PIERO D'ANCONA

MARIAGRAZIA DI FLAVIANO

On a weakly hyperbolic quasilinear mixed problem of second order

On a Weakly Hyperbolic Quasilinear
Mixed Problem of Second Order

PIERO D'ANCONA – MARIAGRAZIA DI FLAVIANO

Abstract. We consider the mixed initial-boundary value problem for a second order hyperbolic quasilinear equation, degenerating at $t = 0$, and we prove the local well posedness in C^∞. The main tools are a priori energy estimates and the Nash-Moser theorem.

Mathematics Subject Classification (2000): 35L80 (primary), 35L20, 35L70 (secondary).

1. – Introduction

The mixed problem for degenerate hyperbolic equations and systems is a difficult one, and indeed a fairly general theory exists only for equations with constant coefficients (see e.g. [17]). Concerning linear problems with variable coefficients, we mention [2], [9], [10], [11], [12], [15] where some cases of special degeneracies for equations of second order are considered (see also [14] for related results on systems).

The study of the mixed problem for nonlinear weakly hyperbolic equations is just at its beginning; to our knowledge, the only available results in this direction are [5] and [3]. These problems combine several difficulties:

- the linerized equation is weakly hyperbolic, thus in general the smoothness of the coefficients is not sufficient to solve it, and additional assumptions on the structure of the equation are required; moreover, lower order terms may have an influence on the solvability of the equation;

- assuming that the linearized equation can be solved, the solution is in general less regular than the data (the "loss of derivatives" phenomenon). This makes it impossible to solve the nonlinear equation by a simple fixed point method, and requires more refined techniques such as the Nash-Moser theorem;

- boundary conditions may interact in a complex way both with the principal part of the operator and with lower order terms, differently from the case of strictly hyperbolic equations.

Here we shall consider the following hyperbolic operator on $[0, T] \times \overline{\Omega}$, $\Omega \subseteq \mathbb{R}^n$ being a bounded open set with smooth boundary:

$$
\begin{aligned}
L(t, x, D)u := u_{tt} &- t^{2k} \sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_i})_{x_j} + t^k \sum_{j=1}^{n} h_j(t,x)u_{tx_j} \\
&+ t^{k-1} \sum_{j=1}^{n} b_j(t,x)u_{x_j} + b_0(t,x)u_t + c(t,x)u
\end{aligned}
\tag{1.1}
$$

where $k \geq 1$ is an integer, and the form

$$
a_{ij}(t,x) = \overline{a_{ji}(t,x)}, \qquad \sum_{ij} a_{ij}(t,x)\xi_i\xi_j \geq \nu_0|\xi|^2, \qquad \nu_0 > 0
\tag{1.2}
$$

is symmetric and strictly positive definite. Thus we see that the operator $L(t, x, D)$ is strictly hyperbolic for $t > 0$ and degenerate only at $t = 0$. Here we are interested in a quasilinear mixed problem of the form

$$
Lu = f(t, x, u, u_t, \nabla u)
\tag{1.3}
$$

$$
u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad u(t,x) = 0 \ \forall (t,x) \in [0,T] \times \partial\Omega
\tag{1.4}
$$

under the conditions

$$
D_x^\alpha u_0(x) = D_x^\alpha u_1(x) = \partial_t^j f(0, x, 0, 0, 0) = 0 \quad \forall x \in \partial\Omega, \alpha \in \mathbb{N}^n, j \geq 0.
\tag{1.5}
$$

REMARK 1.1. We are assuming, for the sake of simplicity, that the data u_0, u_1 vanish of infinite order at the boundary. Clearly, it is possible to relax this assumption by considering an infinite sequence of compatibility conditions at the boundary; see Kubo [11] where such an approach is considered in the *linear case*.

We shall assume that the coefficients a_{ij}, b_j, c are in $C^\infty([0,T] \times \overline{\Omega})$; moreover, we assume that $f(t, x, u, v, p)$ has the following structure:

$$
f(t, x, u, v, p) = h(t, x, u, v) + t^{k-1} g(t, x, u, v, p)
\tag{1.6}
$$

where $h \in C^\infty([0,T] \times \overline{\Omega} \times \mathbb{R}^2)$, $g \in C^\infty([0,T] \times \overline{\Omega} \times \mathbb{R}^{n+2})$.
   Then we can prove:

THEOREM 1.1. *Let $k \geq 1$. Consider the mixed problem* (1.3), (1.4) *with* u_0, u_1, f *satisfying* (1.5), (1.6) *and* $L(t, x, D)$ *defined by* (1.1) *and satisfying* (1.2). *Then there exist $T_0 > 0$ and a local solution $u \in C^\infty([0, T_0] \times \overline{\Omega})$ to the problem.*

REMARK 1.2. The structure assumption (1.6) is very natural, in view of the degeneracy of the principal part of $L(t, x, D)$. Recall, e.g., that the Cauchy problem for the linear equation $u_{tt} - t^{2k}u_{xx} \pm t^\ell u_x = 0$ is well posed in C^∞ if and only if $\ell \geq k - 1$.

