ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

JUNCHENG WEI
LIQUN ZHANG

On a nonlocal eigenvalue problem

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 30, nº 1 (2001), p. 41-61

<http://www.numdam.org/item?id=ASNSP_2001_4_30_1_41_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXX (2001), pp. 41-61

On a Nonlocal Eigenvalue Problem

JUNCHENG WEI - LIQUN ZHANG

Abstract. We consider a nonlocal eigenvalue problem which arises in the study of stability of point-condensation solutions in some reaction-diffusion systems. We give some sufficient (and explicit) conditions for the stability in the general case.

Mathematics Subject Classification (2000): 35B40 (primary), 35B45, 35J40 (secondary).

1. - Introduction

Recently there have been a lot of studies on the so-called point-condensation solutions generated by the Gierer-Meinhardt system from pattern formation [11]

$$\text{(1.1)} \qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + A^p H^{-q} & \text{in } \Omega, \\ \tau H_t = D_H \Delta H - H + A^r H^{-s} & \text{in } \Omega, \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial H}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, the unknowns $A = A(x,t)$ and $H = H(x,t)$ represent the respective concentrations at point $x \in \Omega \subset R^N$ and at time t of the biochemical called an activator and an inhibitor; $\epsilon > 0, \tau > 0, D_H > 0$ are all positive constants; $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^N; Ω is a smooth bounded domain in R^N; $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$. The exponents (p,q,r,s) are assumed to satisfy the condition

$$\text{(1.2)} \qquad p > 1, q > 0, r > 0, s \geq 0, \text{ and } \gamma := \frac{qr}{(p-1)(s+1)} > 1.$$

For backgrounds and recent progress, please see [3], [4], [11], [15], [16], [19], [20], [21], [26], [27], [29], etc.

Pervenuto alla Redazione il 30 novembre 1999 e in forma definitiva il 28 agosto 2000.

If we take $D_H = +\infty$, then we obtain the following so-called *shadow system* of (1.1)

$$
(1.3)\qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + A^p H^{-q} & \text{in } \Omega, \\ \tau \xi_t = -\xi + \dfrac{1}{|\Omega|}\Big(\displaystyle\int_\Omega A^r\Big)\xi^{-s}, & \\ \dfrac{\partial A}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}
$$

For D_H sufficiently large, the full Gierer-Meinhardt system (1.1) can be considered as a quite regular perturbation of its shadow system (1.3). It is well known that the stationary solutions of (1.3) are determined by the equation

$$
(1.4)\qquad \begin{cases} \epsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 \text{ in } \Omega, \ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}
$$

through the substitution $A = \xi^{\frac{q}{p-1}} u(x)$.

It is easy to see that the eigenvalue problem for the the linearization of (1.3) at a solution $A = \xi^{\frac{q}{p-1}} u_\epsilon(x)$, where $\xi^{s+1-\frac{qr}{p-1}} = |\Omega|^{-1}\int_\Omega u_\epsilon^r$ and u_ϵ is a solution to (1.4), reduces to the eigenvalue problem

$$
(1.5)\qquad \begin{cases} \epsilon^2 \Delta \phi_\epsilon - \phi_\epsilon + p u_\epsilon^{p-1}\phi_\epsilon - u_\epsilon^p \dfrac{qr}{s+1+\tau\lambda}\Big(\displaystyle\int_\Omega u_\epsilon^r\Big)^{-1}\int_\Omega u_\epsilon^{r-1}\phi_\epsilon = \lambda_\epsilon \phi_\epsilon & \text{in } \Omega, \\ \dfrac{\partial \phi_\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}
$$

Let u_ϵ be an interior or boundary spike solution. We consider two cases: $\lambda_\epsilon \to 0$ as $\epsilon \to 0$ and $\lambda_\epsilon \to \lambda \neq 0$ as $\epsilon \to 0$. (See Lemma A of [[27], page 359].) As $\epsilon \to 0$, the study of the *nonzero* eigenvalues is reduced to the study of the following nonlocal eigenvalue problem,

$$
(16)\qquad \Delta\phi - \phi + p w^{p-1}\phi - \frac{qr}{s+1+\tau\lambda}\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p = \lambda\phi, \phi \in H^2(R^N).
$$

where

$$
\lambda \in C, \lambda \neq 0,
$$

and w is the unique solution of the following problem

$$
(1.7)\qquad \begin{cases} \Delta w - w + w^p = 0, w > 0 & \text{in } R^N, \\ w(0) = \max\limits_{x \in R^N} w(x), w(x) \to 0 & \text{as } |x| \to \infty. \end{cases}
$$

(See [27], [28] for details on the derivation of (1.6).)

By [27] and [28], if problem (1.6) admits an eigenvalue λ with positive real part, then all single point-condensation solutions are unstable, while if all eigenvalues of problem (1.6) have negative real part, then all single point-condensation solutions are either stable or metastable. (Here we say that a solution is metastable if the eigenvalues of the associated linearized operator either are exponentially small or have strictly negative real parts.) Therefore it is vital to study problem (1.6).

Problem (1.6) can be simplified further. First, we can consider the simplest case $\tau = 0$. (The results work for small τ. For large τ, we refer to [20] and a recent paper by Dancer [2].) Second, let us decompose

$$L^2(R^N) = L_r^2(R^N) \oplus \mathcal{H}$$

where $L_r^2(R^N)$ is the set of radially symmetric L^2 functions on R^N and $\mathcal{H}$ is its orthogonal complement (in $L^2(R^N)$). It is easy to see that the left hand side operator of (1.6), denoted by $\mathcal{L}$, maps $L_r^2(R^N) \cap H^2(R^N)$ into $L_r^2(R^N)$ and so $L_r^2(R^N)$ is invariant under $\mathcal{L}$. On the other hand, if $\phi \in \mathcal{H}$, $\int_{R^N} w^{r-1}\phi = 0$ and hence on this subspace $\mathcal{H}$, $\mathcal{L}\phi = \Delta\phi - \phi + pw^{p-1}\phi$ and it follows easily $\mathcal{L}$ maps $H^2(R^N) \cap \mathcal{H}$ into $\mathcal{H}$. Thus the equation (1.6) is reduced to one on $L_r^2(R^N)$ and one on $\mathcal{H}$. On $\mathcal{H}$, the equation is

$$\Delta\phi - \phi + pw^{p-1}\phi = \lambda\phi, \phi \in \mathcal{H}$$

which has zero as an eigenvalue of multiplicity N and all the other eigenvalues are real and negative. This follows from the fact that the following eigenvalue problem

$$\Delta\phi - \phi + pw^{p-1}\phi = \mu\phi, \phi \in L^2(R^N), \tag{1.8}$$

admits the following set of eigenvalues

$$\mu_1 > 0, \mu_2 = \ldots = \mu_{N+1} = 0, \mu_{N+2} < 0, \ldots$$

where the eigenfunction corresponding to μ_1 is radially symmetric. (See Theorem 2.1 of [17] and Lemma 1.2 of [27].)

