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## Prym Varieties and the Canonical Map of Surfaces of General Type

CIRO CILIBERTO – RITA PARDINI – FRANCESCA TOVENA

**Abstract.** Let  $X$  be a smooth complex surface of general type and let  $\phi : X \rightarrow \mathbb{P}^{p_g(X)-1}$  be the canonical map of  $X$ . Suppose that the image  $\Sigma$  of  $\phi$  is a surface and that  $\phi$  has degree  $\delta \geq 2$ . Let  $\epsilon : S \rightarrow \Sigma$  be a desingularization of  $\Sigma$  and assume that the geometric genus of  $S$  is not zero. Beauville [B2] proved that in this case the surface  $S$  is of general type and  $\epsilon$  is the canonical map of  $S$ . Beauville also constructed the only infinite series of examples that were available in the literature up to now. This construction has been our motivation for introducing the notion of a *good generating pair*, namely of a pair  $(h : V \rightarrow W, L)$  where  $h$  is a finite morphism of surfaces and  $L$  is a nef and big line bundle on  $W$  satisfying certain assumptions. The most important of these are: i)  $|K_V + h^*L| = h^*|K_W + L|$ , and ii) the general curve  $C$  of  $L$  is smooth and non-hyperelliptic. We show that, by means of a construction analogous to that of Beauville's, every good generating pair gives rise to an infinite series of surfaces of general type whose canonical map is 2-to-1 onto a canonically embedded surface. In this way we are able to construct more infinite series of such surfaces (cf. Section 3). In addition, we give bounds on the invariants of good generating pairs and we show that there exist essentially only 2 good generating pairs with  $\dim |L| > 1$ . The key fact that we exploit for obtaining these results is that the Albanese variety  $P$  of  $V$  is a Prym variety and that the Prym map has positive dimensional fibre over the point of the moduli space of p.p.a.v.'s corresponding to  $P$ .

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### 1. – Introduction

Let  $X$  be a smooth surface of general type and let  $\phi : X \rightarrow \Sigma \subseteq \mathbb{P}^{p_g(X)-1}$  be the canonical map of  $X$ , where  $\Sigma$  is the image of  $\phi$ . Suppose that  $\Sigma$  is a surface and that  $\phi$  has degree  $\delta \geq 2$ . Let  $\epsilon : S \rightarrow \Sigma$  be a desingularization of  $\Sigma$ . A classical result, which goes back to Babbage [Bab], and has been more recently proved by Beauville, [B2] (see also [Cat1]), says that either  $p_g(S) = 0$  or  $S$  is of general type and  $\epsilon : S \rightarrow \Sigma$  is the canonical map of  $S$ . In the latter

case we have a dominant rational map  $\psi : X \rightarrow S$  of degree  $\delta$ , which we call a *good canonical cover* of degree  $\delta$  (see Definition 2.3 for a slightly more general definition).

While it is easy to give examples with  $p_g(S) = 0$  (see [B2]), not many examples of good canonical covers can be found in the current literature. There are some “sporadic” examples such that the surface  $X$  is *regular* (see [VdGZ], [B2] Proposition 3.6, [Cat1] Theorem 3.5, [C1], [Pa2]). In addition, there is an interesting construction, due to Beauville (see [Cat2], 2.9 and [MP]), which gives an infinite series of good canonical covers of degree 2 with  $X$  *irregular*, precisely with  $q(X) = 2$ . Beauville’s construction is recalled in Section 4 and in Example 3.1. The corresponding canonical covers have been extensively studied in [MP], where they have been classified in terms of their birational invariants.

In trying to find more examples of canonical covers, we have realized that Beauville’s construction is an instance of a more general situation. This has prompted us to define a *good generating pair* (see 2.4). Roughly speaking, a good generating pair is a pair  $(h : V \rightarrow W, L)$ , where  $h : V \rightarrow W$  is a finite morphism of degree 2,  $V$  is a smooth irreducible surface,  $W$  has isolated double points of type  $A_1$ , and  $L$  is a nef and big line bundle on  $W$ . In addition, one requires that: i)  $\dim |L| \geq 1$  and there is a smooth non-hyperelliptic curve  $C \in |L|$ ; ii)  $h^*K_W = K_V$  (this means that  $h$  is ramified only above the singular points of  $W$ ); iii) the pull-back of the adjoint linear system  $|K_W + L|$  is the complete linear system  $|K_V + h^*L|$ .

The relationship between generating pairs and canonical covers is the following (see Section 2 for more details). Consider the map  $\tilde{h} = h \times Id : V \times \mathbf{P}^1 \rightarrow W \times \mathbf{P}^1$  and the projections  $p_i$ ,  $i = 1, 2$ , of  $W \times \mathbf{P}^1$  onto the two factors. A general surface  $\Sigma \in |p_1^*L \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(n)|$ ,  $n \geq 3$ , has only points of type  $A_1$  as singularities. We set  $X = \tilde{h}^*(\Sigma)$ ,  $\phi = \tilde{h}|_X : X \rightarrow \Sigma$ ,  $\epsilon : S \rightarrow \Sigma$  the minimal desingularization,  $\psi = \epsilon^{-1} \circ \phi$ . Then, using adjunction both on  $V \times \mathbf{P}^1$  and  $W \times \mathbf{P}^1$ , one sees that  $\psi : X \rightarrow S$  is a good canonical cover of degree 2.

General properties of generating pairs  $(h : V \rightarrow W, L)$  are studied in Section 5 (see also Section 8, that contains some information about *higher degree* generating pairs). In particular, we prove that  $V$  and  $W$  have the same Kodaira dimension (see Proposition 5.5) and  $W$  is regular. Then we compute the irregularity of  $V$ , which is always  $> 0$ , in terms of the genus  $g$  of the general curve  $C \in |L|$  and of the degree of  $h$  (see Proposition 5.4). We also give formulas for the invariants of the canonical covers arising from a given generating pair (see Proposition 2.7). It is interesting to notice that Beauville’s example is essentially characterized by the fact that  $V$  and  $W$  have Kodaira dimension  $\kappa = 0$  (see Proposition 8.2 for a more precise statement). The case of Kodaira dimension 1 is also rather restricted, as Proposition 8.3 shows.

Beauville’s example corresponds to the case in which  $V$  is a principally polarized abelian surface,  $W$  is its Kummer surface, and  $L$  is the polarization on  $W$  which lifts to a symmetric principal polarization on  $V$ . Unfortunately, it is not easy to find more examples. In Section 3 we list all the examples we know. Each of these gives rise to an infinite series of good canonical covers,

which possibly deserve to be studied in detail, as has been done for Beauville's examples in [MP].

The difficulty in finding generating pairs is explained in Section 6. This is where *Prym varieties* come into the picture. If  $(h : V \rightarrow W, L)$  is a generating pair, and  $C$  is a general curve of  $|L|$ , then  $h^*C = C' \rightarrow C$  is an unramified double cover, with a related Prym variety  $\text{Prym}(C', C)$ . In Theorem 6.1 we prove that  $\text{Prym}(C', C)$  is naturally isomorphic to the Albanese variety of  $V$ . As a consequence we find that, if the generating pair is good, then the Albanese image of  $V$  is a surface and therefore the Kodaira dimension of  $V$  and  $W$  is non-negative (see Corollary 6.2). Moreover, some general facts about irregular surfaces and isotrivial systems of curves on them, which have been collected in Section 4, imply that the Prym map has an infinite fibre over the point corresponding to the cover  $h : C' \rightarrow C$  (see Proposition 6.6). Using this fact and some results about the fibre of the Prym map due to several authors (see Section 6 for references), we prove that if  $(h : V \rightarrow W, L)$  is a good pair then one has the bounds  $g \leq 12$  for the genus  $g$  of  $C$  and  $q \leq 11$  for the irregularity  $q$  of  $V$  (see Theorem 6.9 and Proposition 6.11). We think it should be possible to improve these bounds if one had a better understanding of the fibres of the Prym map. For instance, one would like to know answers to questions like: when do these fibres contain rational curves? Questions of this type are, of course, of independent interest.

Finally, using Reider's method, we obtain the bound  $L^2 \leq 4$  (see Proposition 7.3), and it becomes clear why there are not so many possibilities for a good generating pair.

We give a complete classification of good pairs with  $L$  ample and  $\dim |L| > 1$ . These satisfy  $\dim |L| \leq 3$  and  $L^2 = 3, 4$ . The only example with  $\dim |L| = 3$  is Beauville's one (see Corollary 7.4). The cases with  $\dim |L| = 2$  are studied in Section 7 (see Theorem 7.10). We show that  $L^2 = 3$  and this corresponds either to Example 3.3 or to a suitable modification of Beauville's example; in the proof, we notice that the general curve  $C$  is trigonal and we apply a global version of the well known *trigonal construction* [Ca2], which is the inverse of the equally famous *Recillas' construction*. A different proof of the same result is sketched in Remark 7.9.

In the *pencil case*  $\dim |L| = 1$  (in which there are examples, like 3.2, but a classification is still lacking) we give strong restrictions for the case  $L^2 = 4$  (see Corollary 7.7).

Using similar ideas, we are able to construct an infinite family of good canonical covers with  $X$  regular. We will return to this in a forthcoming paper.

NOTATION AND CONVENTIONS: all varieties are defined over the field of complex numbers. A map between varieties is a rational map, while a morphism is a rational map that is regular at every point. We do not distinguish between Cartier divisors and line bundles and use the additive and multiplicative notation interchangeably. The Kodaira dimension of a variety  $X$  is denoted by  $\kappa(X)$ . We denote by  $\sim_{num}$  the numerical equivalence between divisors on a smooth surface.

**2. – Canonical covers and generating pairs**

NOTATION 2.1. Let  $S$  be a surface with canonical singularities, i.e. either smooth or with rational double points, so that in particular  $S$  is Gorenstein. We denote by  $K_S$  the canonical divisor of  $S$ , and we let  $p_g(S) = h^0(S, K_S) = h^2(S, \mathcal{O}_S)$  be the *geometric genus* and  $q(S) = h^1(S, \mathcal{O}_S)$  the *irregularity*. If  $p_g(S) \geq 2$ , the *canonical map* of  $S$  is the rational map  $\phi : S \rightarrow \mathbf{P}^{p_g(S)-1}$  defined by the moving part of the canonical system  $|K_S|$  of  $S$ . If  $S_0$  is the smooth locus of  $S$  and  $\epsilon : S' \rightarrow S$  is any desingularization, then  $p_g(S) = p_g(S')$  and  $q(S) = q(S') = h^0(S', \Omega_{S'}^1) = h^0(S_0, \Omega_{S_0}^1)$ . The Albanese map of  $S'$  factors through  $\epsilon$ , since the exceptional locus of  $\epsilon$  is a union of rational curves, and so we can speak of the Albanese map of  $S$ .

Let  $X$  be a smooth surface of general type and let  $\phi : X \rightarrow \Sigma \subseteq \mathbf{P}^{p_g(X)-1}$  be the canonical map of  $X$ , where  $\Sigma$  is the image of  $\phi$ . We assume that  $\Sigma$  is a surface and that  $\phi$  has degree  $d \geq 2$ , and we denote by  $\epsilon : S \rightarrow \Sigma$  a desingularization of  $\Sigma$ . We recall the following theorem due to Beauville, [B2], Theorem 3.4.