2. – The linear problem

We consider here the linear equation $Lu = f(t, x)$ for the operator $L(t, x, D)$ defined in (1.1), satisfying (1.2), on $[0, T] \times \Omega$. Moreover, we assume that the function $f(t, x)$ and all the coefficients are in $C^\infty([0, T] \times \overline{\Omega})$. Our aim is to prove the unique solvability, and a suitable a priori estimate of the solution, for the mixed problem

(2.1) $Lu = f(t, x)$

(2.2) $u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad u(t, x) = 0 \quad \forall (t, x) \in [0, T] \times \partial\Omega$

under the conditions

(2.3) $D_x^\alpha u_0(x) = D^\alpha u_1(x) = \partial_t^j f(0, x) = 0 \quad \forall x \in \partial\Omega, \alpha \in \mathbb{N}^n, j \geq 0$.

This was already proved by Kubo [11]; however we need here much more precise estimates of the solution in terms of the data and the coefficients of the equation, and for this reason we must reprove his result with different techniques.

In the following result it is essential to make a distinction between the norms of space and time derivatives. To this end, we introduce the notation, for $u : K = [0, T] \times \Omega \to \mathbb{C}$,

(2.4)
$$\|u\|_{H^{s,r}(K)} = \sum_{j=0}^{s} \sum_{|\alpha| \leq r} \|\partial_t^j D_x^\alpha u\|_{L^2(K)}$$

while we retain the notation H^s for the usual Sobolev spaces. We shall use the symbol $\mu(s, r)$ to denote the $H^{s,r}$ norm of the coefficients: denoting by $C = \{a_{ij}, h_j, b_j, b_0, c\}$ the set of all the coefficients of the operator L, we write

(2.5)
$$\mu(s, r) = \sum_{g \in C} \|g\|_{H^{s,r}(K)}.$$

To express our result in a compact form we shall finally need the following quantities which contain the norm of the initial data and the norm of the coefficients at $t = 0$:

(2.6) $\lambda_j = \|u_0, u_1\|_{H^j(\Omega)} + \sum_{i+\ell \leq j} \|\partial_t^i f(0, x)\|_{H^\ell(\Omega)}, \quad \Lambda_j = \sum_{\substack{g \in C \\ i+\ell \leq j}} \|\partial_t^i g(0, x)\|_{W^{\ell,\infty}(\Omega)}.$

Then we have:

THEOREM 2.1. *Let $T > 0$ and assume $L(t, x, D)$ has the form* (1.1), (1.2). *Then, for any smooth functions u_0, u_1, f satisfying* (2.3), *Problem* (2.1), (2.2) *has a unique solution. Moreover, there exists an integer s_0, depending only on k and the $W^{1,\infty}$ norm of the coefficients of L, such that for all $s, r \geq s_0$ the following estimate holds on the cylinder $K = [0, T] \times \Omega$:*

(2.7)
$$\|u\|_{H^{s+1,r}} + \|u\|_{H^{s,r+1}}$$
$$\leq C(s, \Lambda_{s+c(r)}) \cdot [\lambda_{s+c(r)} \mu(s + c(r), r) + \|f\|_{H^{s+c(r),r}}]$$

where $c(r) = s_0 + (2k + 1)r$.

PROOF. To simplify the notations, we shall assume that all the functions are real-valued, and that $T \leq 1$; clearly both assumptions are not restrictive. Moreover we can assume that the constant of coerciveness ν_0 in (1.2) is equal to 1.

Let u be a solution of Problem (2.1), (2.2). Fix an integer $N \geq 0$ and introduce the auxiliary function w_N on $[0, T] \times \Omega$ defined as

(2.8) $\quad w_N(t, x) = u_0(x) + tu_1(x) + \dfrac{t^2}{2!} \partial_t^2 u(0, x) + \cdots + \dfrac{t^{N+2}}{(N + 2)!} \partial_t^{N+2} u(0, x)\,.$

Then the function $v(t, x) = u(t, x) - w_N(t, x)$ solves the Cauchy problem

(2.9)
$$Lv = f_0(t, x)$$

(2.10)
$$v(0, x) = v_t(0, x) = 0\,;$$

notice that $f_0(t, x) = f(t, x) - Lw_N$ vanishes at $t = 0$ of order N, i.e., $\partial_t^j f_0(0, x) = 0$ for $j \leq N$, and v vanishes at $t = 0$ of order $N + 2$ (and of course v vanishes at the boundary of Ω). Denoting by $(\cdot, \cdot), \|\cdot\|$ the inner product and the norm in $L^2(\Omega)$ respectively, we consider the energy function

(2.11) $\quad E_0(t) = t^{-d} \left[\|v_t\|^2 + t^{2k}(Av, v) + t^{-2}\|v\|^2 \right], \quad (Av, v) = \sum_{i,j=1}^{n} (a_{ij} v_{x_i}, v_{x_j})$

for an integer $d > 0$ to be chosen. Notice that E_0 is bounded for $t \to 0$, provided $d \leq 2N + 2$. Using equation (2.9) we easily obtain for $t > 0$

$$E_0' = -\frac{d}{t} E_0 + 2t^{2k-d} \sum_{i,j=1}^{n} \left((a_{ij} v_{x_i})_{x_j}, v_t \right)$$

(2.12)
$$- 2t^{k-d} \sum_{j=1}^{n} \left(h_j v_{tx_j}, v_t \right) - 2t^{k-d-1} \sum_{j=1}^{n} \left(b_j v_{x_j}, v_t \right)$$

$$- 2t^{-d}(b_0 v_t + cv + f_0, v_t) + 2kt^{2k-d-1}(Av, v)$$

$$+ t^{2k-d}(\partial_t A\, v, v) + 2t^{2k-d}(Av, v_t) - 2t^{-d-3}\|v\|^2 + 2t^{-d-2}(v, v_t)\,.$$