Thus the eigenvalue problem (1.6) with $\tau = 0$ can be reduced to the following simple form

$$\Delta\phi - \phi + pw^{p-1}\phi - \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p = \lambda\phi, \phi \in L_r^2(R^N), \tag{1.9}$$

where

$$\gamma = \frac{qr}{(p-1)(s+1)}, \lambda \in \mathcal{C}, \lambda \neq 0.$$

From now on, we shall work exclusively with (1.9).

When $\gamma = 0$, problem (1.9) has an eigenvalue $\lambda = \mu_1 > 0$. An important property of (1.9) is that nonlocal term can push the eigenvalues of problem

(1.9) to become negative so that the point-condensation solutions of the Gierer-Meinhardt system become stable or metastable.

We remark that problem (1.9) also arises in the study of generalized Gray-Scott models, see [5], [6], [7], [13], [14], [18], [22], [23], [28], etc.

A major difficulty in studying problem (1.9) is that the operator is not self-adjoint if $r \neq p+1$. Therefore it may have complex eigenvalues or Hopf bifurcations. Many traditional techniques don't work here. We remark that the linear stability analysis for another scalar non-local problem has previously been conducted by Freitas [8], [9] and [10]. In those papers, he considered the linear operator of non-local problem as a perturbation of a local operator. (Similar approach has been used in [1].) Our approach here is not perturbation type. Instead, we work directly with the non-local problem.

In [28] and [27], the eigenvalues of problem (1.9) in the following two cases

$$r = 2, \quad \text{or } r = p+1$$

are studied and the following results are proved.

THEOREM A. (1) *If* (p, q, r, s) *satisfies*

$$\gamma = \frac{qr}{(s+1)(p-1)} > 1, \tag{A}$$

and

$$r = 2, 1 < p < 1 + \frac{4}{N} \text{ or } r = p+1, 1 < p < \left(\frac{N+2}{N-2}\right)_+, \tag{B}$$

where $(\frac{N+2}{N-2})_+ = \frac{N+2}{N-2}$ *when* $N \geq 3$ *and* $(\frac{N+2}{N-2})_+ = +\infty$ *when* $N = 1, 2$.

Then $Re(\lambda) < -c_1 < 0$ *for some* $c_1 > 0$, *where* $\lambda \neq 0$ *is an eigenvalue of problem* (1.9).

(2) *If* (p, q, r, s) *satisfies* (A) *and*

$$r = 2, p > 1 + \frac{4}{N} \text{ and } \gamma < 1 + c_0, \tag{C}$$

for some $c_0 > 0$. *Then problem* (1.9) *has an eigenvalue* $\lambda_1 > 0$.

For general r, the first author in [25] proved the following:

THEOREM B. (1) *If*

$$D(r) := \frac{(p-1)\int_{R^N}\left((L_0^{-1}w^{r-1})w^{r-1}\right)\int_{R^N} w^2}{(\int_{R^N} w^r)^2} > 0 \tag{1.10}$$

where $L_0 = \Delta - 1 + pw^{p-1}$ *(*L_0^{-1} *exists in* $H_r^2(R^N) := \{u \in H^2(R^N) | u(x) = u(|x|)\}$*) and*

$$1 + \frac{1}{\sqrt{1+\rho_0}} < \gamma < 1 + \frac{1}{\sqrt{1-\rho_0}}, \tag{1.11}$$

where $\rho_0 > 0$ is given by

$$\rho_0 := \frac{\int_{R^N} w^{p+1}}{\sqrt{\int_{R^N} w^{2p} \int_{R^N} w^2}} < 1. \tag{1.12}$$

Then for any nonzero eigenvalue λ of problem (1.9), *we have $Re(\lambda) < -c_1 < 0$ for some $c_1 > 0$.*

(2) *If (p, q, r, s) satisfies*

$$1 + \frac{2r}{N} < p < \left(\frac{N+2}{N-2}\right)_+ \quad \text{and } \gamma < 1 + c_0, \tag{1.13}$$

for some $c_0 > 0$. Then problem (1.9) *has a real eigenvalue $\lambda_1 > 0$.*

Generally speaking, it is very difficult to compute $D(r)$. Thus Theorem B does not give us an explicit value for r and p.

The purpose of this paper is to study the general case and to give some explicit conditions on r and p. Our main idea is to start with Theorem A where the cases $r = 2$ and $r = p+1$ are studied, and do a continuation argument for r. To this end, we fix $\gamma > 1$ and $p > 1$. Set

$$F(r) = 1 - \frac{p-1}{2r} N. \tag{1.14}$$

The following theorem gives us some explicit values for r and p.

THEOREM 1. *Suppose $2 < r < p+1$, $F(r) \geq 0$ and*

$$F(r) \geq \frac{\gamma - 2}{\gamma} F(p+1) + \frac{|\gamma - 2|}{\gamma} \sqrt{F(p+1)(F(p+1) - F(2))}, \tag{1.15}$$

then for any nonzero eigenvalue λ of problem (1.9), *we have $Re(\lambda) < -c_1 < 0$ for some $c_1 > 0$.*

REMARK. Condition (1.15) holds if $2 < r < p+1$, $F(2) \geq 0$ (i.e. $1 < p \leq 1 + \frac{4}{N}$) and $1 < \gamma \leq 2$. Thus in this case we obtain the stability of the nonzero eigenvalues of (1.9). This is the first explicit result for the case when $r \notin \{2, p+1\}$. For $\gamma > 2$, we need

$$F(r) \geq \frac{\gamma - 2}{\gamma} \left[F(p+1) + \sqrt{F(p+1)(F(p+1) - F(2))} \right].$$

Theorem 1 follows from the following more general result:

THEOREM 2. *Suppose that there exists an interval* $(r_1, r_2) \subset (1, +\infty)$ *such that either* $2 \in (r_1, r_2)$ *or* $p + 1 \in (r_1, r_2)$, *and for any* $r \in (r_1, r_2)$, *we have*

(i) $\gamma^2 D(r) - F(p+1) - 2\gamma(\gamma-1)F(r) + (\gamma-1)^2 F(2) < 0$,

(ii) $F(p+1) + \gamma F(r) - (\gamma-1)F(2) > 0$,

(iii) $\gamma^2 D(r) > (\gamma-2)^2 F(p+1) - \frac{(\gamma F(r) - (\gamma-2)F(p+1))^2}{F(p+1) - F(2)}$.

Then for any $r \in (r_1, r_2)$ *and any nonzero eigenvalue* λ *of problem* (1.9), *we have* $Re(\lambda) < -c_1 < 0$ *for some* $c_1 > 0$.

REMARKS. 1. Assumption (i) is satisfied at $r = 2$ or $r = p + 1$ for *any* $\gamma > 1$. In the proof of Theorem 1, it is shown that assumption (i) is satisfied if $2 \leq r \leq p + 1$ and $F(r) \geq 0$.

2. If $F(r) \geq 0$, then assumption (ii) is always true, since $\gamma > 1$ and $F(r)$ is an increasing function of r.

3. Assumption (iii) is satisfied for $r = 2$ if and only if $F(2) > 0$. For $r = p+1$ assumption (iii) is also satisfied for *any* $\gamma > 1$. Thus (1) of Theorem A is covered by Theorem 2.

