_2 *depending only on* Ω, m, q, r)*:*

$$\|u\|_p \le C_1 \|D^m u\|_r^{\Theta} \cdot \|u\|_q^{1-\Theta} + C_2 \|u\|_q$$

where

$$\frac{1}{p} = \Theta\left(\frac{1}{r} - \frac{m}{N}\right) + (1-\Theta)\frac{1}{q}$$

for all Θ *in the interval* $[0, 1)$. *The result continues to hold if* Ω *is unbounded and diffeomorphic to a cone, and in this case* $C_2 = 0$.

3. - An extension of Stampacchia's lemma

In this section, we formulate an iteration lemma which will serve as a main ingredient in the subsequent proofs for a waiting time phenomenon. It reads as follows:

LEMMA 3.1 (An extension of Stampacchia's lemma). *Assume that a given nonnegative, nonincreasing function* $G : (0, \varrho_0) \to \mathbb{R}$ *satisfies:*

$$G(\xi) \le \frac{c_0}{(\xi - \eta)^\alpha} \left(G(\eta) + (\varrho_0 - \eta)^\sigma\right)^\beta \tag{3.1}$$

for $0 \le \eta < \xi \le \varrho_0$ *and positive numbers* $c_0, \alpha, \beta, \sigma$ *such that*

$$\beta > 1 \text{ and } \sigma \ge \frac{\alpha}{\beta - 1}. \tag{3.2}$$

Assume further that

$$\varrho_0^\alpha \ge 2^{\frac{\alpha\beta}{\beta-1}} \left(1 + 2^{\frac{\alpha}{\beta-1} - \sigma}\right)^\beta \cdot c_0 \cdot \left(G(0) + \varrho_0^\sigma\right)^{\beta-1}. \tag{3.3}$$

Then

$$G(\varrho_0) = 0.$$

REMARK 3.2. If the term $(\varrho_0 - \eta)^\sigma$ did not occur in (3.1), Lemma 3.1 would be identical to Stampacchia's lemma [27, Lemma 4.1 (i)].

PROOF. Let $\varepsilon = 2^{\frac{\alpha}{\beta-1}-\sigma}$, and consider for $k \in \mathbb{N}$ the points $s_k := \varrho_0 - \frac{\varrho_0}{2^k}$. It will be sufficient to prove that

$$G(s_k) \leq 2^{\frac{\alpha k}{1-\beta}} \left(G(0) + \varrho_0^\sigma\right), \quad k \in \mathbb{N}. \tag{3.4}$$

Indeed, together with the nonnegativity of G and (3.2) the assertion follows immediately. For $k = 1$, we observe using (3.3):

$$\begin{aligned} G(s_1) &\leq \frac{2^\alpha \cdot c_0 \cdot \left(G(0) + \varrho_0^\sigma\right)^\beta}{\varrho_0^\alpha} \\ &\leq \frac{2^{\alpha - \frac{\alpha\beta}{\beta-1}} \left(G(0) + \varrho_0^\sigma\right)}{(1+\varepsilon)^\beta} \\ &\leq 2^{\frac{\alpha}{1-\beta}} \left(G(0) + \varrho_0^\sigma\right), \end{aligned}$$

hence (3.4) for $k = 1$. It remains to verify this relation in the induction step $k \to k+1$. It holds:

$$\begin{aligned} G(s_{k+1}) &\leq \frac{2^{(k+1)\alpha}}{\varrho_0^\alpha} \cdot c_0 \cdot \left(2^{\frac{\alpha k}{1-\beta}} \left(G(0) + \varrho_0^\sigma\right) + 2^{-k\sigma} \varrho_0^\sigma\right)^\beta \\ &\leq \frac{2^{(k+1)\alpha + \frac{\alpha\beta}{1-\beta}} \left(2^{\frac{\alpha k}{1-\beta}} \left(G(0) + \varrho_0^\sigma\right) + 2^{-k\sigma} \varrho_o^\sigma\right)^\beta}{(1+\varepsilon)^\beta \left(G(0) + \varrho_0^\sigma\right)^{\beta-1}} \\ &\leq \frac{2^{(k+1)\alpha + \frac{(k+1)\alpha\beta}{1-\beta}} \left(G(0) + \varrho_0^\sigma (1 + 2^{\frac{\alpha k}{\beta-1} - k\sigma})\right)^\beta}{(1+\varepsilon)^\beta \left(G(0) + \varrho_0^\sigma\right)^{\beta-1}} = (*). \end{aligned}$$

Assumption (3.2) implies

$$1 + 2^{\frac{\alpha k}{\beta-1} - k\sigma} = 1 + \varepsilon^k \leq 1 + \varepsilon;$$

hence

$$(*) \leq \frac{2^{\frac{(k+1)\alpha}{1-\beta}} \cdot (1+\varepsilon)^\beta \left(G(0) + \varrho_0^\sigma\right)^\beta}{(1+\varepsilon)^\beta \left(G(0) + \varrho_0^\sigma\right)^{\beta-1}} = 2^{\frac{\alpha(k+1)}{1-\beta}} \left(G(0) + \varrho_0^\sigma\right),$$

which proves the lemma. □

4. – Waiting time phenomenon for $0 < n < 2$: one space dimension

Let us assume Ω to be given as an interval $(-a, a)$ with $0 < a \leq \infty$ and let us suppose that the following hypothesis on the initial datum u_0 holds:

- $u_0 \equiv 0$ on $(x_0 - 3r_0, x_0]$ for a certain positive number $r_0 < \frac{a+x_0}{3}$;
- a positive constant γ exists such that

$$(W1) \qquad \limsup_{r \to 0} r^{-\gamma(\alpha+1)} \fint_{B(x_0,r)} u_0(x)^{\alpha+1} < \infty$$

for a number $\alpha \in I_n := (\max\{0, \frac{1}{2} - n\}, 2 - n)$.

Then, the following theorem holds:

THEOREM 4.1. *Let $N = 1$, $0 < n < 2$ and let u be a strong solution of* (1.1) *with initial datum u_0. Assume moreover that* $(W1)$ *holds with*

$$\gamma \geq \frac{4}{n}. \tag{4.1}$$

Then, a positive time $T^\star = T^\star(n, \alpha, \gamma, u_0)$ exists such that

$$u(t, \cdot)\Big|_{(x_0 - r_0, x_0]} \equiv 0$$

for $0 < t < T^\star$.