THEOREM 2.2. *Under the above assumptions, either:*

- (i)  $p_g(S) = 0$  or;
- (ii)  $S$  is of general type and  $\epsilon : S \rightarrow \Sigma$  is the canonical map of  $S$ .

We introduce some terminology for surfaces that satisfy condition (ii) of Theorem 2.2:

DEFINITION 2.3. Let  $X, S$  be smooth surfaces of general type. Let  $\psi : X \rightarrow S$  be a dominant rational map of degree  $d \geq 2$ . Assume that:

- (CC1)  $p_g(X) = p_g(S)$ ;
- (CC2) the canonical image of  $S$  is a surface  $\Sigma$ .

In this case the canonical map  $\phi : X \rightarrow \Sigma$  of  $X$  is the composition of  $\psi$  and the canonical map  $\epsilon : S \rightarrow \Sigma$  of  $S$ , and we say that  $\psi : X \rightarrow S$  is a *canonical cover of degree  $d$* . If  $\epsilon : S \rightarrow \Sigma$  is birational, then we say that the canonical cover is *good*.

A few sporadic examples of canonical covers are known in the literature ([VdGZ], [B2] Proposition 3.6, [Cat1] Theorem 3.5, [C1], [Pa2]). However, so far, there is only one construction, due to Beauville (see [Cat2], 2.9 and [MP]), which gives an infinite series of such covers. We recall it next.

Let  $V$  be a principally polarized abelian surface such that the principal polarization  $D$  is irreducible, and let  $h : V \rightarrow W$  be the quotient map onto the Kummer surface  $W = V / \langle -1 \rangle$ . The surface  $W$  can be embedded into  $\mathbf{P}^3$  as a quartic surface via a complete linear system  $|L|$  such that  $h^*|L| = |2D|$ . Consider the map  $\tilde{h} = h \times Id : V \times \mathbf{P}^1 \rightarrow W \times \mathbf{P}^1$  and the projections  $p_i, i = 1, 2$ , of  $W \times \mathbf{P}^1$  onto the two factors. A general surface  $\Sigma \in |p_1^*L \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(n)|$ ,  $n \geq 3$ , has only points of type  $A_1$  as singularities. We set  $X = \tilde{h}^*(\Sigma)$ ,  $\phi =$

$\tilde{h}|_X : X \rightarrow \Sigma$ ,  $\epsilon : S \rightarrow \Sigma$  the minimal desingularization,  $\psi = \epsilon^{-1} \circ \phi : X \rightarrow S$ . Then it is easy to check, using adjunction both on  $V \times \mathbf{P}^1$  and  $W \times \mathbf{P}^1$ , that  $\psi : X \rightarrow S$  is a good canonical cover of degree 2.

We wish to study to what extent this construction can be generalized. We introduce a class of pairs  $(h : V \rightarrow W, L)$ , where  $h : V \rightarrow W$  is a finite morphism of surfaces and  $L$  is a line bundle on  $W$ , such that by applying the above construction to  $(h : V \rightarrow W, L)$  one gets an infinite series of canonical covers.

DEFINITION 2.4. Consider a pair  $(h : V \rightarrow W, L)$ , where  $h$  is a finite morphism of degree  $d \geq 2$  between irreducible surfaces,  $V$  smooth,  $W$  with at most canonical singularities and  $L$  is a line bundle on  $W$ , such that:

- (GP1)  $K_V = h^*K_W$ ;
- (GP2)  $h^0(W, L) \geq 2$  and  $L$  is big, i.e.  $L^2 > 0$ ;
- (GP3) the general curve  $C$  of  $|L|$  is smooth of genus  $g \geq 2$  and the curve  $C' := h^*C$  is not hyperelliptic;
- (GP4)  $p_g(V) = p_g(W)$ ,  $h^0(V, K_V + h^*L) = h^0(W, K_W + L) > 0$ .

We call  $(h : V \rightarrow W, L)$  a *degree  $d$  and genus  $g$  generating pair* of canonical covers, and we denote by  $L'$  the line bundle  $h^*L$  on  $V$ . The pair is said to be *minimal* if both  $V$  and  $W$  are minimal.

The generating pair is called *good* if the general  $C$  of  $|L|$  is not hyperelliptic (hence  $g \geq 3$  in this case).

Notice that condition (GP1) is equivalent to the fact that  $h$  is ramified only over the singular points of  $W$ . Condition (GP3) and Bertini's theorem imply that the general curve  $C$  in  $|L|$  is smooth and irreducible, hence  $L$  is also nef, i.e.  $LD \geq 0$  for every effective divisor  $D$  on  $V$ . The assumption that  $C'$  is not hyperelliptic is a technical condition whose meaning will be clearer later (cf. for instance Theorem 6.1). Finally, the base points of  $|L|$ , if any, are smooth points of  $W$ .

In the rest of this section we show that by applying the original construction of Beauville to a (good) generating pair one obtains an infinite series of (good) canonical covers, and we compute the invariants of such canonical covers. In order to do this, we need the following result, that will be proven later (cf. Proposition 5.4):

PROPOSITION 2.5. *If  $(h : V \rightarrow W, L)$  is a generating pair, then  $q(W) = 0$ .*

We introduce now some more notation:

NOTATION 2.6. Given a generating pair  $(h : V \rightarrow W, L)$  of degree  $d$  and genus  $g$ , we denote by  $p_i, i = 1, 2$ , the projections of  $W \times \mathbf{P}^1$  onto the two factors and we write  $\tilde{h} = h \times Id : V \times \mathbf{P}^1 \rightarrow W \times \mathbf{P}^1$ . We denote by  $\mathcal{L}(n)$  the line bundle  $p_1^*L \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(n)$ , where  $n$  is a positive integer.

In addition, we let  $\Sigma \in |\mathcal{L}(n)|$  be a general surface,  $Y = \tilde{h}^*(\Sigma)$ . We denote by  $\epsilon : S \rightarrow \Sigma$  and  $\epsilon' : X \rightarrow Y$  the minimal desingularizations, by  $f$  the map  $\tilde{h}|_Y : X \rightarrow \Sigma$ , and by  $\psi$  the map  $\epsilon^{-1} \circ f \circ \epsilon' : X \rightarrow S$ .

PROPOSITION 2.7 (We use Notation 2.6). *Let  $(h : V \rightarrow W, L)$  be a generating pair of degree  $d$  and genus  $g$ . If  $n \geq 3$ , then one has:*

- i) *let  $\Sigma \in |\mathcal{L}(n)|$  be general and let  $Y = \tilde{h}^*\Sigma$ ;  $\Sigma$  and  $Y$  are surfaces of general type with at most canonical singularities. If in addition  $L$  is ample, then  $S$  and  $X$  are both minimal;*
- ii)  *$\psi : X \rightarrow S$  is a canonical cover of degree  $d$ , that is said to be  $n$ -related to the generating pair  $(h : V \rightarrow W, L)$ . If the generating pair is good, then  $\psi : X \rightarrow S$  is a good canonical cover, while if the generating pair is not good then the canonical map of  $S$  is 2-to-1 onto a rational surface;*
- iii) *the invariants of  $S$  are:  $p_g(S) = np_g(W) + (n - 1)g$ ,  $q(S) = 0$ ,  $K_S^2 = n(K_W^2 - L^2) + 8(n - 1)(g - 1)$ ;*
- iv) *the invariants of  $X$  are:  $p_g(X) = p_g(S)$  and  $q(X) = (d - 1)(g - 1)$ ,  $K_X^2 = d K_S^2$ .*

PROOF. Recall that by condition (GP3) of Definition 2.4 the general curve of  $|L|$  is smooth, and thus, in particular,  $|L|$  has no fixed components. It follows that also the linear system  $|\mathcal{L}(n)|$  has no fixed components and is not composed with a pencil, and thus the general surface  $\Sigma \in |\mathcal{L}(n)|$  is irreducible. Moreover the set of base points of  $|\mathcal{L}(n)|$  is the inverse image via  $p_1$  of the set of base points of  $|L|$  and thus it is a finite union of fibres of  $p_1$ . Using Bertini's theorem and the fact that the general curve of  $|L|$  is smooth, one proves that the singularities of the general  $\Sigma \in |\mathcal{L}(n)|$  at points of the fixed locus of  $|\mathcal{L}(n)|$  are finitely many rational double points of type  $A_r$ . Now, the projection  $p_1$  restricts to a generically finite map  $p : \Sigma \rightarrow W$  of degree  $n$  which, by Bertini's theorem again, is unramified over the singular points of  $W$ . So the general  $\Sigma$  has, over each singular point  $x$  of  $W$ ,  $n$  singularities which are analytically equivalent to the one  $W$  has in  $x$  (i.e.  $n$  canonical singularities) and it is smooth at points that are smooth for  $W \times \mathbb{P}^1$  and are not base points of  $|\mathcal{L}(n)|$ . To describe the singularities of  $Y = \tilde{h}^*(\Sigma)$ , we notice that the restriction  $Y \rightarrow \Sigma$  of  $\tilde{h}$  is ramified precisely over the singularities of  $\Sigma$  that occur at singular points of  $W \times \mathbb{P}^1$ ; so  $Y$  has  $d$  singularities analytically isomorphic to those of  $\Sigma$  over each of those singular points of  $\Sigma$  that occur at base points of  $|\mathcal{L}(n)|$  and it is smooth elsewhere, since it is general in  $\tilde{h}^*|\mathcal{L}(n)|$ . In conclusion the singularities of  $Y$  and  $\Sigma$  are canonical, and their invariants, which we now compute, are equal to those of  $X, S$ , respectively.

By the adjunction formula and condition (GP1) in Definition 2.4, one has  $K_\Sigma = (K_{W \times \mathbb{P}^1} + \Sigma)|_\Sigma = (p_1^*K_W + \mathcal{L}(n - 2))|_\Sigma$  and  $K_Y = (K_{V \times \mathbb{P}^1} + X)|_X = \tilde{h}^*(K_{W \times \mathbb{P}^1} + \Sigma)|_X = f^*(K_\Sigma)$ , and thus  $K_S^2 = K_\Sigma^2 = n(K_W^2 - L^2) + 8(n - 1)(g - 1)$ ,  $K_X^2 = dK_S^2$ . To compute the remaining invariants of  $S, \Sigma$  and  $X$ , one considers the long cohomology sequences associated to the restriction sequences

$$0 \rightarrow K_{W \times \mathbb{P}^1} \rightarrow K_{W \times \mathbb{P}^1} + \mathcal{L}(n) \rightarrow K_\Sigma \rightarrow 0$$

and

$$0 \rightarrow K_{V \times \mathbb{P}^1} \rightarrow K_{V \times \mathbb{P}^1} + \tilde{h}^*\mathcal{L}(n) \rightarrow K_Y \rightarrow 0.$$

By Kawamata-Viehweg’s vanishing theorem, we have  $h^i(W \times \mathbf{P}^1, K_{W \times \mathbf{P}^1} + \mathcal{L}(n)) = h^i(V \times \mathbf{P}^1, K_{V \times \mathbf{P}^1} + \tilde{h}^* \mathcal{L}(n)) = 0$  for  $i > 0$ . Hence:

$$p_g(S) = h^0(\Sigma, K_\Sigma) = h^0(W \times \mathbf{P}^1, K_{W \times \mathbf{P}^1} + \mathcal{L}(n)) + h^1(W \times \mathbf{P}^1, K_{W \times \mathbf{P}^1}) = h^0(W, K_W + L)(n - 1) + p_g(W) = np_g(W) + (n - 1)g$$

where the last equality follows again from Kawamata–Viehweg’s vanishing and the last equality but one follows from  $q(W) = 0$ . Therefore, by the definition of generating pair:

$$p_g(X) = h^0(V, K_V + L)(n - 1) + p_g(V) = p_g(S).$$

A similar computation gives  $q(S) = q(W) = 0$ ,  $q(X) = q(V) = (d - 1)(g - 1)$ .