4. Note that if $\gamma = 2$ and $D(r) > 0$, assumption (iii) is always satisfied. It can be shown that if $2 \leq r \leq p+1$, assumption (i) and (ii) are always satisfied if $D(r) > 0$. Thus Theorem 2 covers the result of (1) of Theorem B.

5. Let $(\gamma(t), r(t)), 0 \leq t \leq 1$ be a path in $(1, +\infty) \times (1, +\infty)$ with the property that $r(0) = p + 1$ or $r(0) = 2$ and the assumptions (i), (ii) and (iii) hold for $r = r(t), \gamma = \gamma(t), 0 \leq t \leq 1$. Then the result of Theorem 2 is true along the path.

6. Let $F(r) \geq 0, 2 \leq r \leq p + 1$. Then the condition (1.15) of Theorem 1, in general, can be replaced by assumption (iii) of Theorem 2. Namely if there exists $(r_1, r_2) \subset [2, p + 1]$ with property that for all $r \in (r_1, r_2)$ we have $F(r) \geq 0$ and

$$\gamma^2 D(r) > (\gamma-2)^2 F(p+1) - \frac{(\gamma F(r) - (\gamma-2)F(p+1))^2}{F(p+1) - F(2)},$$

then the conclusion of Theorem 1 still holds.

Finally, we remark that the condition that $\gamma > 1$ is necessary for the stability of eigenvalues of (1.9). In fact, for $\gamma < 1$, we have

THEOREM 3. *If* $\gamma < 1$, *then problem* (1.9) *has a positive eigenvalue* $\lambda_0 > 0$.

In the rest of the paper, we prove Theorem 2 in Section 3, Theorem 1 in Section 4 and Theorem 3 in Section 5. We collect some preliminary results in Section 2. Our main idea is a continuation argument. We find a quadratic functional which depends on r and is positive definite along a path from r to $p + 1$ or from r to 2.

ACKNOWLEDGMENTS. The research of the first author is supported by an Earmarked Research Grant from RGC of Hong Kong. The result is obtained when the second author is visiting the IMS of the Chinese University of Hong Kong. He thanks the hospitality of IMS and the Department of Mathematics. We thank the referee for his/her carefully reading the manuscript and many valuable suggestions.

2. – Some preliminaries

Let w be the unique solution of (1.7). We now collect some useful results. We first recall the following lemma.

LEMMA 4 (Lemma 4.1 of [24]). (1) *The linear operator*

$$L_0\phi := \Delta\phi - \phi + pw^{p-1}\phi, \phi \in H^2(R^N),$$

is an invertible map from $H_r^2(R^N)$ to $L_r^2(R^N)$, where $H_r^2(R^N)$ (or $L_r^2(R^N)$) consists of radially symmetric functions in $H^2(R^N)$ (or $L^2(R^N)$).

(2) *The eigenvalue problem*

$$\Delta\phi - \phi + \alpha w^{p-1}\phi = 0, \phi \in H^2(R^N),$$

admits the following set of eigenvalues

$$\alpha_1 = 1,\ V_1 = \text{span } \{w\},$$

$$\alpha_2 = \ldots = \alpha_{N+1} = p,\ V_2 = \text{span } \left\{\frac{\partial w}{\partial x_j} | j = 1, ..., N\right\},$$

$$\alpha_{N+2} > p.$$

In particular, we have

$$\int_{R^N} [|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2] \geq \alpha_{N+2} \int_{R^N} \phi^2, \tag{2.1}$$

for all $\phi \in H_r^1(R^N) = \{u \in H^1(R^N) | u(x) = u(|x|)\}$ and $\int_{R^N} w^p\phi = 0$.

Let

$$\mathcal{L}_0(\varphi, \phi) = \int_{R^N} [\nabla\varphi\nabla\phi + \varphi\phi - pw^{p-1}\varphi\phi], \varphi, \phi \in H^1(R^N), \tag{2.2}$$

and $V = L_0^{-1} w^{r-1}$. So V satisfies

$$\Delta V - V + pw^{p-1}V = w^{r-1}, V \in H_r^2(R^N). \tag{2.3}$$

It is easy to see that

$$\int_{R^N} w^p V = \frac{1}{p-1}\int_{R^N} w^r. \tag{2.4}$$

Note that $L_0(w + \frac{p-1}{2}x \cdot \nabla w) = (p-1)w$. So we have

$$\int_{R^N} (p-1)wV = \left(1 - \frac{N(p-1)}{2r}\right)\int_{R^N} w^r = F(r)\int_{R^N} w^r. \tag{2.5}$$

By Pohozaev identity we also have that

$$F(p+1) = \frac{\int_{R^N} w^2}{\int_{R^N} w^{p+1}}. \tag{2.6}$$

Finally we recall the following result.

Lemma 5 (Lemma 5.1, Lemma 5.2 and Lemma 5.3 of [27]).
(1) *If* $r = 2$, $F(2) > 0$, *then there exists a positive constant* $a_1 > 0$ *such that*

$$\begin{aligned}\int_{R^N} (|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2) &+ \frac{2(p-1)\int_{R^N} w^p\phi \int_{R^N} w\phi}{\int_{R^N} w^2} \\ &- (p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}\left(\int_{R^N} w\phi\right)^2 \\ &\geq a_1 d^2_{L^2(R^N)}(\phi, X_1),\ \forall \phi \in H^1(R^N),\end{aligned}$$

where $X_1 = \operatorname{span}\{w, \frac{\partial w}{\partial x_j}, j = 1, ..., N\}$ *and* $d_{L^2(R^N)}$ *means the distance in* L^2-*norm.*

(2) *If* $r = p+1$, *then there exists a positive constant* $a_2 > 0$ *such that*

$$\begin{aligned}\int_{R^N} (|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2) &+ \frac{(p-1)(\int_{R^N} w^p\phi)^2}{\int_{R^N} w^{p+1}} \\ &\geq a_2 d^2_{L^2(R^N)}(\phi, X_1),\ \forall \phi \in H^1(R^N).\end{aligned}$$

3. – Proof of Theorem 2

In this section, we shall prove Theorem 2. As we remarked earlier, we introduce a quadratic form which is positive definite at $r = p + 1$ and $r = 2$. Then we use a continuation argument for r.

We first introduce a quadratic form.