PROOF. Without loss of generality we assume that $x_0 = 0$. Let us describe the outline of the proof. After an appropriate choice of the cut-off function ζ in estimate (2.4), Gagliardo-Nirenberg's inequality leads for sufficient small numbers $0 < \varrho < r$ to an estimate of $F(\varrho) := \int_0^T \int_{-r_0}^{\varrho} u^{\alpha+n+1}$ mainly in terms of $F(r)$ (cf. inequality (4.7)). Combining this estimate with the iteration Lemma 3.1, the result follows.

Following this strategy, let us first make explicit our choice of cut-off function ζ. Recalling the property of finite speed of propagation (FSP), we infer the existence of a time $T_0 > 0$ such that

$$u(t, \cdot)|_{(-2r_0, -r_0)} \equiv 0 \text{ for } 0 < t < T_0 .$$

Let us choose a function $\phi \in C^\infty(\Omega; \mathbb{R}_0^+)$ such that $\phi|_{(-a,-2r_0)} \equiv 0$ and $\phi|_{(-r_0,a)} \equiv 1$. For a positive number $r < a$, we take $\zeta_r(x) := (r - x)_+ \phi(x)$ as test function in (2.4); we observe that for $0 < T < T_0$ $\operatorname{supp}(\zeta_r(\cdot) u(t, \cdot)) \subset (-r_0, r]$ and that both $|(\zeta_r)_x|$ and $\zeta_r |(\zeta_r)_{xx}|$ are bounded in $L^\infty(\Omega)$ independently of the particular choice of $r > 0$. This yields for $T < T_0$:

$$\begin{aligned} \sup_{t \in (0,T)} \int_{-r_0}^{r} (r - x)^4 u^{\alpha+1}(t) + \int_0^T \int_{-r_0}^{r} (r - x)^4 |(u^{\frac{\alpha+n+1}{2}})_{xx}|^2 & \\ \leq C \left(\int_0^r (r - x)^4 u_0^{\alpha+1} + \int_0^T \int_{-r_0}^{r} u^{\alpha+n+1} \right) . & \end{aligned} \tag{4.2}$$

For arbitrary $0 < \varrho < r$, this can be rewritten in the following way:

$$\sup_{t\in(0,T)} \int_{-r_0}^{\varrho} u^{\alpha+1}(t) + \int_0^T \int_{-r_0}^{\varrho} |(u^{\frac{\alpha+n+1}{2}})_{xx}|^2 \leq \frac{C}{(r-\varrho)^4} \left(\int_0^r (r-x)^4 u_0^{\alpha+1} + \int_0^T \int_{-r_0}^{r} u^{\alpha+n+1} \right). \tag{4.3}$$

Let us introduce the function

$$w(t,x) := \begin{cases} u^{\frac{\alpha+n+1}{2}}(t,x) & \text{if } (t,x) \in (0,T) \times (-r_0, a) \\ 0 & \text{if } (t,x) \in (0,T) \times (-\infty, -r_0]. \end{cases}$$

Due to (FSP), $w \in L^2((0,T); H^2(-\infty,\varrho))$. Choosing $q = \frac{2(\alpha+1)}{\alpha+n+1}$, (4.3) can equivalently be written as

$$\sup_{t\in(0,T)} \int_{-\infty}^{\varrho} w^q(t) + \int_0^T \int_{-\infty}^{\varrho} |w_{xx}|^2 \leq \frac{C}{(r-\varrho)^4} \left(\int_0^r (r-x)^4 w_0^q + \int_0^T \int_{-\infty}^{r} w^2 \right). \tag{4.4}$$

We estimate $\int_0^T \int_{-\infty}^{\varrho} w^2$ by Gagliardo-Nirenberg's inequality (cf. Theorem 2.3):

$$\left(\int_{-\infty}^{\varrho} w^2 \right)^{\frac{1}{2}} \leq K_1 \left(\int_{-\infty}^{\varrho} |w_{xx}|^2 \right)^{\frac{\Theta}{2}} \left(\int_{-\infty}^{\varrho} w^q \right)^{\frac{1-\Theta}{q}} \tag{4.5}$$

with $\Theta = \frac{2-q}{3q+2} < 1$. As a consequence:

$$\begin{aligned} \int_0^T \int_{-\infty}^{\varrho} w^2 &\leq K_1 \int_0^T \left(\int_{-\infty}^{\varrho} |w_{xx}|^2 \right)^{\Theta} \left(\int_{-\infty}^{\varrho} w^q \right)^{\frac{2(1-\Theta)}{q}} \\ &\leq K_1 \cdot T^{1-\Theta} \cdot \sup_{t\in(0,T)} \left(\int_{-\infty}^{\varrho} w^q \right)^{\frac{2(1-\Theta)}{q}} \cdot \left(\int_0^T \int_{-\infty}^{\varrho} |w_{xx}|^2 \right)^{\Theta}. \end{aligned} \tag{4.6}$$

Combining this inequality with (4.4) yields

$$\begin{aligned} \int_0^T \int_{-r_0}^{\varrho} w^2 &\leq C \cdot T^{1-\Theta} \times \\ &\times \left(\frac{1}{(r-\varrho)^4} \left(\int_0^r (r-x)^4 w_0^q + \int_0^T \int_{-r_0}^{r} w^2 \right) \right)^{1+(\frac{2}{q}-1)(1-\Theta)}. \end{aligned} \tag{4.7}$$

Assumption $(W1)$ on the growth of u_0 near the free boundary point $x_0 = 0$ implies the existence of positive numbers ϱ_0 and C such that for all $0 < r \leq \varrho_0$

$$\int_0^r (r-x)^4 w_0^q \leq r^4 \int_0^r u_0^{\alpha+1} \leq C \cdot r^{5+\gamma(\alpha+1)}.$$

Let us introduce for $0 \leq \xi \leq \varrho_0$ the decreasing nonnegative function

$$G(\xi) := \int_0^T \int_{-\infty}^{\varrho_0-\xi} w^2 dx\, dt.$$

Specifying $\xi = \varrho_0 - \varrho$ and $\eta = \varrho_0 - r$, we may rewrite (4.7) in the following way:

$$G(\xi) \leq \frac{C \cdot T^{1-\Theta}}{(\xi-\eta)^{4(1+\delta)}} \left(G(\eta) + (\varrho_0 - \eta)^\sigma\right)^{1+\delta} \text{ for } 0 \leq \eta < \xi \leq \varrho_0,$$

where

$$\delta = \frac{n(1-\Theta)}{\alpha+1} \text{ and } \sigma = 5 + \gamma(\alpha+1).$$