The linear system  $|K_\Sigma|$  contains the restriction of the system  $|p_1^* K_W + \mathcal{L}(n - 2)|$ , whose fixed locus is the inverse image via  $p_1$  of the fixed locus of  $|K_W + L|$ . Let  $C \in |L|$  be a smooth curve; since  $W$  is regular, the linear system  $|K_W + L|$  restricts to the complete canonical system  $|K_C|$ . Thus  $C$  does not contain any base point of  $|K_W + L|$ . If  $L$  is ample, this implies that  $|K_W + L|$  has a finite number of base points, none of which is also a base point of  $|L|$ . Thus in this case the fixed locus of  $|p_1^* K_W + \mathcal{L}(n - 2)|$  intersects the general  $\Sigma$  in a finite number of points and, a fortiori, the canonical system of  $\Sigma$  has no fixed components and the surfaces  $X, S$  are minimal.

Notice now that  $|p_1^* K_W + \mathcal{L}(n - 2)|$  separates the fibres of  $p_2|_\Sigma$ , since  $n \geq 3$ . A fibre  $F$  of  $p_2|_\Sigma$  is identified by  $p_1$  with a curve  $C \in |L|$  and the restriction of  $|p_1^* K_W + \mathcal{L}(n - 2)|$  to  $F$  is identified with the restriction of  $|K_W + L|$  to  $C$ , which is the complete canonical system  $|K_C|$ , since  $W$  is regular. Thus, if the general  $C$  is not hyperelliptic, then the canonical map of  $S$  is birational and  $\psi : X \rightarrow S$  is a good canonical cover, while if the general  $C$  is hyperelliptic then the canonical map of  $S$  is of degree 2 onto a rational surface and  $\psi : X \rightarrow S$  is a non good canonical cover.  $\square$

Since we aim at a classification of generating pairs, we introduce a notion of blow-up. We will show (cf. Corollary 6.3) that almost every generating pair is obtained from a minimal one by a sequence of blow-ups.

DEFINITION 2.8. Let  $(h : V \rightarrow W, L)$  be a generating pair of degree  $d$  and genus  $g$ . Let  $x \in W$  be a smooth point. Then we can consider the cartesian square:

$$\begin{array}{ccc} V' & \longrightarrow & V \\ h' \downarrow & & \downarrow h \\ W' & \xrightarrow{f} & W \end{array}$$

where  $f : W' \rightarrow W$  is the blow-up of  $W$  at  $x$ , with exceptional divisor  $E$  and, accordingly,  $V'$  is the blow-up of  $V$  at the  $d$  points  $x_1, \dots, x_d$  of the fibre of  $h$  over  $x$ . Fix  $m = 0$  or  $1$  and assume that:



- (i)  $L^2 > m^2$ ;
- (ii)  $h^0(W', f^*L - mE) \geq 2$  and the general curve  $C \in |f^*L - mE|$  is smooth.

Then the pair  $(h' : V' \rightarrow W', f^*L - mE)$  is again a generating pair, and we say that it is obtained from  $(h : V \rightarrow W, L)$  by a *simple blow-up*. The blow-up is said to be *essential* if  $m = 1$  and *inessential* if  $m = 0$ .

The reason why we only consider  $m \leq 1$  in the above definition is that generating pairs satisfy the inequality  $L^2 \leq 4$  (cf. Proposition 7.3).

### 3. – Examples of generating pairs

In this section we describe some examples of generating pairs.

**EXAMPLE 3.1.** *Beauville’s example.* (see [Cat2], 2.9, [MP], Example 4 in Section 3). This example has already been described in Section 2:  $V$  is a principally polarized abelian surface with an irreducible polarization  $D$ ,  $W$  is the Kummer surface of  $V$ ,  $h : V \rightarrow W$  is the projection onto the quotient, and  $L$  is an ample line bundle on  $W$  such that the class of  $L' = h^*L$  is equal to  $2D$ . This generating pair is good and therefore so is any related canonical cover. More precisely, by Proposition 2.7, an  $n$ -related canonical cover  $\psi : X \rightarrow S$  is minimal, with geometric genus  $4n - 3$ . The invariants of  $S$  and  $X$  satisfy the relations:

$$K_S^2 = 3p_g(S) - 7; \quad K_X^2 = 6p_g(X) - 14; \quad q(X) = 2.$$

According to [MP], Theorem 4.1, this is the only good generating pair such that the related canonical covers satisfy  $K_X^2 = 6p_g(X) - 14$  and  $K_X$  is ample.

Notice that if the polarization  $D$  on  $V$  is reducible, then  $(h : V \rightarrow W, L)$  is a non good generating pair (cf. also [MP], Example 2 in Section 3), that we call the *non good Beauville’s example*.

**EXAMPLE 3.2.** *A good generating pair of degree 2 and genus 3* (cf. also [C2], Example (c), page 70). Let  $A$  be an abelian surface with an irreducible principal polarization  $D$ , let  $p : V \rightarrow A$  be the double cover branched on a symmetric divisor  $B \in |2D|$  and such that  $p_*\mathcal{O}_V = \mathcal{O}_A \oplus \mathcal{O}_A(-D)$ . Since  $K_V = p^*(D)$ , the invariants of the smooth surface  $V$  are:  $p_g(V) = 2$ ,  $q(V) = 2$ ,  $K_V^2 = 4$ . By the symmetry of  $B$ , multiplication by  $-1$  on  $A$  can be lifted to an involution  $i$  of  $V$  that acts as the identity on  $h^0(V, K_V)$ . We denote by  $h : V \rightarrow W = V / \langle i \rangle$  the projection onto the quotient. We observe that  $p_g(W) = 2$ ,  $q(W) = 0$ ,  $K_W^2 = 2$  and the only singularities of the surface  $W$  are 20 ordinary double points. In addition,  $h^0(W, 2K_W) = \chi(\mathcal{O}_W) + K_W^2 = 5 = h^0(V, 2K_V)$ , so that the bicanonical map of  $V$  factors through  $h : V \rightarrow W$ . An alternative description of  $W$  is as follows. One embeds, as usual, the Kummer surface  $Kum(A)$  of  $A$  as a quartic surface in  $\mathbf{P}^3 = \mathbf{P}(H^0(A, 2D)^*)$ . The surface  $W$

is a double cover of  $Kum(A)$  branched over the smooth plane section  $H$  of  $Kum(A)$  corresponding to  $B$  and on 6 nodes (corresponding to the six points of order 2 of  $A$  lying on  $D$ ). The ramification divisor  $R$  of  $W \rightarrow Kum(A)$  is a canonical curve isomorphic to  $H$ , and thus it is not hyperelliptic. This completes the proof that  $(h : V \rightarrow W, K_W)$  is a good generating pair. Under suitable generality assumptions,  $K_W$  is ample, as well as  $K_V$ . An  $n$ -related canonical cover  $\psi : X \rightarrow S$  has geometric genus  $5n - 3$  and is, in general, minimal. The invariants of  $S$  and  $X$  satisfy the relations:

$$(1) \quad 5K_S^2 = 16p_g(S) - 32; \quad 5K_X^2 = 32p_g(X) - 64; \quad q(X) = 2.$$

EXAMPLE 3.3. *A good generating pair of degree 2 and genus 4* (cf. [C2], Example 3.13). Let  $\Gamma$  be a non-hyperelliptic curve of genus 3 and let  $V := Sym^2(\Gamma)$ . The surface  $V$  is smooth minimal of general type with invariants:  $K_V^2 = 6$ ,  $p_g(V) = q(V) = 3$ . If we embed  $\Gamma$  into  $\mathbb{P}^2$  via the canonical system, then the canonical map of  $V$  sends the unordered pair  $\{p, q\}$  of points of  $V$  to the line  $\langle p, q \rangle \in \mathbb{P}^{2*}$ , hence it is a degree 6 morphism onto the plane. There is an involution  $i$  on  $V$  that maps  $\{p, q\} \in V$  to  $\{r, s\}$ , where  $\langle p, q \rangle \cap \Gamma = p + q + r + s$ . The fixed points of  $i$  correspond to the 28 bitangents of  $\Gamma$  and the canonical map of  $V$  clearly factors through the quotient map  $h : V \rightarrow W = V / \langle i \rangle$ . Hence the invariants of  $W$  are:  $p_g(W) = p_g(V) = 3$ ,  $K_W^2 = K_V^2 / 2 = 3$ ,  $\chi(W) = (\chi(V) + 7) / 2 = 4$ , and thus  $q(W) = 0$ . In addition we have  $h^0(W, 2K_W) = \chi(\mathcal{O}_W) + K_W^2 = 7 = h^0(V, 2K_V)$  and thus  $|2K_V| = h^*|2K_W|$ . So, the pair  $(h : V \rightarrow W, K_W)$  is a good generating pair, since the restriction of  $|K_W|$  to  $C$  is a base-point free  $g_3^1$  and the general canonical curve  $C$  of  $W$  is not hyperelliptic. Notice that  $K_W$  and  $K_V$  are ample. An  $n$ -related canonical cover  $\psi : X \rightarrow S$  is minimal, of geometric genus  $7n - 4$ . The invariants of  $S$  and  $X$  satisfy the relations:

$$(2) \quad 7K_S^2 = 24p_g(S) - 72; \quad 7K_X^2 = 48p_g(X) - 144; \quad q(X) = 3.$$

An interesting question, concerning this example and the previous one, is whether these are the only generating pairs such that the related canonical covers have invariants satisfying (1) and (2).