To this end, let us suppose that (λ, ϕ) is a solution of (1.9) with $\lambda \neq 0$. Set $\lambda = \lambda_R + i\lambda_I$ and $\phi = \phi_R + i\phi_I$. Then we obtain two equations

$$L_0\phi_R - (p-1)\gamma \frac{\int_{R^N} w^{r-1}\phi_R}{\int_{R^N} w^r} w^p = \lambda_R\phi_R - \lambda_I\phi_I, \tag{3.1}$$

$$L_0\phi_I - (p-1)\gamma \frac{\int_{R^N} w^{r-1}\phi_I}{\int_{R^N} w^r} w^p = \lambda_R\phi_I + \lambda_I\phi_R. \tag{3.2}$$

Multiplying (3.1) by ϕ_R and (3.2) by ϕ_I and adding them together, we obtain

$$\begin{aligned} &-\lambda_R \int_{R^N} (\phi_R^2 + \phi_I^2) \\ &= \int_{R^N} (|\nabla\phi_R|^2 + \phi_R^2) - p\int_{R^N} w^{p-1}\phi_R^2 + \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi_R \int_{R^N} w^p\phi_R}{\int_{R^N} w^r} \\ &+ \int_{R^N} (|\nabla\phi_I|^2 + \phi_I^2) - p\int_{R^N} w^{p-1}\phi_I^2 + \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi_I \int_{R^N} w^p\phi_I}{\int_{R^N} w^r}. \end{aligned} \tag{3.3}$$

Multiplying (3.1) by w and (3.2) by w we obtain

$$\begin{aligned} &(p-1)\int_{R^N} w^p\phi_R - \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi_R}{\int_{R^N} w^r}\int_{R^N} w^{p+1} \\ &= \lambda_R \int_{R^N} w\phi_R - \lambda_I \int_{R^N} w\phi_I, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &(p-1)\int_{R^N} w^p\phi_I - \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi_I}{\int_{R^N} w^r}\int_{R^N} w^{p+1} \\ &= \lambda_R \int_{R^N} w\phi_I + \lambda_I \int_{R^N} w\phi_R. \end{aligned} \tag{3.5}$$

For $t > -1$, let us set

$$\begin{aligned} I^t(\varphi) &= \mathcal{L}_0(\varphi,\varphi) \\ &+ \frac{(p-1)\gamma}{\int_{R^N} w^r}\int_{R^N} w^{r-1}\varphi \int_{R^N} w^p\varphi - t\frac{(p-1)}{\int_{R^N} w^2}\int_{R^N} w^p\varphi\int_{R^N} w\varphi \\ &+ t\frac{\gamma(p-1)}{\int_{R^N} w^r \int_{R^N} w^2}\int_{R^N} w^{p+1}\int_{R^N} w^{r-1}\varphi\int_{R^N} w\varphi, \end{aligned} \tag{3.6}$$

where $\mathcal{L}_0$ is defined by (2.2).

From (3.3), (3.4) and (3.5) we obtain that

$$\begin{aligned}&I^t(\phi_R) + I^t(\phi_I)\\ &= -\lambda_R \left[\int_{R^N} (|\phi_R|^2 + |\phi_I|^2) + t \frac{(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2}{\int_{R^N} w^2} \right].\end{aligned} \tag{3.7}$$

To prove Theorem 2, our main idea is to find a continuous function $t = t(r) > -1$ such that $I^{t(r)}$ is positive definite. That is the following lemma.

LEMMA 6. *Suppose that for all $r \in (r_1, r_2)$ assumptions* (i), (ii) *and* (iii) *hold. Moreover either $2 \in (r_1, r_2)$ or $p+1 \in (r_1, r_2)$. Then there exists a continuous function $t = t(r) > -1$, $r \in (r_1, r_2)$ such that $I^{t(r)}(\varphi) > 0$ for any $\varphi \in H_r^1(R^N)$, $\varphi \not\equiv 0$.*

PROOF. We first note that

$$I^t(\varphi) = ((-L^t)\varphi, \varphi)$$

where

$$\begin{aligned}L^t\varphi := L_0\varphi &- \frac{(p-1)\gamma}{2\int_{R^N} w^r} \left(w^p \int w^{r-1}\varphi + w^{r-1} \int_{R^N} w^p\varphi \right)\\ &+ t\frac{p-1}{2\int w^2} \left(w^p \int_{R^N} w\varphi + w \int_{R^N} w^p\varphi \right)\\ &- t\frac{(p-1)\gamma}{2\int_{R^N} w^r \int_{R^N} w^2} \int_{R^N} w^{p+1} \left(w^{r-1} \int_{R^N} w\varphi + w \int_{R^N} w^{r-1}\varphi \right).\end{aligned}$$

Since L^t is self-adjoint, it is easy to see that I^t is positive definite if and only if L^t has only negative eigenvalues.

We now study the following zero eigenvalue problem for L^t on $L_r^2(R^N)$:

$$L^t\varphi = 0,\ \varphi \in L_r^2(R^N),\ \varphi \not\equiv 0. \tag{3.8}$$

It is easy to see that $\varphi \in H_r^2(R^N)$. By (1) of Lemma 4, we have

$$\begin{aligned}\varphi = &\left(\frac{\gamma \int_{R^N} w^{r-1}\varphi}{2\int_{R^N} w^r} - \frac{t\int_{R^N} w\varphi}{2\int_{R^N} w^2} \right) w\\ &+ \left(\frac{(p-1)\gamma \int_{R^N} w^p\varphi}{2\int_{R^N} w^r} + \frac{t\gamma(p-1)\int_{R^N} w^{p+1} \int_{R^N} w\varphi}{2\int_{R^N} w^r \int_{R^N} w^2} \right) V\\ &+ \left(\frac{t\gamma \int_{R^N} w^{p+1} \int_{R^N} w^{r-1}\varphi}{2\int_{R^N} w^r \int_{R^N} w^2} - \frac{t\int_{R^N} w^p\varphi}{2\int_{R^N} w^2} \right) \left(w + \frac{p-1}{2} x \cdot \nabla w \right).\end{aligned} \tag{3.9}$$

Set $A=\int_{R^N} w\varphi,\ B=\int_{R^N} w^p\varphi,\ C=\int_{R^N} w^{r-1}\varphi$. Then we have

$$(3.10)\quad A=\frac{\gamma\int_{R^N} w^2}{2\int_{R^N} w^r}C-\frac{t}{2}A+\left(\frac{\gamma}{2}B+\frac{\gamma t}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}A\right)F(r)$$
$$+\left(\frac{\gamma t\int_{R^N} w^{p+1}}{2\int_{R^N} w^r}C-\frac{t}{2}B\right)F(2),$$

$$(3.11)\quad B=\frac{\gamma}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^r}C-\frac{t}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}A+\left(\frac{\gamma}{2}B+\frac{\gamma t}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}A\right)$$
$$+\left(\frac{\gamma t\int_{R^N} w^{p+1}}{2\int_{R^N} w^r}C-\frac{t}{2}B\right),$$

$$(3.12)\quad C=\frac{\gamma}{2}C-\frac{t}{2}\frac{\int_{R^N} w^r}{\int_{R^N} w^2}A+\left(\frac{\gamma}{2}B+\frac{\gamma t}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^r}A\right)\frac{(p-1)\int_{R^N} Vw^{r-1}}{\int_{R^N} w^r}$$
$$+\left(\frac{rt\int_{R^N} w^{p+1}}{2\int_{R^N} w^2}C-\frac{t}{2}\frac{\int_{R^N} w^r}{\int_{R^N} w^2}B\right)F(r).$$