Applying the iteration Lemma 3.1, we observe that $G(\varrho_0) = 0$ provided

$$\varrho_0^{4(1+\delta)} \geq C \cdot T^{1-\Theta} \left(G(0) + \varrho_0^\sigma\right)^\delta$$

and

$$\sigma \geq \frac{4(1+\delta)}{\delta}.$$

For sufficiently small $T^\star = T^\star(\alpha, n, \varrho_0)$, the former condition can be satisfied. The latter imposes a constraint on the growth exponent γ of u_0 near the free boundary. Since $\Theta = \frac{n}{n+4(\alpha+1)}$, a lower bound on γ is given by the inequality

$$\begin{aligned} 5 + \gamma(\alpha+1) \geq \frac{4(1+\delta)}{\delta} &= 4 + \frac{4(\alpha+1)}{n(1-\Theta)} \\ &= 4 + \frac{n+4(\alpha+1)}{n} = 5 + \frac{4}{n}(\alpha+1). \end{aligned}$$

As a consequence, a time $T^\star = T^\star(\alpha, n, \varrho_0)$ exists such that

$$\int_0^{T_0} \int_{-r_0}^0 u^{\alpha+n+1} dx dt \equiv 0$$

provided $\gamma \geq \frac{4}{n}$. □

5. – Waiting time phenomenon for $\frac{1}{8} < n < 2$: higher space dimensions

In order to extend the results of Section 4 to higher space dimensions, we consider points x_0 of $\partial[\mathrm{supp}(u_0)]$ for which an "external cone property" is satisfied. For $x \in \mathbb{R}^N$, let

$$\overline{x}_N = (x_1, \dots, x_{N-1}),$$

and let $\mathcal{C}(y, \theta)$ denote a cone with vertex in y, opening angle θ and symmetry axis parallel to the x_N-axis:

$$\mathcal{C}(y, \theta) = \left\{ x \in \mathbb{R}^N : \; |\overline{y}_N - \overline{x}_N| < \tan\theta \, (y_N - x_N) \right\}.$$

Without loss of generality, we assume that:

- there exists $\theta \in (0, \frac{\pi}{4})$ and $r_0 > 0$ such that $B(x_0, 8r_0) \subset \Omega$ and

$$\mathrm{supp}(u_0) \cap \mathcal{C}(x_0, 2\theta) \cap B(x_0, 8r_0) = \emptyset;$$

(W2) • a positive constant γ exists such that

$$\limsup_{r \to 0} r^{-\gamma(\alpha+1)} \fint_{B(x_0, r)} u_0^{\alpha+1} < \infty$$

for a number $\alpha \in I_n$.

Note that $(W2)$ is a *local* condition for $\partial[\mathrm{supp}(u_0)]$ at x_0, and that it does not require convexity of the support. The result reads as follows.

THEOREM 5.1. *Let $N \in \{2, 3\}$, $\frac{1}{8} < n < 2$, and let u be a strong solution of* (1.1) *with initial datum u_0. Assume that $(W2)$ holds at some point $x_0 \in \partial[\mathrm{supp}(u_0)]$ with*

$$\gamma \geq \frac{4}{n}.$$

Then, a positive time $T^\star = T^\star(n, \alpha, \gamma, u_0)$ exists such that

$$\mathrm{supp}(u(t, \cdot)) \cap \mathcal{C}\,(x_0, \theta) \cap B(x_0, 4r_0) = \emptyset$$

for almost every $t \in (0, T^\star)$.

PROOF. Without loss of generality we may assume $x_0 = 0$. By assumption $(W2)$

$$\mathrm{supp}(u_0) \cap \mathcal{C}(0, 2\theta) \cap B(0, 8r_0) = \emptyset$$

for some $\theta \in (0, \frac{\pi}{4})$. Let e_N be the canonical basis vector parallel to the x_N-axis. Property (FSP) implies that there exists $T_0 > 0$ such that

$$u(t, \cdot) = 0 \text{ a.e. in } \mathcal{C}(-r_0 e_N, 2\theta) \cap B(0, 4r_0) \quad \forall t < T_0. \tag{5.1}$$

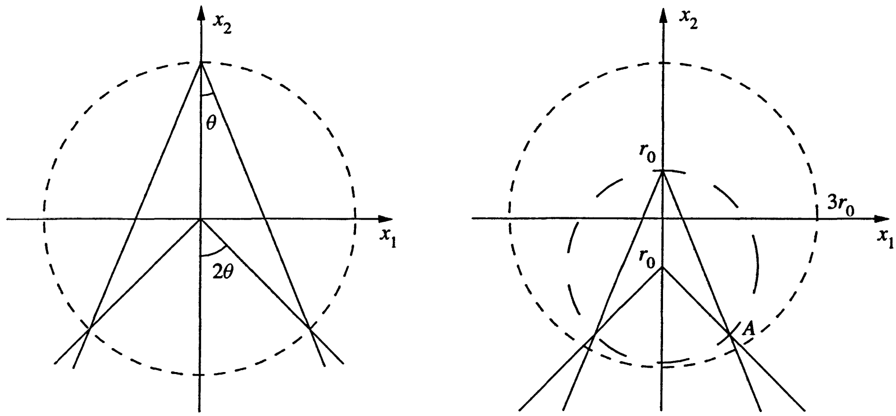

Fig. 1. A visual proof of (5.2) and (5.3). In particular, letting $B = (0, -r_0)$, note that $\overline{OA} < \overline{OB} + \overline{BA} = 3r_0$ by straightforward geometric reasoning.

We consider the one-parameter family of nested cones

$$\mathcal{C}(r) := \mathcal{C}(re_N, \theta)\ .$$

With this choice we have by geometric arguments (see fig. 1) that

$$\mathcal{C}(r) \setminus \mathcal{C}(0, 2\theta) \subset B(0, r) \quad \forall r < r_0\,; \tag{5.2}$$

moreover, a positive constant $\varepsilon = \varepsilon(r_0, \theta)$ exists such that

$$\mathcal{C}(r) \setminus \mathcal{C}(-r_0 e_N, 2\theta) \subseteq \mathcal{C}(r_0) \setminus \mathcal{C}(-r_0 e_N, 2\theta) \subseteq \overline{B}(0, 3r_0 - \varepsilon) \quad \forall r \leq r_0\,. \tag{5.3}$$

Our argumentation will be inspired by the proof of Theorem 4.1 — instead of half-lines we use cones $\mathcal{C}(r)$. To this aim, we introduce the test-functions

$$\zeta_r(x) := \begin{cases} (r - x_N)\left[\beta^2 - \dfrac{|\overline{x}_N|^2}{(r - x_N)^2}\right] & x \in \mathcal{C}(r) \\ 0 & \text{else,} \end{cases} \qquad \beta := \tan\theta$$