Since v, v_t vanish at $\partial\Omega$, we can write

$$2t^{k-d}\sum_{j=1}^{n}\left(h_j v_{tx_j}, v_t\right) = t^{k-d}\sum_{j=1}^{n}\left(h_j, \partial_{x_j}(v_t^2)\right) = -t^{k-d}\sum_{j=1}^{n}\left(h_{jx_j}, v_t^2\right).$$

Then equation (2.12) becomes

$$E_0' = -\frac{d}{t}E_0 + t^{k-d}\sum_{j=1}^{n}\left(h_{jx_j}, v_t^2\right) - 2t^{k-d-1}\sum_{j=1}^{n}\left(b_j v_{x_j}, v_t\right)$$

$$- 2t^{-d}(b_0 v_t, v_t) - 2t^{-d}(cv, v_t) + 2t^{-d}(f_0, v_t) + 2kt^{2k-d-1}(Av, v)$$

$$+ t^{2k-d}(\partial_t A\, v, v) - 2t^{-d-3}\|v\|^2 + 2t^{-d-2}(v, v_t),$$

and this implies easily

$$(2.13) \qquad\qquad E_0' \leq (c_0 - d)t^{-1}E_0 + 2t^{-d}(f_0, v_t)$$

where c_0 is a constant depending only on k and the $W^{1,\infty}$ norms of the coefficients. If we assume d satisfies the condition

$$(2.14) \qquad\qquad d > c_0 + 1$$

we obtain from (2.13)

$$(2.15) \qquad\qquad E_0' \leq t^{-d}\|f_0\|^2.$$

This formal estimate will be used later to give an a priori bound on the first derivatives of the solution.

We proceed now to estimate higher order derivatives of u with respect to time. For the sake of simplicity we assume that the coefficients a_{ij}, h_j and b_j do not depend on time; the general case requires only minor modifications. Notice that the complete coefficients of the equation still depend on time because of the degenerate coefficients t^{2k}, t^k, t^{k-1}. We apply the operator $\partial^j = \partial_t^j$ to the equation (2.9) and we obtain

$$(2.16) \qquad\qquad L(\partial^j v) = [L, \partial^j]v + \partial^j f_0,$$

then we multiply by $\partial_t^j v_t$. If we define the higher order energy

$$(2.17) \qquad E_j = t^{-d}\left[\|\partial_t^j v_t\|^2 + t^{2k}(A\partial_t^j v, \partial_t^j v) + t^{-2}\|\partial_t^j v\|^2\right]$$

and we apply the above computations we obtain easily (as in (2.13))

$$(2.18) \qquad E_j' \leq (c_0 - d)t^{-1}E_j + 2t^{-d}(\partial^j f_0, \partial^j v_t) + 2t^{-d}([L, \partial^j]v, \partial^j v_t).$$

We can write

$$t^{-d}([L, \partial^j]v, \partial^j v_t)) = I + II_b + II_h + III + IV,$$

where

$$I = -\sum_{i=1}^{2k \wedge j} \frac{(2k)!}{(2k-i)!} (A\partial^{j-i}v, \partial^j v_t) t^{2k-i-d},$$

$$II_b = \sum_{i=1}^{(k-1) \wedge j} \frac{(k-1)!}{(k-1-i)!} \sum_r (b_r \partial^{j-i} v_{x_r}, \partial^j v_t) t^{k-1-i-d},$$

$$II_h = \sum_{i=1}^{k \wedge j} \frac{k!}{(k-i)!} \sum_r (h_r \partial^{j-i+1} v_{x_r}, \partial^j v_t) t^{k-i-d},$$

$$III = \sum_{i=1}^{j} \binom{j}{i} (\partial^i b_0 \partial^{j-i} v_t, \partial^j v_t) t^{-d},$$

$$IV = \sum_{i=1}^{j} \binom{j}{i} \sum_r (\partial^i c \partial^{j-i} v, \partial^j v_t) t^{-d}.$$

Using the definition (2.17) of E_j, the terms $II - IV$ are readily estimated as follows (we are using $t \le 1$): denoting by ρ_j the $W^{j,\infty}$ norm of the coefficients on the cylinder $K = [0, T] \times \Omega$, we have

$$III + IV \le c(j) \sum_{i=1}^{j} \rho_i \sqrt{E_{j-i}} \sqrt{E_j},$$

$$II_b + II_h \le c(k)\rho_0 \sum_{i=1}^{j} t^{-1-i} \sqrt{E_{j-i}} \sqrt{E_j}.$$

The estimate of the term I is more delicate. We can write

$$I = -\sigma_j'(t) + I_1 + I_2$$

where

$$\sigma_j(t) = t^{-d} \sum_{i=1}^{(2k) \wedge j} \frac{(2k)!}{(2k-i)!} t^{2k-i} (A\partial^{j-i}v, \partial^j v),$$

(2.19)

$$I_1 = \sum_{i=1}^{(2k) \wedge j} \frac{(2k)!}{(2k-i)!} \partial_t (t^{-d+2k-i}) (A\partial^{j-i}v, \partial^j v)$$

and

$$I_2 = \sum_{i=1}^{(2k) \wedge j} \frac{(2k)!}{(2k-i)!} t^{-d+2k-i} (A\partial^{j-i+1}v, \partial^j v).$$

Proceeding as above we have

$$I_1 + I_2 \leq c(k,d)\rho_0 \sum_{i=0}^{(2k)\wedge j} t^{-1-i}\sqrt{E_{j-i}}\sqrt{E_j}\,.$$