EXAMPLE 3.4. *A non good generating pair of degree 3 and genus 2*. Let  $p : W \rightarrow \mathbb{P}^2$  be the double cover of  $\mathbb{P}^2$  ramified on an irreducible sextic  $B$  with 9 cusps ( $B$  is the dual of a smooth cubic). The surface  $W$  is a K3 surface whose singularities are 9 double points of type  $A_2$ . According to [BdF] (cf. also [BL2], [Ba]), there exists a smooth cover  $h : V \rightarrow W$  of degree 3 ramified only at the 9 double points. Notice that the surface  $V$  is an abelian surface. We set  $L = p^*(\mathcal{O}_{\mathbb{P}^2}(1))$  and we remark that the pair  $(h : V \rightarrow W, L)$  is a non good generating pair, since  $h^0(W, L) = \chi(\mathcal{O}_W) + \frac{1}{2}L^2 = 3 = h^0(V, h^*L)$  by the ampleness of  $L$ . An  $n$ -related, minimal, canonical cover  $\psi : X \rightarrow S$  has geometric genus  $3n - 2$ . The invariants of  $S$  and  $X$  satisfy the relations:

$$K_S^2 = 2p_g(S) - 4; \quad K_X^2 = 6p_g(X) - 12; \quad q(X) = 2.$$

The surfaces  $S$  thus obtained have invariants lying on the Noether's line  $K_S^2 = 2p_g(S) - 4$ . It would be interesting to know whether there are other canonical covers with so low geometric genus.

**EXAMPLE 3.5.** *A series of non good generating pairs of degree 2 with unbounded invariants.* For  $i = 1, 2$ , let  $C_i$  be a smooth curve of genus  $g_i > 0$  with a double cover  $\phi_i : C_i \rightarrow \mathbf{P}^1$ , and let  $\sigma_i$  be the involution on  $C_i$  induced by  $\phi_i$ . We set  $V = C_1 \times C_2$ ,  $W = V / \langle \sigma_1 \times \sigma_2 \rangle$  and we denote by  $h : V \rightarrow W$  the projection onto the quotient. We remark that there exists a double cover  $f : W \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  such that  $\phi_1 \times \phi_2 : V \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  factors as  $\phi_1 \times \phi_2 = f \circ h$ . We denote by  $H$  a divisor of type  $(1, 1)$  on  $\mathbf{P}^1 \times \mathbf{P}^1$  and we set  $L = f^*H$ . Both systems  $|K_V|$  and  $|K_V + h^*L|$  are pull-back via  $\phi_1 \times \phi_2 : V \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ . This immediately implies that  $(h : V \rightarrow W, L)$  is a non good generating pair of degree 2 and genus  $g_1 + g_2 + 1$ . One has:  $p_g(W) = g_1g_2$ ,  $q(V) = g_1 + g_2$ . An  $n$ -related canonical cover  $\psi : X \rightarrow S$  has geometric genus  $ng_1g_2 + (n-1)(g_1 + g_2 + 1)$  and moreover  $q(X) = g_1 + g_2$ .

Notice that, if  $g_1 = g_2 = 1$ , we find again the non good Beauville's example (cf. Example 3.1).

#### 4. – Auxiliary results on irregular surfaces

In this section we collect a few general facts on irregular surfaces that will be used in the rest of the paper. We use Notation 2.1.

**PROPOSITION 4.1.** *Let  $h : V \rightarrow W$  be a finite morphism of surfaces with canonical singularities such that  $K_V = h^*K_W$  and  $p_g(V) = p_g(W)$ . If  $q(V) > q(W) > 0$ , then the Albanese image of  $V$  is a curve.*

**PROOF.** The critical set  $\Delta$  of  $h$  is finite by assumption. We let  $W_0 = W \setminus (\text{Sing}(W) \cup \Delta)$  and  $V_0 = h^{-1}W_0$ , so that the restricted map  $h : V_0 \rightarrow W_0$  is a finite étale map between smooth surfaces. In particular  $h$  is flat, and there is a canonical vector bundle isomorphism  $h_*\mathcal{O}_{V_0} \cong \mathcal{O}_{W_0} \oplus E$ , where  $E$  is locally free of rank  $(\deg h - 1)$ . Since  $\Omega_{V_0}^i = h^*\Omega_{W_0}^i$ ,  $i = 1, 2$ , one has  $h_*\Omega_{V_0}^i = \Omega_{W_0}^i \otimes h_*\mathcal{O}_{V_0} = \Omega_{W_0}^i \oplus (\Omega_{W_0}^i \otimes E)$ . Notice that this decomposition as a direct sum is canonical. We set  $M_+^i = H^0(W_0, \Omega_{W_0}^i)$  and  $M_-^i = H^0(W_0, \Omega_{W_0}^i \otimes E)$ . We deduce that  $H^0(V_0, \Omega_{V_0}^i) = H^0(W_0, h_*\Omega_{V_0}^i) = M_+^i \oplus M_-^i$  and we denote by  $\pi_+^i$  the projection onto the first factor of this decomposition. To ensure that the Albanese image is a curve, we show that  $\tau_1 \wedge \tau_2 = 0$  for every choice of  $\tau_1, \tau_2 \in H^0(V_0, \Omega_{V_0}^1) = M_+^1 \oplus M_-^1$ . Noticing that both  $M_+^1$  and  $M_-^1$  are non-zero (since  $q(V) > q(W) > 0$ ), we only need to show that  $h^*\sigma \wedge \tau = 0$  for every choice of  $\sigma \in M_+^1$  and  $\tau \in M_-^1$ . Indeed, to show that  $\wedge^2 M_+^1 = 0$  we fix  $(0 \neq) \tau \in M_-^1$ : if  $\sigma_1, \sigma_2 \in M_+^1$ , the vanishing  $h^*\sigma_i \wedge \tau = 0$  ( $i = 1, 2$ ) means that  $h^*\sigma_i$  is pointwise proportional to  $\tau$  ( $i = 1, 2$ ), so that  $h^*\sigma_1$  and  $h^*\sigma_2$  are mutually pointwise proportional. Similarly one proves that  $\wedge^2 M_-^1 = 0$ .

Since  $p_g(V) = p_g(W)$ ,  $\pi_+^2$  is an isomorphism. Notice also that  $\pi_+^i(h^*\sigma) = \sigma$  for any  $\sigma \in M_+^i$ , and that  $\pi_+^2(h^*\sigma \wedge \tau) = \sigma \wedge \pi_+^1(\tau)$  for  $\sigma \in M_+^1$  and  $\tau \in H^0(V_0, \Omega_{V_0}^1)$ . Therefore  $h^*\sigma \wedge \tau = 0$  for any  $\sigma \in M_+^1$  and  $\tau \in \ker \pi_+^1 = M_-^1$ , as required. □

We recall the following results:

PROPOSITION 4.2 (Serrano, [Se], Section 1). *Let  $V$  be a smooth surface, let  $C$  be a smooth curve, and let  $p : V \rightarrow C$  be an isotrivial fibration with fibre  $D$ . Then there exist a curve  $B$ , a finite group  $G$  acting both on  $B$  and  $D$ , an isomorphism  $f : C \rightarrow B/G$ , and a birational map  $r : V \rightarrow (D \times B)/G$ , where  $G$  acts diagonally on  $D \times B$ , such that the following diagram commutes:*

$$\begin{array}{ccc}
 V & \xrightarrow{r} & (D \times B)/G \\
 p \downarrow & & \downarrow p'' \\
 C & \xrightarrow{f} & B/G
 \end{array}$$

where  $p''$  is the map induced by the projection  $D \times B \rightarrow B$ . The irregularity  $q(V)$  is equal to  $g(C) + g(D/G)$ . In particular, if  $q(V) > 0$  and  $g(C) = 0$ , then the Albanese image of  $V$  is a curve isomorphic to  $D/G$  and the Albanese pencil is given by the composition  $p' \circ r$ , where  $p'$  is the map induced by the projection  $D \times B \rightarrow D$ .

PROPOSITION 4.3 (Xiao, [Xi], Theorem 1). *Let  $p : V \rightarrow \mathbf{P}^1$  be a fibration with fibres of genus  $\gamma$ . If  $p$  is not isotrivial, then  $\gamma \geq 2q(V) - 1$ .*

The next proposition combines the previous results.

PROPOSITION 4.4. *Let  $V$  be a smooth surface with a pencil  $|D|$  such that the general curve  $D$  of  $|D|$  is smooth and irreducible of genus  $\gamma > 1$ ; if the Albanese image of  $V$  is a curve, then one (and only one) of the following holds:*

- i) *there exists a birational map  $r : V \rightarrow D \times \mathbf{P}^1$  such that  $D$  is the strict transform via  $r$  of a fibre of the projection  $D \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$ . In this case  $\gamma = q(V)$ ;*
- ii) *there exist an hyperelliptic curve  $B$ , a free involution  $i$  on  $D$ , and a birational map  $r : V \rightarrow (D \times B)/\mathbf{Z}_2$ , where  $\mathbf{Z}_2$  acts on  $B$  as the hyperelliptic involution, on  $D$  via  $i$  and diagonally on  $D \times B$ , such that  $D$  is the strict transform via  $r$  of a fibre of the projection  $(D \times B)/\mathbf{Z}_2 \rightarrow B/\mathbf{Z}_2 = \mathbf{P}^1$ . In this case  $\gamma = 2q(V) - 1$ .*
- iii)  $\gamma > 2q(V) - 1$ .

*In particular, if  $p$  is not isotrivial, then iii) holds.*

PROOF. Since the statement is essentially birational, we may assume that  $D$  defines a morphism  $p : V \rightarrow \mathbf{P}^1$ , up to blowing up the base locus of  $|D|$ . Denote by  $\alpha : V \rightarrow C$  the Albanese pencil. If  $p$  is not isotrivial, then  $\gamma \geq 2q(V) - 1$  holds by Proposition 4.3. If  $\gamma = 2q(V) - 1$ , then by the Hurwitz formula the restriction of  $\alpha$  to a smooth curve  $D$  is an étale cover of  $C$ , whose degree is 2. Thus  $p$  is isotrivial, contradicting the previous assumption.

Assume now that  $p$  is isotrivial. By Proposition 4.2, there is a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{r} & (D \times B)/G \\ p \downarrow & & \downarrow p'' \\ \mathbf{P}^1 & \xrightarrow{f} & B/G \end{array}$$

where  $B$  is a curve,  $G$  is a finite group acting on  $B$  and on  $D$  and acting diagonally on  $D \times B$ ,  $r$  is a birational map and  $f : \mathbf{P}^1 \rightarrow B/G$  is an isomorphism. Again by Proposition 4.2, the Albanese image of  $V$  is isomorphic to  $D/G$ . So we have either  $G = \{1\}$ , corresponding to case i), or  $2q(V) - 1 \leq \gamma$ , with equality if and only if  $G = \mathbf{Z}_2$  acts freely on  $D$ . The latter case corresponds to case ii). □

### 5. – General properties of generating pairs

In this section we give some useful information on the degree, genus and Kodaira dimension of a generating pair.

NOTATION 5.1. If  $(h : V \rightarrow W, L)$  is a generating pair of degree  $d$  and genus  $g$ , we write  $C$  for a general curve of  $|L|$  and  $C' = h^*C$ , so that  $C$  and  $C'$  are smooth curves of genera  $g$  and  $d(g - 1) + 1$  respectively, and  $h$  restricts to an unramified cover  $\pi : C' \rightarrow C$  of degree  $d$ .