(see fig. 2). By definition of $\mathcal{C}(r)$, ζ_r are nonnegative and continuous. In addition, a straightforward computation shows that

$$\left.\begin{aligned} |\nabla\zeta_r(x)|^2 &= \beta^4 + (4 + 2\beta^2)\frac{|\overline{x}_N|^2}{(r - x_N)^2} + \frac{|\overline{x}_N|^4}{(r - x_N)^4}\,, \\ \Delta\zeta_r(x) &= -\frac{2}{(r - x_N)}\left[N - 1 + \frac{|\overline{x}_N|^2}{(r - x_N)^2}\right]\,, \end{aligned}\right\} \quad x \in \mathcal{C}(r)\,.$$

Since

$$\frac{|\overline{x}_N|}{(r - x_N)} < \beta \text{ in } \mathcal{C}(r)\,,$$

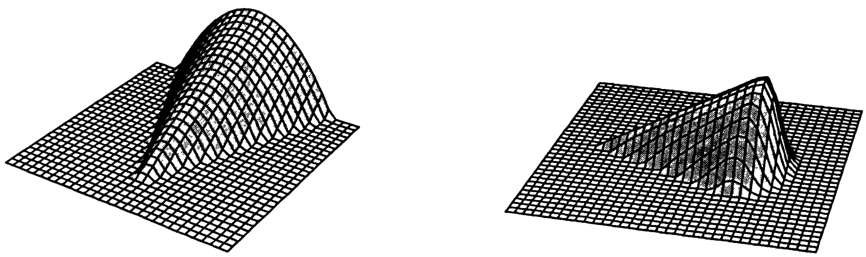

Fig. 2. The functions ζ_r and $\overline{\zeta}_r$ in space dimension $N = 2$.

a positive constant $C = C(N, \theta)$ exists such that

$$|\nabla \zeta_r|^4 \le C, \quad |\zeta_r \Delta \zeta_r|^2 \le C\,. \tag{5.4}$$

Let us introduce a cut-off $\varphi \in C_0^\infty((-4,4))$ such that $\varphi \equiv 1$ in $(-3,3)$. We let

$$\overline{\zeta}_r(x) := \varphi\left(\frac{|x|}{r_0}\right)\zeta_r(x)$$

(see fig. 2), and in view of (5.4) we have $\overline{\zeta}_r \in W_0^{1,\infty}(\Omega)$ with $\overline{\zeta}_r \Delta \overline{\zeta}_r \in L^\infty(\Omega)$. Hence $\overline{\zeta}_r$ are admissible test-functions in (2.4).

Observing that

$$\begin{aligned}\operatorname{supp}(\overline{\zeta}_r) \cap \operatorname{supp}(u(t)) &= \mathcal{C}(r) \cap B(0, 4r_0) \cap \operatorname{supp}(u(t)) \\ &\overset{(5.1)}{\subseteq} \mathcal{C}(r) \setminus \mathcal{C}(y_{-r_0}, 2\theta) \overset{(5.3)}{\subset} \overline{B}(0, 3r_0 - \varepsilon)\,,\end{aligned} \tag{5.5}$$

we obtain for $r \in [0, r_0]$ and $t < T_0$

$$\begin{aligned}\sup_{t\in(0,T)} \int_{\mathcal{C}(r)\cap B(0,3r_0)} \zeta_r^4 u^{\alpha+1}(t) + C^{-1} \int_0^T \int_{\mathcal{C}(r)\cap B(0,3r_0)} \zeta_r^4 |D^2 u^{\frac{\alpha+n+1}{2}}|^2 \\ \le \int_{\mathcal{C}(r)\cap B(0,3r_0)} \zeta_r^4 u_0^{\alpha+1} + C \int_0^T \int_{\mathcal{C}(r)\cap B(0,3r_0)} u^{\alpha+n+1}\,.\end{aligned}$$

It also follows from (5.5) that for any $t \in (0, T_0)$

$$u(t,\cdot)\Big|_{\mathcal{C}(r)\cap B(0,3r_0)} = 0 \text{ a.e. in } (\mathcal{C}(r) \cap B(0,3r_0)) \setminus \overline{B}(0, 3r_0 - \varepsilon).$$

Thus, we may extend $u(t,\cdot)$ to zero on $\mathcal{C}(r) \setminus B(0, 3r_0)$: We choose

$$w(t,x) := \begin{cases} u(t,x)^{\frac{\alpha+n+1}{2}} & x \in \mathcal{C}(r) \cap B(0,3r_0) \\ 0 & x \in \mathcal{C}(r) \setminus B(0,3r_0) \end{cases}$$

and we conclude that

$$(5.6)\quad \sup_{t\in(0,T)}\int_{C(r)}\zeta_r^4 w^q(t)+C^{-1}\int_0^T\int_{C(r)}\zeta_r^4|D^2w|^2\le\int_{C(r)}\zeta_r^4 w(0)^q+C\int_0^T\int_{C(r)}w^2,$$

with $q=\frac{2(\alpha+1)}{\alpha+n+1}$. We need to estimate ζ_r from below on any nested cone $C(\varrho)$, $\varrho<r$. Since $\Delta\zeta_r<0$ in $C(\varrho)$, it is sufficient to evaluate ζ_r on its boundary:

$$\partial C(\varrho)=\left\{x\in\mathbb{R}^N:\ \beta(\varrho-x_N)=|x_N|\right\}.$$

To this purpose, consider

$$g_r(x_N)=\zeta_r\Big|_{\partial C(\varrho)}=\frac{\beta^2(r-\varrho)}{r-x_N}(r+\varrho-2x_N),\quad x_N<\varrho;$$

observe that $g_r'(x_N)<0$, and therefore

$$(5.7)\qquad \zeta_r(x)\ge\beta^2(r-\varrho)=g_r(\varrho)\quad\forall x\in C(\varrho).$$

We use (5.7) to minorize the left-hand side of (5.6), which yields

$$(5.8)\quad \sup_{(0,T)}\int_{C(\varrho)}w^q(t)+\int_0^T\int_{C(\varrho)}|D^2w|^2\le\frac{C}{(r-\varrho)^4}\left(\int_{C(r)}\zeta_r^4w(0)^q+\int_0^T\int_{C(r)}w^2\right).$$

Gagliardo-Nirenberg inequality gives

$$\|w\|_2\le C\left\|D^2w\right\|_2^{\Theta}\|w\|_q^{1-\Theta},\quad \Theta=\frac{nN}{nN+4(\alpha+1)}.$$