Thus, summing up, and applying Schwartz' inequality, we obtain from (2.18) the following:

(2.20)
$$\begin{aligned} E'_j + \sigma'_j \leq {}& (c_0 - d)t^{-1}E_j \\ & + 2t^{-d}(\partial^j f_0, \partial^j v_t) + c(k,d)t^{-1}E_j \\ & + c(j,k,d)\sum_{i=1}^{j} t^{-2i-1}\rho_i^2 E_{j-i}\,. \end{aligned}$$

If we choose d so large that

(2.21)
$$d \geq c_0 + 1 + c(k,d)$$

(and this is our final choice for d) we obtain

(2.22)
$$(E_j + \sigma_j)' \leq t^{-d}\|\partial_j f_0\|^2 + c(j,k,d)\sum_{i=1}^{j} t^{-2i-1}\rho_i^2 E_{j-i}\,.$$

We define now inductively the quantities F_j as follows: $F_0 = E_0$, and for $j \geq 1$

(2.23)
$$F_j = E_j + \sigma_j + \sum_{i=1}^{j}\gamma_{j,i}t^{-2i}F_{j-i}$$

where the positive constants $\gamma_{\ell,j}$ will be chosen in a moment. First of all, from the definition (2.19) of σ_j we have immediately

$$\sigma_j \leq \frac{1}{2}E_j + c(k)\sum_{i=1}^{j} E_{j-i}t^{-2i}\,,$$

hence

$$F_j \geq \frac{1}{2}E_j + \sum_{i=1}^{j}(\gamma_{j,i}F_{j-i} - c(k)E_{j-i})t^{-2i}\,;$$

thus, if we impose the condition

(2.24)
$$\gamma_{j,i} \geq c(k) \qquad \forall i,j$$

we see by induction that

(2.25) $$F_j \geq \frac{1}{2} E_j \quad \forall j .$$

If in addition to (2.24) we require

(2.26) $$\gamma_{j,i} \geq 2c(j, k, d) \rho_i^2$$

we have

(2.27) $$\gamma_{j,i} F_{j-i} \geq c(j, k, d) \rho_i^2 E_{j-i}$$

and this implies, by (2.22),

(2.28) $$F_j' \leq t^{-d} \|\partial^j f_0\|^2 + \sum_{i=1}^{j} \gamma_{i,j} t^{-2i} F_{j-i}' .$$

Now it is easy, using (2.28), and the basic estimate (2.15), to prove by induction the following estimate

(2.29) $$F_j' \leq c_j t^{-d} (\|\partial^j f_0\|^2 + t^{-2} \rho_1^2 \|\partial^{j-1} f_0\|^2 + \cdots + t^{-2j} \rho_j^2 \|f_0\|^2)$$

where we have also used the Gagliardo-Nirenberg inequalities

$$\rho_h \rho_k \leq \rho_{h+k} .$$

If $v(t, x)$, $f_0(t, x)$ vanish of order high enough at $t = 0$, i.e., if N is chosen as follows

(2.30) $$N = j + [d/2] + 1 ,$$

we can integrate from 0 to t and obtain, using (2.27), the estimate

(2.31) $$E_j(t) \leq c_j \int_0^t \sum_{i=0}^{j} s^{-d-2i} \|\partial^{j-i} f_0\|^2 ds .$$

This implies, recalling notation (2.4),

(2.32) $$\|v\|_{H^{j+1,0}} + \|v\|_{H^{j,1}} \leq c_j \int_0^t \sum_{i=0}^{j} s^{-d-2i} \|\partial^{j-i} f_0\|^2 ds.$$

By Taylor's formula, since $\partial^{j-i} f_0$ vanishes of order $N - (j - i) = i + [d/2] + 1$ at $t = 0$, we have

$$\partial^{j-i} f_0(s, x) = c(i) \int_0^s (s - \tau)^{i+1+[d/2]} \partial_t^{j+2+[d/2]} f_0(\tau, x) d\tau .$$

Introducing this into (2.32) we get easily on $K = [0, T] \times \Omega$

$$\|v\|_{H^{j+1},0} + \|v\|_{H^{j},1} \leq c_j \cdot \|f_0\|_{H^{j+2+[d/2]},0}$$

whence, recalling that $v = u - w_N$, $f_0 = f - Lw_N$

(2.33) $$\|u\|_{H^{j+1},0} + \|u\|_{H^{j},1} \leq \|w_N\|_{H^{j+1},0} + \|w_N\|_{H^{j},1}$$
$$+ c_j \cdot [\|f\|_{H^{j+2+[d/2]},0} + \|Lw_N\|_{H^{j+2+[d/2]},0}] .$$