LEMMA 5.2. *Let  $(h : V \rightarrow W, L)$  be a generating pair of degree  $d$  and genus  $g$ . If the Albanese image of  $V$  is a curve, then  $d(g - 1) + 1 > 2q(V) - 1$ .*

PROOF. According to Proposition 4.4, we distinguish three cases. Setting  $D = C'$ ,  $\gamma = d(g - 1) + 1$  and keeping the rest of notation of Proposition 4.4, we only need to exclude the first two cases:

- i)  $V$  is ruled and  $C'$  is a section: in this case the adjoint system  $|K_V + L|$  is empty, contradicting assumption (GP4) of Definition 2.4;
- ii) there are two subcases:
  - ii-a) Assume  $B = \mathbf{P}^1$ . Then  $V$  is ruled over  $C'/\mathbf{Z}_2$  and  $C'$  is a bisection of  $V$  meeting each fibre of the map  $p : V \rightarrow C'/\mathbf{Z}_2$  in two distinct points interchanged by the free  $\mathbf{Z}_2$  action. By repeatedly blowing down  $-1$  curves  $E$  such that  $EL' \leq 1$ , one obtains a map  $f : V \rightarrow V'$  such that  $V'$  is minimal and the map  $p$  factors as  $p' \circ f$ , where  $p' : V' \rightarrow C'/\mathbf{Z}_2$ . The curve  $C'' = f(C')$  is smooth and the induced map  $f : C' \rightarrow C''$  is an isomorphism. Moreover, the map  $p' : V' \rightarrow C'/\mathbf{Z}_2$  is a projective bundle, i.e. there exists a rank 2 vector bundle  $M$  on  $C'/\mathbf{Z}_2$  such that  $V' = \text{Proj}_{C'/\mathbf{Z}_2}(M)$ , and  $C''$  meets each fibre of  $p'$  in two distinct points interchanged by the free  $\mathbf{Z}_2$ -action.

If we denote by  $H$  the tautological section of  $V'$  and by  $L''$  the line bundle determined by  $C''$  on  $V'$ , then the condition that the projection map  $C' \rightarrow C'/\mathbf{Z}_2$  is unramified of degree 2 is equivalent to  $L''$  being numerically equivalent to  $2H - \text{deg}(M)F$ , and thus we have  $L''^2 = 0$ . This would imply  $L'^2 \leq 0$ , contradicting the fact that  $L'$  is big.

ii-b) Assume that  $B$  is not rational. Notice that  $(C' \times B)/\mathbf{Z}_2$  is the quotient of  $C' \times B$  by a free  $\mathbf{Z}_2$ -action. Hence it is smooth. In addition it is minimal, since it is a free quotient of the minimal surface  $C' \times B$ . This implies that the birational map  $r : V \rightarrow (C' \times B)/\mathbf{Z}_2$  is a morphism. Let  $C''$  be a fibre of the morphism  $(C' \times B)/\mathbf{Z}_2 \rightarrow B/\mathbf{Z}_2 = \mathbf{P}^1$ . Since, by Proposition 4.2,  $C'$  is the strict transform of  $C''$  via  $r$  and since  $C''^2 = 0$ , we have again that  $L'^2 \leq 0$ , which is impossible since  $L'$  is big. □

LEMMA 5.3. *Let  $(h : V \rightarrow W, L)$  be a generating pair of degree  $d$  and genus  $g$ . Then  $d(g - 1) + 1 \geq 2q(V) - 1$ , and if equality holds then the Albanese image of  $V$  is a surface.*

PROOF. Consider a pencil  $\mathcal{P} \subset |L|$  such that the general curve is smooth and irreducible. Up to blowing up, we may assume that the pull-back of  $\mathcal{P}$  on  $V$  via  $h$  is a base point free pencil. If the corresponding fibration is not isotrivial, then the claim holds by Proposition 4.3. If the fibration is isotrivial, then the Albanese image of  $V$  is a curve according to Proposition 4.2, and we have  $d(g - 1) + 1 > 2q(V) - 1$  by Lemma 5.2. □

PROPOSITION 5.4. *If  $(h : V \rightarrow W, L)$  is a generating pair of degree  $d$ , then  $q(W) = 0$ , and the list of possibilities is as follows:*

- i)  $d = 2, q(V) = g - 1,$
- ii)  $d = 3, g \leq 3, q(V) = 2(g - 1)$
- iii)  $d = 4, g = 2, q(V) = 3.$

*If the pair is good, then case i) holds; in case ii),  $g = 3$ , and in case iii) the Albanese image of  $V$  is a surface.*

PROOF. By Kawamata-Viehweg's vanishing theorem one has  $h^0(W, K_W + L) = \chi(W) + g - 1$ . Analogously, one has  $h^0(V, K_V + L') = \chi(V) + d(g - 1)$ , and thus  $q(V) - q(W) = (d - 1)(g - 1) > 0$  by condition (GP3) of Definition 2.4. Assume that  $q(W) > 0$ . By Proposition 4.1, the Albanese image of  $V$  is a curve and Lemma 5.2 implies that  $d(g - 1) + 1 > 2q(V) - 1 = 2(d - 1)(g - 1) + 2q(W) - 1$ , but this is impossible, since  $d, g \geq 2$ . So,  $q(W) = 0$  and, according to Lemma 5.3, one has  $d(g - 1) + 1 \geq 2q(V) - 1 = 2(d - 1)(g - 1) - 1$ . The statement includes all possible solutions. In cases ii) with  $g = 3$  and iii) we also apply Lemma 5.3.

Assume now that the pair is good. By the above discussion, we have  $d \leq 3$ . By ([B2], Proposition 4.1 and Remark 4.2), if  $\psi : X \rightarrow S$  is a good canonical cover of degree 3, then  $q(X) \leq 3$ . On the other hand, by Proposition 2.7, canonical covers arising from a good generating pair of degree 3 and genus  $g$  satisfy  $q(X) = 2(g - 1) \geq 4$ . □

PROPOSITION 5.5. *Let  $(h : V \rightarrow W, L)$  be a generating pair: then  $\kappa(V) = \kappa(W)$ .*

PROOF. Since  $\kappa(V) \geq \kappa(W)$ , we may assume  $\kappa(W) \leq 1$ . Consider the following commutative diagram:

$$\begin{array}{ccc} V' & \xrightarrow{b} & V \\ h' \downarrow & & \downarrow h \\ W' & \xrightarrow{f} & W \end{array}$$

where  $f : W' \rightarrow W$  is a minimal desingularization and  $h' : V' \rightarrow W'$  is obtained by taking base change, normalizing and finally solving the singularities of the surface thus obtained. We notice the following facts:

- (i)  $b$  is a sequence of blow-ups, since  $V$  and  $V'$  are smooth surfaces. Thus  $K_{V'} = b^*K_V + E$ , where  $E$  is an effective divisor supported on the  $b$ -exceptional locus, and we have  $|mK_{V'}| = b^*|mK_V| + mE$ , for every  $m \geq 1$ .
- (ii)  $K_{W'} = f^*K_W$ , since  $W$  has only canonical singularities. Therefore we have  $b^*K_V = b^*(h^*K_W) = h'^*K_{W'}$ .

Suppose that  $\kappa(W) = -\infty$ , i.e.  $W$  is rational by Proposition 5.4. Hence also  $W'$  is rational, and therefore there is an effective irreducible big divisor  $D$  on  $W'$  such that  $DK_{W'} < 0$ . By remark (ii) above, there is an effective big divisor  $D'$  on  $V'$  such that  $D'(b^*K_V) < 0$ . This, together with remark (i), shows that  $\kappa(V') = \kappa(V) = -\infty$ .

Assume now that  $\kappa(W) = 0$ ; then there exists a nef and big line bundle  $H$  on  $W'$  such that  $HK_{W'} = 0$ . Thus  $(h'^*H)(b^*K_V) = (h'^*H)(h'^*K_{W'}) = 0$ , and thus  $h'^*H$  is a nef and big divisor that has zero intersection with the moving part of any pluricanonical system. Thus it follows that  $\kappa(V') \leq 0$ . If  $\kappa(W') = 1$ , then there exists a fibration  $f : W' \rightarrow D$ , where  $D$  is a smooth curve, such that the general fibre  $E$  of  $f$  is an elliptic curve. So  $(h'^*E)(b^*K_V) = 0$ , and thus the maps given by the pluricanonical systems are all composed with the fibration  $f' = f \circ h$ . This shows that  $\kappa(V) \leq 1$ . □

By the previous proposition, one can speak of the *Kodaira dimension* of a generating pair. Generating pairs of degree 2 and Kodaira dimension 0 are completely described in Proposition 8.2.

### 6. – Pairs of degree 2 and Prym varieties

We consider here the case of generating pairs of degree 2. The relevance of this case is underlined in Proposition 5.4, where it is shown that all good

generating pairs have degree 2 and that all generating pairs of degree  $> 2$  have genus  $\leq 3$ .

If  $C \in |L|$  is a general curve and  $C' = h^*C$ , then the map  $h$  induces an étale double cover  $\pi = h|_{C'} : C' \rightarrow C$ . If one denotes by  $J$  (resp.  $J'$ ) the Jacobian of  $C$  (resp.  $C'$ ), then the connected component containing the origin of the kernel of the norm map  $\pi_* : J' \rightarrow J$  is a  $(g - 1)$ -dimensional abelian subvariety, on which the principal polarization of  $J'$  induces the double of a principal polarization. This principally polarized abelian variety is called the *Prym variety* of  $C' \rightarrow C$  and it is denoted by  $Prym(C', C)$ . The connection between generating pairs and Prym varieties is explained in the following theorem.

**THEOREM 6.1.** *Let  $(h : V \rightarrow W, L)$  be a generating pair of degree 2. Let  $C \in |L|$  be a general curve. Then there is a natural isomorphism  $\varphi : Prym(C', C) \rightarrow A$ , where  $A = Alb(V)$  is the Albanese variety of  $V$ . In particular  $Prym(C', C)$  does not depend on  $C \in |L|$ .*

**PROOF.** By the definition of generating pair, the singular points of  $W$  are  $t$  ordinary double points, where  $t$  satisfies the relation  $\chi(V, \mathcal{O}_V) = 2\chi(W, \mathcal{O}_W) - t/4$ . Evaluating the Euler characteristic of  $V$  and  $W$  as in Proposition 5.4, one deduces that  $t = 4(g + p_g(W)) > 0$ . So, one can choose a ramification point  $x_0$  for  $h$  in  $V$ . Since  $W$  is regular by proposition 5.4, the Albanese map of  $W$  with base point  $h(x_0)$  is the zero map. The Albanese map of  $V$  with base point  $x_0$ , denoted by  $\alpha : V \rightarrow A$ , is equivariant with respect to the involution induced by  $h$  and the multiplication by  $-1$  on  $A$ . In particular, the restriction  $\alpha|_{C'} : C' \rightarrow A$  is also equivariant.

Now we use the universal property of Prym varieties (cf. [BL1], page 382). Let  $\beta : C' \rightarrow Prym(C', C)$  be the Abel-Prym map with respect to a point  $c' \in C'$  and let  $\tau : A \rightarrow A$  be the translation by  $\alpha(c')$ . Then there is a unique homomorphism  $\varphi : Prym(C', C) \rightarrow A$ , independent of  $c' \in C'$ , such that  $\alpha|_{C'} = \tau \circ \varphi \circ \beta$ .