Therefore, we follow the lines of the proof of Theorem 4.1 and we infer that

$$(5.9)\quad \begin{gathered}\int_0^T\int_{C(\varrho)}w^2\le\frac{C\cdot T^{1-\Theta}}{(r-\varrho)^{4(1+\delta)}}\left(\int_{C(r)}\zeta_r^4w(0)^q+\int_0^T\int_{C(r)}w^2\right)^{1+\delta},\\ \delta=\frac{n(1-\Theta)}{\alpha+1}.\end{gathered}$$

By assumption $(W2)$, there exist $\varrho_0\in(0,r_0)$ and $C>0$ such that

$$\int_{B(0,r)}u_0^{\alpha+1}<C\cdot r^{N+\gamma(\alpha+1)}\quad\forall r<\varrho_0;$$

hence, by (5.2),

$$\int_{C(r)}\zeta_r^4w(0)^q\le\int_{B(0,r)}\zeta_r^4w(0)^q\le C\cdot r^{4+N+\gamma(\alpha+1)}\quad\forall r<\varrho_0.$$

Using this estimate in (5.9), we conclude that

$$G(\xi)\le\frac{C\cdot T^{1-\Theta}}{(\xi-\eta)^{4(1+\delta)}}\left[G(\eta)+(\varrho_0-\eta)^{4+N+\gamma(\alpha+1)}\right]^{1+\delta}\quad\forall 0<\eta<\xi<\varrho_0,$$

where

$$G(\xi):=\int_0^T\int_{C(\varrho_0-\xi)}w^2.$$

An appeal to Lemma 3.1 yields the existence of a time $T^\star\in(0,T_0)$ such that

$$w=0\quad\text{in } C(0)\times(0,T^\star);$$

recalling the definition of w and (5.1), (5.3) completes the proof of the theorem. □

6. – Waiting time phenomenon for $2 \leq n < 3$: one space dimension

In this section, we prove the existence of a waiting time if the diffusion growth coefficient n belongs to $[2, 3)$. Our approach is based on the weighted energy estimate (2.5): for this reason we confine ourselves to strong solutions of the Cauchy problem in one space dimension (i.e. $\Omega \equiv \mathbb{R}$), and we assume a growth condition for the derivative of initial data in a neighbourhood of the free boundary:

- $u_0 \equiv 0$ on $(-\infty, x_0]$;
- a positive constant γ exists such that

$$(W3) \qquad \limsup_{r\to 0} r^{2-2\gamma} \fint_{B(x_0,r)} u_{0x}^2 dx < \infty .$$

The following theorem holds:

THEOREM 6.1. *Let* $2 \leq n < 3$, $\Omega \equiv \mathbb{R}$ *and let* u *be a strong solution to* (1.1) *with initial datum* u_0. *Assume moreover that* $(W3)$ *holds with*

$$\gamma \geq \frac{4}{n} . \tag{6.1}$$

Then, a positive time $T^\star = T^\star(n, \gamma, u_0)$ *exists such that*

$$u(t,\cdot)\Big|_{(-\infty,x_0]} \equiv 0$$

for $0 < t < T^\star$.

REMARK 6.2. Growth condition $(W3)$ is stronger than the corresponding one $(W1)$ in the case $n < 2$. Indeed, via Sobolev inequality one obtains

$$r^{-\gamma(\alpha+1)} \fint_{B(x_0,r)} u_0^{\alpha+1} \leq r^{-\gamma(\alpha+1)} r^{\frac{\alpha+3}{2}} \left(\fint_{B(x_0,r)} u_{0x}^2 \right)^{\frac{\alpha+1}{2}}$$
$$= \left(r^{-2(\gamma-1)} \fint_{B(x_0,r)} u_{0x}^2 dx \right)^{\frac{\alpha+1}{2}} .$$

Nevertheless, the two conditions coincide if $u_0(x) = C(x - x_0)_+^\gamma$ in a neighbourhood of x_0, and both yield $\gamma = 4/n$ as critical growth exponent.

PROOF. Without loss of generality we assume $x_0 = 0$. Combining the weighted energy estimate (2.5) with Hardy's inequality

$$\int_{-\infty}^{r} (r-x)^4 u^2 \leq C \int_{-\infty}^{r} (r-x)^6 u_x^2 ,$$

we obtain:

$$\sup_{t\in(0,T)}\int_{-\infty}^{r}(r-x)^4u^2(t)+\int_0^T\int_{-\infty}^{r}(r-x)^6(u^{\frac{n+2}{2}})_{xxx}^2 \le C\int_0^T\int_{-\infty}^{r}u^{n+2}+\int_0^r(r-x)^6u_{0x}^2\,. \tag{6.2}$$

For arbitrary positive numbers $\varrho<r$, we find $r-x>r-\varrho$ on $(-\infty,\varrho)$. Hence

$$\sup_{t\in(0,T)}\int_{-\infty}^{\varrho}u^2(t)+(r-\varrho)^2\int_0^T\int_{-\infty}^{\varrho}(u^{\frac{n+2}{2}})_{xxx}^2 \le \frac{C}{(r-\varrho)^4}\left(\int_0^T\int_{-\infty}^{r}u^{n+2}+\int_0^r(r-x)^6u_{0x}^2\right). \tag{6.3}$$

We estimate $\int_{-\infty}^{r}u^{n+2}$ using Theorem 2.3:

$$\left(\int_{-\infty}^{r}u^{n+2}\right)^{\frac12}\le K_1\left(\int_{-\infty}^{r}(u^{\frac{n+2}{2}})_{xxx}^2\right)^{\frac{n}{2(n+12)}}\left(\int_{-\infty}^{r}u^2\right)^{\frac{3(n+2)}{n+12}}.$$

Hölder's inequality gives

$$\int_0^T\int_{-\infty}^{r}u^{n+2}\le K_1\left(\int_0^T\int_{-\infty}^{r}(u^{\frac{n+2}{2}})_{xxx}^2\right)^{\frac{n}{n+12}}\left(\int_0^T\left(\int_{-\infty}^{r}u^2\right)^{\frac{n+2}{2}}\right)^{\frac{12}{n+12}}, \tag{6.4}$$

and Young's inequality implies that

$$\begin{aligned}\frac{1}{(r-\varrho)^4}\int_0^T\int_{-\infty}^{r}u^{n+2}&\le(r-\varrho)^{\frac{2n}{n+12}}K_1\left(\int_0^T\int_{-\infty}^{r}(u^{\frac{n+2}{2}})_{xxx}^2\right)^{\frac{n}{n+12}}\\&\quad\times(r-\varrho)^{-4-\frac{2n}{n+12}}\left(\int_0^T\left(\int_{-\infty}^{r}u^2\right)^{\frac{n+2}{2}}\right)^{\frac{12}{n+12}}\\&\le\varepsilon(r-\varrho)^2\int_0^T\int_{-\infty}^{r}(u^{\frac{n+2}{2}})_{xxx}^2\\&\quad+C_\varepsilon(r-\varrho)^{-(4+\frac n2)}\int_0^T\left(\int_{-\infty}^{r}u^2\right)^{\frac{n+2}{2}}.\end{aligned} \tag{6.5}$$