Now, writing

(2.34) $$u_i(x) = \partial_t^i u(0, x)$$

we know that w_N is defined as

$$w_N = \sum_{i=0}^{N} u_i(x) \cdot t^i / i!$$

with $N = j + [d/2] + 1$, hence if we set

(2.35) $$s_1 = [d/2] + 1$$

we have for any s, r

(2.36) $$\|w_N\|_{H^{s,r}} \leq C(j) \sum_{i=0}^{s_1+j} \|u_i\|_{H^r} .$$

While u_0, u_1 are given, the functions u_i for $i \geq 2$ can be computed recursively from the equation for u at $t = 0$; we have easily a representation of the form

$$u_\ell(x) = I + II + III ,$$

where

$$I = \sum_{\substack{0 \leq \nu \leq \ell-1 \\ |\alpha_1|+\ldots+|\alpha_\nu|+|\alpha| \leq \ell}} C D_{t,x}^{\alpha_1} A_1(0, x) \cdot \ldots \cdot D_{t,x}^{\alpha_\nu} A_\nu(0, x) D_x^\alpha u_0(x) ,$$

$$II = \sum_{\substack{0 \leq \nu \leq \ell \\ |\alpha_1|+\ldots+|\alpha_\nu|+|\alpha| \leq \ell-1}} C D_{t,x}^{\alpha_1} A_1(0, x) \cdot \ldots \cdot D_{t,x}^{\alpha_\nu} A_\nu(0, x) D_x^\alpha u_1(x) ,$$

$$III = \sum_{\substack{0 \leq \nu \leq \ell-2 \\ |\alpha_1|+\ldots+|\alpha_\nu|+|\alpha|+i \leq \ell-2}} C D_{t,x}^{\alpha_1} A_1(0, x) \cdot \ldots \cdot D_{t,x}^{\alpha_\nu} A_\nu(0, x) D_x^\alpha \partial_t^i f(0, x)$$

while A_i can be any of the coefficients of the equation, and C are numeric constants with a common bound $C(\ell)$. Recalling notation (2.6) we have immediately

$$\|u_i\|_{H^r(\Omega)} \leq C(i+r, \Lambda_{i+r})\lambda_{i+r}$$

and hence by (2.36)

(2.37) $$\|w_N\|_{H^{s,r}} \leq C(j+s_1+r, \Lambda_{j+s_1+r})\lambda_{j+s_1+r}\,.$$

It remains to estimate the norm of Lw_N; recalling notation (2.5) and using Sobolev' immersion we have

(2.38) $$\|Lw_N\|_{H^{s,r}} \leq \mu(s,r)\|w_N\|_{W^{s+r,\infty}} \leq C(j+s_2+r, \Lambda_{j+s_2+r})\lambda_{j+s_2+r}\cdot\mu(s,r)$$

provided

(2.39) $$s_2 = s_1 + [n/2] + 1 = [d/2] + [n/2] + 2\,.$$

In conclusion, (2.33) gives

(2.40) $$\|u\|_{H^{j+1,0}} + \|u\|_{H^{j,1}} \leq C(j, \Lambda_{j+s_0}) \cdot [\lambda_{j+s_0}\mu(j+s_0, 0) + \|f\|_{H^{j+s_0,0}}]$$

provided we choose

(2.41) $$s_0 = [d/2] + [n/2] + 3\,.$$

We have thus estimated the time derivatives of any order of u and $\nabla_x u$. To estimate the higher order space derivatives of u and conclude the proof of (2.7), we resort to equation (2.1). Indeed, we can write it in the form

(2.42) $$t^{2k}\sum a_{ij}u_{x_ix_j} = g(t,x)$$

where in $g(t,x)$ we have collected all the terms containing at most one spatial derivative (and $f(t,x)$), whose L^2 norm we have already estimated. From (2.42), since u is a smooth function, we deduce that $\partial_t^j g(0,x)$ vanishes for $j = 0,\ldots,2k-1$ and hence can be written as

$$g(t,x) = c_k \int_0^t (t-s)^{2k-1}\partial_t^{2k}g(s,x)ds$$

and this implies

$$\|g(t,x)\|_{L^2(K)} \leq c_k t^{2k}\|\partial_t^{2k}g\|_{L^2(K)}\,.$$

Thus, by (2.42) and the estimates already proved, we are in position to estimate the L^2 norm of $\sum a_{ij}u_{x_ix_j}$; since $\sum a_{ij}\partial_{x_ix_j}^2$ is a uniformly strictly elliptic operator, by elliptic regularization we obtain an estimate of the L^2 norm of the second space derivatives of u. Notice that we have a loss of $2k$ time derivatives

because of the degeneracy of the coefficient t^{2k}. The procedure can be iterated, and finally we obtain estimate (2.7).

To conclude the proof of Theorem 2.1, we must construct a solution to problem (2.1)-(2.3). This can be done in several ways. The simplest one is to approximate the initial data with functions in $C_0^\infty(\Omega)$, i.e., with supports at a finite distance δ from the boundary of Ω (with $\delta \to 0$ in the limit). Moreover, we can prolonge the coefficients of the equation as C^∞ functions on $[0, T] \times \mathbb{R}^n$, and solve the Cauchy problem on $\mathbb{R}^n$. This problem has a unique smooth solution by standard theory (see e.g. [16]), with finite speed of propagation, and for a short time $0 \le t \le \epsilon$ the solution thus obtained is also a local solution to the original mixed problem. Since for $t \ge \epsilon$ our problem is strictly hyperbolic, the solution can be prolonged up to time T by the standard theory of mixed initial value strictly hyperbolic equations (see e.g. [13]). In conclusion, with our choice of approximate initial data we have a smooth global solution. We can apply (2.7) to these approximate solutions and by a compactness argument we obtain the required solution, satisfying estimate (2.7).   □

3. – The Nash-Moser implicit function theorem

We recall the basics of the Nash-Moser theory. We follow the exposition given in [6], which is particularly suitable for applications to initial value problems. (For the proofs of the following results and many more details see [6]).

We shall work in a subcategory of the graded Fréchet spaces, the *tame* spaces, defined as follows. We recall that a *graded* (*Fréchet*) space is a Fréchet space whose topology is generated by a *grading*, i.e. an increasing sequence of seminorms $|\cdot|_n$, $|f|_n \le |f|_{n+1}$ for all $f \in F$ and $n = 0, 1, \cdots$.