Denote by  $J'$  the Jacobian of  $C'$ . Let  $j : C' \rightarrow J'$  be the Abel map with base point  $c'$  and  $\gamma : J' \rightarrow Prym(C', C)$  the map such that  $\beta = \gamma \circ j$ . Let  $i_* : J' \rightarrow A$  be the homomorphism induced by the inclusion  $i : C' \rightarrow V$  and the choice of  $c' \in C'$ . Notice that, up to a translation, we have  $\alpha|_{C'} = i_* \circ j$ . Then it is clear that  $i_*$  factors, up to a translation, as  $\varphi \circ \gamma$ . The differential of  $i_*$  at the origin of  $J'$  is dual to the map  $H^1(V, \mathcal{O}_V) \rightarrow H^1(C', \mathcal{O}_{C'})$ , which is injective since  $H^1(V, \mathcal{O}_V(-L')) = 0$  because  $L'$  is big and nef. So  $i_*$  is surjective and  $\varphi$  is an isogeny since  $A$  and  $Prym(C', C)$  both have dimension  $g - 1$  by Proposition 5.4. To show that  $\varphi$  is an isomorphism, it is enough to prove that  $i_*$  has connected fibres. In turn, this follows if we show that the map  $H_1(C', \mathbf{Z}) \rightarrow H_1(V, \mathbf{Z})$  induced by the inclusion  $i : C' \rightarrow V$  is surjective. The system  $|L'|$  has no fixed part by assumption, so by Theorem 6.2 of [Za] there exists an integer  $k$  such that  $|kL|$  gives a morphism  $\psi : V \rightarrow \mathbf{P}^N$ ; the image of  $g$  is a surface, since  $L'$  is big. So there exists a hyperplane  $H$  in  $\mathbf{P}^N$  such that  $\psi^{-1}H = C'$  as sets. By Theorem 1.1, page 150, of [GM], the map  $\pi_1(C) \rightarrow \pi_1(V)$  is surjective, and thus  $H_1(C', \mathbf{Z}) \rightarrow H_1(V, \mathbf{Z})$  is surjective too. □



**COROLLARY 6.2.** *Let  $(h : V \rightarrow W, L)$  be a generating pair of degree 2 and genus  $g$ ; then the Albanese image of  $V$  is a surface. In particular, the Kodaira dimension of the pair is non-negative.*

**PROOF.** Assume that the Albanese image of  $V$  is a curve  $\Gamma$ . Then  $\Gamma$  has genus  $g - 1$ . On the other hand, by Theorem 6.1, the Albanese image of  $V$  contains the Abel-Prym image of  $C'$ , which is isomorphic to  $C'$  (cf. [BL1], Proposition 12.5.2), since  $C'$  is not hyperelliptic. This is a contradiction and thus the claim is proven.  $\square$

**COROLLARY 6.3.** *Let  $(h : V \rightarrow W, L)$  be a generating pair of degree 2; then  $(h : V \rightarrow W, L)$  is obtained from a minimal pair by a sequence of simple blow-ups of weight 0 or 1.*

**PROOF.** Denote by  $\iota : V \rightarrow V$  the involution induced by  $h$  and let  $E$  be a  $-1$  curve of  $V$ . We claim that either  $L'E = 0$  or  $L'E = 1$ . Let  $\epsilon : V \rightarrow V_0$  be the blow-down of  $E$ , let  $C' \in |L'|$  be smooth and let  $C_0 = \epsilon(C')$ ; notice that  $C_0$  is singular if and only if  $L'E > 1$ . Let  $\alpha : V \rightarrow A$  be the Albanese map of  $V$ ;  $A$  is also the Albanese variety of  $V_0$  and, if we denote by  $\alpha_0 : V_0 \rightarrow A$  the Albanese map of  $V_0$ , one has  $\alpha = \alpha_0 \circ \epsilon$ . Thus  $\alpha(C') = \alpha_0(C_0)$ ; by Theorem 6.1,  $\alpha(C')$  is isomorphic to  $C'$ , since  $C'$  is not hyperelliptic, and thus  $C_0$  is smooth and  $L'E \leq 1$ . Let  $E'$  be the image of  $E$  via  $\iota$ ;  $E'$  is also a  $-1$ -curve and thus, since  $\kappa(V) \geq 0$  by Corollary 6.2, either  $E = E'$  or  $E$  and  $E'$  are disjoint. If  $E = E'$ , then  $E$  contains precisely 2 fixed points of  $\iota$ , but this contradicts the fact that  $E^2$  is odd. So  $E \neq E'$  and  $F = h(E) = h(E')$  is a  $-1$  curve contained in the smooth part of  $W$ . Let  $V'$  be the surface obtained by blowing down  $E$  and  $E'$ , let  $W'$  be the surface obtained by blowing down  $F$  and let  $h' : V' \rightarrow W'$  be the double cover induced by  $h$ ; if one denotes by  $M$  the direct image of  $L$ , then it is easy to check that  $(h' : V' \rightarrow W', M)$  is also a generating pair. By iterating this process finitely many times, one eventually obtains a generating pair with  $V$  minimal. Thus  $K_V = h^*K_W$  is nef, and it follows that  $K_W$  is also nef and  $W$  is minimal, too.  $\square$

**COROLLARY 6.4.** *Let  $(h : V \rightarrow W, L)$  be a generating pair of genus  $g$  and degree 2. Then:*

- (i)  $p_g(V) = p_g(W) \geq g - 2 > 0$ ;
- (ii) *if the Kodaira dimension of the pair is 2, then  $p_g(V) = p_g(W) \geq \max\{g - 1, 2g - 6\}$ ; if  $p_g(V) = 2g - 6$  then  $V$  is birational to the product of a curve of genus 2 and a curve of genus  $g - 3$ .*

**PROOF.** By Corollary 6.2,  $g - 1 = q(V) > 1$ . Thus we have  $\chi(V) \geq 0$ ,  $p_g(W) = p_g(V) \geq q(V) - 1 = g - 2 > 0$ . The case of Kodaira dimension 2 follows from the theorem at pg. 345 of [B4].  $\square$

**COROLLARY 6.5.** *Let  $(h : V \rightarrow W, L)$  be a generating pair of genus  $g$  and degree 2. If  $|C'| \subset |L'|$  is a pencil containing a smooth curve, then  $|C'|$  is not isotrivial.*

**PROOF.** Follows from Corollary 6.2 and Proposition 4.2.  $\square$

If we denote by  $\mathcal{R}_g$  the moduli space of étale double covers of curves of genus  $g$  and by  $\mathcal{A}_{g-1}$  the moduli space of principally polarized abelian varieties of dimension  $g - 1$ , then the Prym map  $\mathcal{P}_g : \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}$  associates to every isomorphism class of étale double covers the corresponding Prym variety. The geometry of Prym varieties has been extensively studied by many authors. We are going to use some of these results in order to give a bound on the genus of good generating pairs.

**PROPOSITION 6.6.** *Let  $(h : V \rightarrow W, L)$  be a generating pair of genus  $g$  and degree 2. Let  $C \in |L|$  be general and let  $C' = h^*C$ . Then the fibre of the Prym map  $\mathcal{P}_g : \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}$  at the point of  $\mathcal{R}_g$  corresponding to the double cover  $C' \rightarrow C$  has positive dimension.*

**PROOF.** Follows from Theorem 6.1 and Corollary 6.5. □

It is known that the Prym map is generically finite for  $g \geq 6$  (cf. the survey [B5] and the references quoted therein). However there exist positive dimensional fibres of  $\mathcal{P}_g$  for any value of  $g$ . In order to state Naranjo's Theorem 6.7 that characterizes the positive dimensional fibres of  $\mathcal{P}_g$  for high values of  $g$ , we recall that a curve  $C$  is called bi-elliptic if and only if it admits a double cover  $C \rightarrow E$  onto an elliptic curve  $E$ .

**THEOREM 6.7** (Naranjo, see [Na2], page 224 and [Na1], Theorem (10.10)). *Let  $C' \rightarrow C$  be an unramified double cover of a genus  $g$  curve  $C$ .*

- (i) *If  $g \geq 13$ , then the fibre of  $\mathcal{P}_g$  at the point of  $\mathcal{R}_g$  corresponding to  $C' \rightarrow C$  is positive dimensional if and only if  $C$  is either hyperelliptic or bi-elliptic. In the latter case, if  $C \rightarrow E$  is a double cover of an elliptic curve, then the Galois group of the composition  $C' \rightarrow C \rightarrow E$  is  $G = \mathbf{Z}_2 \times \mathbf{Z}_2$  and each quotient of  $C'$  under an element of  $G$  has genus strictly greater than 1.*
- (ii) *Assume that  $g \geq 10$ , that the fibre of  $\mathcal{P}_g$  at the point of  $\mathcal{R}_g$  corresponding to  $C' \rightarrow C$  is positive dimensional and that  $C$  is bi-elliptic. Then the Galois group of the composition  $C' \rightarrow C \rightarrow E$  is  $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ , and each quotient of  $C'$  under an element of  $G$  has genus strictly greater than 1.*

From the point of view of generating pairs, the hyperelliptic case in Theorem 6.7, i), corresponds to the case of generating pairs of degree 2 which are not good, and Example 3.5 shows that these exist for arbitrary values of  $g$ . On the other hand the bielliptic case can be excluded for good generating pairs with  $g$  large, as Theorem 6.9 below shows.

We recall some general and elementary properties of bi-elliptic curves and bi-double covers, i.e. finite covers with Galois group  $\mathbf{Z}_2 \times \mathbf{Z}_2$  (cf. [Na1], page 50 and ff.; [Pa1]). If  $C$  is bi-elliptic, then the double cover  $C \rightarrow E$  with  $E$  elliptic is unique up to automorphisms of  $E$  if  $g \geq 6$ . Analogously, a bi-elliptic curve  $C$  is not hyperelliptic if  $g \geq 4$  and it is not trigonal if  $g \geq 6$ .

If  $C' \rightarrow C$  is an étale double cover of a bi-elliptic curve  $C \rightarrow E$ , then the composition  $C' \rightarrow C \rightarrow E$  is a degree 4 cover of  $E$  whose Galois group  $G$  contains  $\mathbf{Z}_2$ . Assume that  $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ , denote by  $\sigma$  the element of  $G$  such

that  $C' / \langle \sigma \rangle = C$  and by  $\sigma_i$  ( $i = 1, 2$ ) the remaining non trivial elements. For  $i = 1, 2$ , set  $p_i : C' \rightarrow C_i = C' / \langle \sigma_i \rangle$  the corresponding projection and notice that  $C_i$  is a smooth curve of genus  $g_i$ , where  $g_1 + g_2 = g + 1$ . Then there exists a cartesian diagram:

$$(3) \quad \begin{array}{ccc} C' & \xrightarrow{\pi_2} & C_2 \\ \pi_1 \downarrow & & \downarrow \phi_2 \\ C_1 & \xrightarrow{\phi_1} & E \end{array}$$

where, for  $i = 1, 2$ ,  $\phi_i : C_i \rightarrow E$  is a double cover, and the branch loci  $\Delta_i$  of  $\phi_i$ ,  $i = 1, 2$ , are disjoint. Moreover,  $\sigma = \sigma_1 \circ \sigma_2$ . The group  $G$  also acts on  $\text{Prym}(C', C)$ . We denote by  $P_i$  the connected component containing the origin of the fixed locus of the action of  $\sigma_i$  on  $\text{Prym}(C', C)$  ( $i = 1, 2$ ), and we observe that  $(P_1, P_2)$  is a pair of complementary abelian subvarieties of  $\text{Prym}(C', C)$  of dimensions  $g_1 - 1$  and  $g_2 - 1$ , respectively.