Hence, we obtain for arbitrary $\varepsilon>0$ and $0<\varrho<r$ the estimate

$$\begin{aligned}\sup_{t\in(0,T)}\int_{-\infty}^{\varrho}u^2(t)+(r-\varrho)^2\int_0^T\int_{-\infty}^{\varrho}(u^{\frac{n+2}{2}})_{xxx}^2&\le\varepsilon(r-\varrho)^2\int_0^T\int_{-\infty}^{r}(u^{\frac{n+2}{2}})_{xxx}^2\\&\quad+\frac{C_\varepsilon}{(r-\varrho)^{4+\frac n2}}\int_0^T\left(\int_{-\infty}^{r}u^2\right)^{\frac{n+2}{2}}\\&\quad+\frac{C}{(r-\varrho)^4}\int_0^r(r-x)^6u_{0x}^2\,.\end{aligned} \tag{6.6}$$

Let us introduce the quantities

$$V(\varrho) := \sup_{t\in(0,T)} \int_{-\infty}^{\varrho} u^2(t)\,, \tag{6.7}$$

$$U(\varrho) := \int_0^T \int_{-\infty}^{\varrho} (u^{\frac{n+2}{2}})^2_{xxx}\,, \tag{6.8}$$

and

$$F_\varepsilon(\varrho, r) := \frac{C_\varepsilon}{(r-\varrho)^{4+\frac{n}{2}}} \int_0^T \left(\int_{-\infty}^{r} u^2\right)^{\frac{n+2}{2}} + \frac{C}{(r-\varrho)^4}\int_0^r (r-x)^6 u_{0x}^2\,. \tag{6.9}$$

Then (6.6) can be rewritten as follows:

$$V(\varrho) + (r-\varrho)^2 U(\varrho) \le \varepsilon (r-\varrho)^2 U(r) + F_\varepsilon(\varrho, r) \quad \forall \varepsilon > 0,\ \forall 0 < \varrho < r\,.$$

To take full advantage of this inequality, we prove the following auxiliary result.

LEMMA. *Assume that*

$$V(\varrho') + (r'-\varrho')^2 U(\varrho') \le \varepsilon (r'-\varrho')^2 U(r') + F_\varepsilon(\varrho', r') \tag{6.10}$$

for arbitrary $0 < \varrho < \varrho' < r' < r$ *and* $\varepsilon > 0$ *sufficiently small. Then a positive constant* K_ε *exists such that*

$$V(\varrho) + \frac{(r-\varrho)^2}{4} U(\varrho) \le K_\varepsilon \cdot F_\varepsilon(\varrho, r)\,. \tag{6.11}$$

PROOF OF THE AUXILIARY LEMMA. It is inspired by the iteration method presented in [21]. For given $0 < \varrho < r$, let us consider sequences $(\varrho_k)_{k\in\mathbb{N}}$ and $(r_k)_{k\in\mathbb{N}}$ given by $\varrho_k = r - \frac{r-\varrho}{2^{k-1}}$ and $r_k = r - \frac{r-\varrho}{2^k}$. Note that

$$r_k = \varrho_{k+1} \quad \text{and} \quad r_k - \varrho_k = \frac{r-\varrho}{2^k} \quad \text{for } k \in \mathbb{N}\,. \tag{6.12}$$

Let us prove by induction that for arbitrary $M \in \mathbb{N}$

$$V(\varrho) + \frac{(r-\varrho)^2}{4} U(\varrho) \le \frac{\varepsilon^M}{4} (r-\varrho)^2 U(r_M) + \sum_{k=1}^{M} (4\varepsilon)^{k-1} F_\varepsilon(\varrho_k, r_k)\,. \tag{6.13}$$

For $M = 1$ this relation follows immediately from (6.10) since $(r_1 - \varrho_1) = (r-\varrho)/2$. For an index $M+1$, we have by assumption:

$$V(\varrho_{M+1}) + (r_{M+1} - \varrho_{M+1})^2 U(\varrho_{M+1}) \le \varepsilon (r_{M+1} - \varrho_{M+1})^2 U(r_{M+1}) + F_\varepsilon(\varrho_{M+1}, r_{M+1})\,.$$

By (6.12):

$$V(r_M) + \frac{(r-\varrho)^2}{4^{M+1}} U(r_M) \le \frac{\varepsilon (r-\varrho)^2}{4^{M+1}} U(r_{M+1}) + F_\varepsilon(\varrho_{M+1}, r_{M+1}),$$

which implies that

$$\frac{\varepsilon^M}{4}(r-\varrho)^2 U(r_M) \le \frac{\varepsilon^{M+1}(r-\varrho)^2}{4} U(r_{M+1}) + (4\varepsilon)^M F_\varepsilon(\varrho_{M+1}, r_{M+1}).$$

On the other hand

$$V(\varrho) + \frac{(r-\varrho)^2}{4} U(\varrho) \le \frac{\varepsilon^M}{4}(r-\varrho)^2 U(r_M) + \sum_{k=1}^{M} (4\varepsilon)^{k-1} F_\varepsilon(\varrho_k, r_k),$$

and therefore

$$V(\varrho) + \frac{(r-\varrho)^2}{4} U(\varrho) \le \frac{\varepsilon^{M+1}}{4}(r-\varrho)^2 U(r_{M+1}) + \sum_{k=1}^{M+1} (4\varepsilon)^{k-1} F_\varepsilon(\varrho_k, r_k)$$

which proves (6.13). Let us pass to the limit $M \to \infty$ on the right-hand side of (6.13). The first term tends to zero; for the second term, recalling (6.9) we write

$$F_\varepsilon(\varrho_k, r_k) \le 2^{k(4+\frac{n}{2})} \frac{C_\varepsilon}{(r-\varrho)^{4+\frac{n}{2}}} \int_0^T \left(\int_{-\infty}^r u^2\right)^{\frac{n+2}{2}} + 2^{4k} \frac{C}{(r-\varrho)^4} \int_0^r (r-x)^6 u_{0x}^2 .$$