DEFINITION 3.1. A linear map $L : F \to G$ of one graded space into another is a *tame linear map* if for some $r, b \in \mathbb{N}$ the following estimate holds

$$(3.1) \qquad |Lf|_n \le c_n |f|_{n+r} \qquad f \in F, \quad n \le b$$

where the constant c_n depends only on n. The number b is called the base and r the degree of the tame estimate (3.1).

Note that tameness implies boundedness.

We recall also that the space of *exponentially decreasing sequences* $\Sigma(B)$ on a Banach space B is the graded space of all sequences of vectors in B, such that, for $n \ge 0$,

$$(3.2) \qquad |\{v_k\}|_n \equiv \sum_{k=0}^{\infty} e^{nk} |v_k|_B < \infty$$

endowed with the grading $|\cdot|_n$ defined in (3.2).

DEFINITION 3.2. A graded space is *tame* if, for some Banach space B, there exist two tame linear maps $L_1 : F \to \Sigma(B)$, and $L_2 : \Sigma(B) :\to F$ such that $L_2 L_1$ is the identity on F. (Shortly, a tame space is a tame direct summand of the space of exponentially decreasing sequences on some Banach space).

The tameness property is stable under usual operations. In particular, direct sums and products of tame spaces are tame, and a closed subspace of a tame space is tame under the induced grading. The most important examples of tame spaces are the spaces of C^∞ functions on manifolds, in fact

PROPOSITION 3.1. *Let X be a smooth compact manifold, with or without boundary. Then $C^\infty(X)$, equipped with one of the gradings*

$$|f|_n = \sup_{|\alpha| \leq n} \sup_{x \in X} |D^\alpha f(x)| \text{ or } |f|_n^2 = \sum_{|\alpha| \leq n} \|D^\alpha f(x)\|_{L^2(X)}^2$$

is a tame space.

In an analogous way, it is not difficult to see that $C^\infty([0,T]; X)$, X a Hilbert space, is a tame space with the obvious grading.

The definition of a *nonlinear* tame map runs as follows:

DEFINITION 3.3. Let $P : U \subseteq F \to G$ be a nonlinear map from an open subset U of the graded space F to the graded space G. P satisfies a *tame estimate* of degree r and base b if, for any $f \in U$, $n \geq b$,

$$(3.3) \qquad\qquad |Pf|_n \leq c_n(1 + |f|_{n+r})$$

for some constant c_n depending only on n. P is said to be *tame* if it continuous and satisfies a tame estimate in the neighbourhood of each point, with some constants r, b and c_n (which may depend on the neighbourhood).

The basic definition is the following:

DEFINITION 3.4. Let F, G be graded spaces, U an open subset of F. A map $P : U \to G$ is smooth tame if it is C^∞ and its Fréchet derivatives $D^n P$ are tame for all $n \geq 0$.

Sums and compositions of smooth tame maps are smooth tame. Moreover, linear and nonlinear partial differential operators on $C^\infty(X)$ are smooth tame maps, for X smooth manifold with or without boundary ([6]).

Now we are able to state the fundamental result of the Nash-Moser theory.

THEOREM 3.1. *Let F, G be tame spaces, U an open subset of F, $P : U \to G$ a smooth tame map. Assume that the equation $DP(u)h = k$ has a unique solution $h \equiv VP(u)k$ for all $u \in U, k \in G$, and that $VP : U \times G \to F$, thus defined, is smooth tame. Then P is locally invertible, and each local inverse is smooth tame.*

4. – Proof of Theorem 1.1

In order to slightly simplify the computations we shall assume that the right hand member of (1.3) does not depend on u_t, i.e., $f = f(t, x, u, \nabla u)$. The general case requires only minor (and obvious) modifications.

Clearly it is sufficient to prove the theorem in the case both the initial data vanish; indeed, defining $v = u - u_0 - t u_1$ the Cauchy problem is equivalent to a similar problem for v, with initial data equal to zero and a different function f.

Fixed any integer $r > n/2$, we shall apply the Nash-Moser theorem in the Fréchet space

$$F = \{u \in C^\infty([0, T]; H^r(\Omega)) :$$
$$u(0, x) = u_t(0, x) = 0 \; \forall x \in \Omega, \; u(t, x) = 0 \; \forall (t, x) \in [0, T] \times \partial\Omega\}$$

graded by

$$|u|_m = \|u\|_{H^{m+1,r}([0,T]\times\bar\Omega)} + \|u\|_{H^{m,r+1}([0,T]\times\bar\Omega)}$$

(see definition (2.4)). This is a tame Fréchet space, since it is a closed subspace of the tame space $C^\infty([0, T]; H^r(\Omega))$ (see Section 3 and [6]). We define a nonlinear map $P : F \to F$ as follows:

$$(Pu)(t, x) = u(t, x) - \int_0^t (t - s) s^{2k} \sum_{i,j=1}^n (a_{ij}(s, x) u_{x_i})_{x_j} ds$$

$$+ \int_0^t (t - s) s^k \sum_{j=1}^n h_j(s, x) u_{tx_j} ds + \int_0^t (t - s) s^{k-1} \sum_{j=1}^n b_j(s, x) u_{x_j} ds$$

$$+ \int_0^t (t - s) b_0(s, x) u_t(s, x) ds + \int_0^t (t - s) c(s, x) u(s, x) ds$$

$$- \int_0^t (t - s) f(s, x, u(s, x), u_x(s, x)) ds .$$

Notice that, in order to find a local solution to Problem (1.3)-(1.5), it is sufficient to show that the image of P contains a function vanishing for $0 \leq t \leq \varepsilon$, for some $\varepsilon > 0$.