Recall that

$$D(r):=\frac{(p-1)\int_{R^N} Vw^{r-1}\int_{R^N} w^2}{(\int_{R^N} w^r)^2}.$$

Since $A^2+B^2+C^2\neq 0$ (otherwise, by (3.9), $\varphi\equiv 0$), we have by (3.10), (3.11) and (3.12) that

$$\begin{vmatrix} \gamma tF(r)-F(p+1)(t+2) & \gamma F(r)-tF(2) & \gamma F(p+1)+\gamma F(2)t \\ (\gamma-1)t & \gamma-2-t & \gamma+\gamma t \\ (\gamma D(r)-F(p+1))t & \gamma D(r)-F(r)t & (\gamma-2)F(p+1)+\gamma F(r)t \end{vmatrix}=0.$$

That is

$I_1(t):=$

$$\begin{vmatrix} (\gamma F(r)-F(p+1))t-2F(p+1) & \gamma F(r)-tF(2) & \gamma F(p+1)+\gamma^2F(r) \\ (\gamma-1)t & \gamma-2-t & \gamma^2-\gamma \\ (\gamma D(r)-F(p+1))t & \gamma D(r)-F(r)t & \gamma^2 D+(\gamma-2)F(p+1) \end{vmatrix}=0.$$

It is easy to check that

$$I_1(0)=2F(p+1)\big(\gamma^2D(r)-(\gamma-2)^2F(p+1)\big),$$
$$I'_1(0)=4F(p+1)\big(\gamma^2D(r)+(\gamma-2)F(p+1)-\gamma(\gamma-1)F(r)\big),$$
$$I''_1(0)=4F(p+1)\big(\gamma^2D(r)-F(p+1)-2\gamma(\gamma-1)F(r)+(\gamma-1)^2F(2)\big).$$

Thus we obtain that L^t has a zero eigenvalue if and only if

$$I_1(t) = \frac{1}{2} I''_1(0)t^2 + I_1'(0)t + I_1(0) = 0.$$

Note that $I_1(-1) = 2(\gamma - 1)^2 F(p+1)(F(2) - F(p+1))$. Assumption (i) implies that $I_1(t)$ is concave while assumption (ii) implies that the maximum point

$$t_{\max} := -\frac{I_1'(0)}{I_1''(0)}$$

is greater than -1. Finally simple computations show that

$$\begin{aligned} I_1(t_{\max}) &= I_1(0) - \frac{(I_1'(0))^2}{2I_1''(0)} \\ &= \frac{(4(\gamma-1)F(p+1))^2}{2I_1''(0)} \Big[(\gamma^2 D(r) - (\gamma-2)^2 F(p+1))(F(2) \\ &\quad - F(p+1)) - (\gamma F(r) - (\gamma - 2)F(p+1))^2 \Big] > 0 \end{aligned}$$

by assumption (iii).

Let (r_1, r_2) be defined in Lemma 6 or Theorem 2. Without loss of generality, we may assume that $2 \in (r_1, r_2)$. Let us now choose

$$t(r) := t_{\max} = -\frac{I_1'(0)}{I''_1(0)}. \tag{3.13}$$

Then $t(r) > -1$ and $I_1(t(r)) > 0$.

We first prove Lemma 6 for $r = 2$. We need to show that

$$I^{t(2)}(\varphi) > 0, \forall\ \varphi \in H_r^1(R^N), \varphi \not\equiv 0. \tag{3.14}$$

To this end, we use a continuation argument. By Lemma 5 (1), if $F(2) > 0$, then $I^{\gamma-2}$ is positive definite which implies that $L^{\gamma-2}$ has no nonnegative eigenvalues. Moreover, when $r = 2$,

$$I_1(\gamma - 2) = 8F(p+1)(\gamma-1)^2 F(2) > 0 \tag{3.15}$$

and

$$t(2) = \frac{\gamma F(2) + (\gamma-2)F(p+1)}{F(p+1) - F(2)} > \gamma - 2. \tag{3.16}$$

Since $I_1(t)$ is concave, we have that $I_1(t) > 0$ for $t \in [\gamma - 2, t(2)]$.

Let us now vary t. We claim that

$$I^t(\varphi) > 0, \forall t \in [\gamma - 2, t_{\max}], \text{ and } \varphi \in H_r^1(R^N), \varphi \not\equiv 0. \tag{3.17}$$

In fact, suppose not. Then at some point $t = t_0 \in (\gamma - 2, t_{\max}]$, we must have that L^{t_0} has a zero eigenvalue, which implies that $I_1(t_0) = 0$. This is impossible.

So (3.14) is proved.

Next we vary r. Assume that $r = r_0 > 1$ is the first value for which $I^{t(r)}(\phi) = 0$ and that r_0 satisfies assumptions (i)-(iii). Then at $r = r_0$, $L^{t(r_0)}$ must have a zero eigenvalue which implies that $I_1(t(r_0)) = 0$. This is in contradiction to the fact that $I_1(t(r_0)) > 0$. Thus we deduce that $I^{t(r)}(\varphi) > 0$ for any $\varphi \in H_r^1(R^N)$ and r satisfying the assumptions (i)-(iii).

Similarly we can prove the case when $p + 1 \in (r_1, r_2)$.

Lemma 6 is thus proved. □

Finally, Theorem 2 follows directly from Lemma 6 and (3.7).

4. – Proof of Theorem 1

In this section, we prove Theorem 1. Through this section, we assume that $2 < r < p + 1$.

We first estimate the value $D(r)$ under the assumption that $F(r) \geq 0$. Note that $F(r)$ is easy to compute while $D(r)$ is not. The next lemma relates $D(r)$ with $F(r)$, which is of independent interest. (A similar idea was used in [30].)

LEMMA 7. *Suppose that* $2 < r < p + 1$ *and* $F(r) \geq 0$, *then* $D(r) > 0$.

PROOF. Note that for $r = p + 1$, $F(r) > 0$ and $D(r) > 0$. Let r_1 be the least value in $[2, p+1)$ such that $D(r) > 0$ while $F(r) \geq 0$ for $r_1 < r < p+1$. If $r_1 = 2$, we are done. Suppose that $2 < r_1 < p + 1$. Then it follows that $D(r_1) = 0$ while $F(r_1) \geq 0$. We shall derive a contradiction by claiming that $V = L_0^{-1} w^{r_1 - 1}$ cannot change sign. In the following, we still denote r_1 by r.

We first claim that V changes sign at most twice. In fact, if V changes sign more than twice, then there exist intervals (s_1, s_2) and (s_3, s_4) such that $V(x) > 0$ for $|x| \in (s_1, s_2)$ or $|x| \in (s_3, s_4)$ and $V(x) = 0$ for $|x| = s_i$, $i = 1, 2, 3, 4$. Put

$$V_1(x) = \begin{cases} V(x), & |x| \in (s_1, s_2), \\ 0, & \text{otherwise}, \end{cases}$$

$$V_2(x) = \begin{cases} V(x), & |x| \in (s_3, s_4), \\ 0, & \text{otherwise}. \end{cases}$$

Let α be such that $\varphi = V_1 + \alpha V_2$ satisfies

$$\int_{R^N} w^p \varphi = 0. \tag{4.1}$$

Integrating over $(B_{s_2}(0)\backslash B_{s_1}(0)) \cup (B_{s_4}(0)\backslash B_{s_3}(0))$, we obtain that

$$\int_{R^N} [|\nabla\varphi|^2 + \varphi^2 - pw^{p-1}\varphi^2] = -\int_{R^N} w^{r-1}(V_1 + \alpha^2 V_2) < 0$$

which, by (4.1), contradicts (2.1) of Lemma 4.