Hence, for sufficiently small $\varepsilon > 0$ we obtain

$$\sum_{k=1}^{M} (4\varepsilon)^{k-1} F_\varepsilon(\varrho_k, r_k) \le \frac{K_\varepsilon}{(r-\varrho)^{4+\frac{n}{2}}} \left(C_\varepsilon \int_0^T \left(\int_{-\infty}^r u^2\right)^{\frac{n+2}{2}} + (r-\varrho)^{\frac{n}{2}} C \int_0^r (r-x)^6 u_{0x}^2 \right)$$

which proves the auxiliary lemma. □

Retranslating (6.11) into terms of u, we find the existence of a positive constant C such that

(6.14)
$$\sup_{t\in(0,T)} \int_{-\infty}^{\varrho} u^2 + (r-\varrho)^2 \int_0^T \int_{-\infty}^{\varrho} (u^{\frac{n+2}{2}})_{xxx}^2 \le \frac{C}{(r-\varrho)^{4+\frac{n}{2}}} \left[\int_0^T \left(\int_{-\infty}^r u^2\right)^{\frac{n+2}{2}} + (r-\varrho)^{\frac{n}{2}} \int_0^r (r-x)^6 u_{0x}^2 \right].$$

Together with the estimate

$$\int_0^T \left(\int_{-\infty}^{\varrho} u^2\right)^{\frac{n+2}{2}} \le T \cdot \sup_{t\in(0,T)} \left(\int_{-\infty}^{\varrho} u^2\right)^{\frac{n+2}{2}},$$

and the assumption $(W3)$ on the growth of u_{0x} near the free boundary we easily deduce that

$$\int_0^T \left(\int_{-\infty}^{\varrho} u^2\right)^{\frac{n+2}{2}} \le \frac{C\cdot T}{(r-\varrho)^{(4+\frac{n}{2})(1+\frac{n}{2})}} \left[\int_0^T \left(\int_{-\infty}^{r} u^2\right)^{\frac{n+2}{2}} + r^{5+2\gamma+\frac{n}{2}}\right]^{1+\frac{n}{2}}. \tag{6.15}$$

Following the lines of argumentation in Section 4, we conclude that a waiting time $T^\star$ exists provided

$$5+2\gamma+\frac{n}{2} \ge \frac{\left(4+\frac{n}{2}\right)\left(1+\frac{n}{2}\right)}{\frac{n}{2}}.$$

A straightforward calculation shows that this is equivalent to $\gamma \ge \frac{4}{n}$, and completes the proof of the theorem. □

7. – Questions of optimality

In this section, we will provide numerical evidence that the critical growth exponent $\gamma_0 = \frac{4}{n}$ found in the previous sections should be optimal. More precisely, if $u_0(x) \ge c|x-x_0|^\gamma$ for some $\gamma < 4/n$ at a Lipschitz-regular boundary point $x_0 \in \partial\{u_0 > 0\}$, then no waiting time phenomenon occurs. That $4/n$ is a good candidate to be the critical exponent, might also be heuristically predicted via the ansatz

$$u(x,t) \sim [x]_+^\alpha g(t) \text{ as } x \to 0,$$

which immediately yields $\alpha = 4/n$. We will also compare our results with related ones for the porous media equation.

The numerical experiments are performed by use of the entropy consistent finite element scheme that was recently developed and analysed by Martin Rumpf and the third author in [19]. This scheme proved to be very efficient in tracing the free boundary. In particular, a comparison with explicitly known selfsimilar source-type solutions showed that the distance between discrete and continuous free boundary is bounded by the grid size (cf. [19]).

For different values of the diffusion growth coefficient n in (1.1) and various positive real numbers γ, we solve equation (1.1) on the interval $\Omega = (0, 10^{-2})$ for initial data

$$u_{0\gamma}(x) := (0.9 \cdot 10^{-2} - x)_+^\gamma .$$

We choose this comparatively small interval both to emphasize the purely local character of the criteria formulated in the previous sections and to increase the sensitivity of initial data to changes in the exponent γ. In each experiment, we discretize $\Omega = (0, 10^{-2})$ uniformly by $M = 501$ grid points, and for discrete solutions U_γ we define the discrete waiting time T_n^γ by

$$T_n^\gamma := \max\left\{T_i : \int_{\bar{x}}^{0.01} U_\gamma(T_i, x)dx \leq 0 \text{ and } U_\gamma(T_i, \bar{x}) \leq 0\right\} .$$

Here, $(T_i)_{i\in I\subset\mathbb{N}}$ denotes the set of time steps, and $\bar{x}$ is the smallest spatial grid point that satisfies $\bar{x} \geq 0.9 \cdot 10^{-2}$. Note that qualitatively the results are not changed by different choices of M, provided M is sufficiently large.

In the algorithm, the continuous diffusion coefficient $m(u) = u^n$ has to be replaced by a certain discrete approximation which is characterized by a regularization parameter σ (for details, cf. [19, Section 6]). This approximation is different if $n \geq 1$ or if $n < 1$. In the former case, we choose the relevant regularization parameter σ as $\sigma = 10^{-8}$, in the latter case we take $\sigma = 10^{-12}$.

In figures 3, 4 and 5, we plot the dependency of the waiting time T on the growth exponent γ of initial data for values of $n = \frac{1}{2}$, $n = \frac{3}{2}$ and $n = \frac{5}{2}$, respectively. By a vertical line the critical number $\gamma_0 = \frac{4}{n}$ is emphasized. Especially for $n = \frac{5}{2}$, figure 6 gives a characteristic picture of the dependency of waiting time on γ in a small neighbourhood of this value $\gamma_0 = \frac{4}{n}$.

Nevertheless, it is worthwhile to compare our analytical results with the related ones for the porous medium equation

$$u_t = \operatorname{div}(u^m \nabla u).$$

Here, it is well known (cf. [22], [2], [1], [28] and the references therein) that a waiting time phenomenon occurs if and only if

$$\limsup_{r\to 0} r^{-\frac{2}{m}} \fint_{B(x_0,r)} u_0(x)\, dx < \infty .$$

Moreover, the propagation velocity of the free boundary is proportional to the quantity $\lim_{x\to x_0} u^{m-1} \frac{\partial}{\partial \nu} u|_{x\in \text{supp}(u)}$. For the thin film equation, it is proven for special cases [19] and it is conjectured in general that the velocity of the free boundary is given by $\lim_{x\to x_0} u^{n-1} \frac{\partial}{\partial \nu} \Delta u|_{x\in\text{supp}(u)}$. It is very instructive to insert the critical profile $x^{\frac{2}{m}}$ or $x^{\frac{4}{n}}$, respectively into the corresponding velocity expressions, i.e. to assume u_0 to be given by $x^{\frac{2}{m}}$ or $x^{\frac{4}{n}}$, respectively. A straightforward computation shows that in both cases we obtain an expression proportional to $f(x) = x$.