In order to apply the Nash-Moser theorem, we must check several assumptions. First of all, P is clearly a smooth tame map, being a composition of integrations and nonlinear differential operators. Secondly, we need to prove that the equation

$$(4.1) \hspace{4cm} DP(u)v = w$$

can be solved in v for any $u, w \in F$ and the solution $v = VP(u)w$ is a smooth tame map of u, w. The explicit form of the linearized operator $DP(u)$ is the

following:

$$DP(u)v = v(t, x) - \int_0^t (t - s)s^{2k} \sum_{i,j=1}^n (a_{ij}v_{x_i})_{x_j} ds$$

$$+ \int_0^t (t - s)s^k \sum_{j=1}^n h_j v_{tx_j} ds + \int_0^t (t - s)s^{k-1} \sum_{j=1}^n b_j v_{x_j} ds$$

$$+ \int_0^t (t - s)b_0(s, x)v_t ds + \int_0^t (t - s)c(s, x)v(s, x)ds$$

$$- \int_0^t (t - s)f_u[u]v ds - \int_0^t (t - s) \sum_{j=1}^n f_{u_{x_j}}[u]v_{x_j} ds \,,$$

where we are writing for brevity

$$g[u] = g(s, x, u(s, x), \nabla u(s, x)) \,,$$

and, hence, equation (4.1) is equivalent to the Cauchy problem

(4.2) $$Lv = f_u(t, x, u, u_x)v + \sum_{j=1}^n f_{u_{x_j}}(t, x, u, u_x)v_{x_j} + w_{tt}$$

(4.3) $$v(0, x) = 0$$

(4.4) $$v_t(0, x) = 0$$

as it is readily seen (of course to be solved for $v \in F$, which includes the boundary conditions of infinite order); recall that the functions in F have the property $w(0, x) = w_t(0, x) = 0$. This is exactly a problem of the type studied in Theorem 2.1, which we know has a unique solution satisfying a suitable estimate. Thus we see that the mapping $VP : F \times F \to F$ is well defined; moreover, estimate (2.7) gives for $v = VP(u)w$

(4.5)

$$\|v\|_{H^{s+1,r}} + \|v\|_{H^{s,r+1}}$$
$$\leq C(j, \Lambda_{s+c(r)}) \cdot [\lambda_{s+c(r)}\mu(s + c(r), r) + \|w_{tt}\|_{H^{s+c(r),r}}]$$

with

$$c(r) \doteq s_0 + (2k + 1)r \,.$$

The quantities λ_j, Λ_j can be computed in terms of the fixed initial data u_0, u_1 (and of the function f), hence we can write more simply

(4.6) $$\|v\|_{H^{s+1,r}} + \|v\|_{H^{s,r+1}} \leq C(s)[\mu(s + c(r), r) + \|w_{tt}\|_{H^{s+c(r),r}}] \,.$$

Here $\mu(s, r)$ is the $H^{s,r}$ norm of the coefficients of equation (4.2), which include not only the coefficients of the operator L but also the functions $f_u(t, x, u, u_x)$ and $f_{u_{x_j}}(t, x, u, u_x)$. By standard Moser estimates we have

$$\| f_u(t, x, u, \nabla u) \|_{H^s} \leq C(\| u, \nabla u \|_{L^\infty}) \cdot \| u \|_{H^{s+1}}$$

and it is easy to extend this estimate as follows:

$$\| f_u(t, x, u, \nabla u) \|_{H^{s,r}} \leq C(\| u, \nabla u \|_{L^\infty}) \cdot [\| u \|_{H^{s+1,r}} + \| u \|_{H^{s,r+1}}] .$$

Hence, by Sobolev embedding, we can write in terms of the grading

$$\mu(s + c(r), r) \leq C(|u|_{s+s_1}) \cdot [1 + |u|_{s+c(r)}]$$

provided

(4.7) $$s_1 > \frac{n}{2} + 1 .$$

Coming back to estimate (4.5) we obtain

(4.8) $$|v|_s = |VP(u)w|_s \leq C(s, |u|_{s_1})[1 + |u|_{s+s(r)} + |w|_{s+s(r)}]$$

where

(4.9) $$s(r) = c(r) + 1 .$$

Notice that $s(r)$ depends on r and on the $W^{1,\infty}$ norms of the coefficients of the linearized equation, thus

$$s(r) = C(r, |u|_{[n/2]+1}) .$$

Recalling Definition 3.3, we see that $VP : F \times F \to F$ is a tame map.

By completely analogous computations, using again estimate (2.7), we can prove that the Fréchet derivatives $D^n VP$ of VP are tame maps. In conclusion, VP is a smooth tame map and the Nash-Moser theorem can be applied. We obtain that $P : F \to F$ is locally invertible, and its local inverses are smooth tame.