If V changes sign exactly twice, then there exist $0 < s_1 < s_2$ such that either

$$V(x) < 0 \text{ for } |x| < s_1 \text{ and } |x| > s_2, \tag{4.2}$$

or

$$V(x) > 0 \text{ for } |x| < s_1 \text{ and } |x| > s_2. \tag{4.3}$$

Case (4.3) can be eliminated by our previous arguments (if we take $s_4 = +\infty$). Thus we only need to consider case (4.2).

Let $w(s_i) = \beta_i$, $i = 1, 2$. Since $2 < r < p + 1$, it is not hard to find two constants α_1, α_2 such that $f(t) := t^{p-1} + \alpha_1 t^{r-2} + \alpha_2$ changes sign exactly at points $t = \beta_i$ $i = 1, 2$ for $t > 0$. In fact we can solve

$$\begin{cases} \beta_1^{p-1} + \alpha_1 \beta_1^{r-2} + \alpha_2 = 0, \\ \beta_2^{p-1} + \alpha_1 \beta_2^{r-2} + \alpha_2 = 0. \end{cases}$$

Since $2 \le r \le p + 1, \beta_2 < \beta_1$, we have $\alpha_2 > 0, \alpha_1 < 0$. By our assumption, $\int_{R^N} w^{r-1} V = 0$, $\int_{R^N} wV \ge 0$, $\int_{R^N} w^p V > 0$, we have

$$\int_{R^N} wV(w^{p-1} + \alpha_1 w^{r-2} + \alpha_2) > 0. \tag{4.4}$$

On the other hand, because of the choices of α_1 and α_2, we have for all $|x|$,

$$V(w^{p-1} + \alpha_1 w^{r-2} + \alpha_2) \le 0, \tag{4.5}$$

which is a contradiction to (4.4).

Thus we have proved that V changes sign at most once.

Next, if V changes sign exactly once at $|x| = s_1$, again we put $w(s_1) = \beta_1$. By our assumption that $D(r) = 0$ and $F(r) \ge 0$, we obtain two inequalities:

$$\int_{R^N} (w^p - \beta_1^{p+1-r} w^{r-1})V > 0, \tag{4.6}$$

and

$$\int_{R^N} (w^{r-1}V - \beta_1^{r-2} wV) \le 0. \tag{4.7}$$

Two cases are considered: if $V < 0$ for $|x| < s_1$, then

$$(w^{p+1-r} - \beta_1^{p+1-r}) w^{r-1} V \le 0, \tag{4.8}$$

which contradicts to (4.6). If $V > 0$ for $|x| < s_1$, we have

$$(w^{r-2} - \beta_1^{r-2})V \ge 0, \tag{4.9}$$

which contradicts to (4.7).

In conclusion, V cannot change sign, contradicting our assumption that $D(r) = 0$.

Thus $D(r) > 0$ when $F(r) \ge 0$. □

Our next lemma gives us an upper bound for $D(r)$.

LEMMA 8. *If* $2 < r < p+1$, $F(r) \geq 0$, *then we have*

$$D(r) < F(p+1) - \frac{\left(F(p+1) - F(r)\right)^2}{F(p+1) - F(2)}. \tag{4.10}$$

PROOF. Note that by Lemma 7, $D(s) > 0$ for $r \leq s \leq p+1$.

We first claim that for $r \leq s \leq p+1$ it holds that

$$\inf\left\{ \int_{R^N} [|\nabla\varphi|^2 + \varphi^2 - pw^{p-1}\varphi^2] \,\middle|\, \varphi \in H_r^1(R^N), \int_{R^N} \varphi^2 = 1 \text{ and } \int_{R^N} w^{s-1}\varphi = 0 \right\} > 0. \tag{4.11}$$

In fact, this is true for $s = p+1$ by (2) of Lemma 4. Suppose that there exists $s \in [r, p+1)$ such that

$$\inf\left\{ \int_{R^N} [|\nabla\varphi|^2 + \varphi^2 - pw^{p-1}\varphi^2] \,\middle|\, \varphi \in H_r^1(R^N), \int_{R^N} \varphi^2 = 1 \text{ and } \int_{R^N} w^{s-1}\varphi = 0 \right\} = 0. \tag{4.12}$$

Then φ satisfies

$$\Delta\varphi - \varphi + pw^{p-1}\varphi = c_1 w^{s-1} + c_2\varphi, \int_{R^N} w^{s-1}\varphi = 0, \int_{R^N} \varphi^2 = 1,$$

for some constants c_1, c_2. By (4.12), we have $c_2 = 0$. So $\varphi = c_1 V$. We note that $c_1 \neq 0$ otherwise $\varphi \equiv 0$ by Lemma 4. Thus $\int_{R^N} w^{s-1} V = 0$, which contradicts to the fact that $D(s) > 0$.

Next we consider the following variational problem:

$$\inf\left\{ \int_{R^N} |\nabla\varphi|^2 + \varphi^2 - pw^{p-1}\varphi^2 \,\middle|\, \varphi \in H_r^1(R^N), \int_{R^N} w^{r-1}\varphi = 1 \right\}. \tag{4.13}$$

We claim that it is achieved by some function φ_0. In fact, we put $\varphi = \frac{1}{\int_{R^N} w^r} w + \psi$, then $\int_{R^N} w^{r-1}\psi = 0$. Therefore by (4.11) there exists a $c_0 > 0$ such that

$$\int_{R^N} [|\nabla\psi|^2 + \psi^2 - pw^{p-1}\psi^2] \geqslant c_0 \int_{R^N} \psi^2.$$

Then by the standard variational method, we can easily show that there exists a φ_0 which achieves (4.13) and satisfies

$$\Delta\varphi_0 - \varphi_0 + pw^{p-1}\varphi_0 = \lambda w^{r-1}$$

where $\lambda = -\mathcal{L}_0(\varphi_0, \varphi_0)$. By uniqueness, $\varphi_0 = \lambda V$ and thus

$$\begin{aligned} D(r) &= \frac{(p-1)\int_{R^N} V w^{r-1} \int_{R^N} w^2}{(\int_{R^N} w^r)^2} \\ &= \frac{(p-1)\int_{R^N} w^2}{(\int_{R^N} w^r)^2} \frac{1}{\lambda}. \end{aligned} \tag{4.14}$$

We now choose some special test functions to compute a lower bound for λ. In fact, we take

$$\varphi = c\left(\lambda_1 w + \lambda_2 \left(w + \frac{p-1}{2} x \cdot \nabla w\right)\right)$$

where

$$c(\lambda_1 + \lambda_2 F(r)) \int_{R^N} w^r = 1, \tag{4.15}$$

and λ_1 and λ_2 are to be chosen later. It follows that

$$\int_{R^N} w^{r-1} \varphi = c\big(\lambda_1 + \lambda_2 F(r)\big) \int_{R^N} w^r = 1.$$