LEMMA 6.8. *Let  $(h : V \rightarrow W, L)$  be a good generating pair of genus  $g \geq 10$ . Then the general curve  $C \in |L|$  is not bielliptic.*

PROOF. By Corollary 6.3 we may assume that the pair is minimal. Suppose, by contradiction, that the general curve  $C \in |L|$  admits an elliptic involution  $C \rightarrow E$ , which, as we saw, is unique up to automorphisms of  $E$ . Moreover, by part (ii) of Theorem 6.7 and by Proposition 6.6, the Galois group  $G$  of the composition  $C' \rightarrow C \rightarrow E$  can be identified with  $\mathbf{Z}_2 \times \mathbf{Z}_2$ . Theorem 6.7 also ensures that there exists a cartesian diagram as in (3), with  $C_i$  of genus  $g_i > 1$ . We wish to extend this construction to  $V$ .

In order to do this, we prove first that we may choose the involutions  $\{\sigma_1, \sigma_2\}$  consistently on the curves  $C' = h^*C$  as  $C$  varies in  $|C|$ . In other words, there is a double cover  $\Psi \rightarrow \Phi$  of the open subset  $\Phi$  of  $|C|$  parametrizing smooth curves, such that its fibre at a general point  $C \in \Phi$  is the pair of involutions  $\{\sigma_1, \sigma_2\}$  acting on  $C' = h^*C$ . We want to prove that  $\Psi$  is the union of two irreducible components both mapping birationally to  $\Phi$ , by showing that there are two sections of  $\Phi \rightarrow \Psi$  mapping the general point  $C \in \Phi$  to  $\sigma_1$ , resp.  $\sigma_2$ , namely that we can rationally distinguish  $\sigma_1$  from  $\sigma_2$ .

Recall that, by Theorem 6.1,  $\text{Prym}(C', C)$  is isomorphic, in a canonical way, to the Albanese variety  $A$  of  $V$ . In this isomorphism, the connected component  $P_i$  of the origin of the fixed locus of the action of  $\sigma_i$  on  $\text{Prym}(C', C)$  maps to an abelian subvariety  $B_i$  of  $A$  ( $i = 1, 2$ ). The pair  $(B_1, B_2)$  of complementary subvarieties can vary only in a discrete set, and therefore it is constant, independent of  $C$ . This proves our claim about the reducibility of  $\Psi$ .

Next we claim that there are involutions  $\tau_i$  on  $V$  inducing  $\sigma_i$  on the general  $C'$ , for  $i = 1, 2$ . Indeed, let  $\mathcal{F}$  be a general pencil inside  $|C|$ . If  $x \in V$  is a general point, define  $\tau_i(x)$  as  $\sigma_i(x)$ , where  $\sigma_i$  is the involution defined on the unique curve  $C'$  in  $h^*(\mathcal{F})$  passing through  $x$ . Since  $V$  is minimal,  $\tau_i$  extends to an automorphism of  $V$ . Notice that  $\tau_i$  is independent of  $\mathcal{F}$ . Otherwise, as  $\mathcal{F}$

varies in a general rational 1-parameter family of pencils, the point  $\tau_i(x)$  would describe a rational curve, for  $x \in V$  general; hence  $\kappa(V)$  would be negative, against Proposition 6.4.

We denote by  $S_i$  the quotient surface  $V / \langle \tau_i \rangle$ , by  $h_i : V \rightarrow S_i$  the projection onto the quotient and by  $C_i$  the image in  $S_i$  of a general  $C'$ . The singularities of  $S_1$  and  $S_2$ , if any, are  $A_1$  points and  $q(S_i) = g_i - 1$ . By Proposition 4.4, if the curves  $C_i$  vary in moduli, then  $g_i \leq 3$ , thus  $g = g_1 + g_2 + 1 \leq 7$ , a contradiction. If the curves  $C_i$  do not vary in moduli, then the Albanese image of  $S_i$  is a curve by Proposition 4.2 and the inequality  $g_i \leq 3$  ( $i = 1, 2$ ) holds by Proposition 4.3, since  $q(S_i) \neq g_i$ .  $\square$

Now we are ready to prove the following basic result:

**THEOREM 6.9.** *Let  $(h : V \rightarrow W, L)$  be a good generating pair of genus  $g$ . Then  $g \leq 12, q(V) \leq 11$ .*

**PROOF.** Suppose, by contradiction, that  $g \geq 13$ . According to Proposition 6.6 and to Theorem 6.7, (i), we can assume that the general  $C \in |L|$  is bi-elliptic. This, on the other hand, contradicts Lemma 6.8.  $\square$

**COROLLARY 6.10.** *Let  $(h : V \rightarrow W, L)$  be a good generating pair of genus  $g$ . Assume that, in addition,  $V$  is of general type. Then  $K_W^2 \leq 529$ .*

**PROOF.** It follows by applying the index theorem to  $L$  and  $K_W$  on  $W$ .  $\square$

A more precise statement is the following:

**PROPOSITION 6.11.** *Let  $(h : V \rightarrow W, L)$  be a good generating pair of genus  $g$ . Then  $g \leq 12$  and:*

- (i) *if the general curve  $C$  in  $|L|$  is bi-elliptic or trigonal, then  $g \leq 9$  and  $q(V) \leq 8$ ;*
- (ii) *if  $10 \leq g \leq 12$  then either the general curve  $C$  in  $|L|$  is a smooth plane sextic (and  $g = 10$ ) or it is not bi-elliptic and has a base point free  $g_4^1$ .*

**PROOF.** The proof follows from Theorem 6.9, Theorem 6.7, and from the following results:

- (i) (Green-Lazarsfeld [GL]) Assume  $g \geq 10$ . If the fibre of the Prym map  $\mathcal{P}_g$  is positive dimensional at the point of  $\mathcal{R}_g$  corresponding to a double cover  $C' \rightarrow C$ , then either  $C$  has a  $g_4^1$  or it is a smooth plane curve of degree six (and genus 10).
- (ii) (Naranjo [Na2]) Assume  $g \geq 10$ . Then the fibre of  $\mathcal{P}_g$  over the point of  $\mathcal{R}_g$  corresponding to a double cover of a trigonal curve  $C$  is finite.  $\square$

## 7. – Good generating pairs with $h^0(W, L) \geq 3$

This section is devoted to the proof of the following:

**THEOREM 7.1.** *Let  $(h : V \rightarrow W, L)$  be a good generating pair such that  $L$  is ample and  $h^0(W, L) \geq 3$ ; then  $(h : V \rightarrow W, L)$  is one of the following:*

- (i) *Example 3.1, and in this case  $h^0(W, L) = 4, L^2 = 4, g = 3$ ;*
- (ii) *a blow-up of weight 1 of case (i) above, and in this case  $h^0(W, L) = 3, L^2 = 3, g = 3$ ;*
- (iii) *Example 3.3, and in this case  $h^0(W, L) = 3, L^2 = 3, g = 4$ .*

Theorem 7.1 will follow from a series of auxiliary results (Proposition 7.5 and Theorem 7.10), containing also some additional information on generating pairs. We also make use of the following result, which is proven in Section 8 (it follows from Proposition 8.2 and 8.3).

**PROPOSITION 7.2.** *If  $(h : V \rightarrow W)$  is a good generating pair with  $\kappa(V) \leq 1$  and  $h^0(W, L) \geq 3$ , then it is obtained from Example 3.1 by a sequence of blow-ups, at most one of which is essential, of weight 1.*

We start by using Reider's method to give an upper bound for  $L^2$  for most generating pairs.

**PROPOSITION 7.3.** *If  $(h : V \rightarrow W, L)$  is a generating pair of degree 2 and non-negative Kodaira dimension, then  $L^2 \leq 4$ .*

**PROOF.** Since  $L^2 = 2L^2$ , it suffices to show that  $L^2 \leq 9$ . We assume that  $L^2 \geq 10$  and we observe that, by the hypothesis, the linear system  $|K_V + L'|$  is not birational on  $V$ . Indeed, if  $x \in W$  is a general point and  $h^{-1}(x) = \{x_1, x_2\}$ , then  $x_1, x_2$  are identified by  $|K_V + L'|$ . According to Reider's theorem ([Re], Theorem 1 and Corollary 2), there exists an effective divisor  $B = B_x$  passing through  $x_1$  and  $x_2$ , such that  $L'B = 1$  or  $2$  and  $B^2 \leq 0$ . Since  $x$  is general we must have  $B^2 = 0$  and, by standard arguments, we may assume that  $B$  moves in a base point free pencil and  $L'D > 0$  for each component of a general  $B$ . Since the general curve  $C'$  of  $|L'|$  is irreducible and meets the general curve  $B_x$  at the points  $x_1$  and  $x_2$ , it follows that  $L'B = 2$ . If the general  $B$  is reducible, then  $B = B_1 + B_2$ , where  $B_1, B_2$  are numerically equivalent irreducible curves. Then  $L'B_i = 1$  and  $V$  is covered by rational curves, contradicting the assumption  $\kappa(V) \geq 0$ . So,  $L'B = 2$  and  $B$  is irreducible. Furthermore, the general fibre of  $h$  is contained in some curve of the pencil described by  $B_x$ , as  $x$  varies in  $W$ . This immediately implies that each curve of this pencil is invariant under the involution  $\iota$  determined by  $h$ . On the other hand,  $|L'|$  cuts out on a general curve  $B$  a  $g_2^1$  which induces on  $B$  the restriction of  $\iota$ . This means that the image of  $B$  via  $h$  is a rational curve on  $W$ , which therefore has a pencil of rational curves. But this contradicts  $\kappa(W) \geq 0$ .  $\square$

**LEMMA 7.4.** *Let  $(h : V \rightarrow W, L)$  be a generating pair such that  $h^0(W, L) \geq 3$ . Then there are the following possibilities:*

- (i)  *$h^0(W, L) = 3$  and  $2 \leq L^2 \leq 4$ ;*
- (ii)  *$h^0(W, L) = 4, L^2 = 4$  and  $|L|$  is base point free.*

PROOF. If  $h^0(W, L) = r$ , then the restriction of the system  $|L|$  to a general  $C$  is a linear system  $|D|$  of dimension  $r - 2 > 0$  and degree  $L^2 \leq 4$ , according to Proposition 7.3. We denote by  $|M|$  the moving part of  $D$ . If  $r = 3$ , then  $L^2 \geq \deg M \geq 2$ , since  $C$  is not rational, and (i) is proven. If  $r > 3$ , then  $4 \geq L^2 \geq \deg M \geq 2 \dim |M| = 2(r - 2) \geq 4$ . Thus  $L^2 = \deg M = r = 4$  and  $|L|$  is base point free.  $\square$

PROPOSITION 7.5. *Let  $(h : V \rightarrow W, L)$  be a good generating pair such that  $h^0(W, L) \geq 3$ . The possible cases are:*

- (i)  $L^2 = 4, h^0(W, L) = 3$  and  $|L|$  is base point free. In particular, the general  $C \in |L|$  is tetragonal.
- (ii)  $h^0(W, L) = 3$  and either  $L^2 = 3$  or  $L^2 = 4$  and  $|L|$  has a simple base point. In particular, the general  $C \in |L|$  is trigonal;
- (iii)  $L^2 = 4, h^0(W, L) = 4$ , and the pair is obtained from Beauville's Example 3.1 via inessential blow-ups.