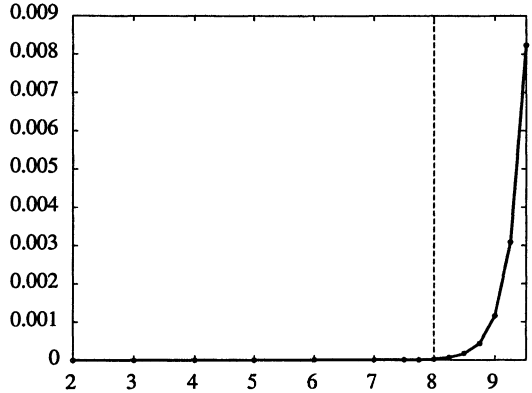

Fig. 3. $n = 1/2$: Waiting time T^* depending on growth exponent γ.

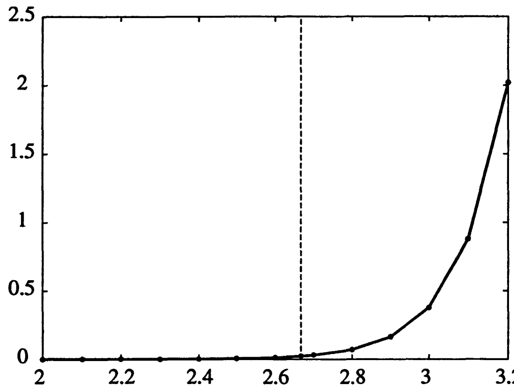

Fig. 4. $n = 3/2$: Waiting time T^* depending on growth exponent γ.

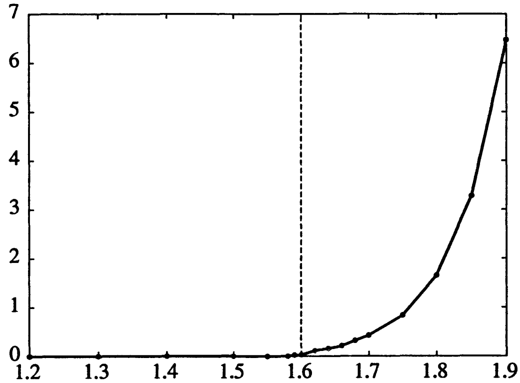

Fig. 5. $n = 5/2$: Waiting time T^* depending on growth exponent γ.

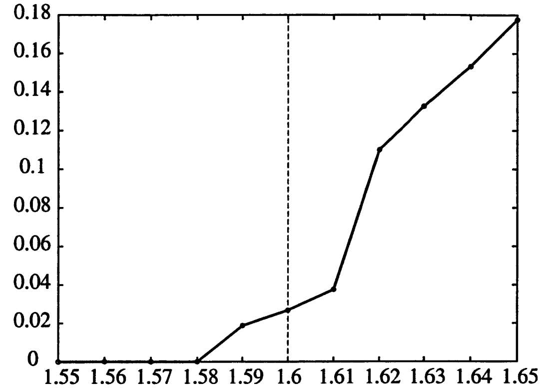

Fig. 6. $n = 5/2$: Waiting time depending on growth exponent γ – Zoom.

It is well-known that the growth exponent of selfsimilar source-type solutions for thin film equations is given by

$$\kappa = \begin{cases} \frac{3}{n} & \text{if } \frac{3}{2} \leq n < 3 \\ 2 & \text{if } 0 < n < \frac{3}{2} \end{cases}$$

(cf. [8] and [16]); hence, figures 3-6 show that for initial data which are slightly smoother no waiting time phenomenon occurs. In the case of the porous medium equation a similar observation can be made: The selfsimilar source-type solution behaves like $|x - x_0|^{\frac{1}{m}}$, whereas the critical exponent for waiting time is given by $\frac{2}{m}$.

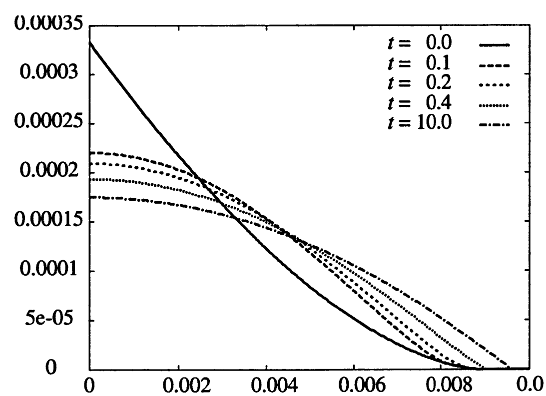

Fig. 7. $n = 5/2, \gamma = 1.7$: Delayed onset of spreading – solution profiles for different times t (number of gridpoints: 300).

Finally, figure 7 depicts profiles of discrete solutions to equation (1.1) with $n = 5/2$ and $u_0(\cdot) = u_{0\gamma}(\cdot)$ with $\gamma = 1.7$. Note that for $t \leq 0.4$

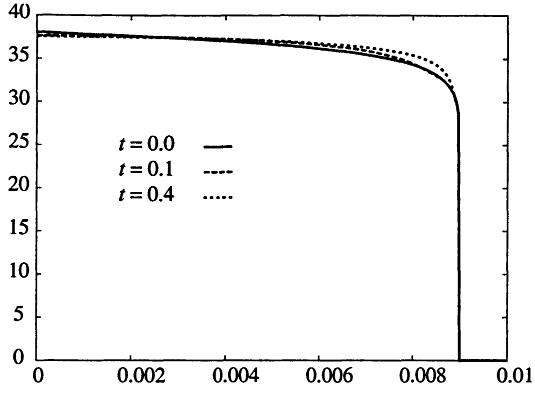

Fig. 8. $n = 5/2$, $\gamma = 1.7$: Delayed onset of spreading – logarithmic blow-up.

$\mathrm{supp}(U(t, \cdot)) = [0, 0.009]$ and that the solution profile steepens before the free boundary starts to propagate. To illustrate that for those t the free boundary indeed remains fixed, figure (8) shows a corresponding logarithmic blow-up that is given by

$$v(t,x) := \begin{cases} \log(10^{20} \cdot u(t,x)) & \text{if } u(t,x) > 10^{-20} \\ 0 & \text{otherwise}. \end{cases}$$

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