As mentioned above, to conclude the proof it is now sufficient to show that the image of P contains a function φ_ε vanishing for $0 \leq t \leq \varepsilon$, for some $\varepsilon > 0$. Now, let $g_0 = u_0, g_1 = u_1$ and for $j \geq 2$

(4.10) $$g_j(x) = \partial_t^{j-2} \{ L[w] + f(t, x, w, w_x) \} |_{t=0} .$$

It is a standard result of calculus that, given a sequence of functions $g_j(x) \in C_0^\infty(K)$ for some compact $K \subset \mathbb{R}^n$, in this case $K = \overline{\Omega}$, there exists a function $w(t, x) \in C_0^\infty([0, T] \times K)$ such that g_j are the traces of w at $t = 0$:

$$g_j(x) = \partial_t^j w(t, x) \Big|_{t=0} , \quad j \geq 1$$

(see Theorem 1.2.6 in [7]). Notice that $w \in F$, and condition (4.10) implies that

(4.11) $$\partial_t^j Pw(t,x)|_{t=0} = 0 \qquad \forall j \in \mathbb{N}$$

as it is readily seen. We have proved that P is a bijection of some neighbourhood W of w onto some neighbourhood U of Pw; by possibly restricting U we may assume that U is a ball in the $|\cdot|_M$ norm, for some M. Then define $\varphi_\varepsilon(t,x)$ as follows:

$$\varphi_\varepsilon(t,x) = \int_0^t (t-s)^M \rho\left(\frac{s}{\varepsilon}\right) \partial_t^{M+1}(Pw(s,x))ds \,,$$

where $\rho(s)$ is a C^∞ function on $\mathbb{R}$ such that $0 \le \rho \le 1$, $\rho \equiv 0$ for $s \le 1$, $\rho \equiv 1$ for $s \ge 2$. Clearly, the function φ_ε vanishes for $0 \le t \le \varepsilon$ and

$$Pw - \varphi_\varepsilon = \int_0^t (t-s)^M \left(1 - \rho\left(\frac{s}{\varepsilon}\right)\right) \partial_t^{M+1}(Pw(s,x))ds \,,$$

and hence, for any $h \le M$ and any α

$$|D_x^\alpha \partial_t^h (Pw - \varphi_\varepsilon)| \le C(\alpha,h)(a,u_0,u_1,f) \cdot \varepsilon \,.$$

If ε is small enough, the last inequalities imply that $\varphi_\varepsilon \in U$, and then $u = P^{-1}\varphi_\varepsilon$ is the required solution.

Uniqueness follows by standard linearization arguments.

We have thus proved the existence of a local solution in $C^\infty([0,T_r];H^r)$ for any r on a time interval depending on r. It may happen that $T_s < T_r$ for $s > r$; but the equation being strictly hyperbolic for $t > \epsilon$, by standard regularity results we see that actually $T_s = T_r$ provided $r > n/2$, hence it is C^∞ both in time and space variables, and this concludes the proof.

REFERENCES

[1] R. A. ADAMS, "Sobolev spaces", Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 65.

[2] R. G. ARAPETJAN, *A mixed problem for second order hyperbolic equations*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. **12** (1977), 32-45, 85.

[3] P. D'ANCONA – M. DI FLAVIANO, *An abstract degenerate hyperbolic equation with application to mixed problems*, Hokkaido Math. J. (2) **29** (2000), 315-328.

[4] P. D'ANCONA – R. MANFRIN, *A class of locally solvable semilinear equations of weakly hyperbolic type*, Ann. Mat. Pura Appl. **168** (1995), 355-372.

[5] P. D'ANCONA – R. RACKE, *Weakly hyperbolic equations in domains with boundaries*, Nonlinear Anal. **33** (1998), 455-472.

[6]   R. S. HAMILTON, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (1) **7** (1982), 65-222.

[7]   L. HÖRMANDER, "The analysis of linear partial differential operators", second ed., Springer-Verlag, Berlin, 1990.

[8]   K. KIMURA, *A mixed problem for weakly hyperbolic equations of second order*, Comm. Partial Differential Equations **6** (1981), 1335-1361.

[9]   M. L. KRASNOV, *Mixed boundary problems for degenerate linear hyperbolic differential equations second order*, Mat. Sb. **49** (1959), 29-84.

[10] A. KUBO, *Mixed problems for some weakly hyperbolic second order equations*, Math. Japonica (5) **29** (1984), 721-751.

[11] A. KUBO, *On the mixed problems for weakly hyperbolic equations of second order*, Comm. Partial Differential Equations (9) **9** (1984), 889-917.

[12] A. KUBO, *Well posedness for the mixed problems of degenerate hyperbolic equations*, Funkcialaj Ekvacioj **34** (1991), 95-102.

[13] J.-L. LIONS – E. MAGENES, "Non-homogeneous boundary value problems and applications", Vol. II. Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York-Heidelberg, 1972.

[14] V. A. MALOVICHKO, *Boundary value problems for degenerate and nondegenerate hyperbolic systems*, Differentsial' nye Uravneniya **25** (1989) 1977-1981, 2022.

[15] O. A. OLEINIK, *The Cauchy problem and the boundary value problem for second order hyperbolic equations degenerating in a domain and on its boundary*, Soviet Math. Dokl. (3) **7** (1966), 969-977.

[16] O. A. OLEINIK, *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math. **23** (1970), 569-586.

[17] R. SAKAMOTO, "Hyperbolic boundary value problems", Cambridge University Press, Cambridge, 1982.

Università di Roma "La Sapienza"
Dipartimento di Matematica
Piazzale A. Moro, 2
I-00185 Roma, Italy
dancona@mat.uniroma1.it

Università del'Aquila
Dipartimento di Matematica pura ed applicata
Via Vetoio
I-67010 Coppito (L'Aquila), Italy
diflavia@univaq.it