Let us compute

$$\begin{aligned} \mathcal{L}_0(\varphi, \varphi) &= c^2 \bigg[\lambda_1^2 \mathcal{L}_0(w, w) + \lambda_2^2 \mathcal{L}_0 \left(w + \frac{p-1}{2} x \cdot \nabla w,\ w + \frac{p-1}{2} x \cdot \nabla w\right) \\ &\quad + 2\lambda_1\lambda_2 \mathcal{L}_0(w, w + \frac{p-1}{2} x \cdot \nabla w)\bigg] \\ &= c^2 \bigg[\lambda_1^2 (1-p) \int_{R^N} w^{p+1} + \lambda_2^2 (1-p) F(2) \int_{R^N} w^2 \\ &\quad + 2\lambda_1\lambda_2 (1-p) \int_{R^N} w^p (w + \frac{p-1}{2} x \cdot \nabla w)\bigg] \\ &= c^2 \left[\lambda_1^2 (1-p) \frac{1}{F(p+1)} + \lambda_2^2 (1-p) F(2) + 2\lambda_1\lambda_2 (1-p)\right] \int_{R^N} w^2 \\ &= \frac{\lambda_1^2 \frac{1}{F(p+1)} + \lambda_2^2 F(2) + 2\lambda_1\lambda_2}{(\lambda_1 + \lambda_2 F(r))^2} \frac{(1-p)\int_{R^N} w^2}{(\int_{R^N} w^r)^2} \\ &= \frac{\lambda_1^2 + \lambda_2^2 F(2) F(p+1) + 2\lambda_1\lambda_2 F(p+1)}{(\lambda_1 + \lambda_2 F(r))^2} \frac{(1-p)\int_{R^N} w^2}{(\int_{R^N} w^r)^2 F(p+1)}. \end{aligned}$$

Set $\frac{\lambda_1}{\lambda_2} = \eta$ and

$$h(\eta) := \frac{(\eta + F(r))^2}{\eta^2 + 2\eta F(p+1) + F(2) F(p+1)} \cdot F(p+1). \tag{4.16}$$

Then we obtain that

$$\mathcal{L}_0(\varphi,\varphi)=\frac{1}{h(\eta)}\frac{(1-p)\int_{R^N} w^2}{(\int_{R^N} w^r)^2}. \tag{4.17}$$

We now choose an optimal η. To this end, we need to compute the minimum of $h(\eta)$. Let $h'(\eta_0)=0$. Then it is easy to check that

$$\eta_0=\frac{F(r)-F(2)}{F(p+1)-F(r)}F(p+1)>0$$

and

$$h(\eta_0)=F(p+1)-\frac{(F(p+1)-F(r))^2}{F(p+1)-F(2)}.$$

Note that

$$h(\eta_0)=F(p+1)-\frac{(F(p+1)-F(r))^2}{F(p+1)-F(2)}>F(r)\geq 0$$

for $2<r<p+1$.

By (4.14) and the definition of λ, we have

$$D(r)=\frac{(p-1)\int_{R^N} w^2}{(\int_{R^N} w^r)^2\lambda}<h(\eta_0), \tag{4.18}$$

which proves Lemma 8. □

LEMMA 9. *For $2<r<p+1$, $F(r)\geq 0$ and $\gamma>1$, there always holds that*

$$\gamma^2 D(r)-F(p+1)-2\gamma(\gamma-1)F(r)+(\gamma-1)^2F(2)<0. \tag{4.19}$$

PROOF. It is enough to show that

$$D(r)<\frac{1}{\gamma^2}F(p+1)+2\left(1-\frac{1}{\gamma}\right)F(p+1)-\left(1-\frac{1}{\gamma}\right)^2F(2).$$

Let $s=\frac{1}{\gamma}\in(0,1]$ and

$$\begin{aligned}\beta(s)&=s^2F(p+1)+2(1-s)F(r)-(1-s)^2F(2)\\&=s^2\big(F(p+1)-F(2)\big)+2s\big(F(2)-F(r)\big)+2F(r)-F(2).\end{aligned}$$

Since

$$\beta'(0)=2\big(F(2)-F(r)\big)\leq 0,\ \beta'(1)=2[F(p+1)-F(r)]\geq 0,$$

$\beta(s)$ has a minimum in $(0,1)$.

Let $0 = \beta'(s_0) = 2s_0\big(F(p+1) - F(2)\big) + 2\big(F(2) - F(r)\big)$. Then

$$s_0 = \frac{F(r) - F(2)}{F(p+1) - F(2)} > 0$$

and simple computations show that

$$\beta(s_0) = F(p+1) - \frac{\big(F(p+1) - F(r)\big)^2}{F(p+1) - F(2)}.$$

Thus the minimum of $\beta(s)$ in $(0, 1]$ is

$$\beta(s_0) = \min_{s\in(0,1]} \beta(s) = F(p+1) - \frac{\big(F(p+1) - F(r)\big)^2}{F(p+1) - F(2)}.$$

By Lemma 8

$$D(r) < F(p+1) - \frac{\big(F(p+1) - F(r)\big)^2}{F(p+1) - F(2)} = \beta(s_0) \leq \beta(s), \quad \forall\ 0 < s \leq 1.$$

Lemma 9 is thus finished. □

Proof of Theorem 1. By Lemma 9, assumption (i) is always satisfied if $2 < r < p+1$ and $F(r) \geq 0$. Assumption (ii) is satisfied if $F(r) \geq 0$. By Lemma 7, $D(r) > 0$. By (1.15), Assumption (iii) is satisfied. Theorem 1 now follows from Theorem 2. □

5. – Proof of Theorem 3

We prove Theorem 3 in this section.

Assume that $\gamma < 1$. To prove Theorem 3, we introduce the following function:

$$\rho(\lambda) := \int_{R^N} w^r - \gamma(p-1)\int_{R^N}((L_0 - \lambda)^{-1} w^p) w^{r-1}. \tag{5.1}$$

Note that $\rho(\lambda)$ is well-defined in $(0, \mu_1)$, where μ_1 is the unique positive eigenvalue of L_0. Let us denote the corresponding eigenfunction by Φ_0. Since μ_1 is a principal eigenvalue, we may assume that $\Phi_0 > 0$. (See Lemma 1.2 of [28].)

It is easy to see that to prove Theorem 3, it is enough to find a positive zero of $\rho(\lambda)$.

First we have

$$\rho(0) = \int_{R^N} w^r - \gamma(p-1)\int_{R^N} L_0^{-1} w^p w^{r-1} = (1-\gamma)\int_{R^N} w^r > 0. \tag{5.2}$$

Set $\Phi_\lambda = (L_0 - \lambda)^{-1} w^p$. Then Φ_λ satisfies

$$(L_0 - \lambda)\Phi_\lambda = w^p. \tag{5.3}$$

Multiplying (5.3) by Φ_0 and integrating by parts, we have

$$(\mu_1 - \lambda)\int_{R^N} \Phi_\lambda \Phi_0 = \int_{R^N} \Phi_0 w^p,$$

which implies that

$$\int_{R^N} \Phi_\lambda \Phi_0 = \frac{1}{\mu_1 - \lambda}\int_{R^N} \Phi_0 w^p.$$

Let

$$\Phi_\lambda = \left(\frac{1}{(\mu_1 - \lambda)\int_{R^N} \Phi_0^2}\int_{R^N} \Phi_0 w^p\right)\Phi_0 + \Phi_\lambda^\perp, \ \Phi_\lambda^\perp \perp \Phi_0. \tag{5.4}$$

Then as $\lambda \to \mu_1, \lambda < \mu_1$, we have that $\|\Phi_\lambda^\perp\|_{L^2(R^N)}$ is uniformly bounded and by (5.4)

$$\int_{R^N} \Phi_\lambda w^{r-1} \to +\infty,$$

which implies that

$$\rho(\lambda) \to -\infty \text{ as } \lambda \to \mu_1, \lambda < \mu_1. \tag{5.5}$$

By (5.2) and (5.5), there is a $\lambda_0 \in (0, \mu_1)$ such that $\rho(\lambda_0) = 0$. Theorem 3 is thus proved.

REFERENCES

[1] A. Bose – G. Kriegsman, *Stability of localized structures in non-local reaction-diffusion equations*, Methods Appl. Anal. **5** (1998), 351-366.

[2] E. N. Dancer, *On stability and Hopf bifurcations for chemotaxis systems*, to appear in: "Proceedings of IMS Workshop on Reaction-Diffusion Systems 2000".

[3] M. del Pino, *A priori estimates and applications to existence-nonexistence for a semilinear elliptic system*, Indiana Univ. Math. J. **43** (1994), 703-728.

[4] M. DEL PINO – P. FELMER – M. KOWALCZYK, *Boundary spikes in the Gierer-Meinhardt system*, Asymp. Anal., to appear.

[5] A. DOELMAN – A. GARDNER – T. J. KAPER, *Stability of singular patterns in the 1-D Gray-Scott model: A matched asymptotic approach*, Physica D. **122** (1998), 1-36.

[6] A. DOELMAN – A. GARDNER – T. J. KAPER, *A stability index analysis of 1-D patterns of the Gray-Scott model*, Technical Report, Center for BioDynamics, Boston University, submitted.

[7] A. DOELMAN – T. KAPER – P. A. ZEGELING, *Pattern formation in the one-dimensional Gray-Scott model*, Nonlinearity **10** (1997), 523-563.

[8] P. FREITAS, *Bifurcation and stability of stationary solutions on nonlocal scalar reaction diffusion equations*, J. Dyn. Differential Equations **6** (1994), 613-629.

[9] P. FREITAS, *A non-local Sturm-Liouville eigenvalue problem*, Proc. Roy. Soc. Edinburg Sect. A **124** (1994), 169-188.

[10] P. FREITAS, *Stability of stationary solutions for a scalar nonlocal reaction diffusion equation*, Quart. J. Mech. Appl. Math. **48** (1995), 557-582.

[11] A. GIERER – H. MEINHARDT, *A theory of biological pattern formation*, Kybernetik (Berlin) **12** (1972), 30-39.

[12] P. GRAY – S. K. SCOTT, *Autocatalytic reactions in the isothermal, continuous stirred tank reactor: isolas and other forms of multistability*, Chem. Eng. Sci. **38** (1983), 29-43.

[13] P. GRAY – S. K. SCOTT, *Autocatalytic reactions in the isothermal, continuous stirred tank reactor: oscillations and instabilities in the system $A + 2B \to 3B$, $B \to C$*, Chem. Eng. Sci. **39** (1984), 1087-1097.

[14] J. K. HALE – L. A. PELETIER – W. C. TROY, *Exact homoclinic and heteroclinic solutions of the Gray-Scott model for autocatalysis*, SIAM J. Appl. Math. **61** (2000), 102-130.

[15] D. IRON – M. J. WARD, *A metastable spike solution for a non-local reaction-diffusion model*, SIAM J. Appl. Math. **60** (2000), 778-802.

[16] D. IRON – M. J. WARD – J. WEI, *The stability of spike solutions to the one-dimensional Gierer-Meinhardt model*, Physica D., to appear.

[17] C.-S. LIN – W.-M. NI, "On the diffusion coefficient of a semilinear Neumann problem", Calculus of variations and partial differential equations (Trento, 1986), 160-174, Lecture Notes in Math., 1340, Springer, Berlin-New York, 1988.

[18] C. B. MURATOV – V. V. OSIPOV, *Spike autosolitions in Gray-Scott model*, J. Phys. A-Math. Gen. **48** (2000), 8893-8916.

[19] W.-M. NI – I. TAKAGI, *Point-condensation generated by a reaction-diffusion system in axially symmetric domains*, Japan J. Industrial Appl. Math. **12** (1995), 327-365.

[20] W.-M. NI – I. TAKAGI – E. YANAGIDA, Tohoku Math. J., to appear.

[21] W.-M. NI, *Diffusion, cross-diffusion, and their spike-layer steady states*, Notices Amer. Math. Soc. **45** (1998), 9-18.

[22] Y. NISHIURA – D. UEYAMA, *A skeleton structure of self-replicating dynamics*, Physica D. **130** (1999), 73-104.

[23] Y. NISHIURA, *Global structure of bifurcation solutions of some reaction-diffusion systems*, SIAM J. Math. Anal. **13** (1982), 555-593.

[24] J. WEI, *On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem*, J. Differential Equations **129** (1996), 315-333.

[25] J. WEI, *On a nonlocal eigenvalue problem and its applications to point-condensations in reaction-diffusion systems*, Int. J. Bifur. and Chaos **10** (2000), 1485-1496.

[26] J. WEI – M. WINTER, *On the two dimensional Gierer-Meinhardt system with strong coupling*, SIAM J. Math. Anal. **30** (1999), 1241-1263.

[27] J. WEI, *On single interior spike solutions of Gierer-Meinhardt system: uniqueness, spectrum estimates and stability analysis*, Europ. J. Appl. Math. **10** (1999), 353-378.

[28] J. WEI, *Existence, stability and metastability of point condensation patterns generated by Gray-Scott system*, Nonlinearity **3** (1999), 593-616.

[29] J. WEI, *Point condensations generated by Gierer-Meinhardt system: a brief survey, book chapter*, In: "New Trends in Nonlinear Partial Differential Equations 2000", Y. Morita – H. Ninomiya – E. Yanagida – S. Yotsutani (eds), pp. 46-59.

[30] L. ZHANG, *Uniqueness of positive solutions of* $\Delta u + \lambda u + u^p = 0$ *in a ball*, Comm. Partial Differential Equations **17** (1992), 1141-1164.

Department of Mathematics
The Chinese University of Hong Kong
Shatin, Hong Kong
wei@math.cuhk.edu.hk

Institute of Mathematics
Academia Sinica, Beijing, China
lqzhang@math03.math.ac.cn