PROOF. We denote by  $|D|$  the restriction of  $|L|$  to a general  $C$  of  $|L|$ . Assume that we are in case (i) of Lemma 7.4: then (i) and (ii) follow by remarking that the moving part of  $|D|$  has degree  $> 2$ , since  $C$  is not hyperelliptic.

Assume that we are in case (ii) of Lemma 7.4. Then Clifford's theorem implies that  $g = 3, |D|$  is the canonical system and  $C$  is embedded by  $|D|$  as a smooth plane quartic. So the linear system  $|L|$  maps the surface  $W$  birationally onto a quartic  $Q \subset \mathbb{P}^3$ . The Kodaira dimension of  $W$  is non-negative by Corollary 6.4 and thus  $\kappa(W) = 0$ . Claim (iii) now follows by Proposition 7.2.  $\square$

PROPOSITION 7.6. *If  $(h : V \rightarrow W, L)$  is a good generating pair of Kodaira dimension 2 such that  $L^2 = 4$  and  $h^0(W, L) \leq 3$ , then  $10 \leq g \leq 12$ .*

PROOF. The inequality  $g \leq 12$  follows from Theorem 6.9.

By Corollary 6.3, it follows that the pair is obtained from a minimal pair by inessential blow-ups. Thus we may assume that the pair is minimal. Write  $h^0(W, L) = 2 + l$ , so that either  $l = 0$  or  $l = 1$ .

Since  $W$  is of general type, one has  $0 < K_W L = 2g - 2 - L^2 = 2g - 6$ , hence  $g \geq 4$ . For a general  $C \in |L|$ , consider the exact sequence:

$$0 \rightarrow H^0(W, K_W - L) \rightarrow H^0(W, K_W) \rightarrow H^0(C, K_C - L|_C)$$

and notice that  $h^0(C, K_C - L|_C) = g - 4 + l$ , by the regularity of  $W$  and by Riemann-Roch applied to  $C$ .

Assume first  $g \leq 6$ . Then by Corollary 6.4, we have  $h^0(W, K_W - L) \geq g - 1 - (g - 4 + l) = 3 - l \geq 2$ . Notice that  $h^0(W, K_W - 2L) = 0$ , since  $(K_W - 2L)L = 2g - 14 < 0$  and  $L$  is nef. Thus  $|K_W - L|$  cuts out on  $C$  a linear series of dimension  $\geq 1$  and of degree  $2g - 10 \leq 2$ , contradicting the assumption that  $C$  is not hyperelliptic.

Therefore we have  $g \geq 7$ . Corollary 6.4 and the above exact sequence yield:  $h^0(W, K_W - L) \geq 2g - 6 - (g - 4 + l) = g - 2 - l$ . Let  $h^0(W, K_W - 2L) = r$ . Then

$|K_W - L|$  cuts out on  $C$  a special linear series of dimension  $\geq g - 3 - r - l$  and degree  $2g - 10$ . By Clifford's theorem and by the fact that  $C$  is not hyperelliptic, we have  $g - 3 - r - l < g - 5$ , namely  $r + l \geq 3$ . Thus we either have  $l = 0$ ,  $r \geq 3$  or  $l = 1$ ,  $r \geq 2$ . If  $g \leq 8$ , then  $L(K_W - 3L) = 2g - 18 < 0$ , and thus  $h^0(W, K_W - 3L) = 0$  and  $|K_W - 2L|$  cuts out on  $C$  a special linear series of dimension at least  $r - 1 \geq 1$  and of degree  $2g - 14 \leq 2$ , contradicting again the assumption that  $C$  is not hyperelliptic.

If  $g = 9$ , then  $L(K - 3L) = 0$  and thus  $h^0(W, K_W - 3L) \leq 1$ , since  $L$  is nef and big. Assume that  $h^0(W, K_W - 3L) = 1$ ; then we have  $K_W^2 - 36 = K_W(K_W - 3L) \geq 0$ , since  $W$  is of general type. On the other hand, the index theorem gives  $K_W^2 \leq 36$ . It follows that  $K_W^2 = 36$  and  $K \sim_{num} 3L$ . Therefore  $K_W = 3L$ , since  $K_W - 3L$  is effective. So  $r = h^0(W, K_W - 2L) = h^0(W, L) = 2 + l$ , and for  $l = 0$  this contradicts the above inequality  $r + l \geq 3$ . If  $l = 1$ , consider the exact sequence:

$$(4) \quad 0 \rightarrow (k - 1)L \rightarrow kL \rightarrow kL|_C \rightarrow 0.$$

By Clifford's theorem we have  $h^0(C, 2L|_C) \leq 4$ . So, for  $k = 2$ , the sequence (4) implies  $h^0(W, 2L) \leq 7$ . Using this and sequence (4) for  $k = 3$ , one gets  $p_g(W) = h^0(W, 3L) \leq 13$ , and thus  $\chi(V) = 2 + p_g(W) - g \leq 6$ . On the other hand, Miyaoka-Yau's inequality would give  $72 = K_V^2 \leq 9\chi(V) = 54$ , a contradiction.

So we are left with the case  $h^0(W, K_W - 3L) = 0$ . If  $l = 0$ , then  $r \geq 3$  and the restriction of  $|K_W - 2L|$  to  $C$  is a  $g_4^2$ , contradicting again the fact that  $C$  is non-hyperelliptic of genus 9. Thus the case  $g = 9$  and  $l = 0$  does not occur.

If  $l = 1$ , then we have  $r = 2$ , since for  $r > 2$  we can argue as above and show that  $|K_W - 2L|$  restricts to a  $g_4^2$  on  $C$ . So we have  $h^0(W, K_W - L) \leq 2 + h^0(C, (K_W - L)|_C) \leq 2 + 4 = 6$ , where the last inequality follows again by Clifford's theorem. In turn,  $p_g(W) \leq 6 + h^0(C, K_W|_C) = 12$ . On the other hand, by Proposition 6.4, one has  $p_g(W) \geq 12$ , with equality holding iff  $V = C_1 \times C_2$ , with  $C_1$  a curve of genus 2 and  $C_2$  a curve of genus 6. Thus  $p_g(W) = 12$  and the restriction map  $H^0(W, K_W) \rightarrow H^0(C, K_W|_C)$  is surjective. Since the canonical map of  $V$  factors through the map  $h : V \rightarrow W$  and  $q(W) = 0$ , the curve  $C_2$  is hyperelliptic, and the canonical map of  $V$  has degree 4. The canonical image  $\Sigma$  of  $V$  (and of  $W$ ) is  $\mathbf{P}^1 \times \mathbf{P}^1$  embedded via the system  $|\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1, 5)|$ . The curves of  $|L|$  are mapped 2-to-1 onto curves  $D$  of  $\Sigma$  with  $D^2 = 2$ . So the curves  $D$  are of type  $(1, 1)$ , and thus rational. It follows that the general curve of  $|L|$  is hyperelliptic, contradicting again the assumption that the pair be good.  $\square$

**COROLLARY 7.7.** *If  $(h : V \rightarrow W, L)$  is a good generating pair such that  $L^2 = 4$  and  $h^0(W, L) = 3$ , then  $10 \leq g \leq 12$ .*

**PROOF.** Proposition 7.2 implies that the Kodaira dimension of the pair is 2. The thesis follows from Proposition 7.2.  $\square$

The next result shows that case (i) of Proposition 7.5 does not occur for  $L$  ample.

**PROPOSITION 7.8.** *Let  $(h : V \rightarrow W, L)$  be a good generating pair with  $h^0(W, L) = 3$ ,  $L^2 = 4$  and  $L$  ample; then  $|L|$  has one simple base point.*

**PROOF.** First of all we remark that the assumption that  $L$  is ample, Corollary 6.3 and Proposition 7.3 imply that the pair is minimal. By Lemma 7.5 we only have to exclude that  $|L|$  is base point free. So we assume that  $|L|$  has no base points and we show that this leads to a contradiction. As usual we denote by  $C$  a general curve of  $|L|$  and by  $C'$  the inverse image of  $C$  via  $h$ ; we denote by  $\phi : W \rightarrow \mathbf{P}^2$  the finite degree 4 morphism given by  $|L|$ . Notice that  $\phi$  is flat, since it is a projective morphism with finite fibres from a normal surface to a smooth one. Our proof requires various steps.

**STEP 1.** *The polarized Abelian variety  $\text{Prym}(C', C)$  is not a Jacobian or a product of Jacobians.* By Theorem 4.10 of [B1] and the assumption that  $C$  is not hyperelliptic, if  $\text{Prym}(C', C)$  is a Jacobian then one of the following holds: (i)  $C$  is trigonal, (ii)  $C$  is bielliptic, (iii)  $g \leq 6$ . Case (iii) is impossible by Corollary 7.6. Since  $C$  has a free  $g_4^1$ , case (i) implies  $g \leq 6$ , and therefore it is also excluded. Finally case (ii) is excluded by Lemma 6.8.

**STEP 2.** *The curves of  $|L'|$  are 2-connected.* Notice that the curves of  $|L'|$  are 1-connected since  $|L'|$  is ample. Assume that  $D \in |L'|$  is not 2-connected, namely that  $D = A + B$  with  $A, B$  effective and  $AB = 1$ . Then  $1 \leq L'A = A^2 + AB = A^2 + 1$  and  $1 \leq L'B = B^2 + 1$ , hence  $A^2 \geq 0$ ,  $B^2 \geq 0$ . If, say,  $A^2 = 0$ , then  $L'A = 1$  and  $A$  is irreducible, since  $L'$  is ample, and rational, since  $|L|$  is base point free. This contradicts  $\kappa(V) = 2$ , hence  $A^2$  and  $B^2$  are both positive. Since  $8 = L'^2 = A^2 + B^2 + 2$ , we have  $A^2 + B^2 = 6$  whereas the index theorem gives  $A^2 B^2 \leq (AB)^2 = 1$ .

**STEP 3.** *The branch divisor  $Z$  of  $\phi$  in  $\mathbf{P}^2$  is not a union of lines.* Here we need to consider the intersection number of Weil divisors on  $W$ . We recall that, since the singularities of  $W$  are  $A_1$  points, given Weil divisors  $A, B$  on  $W$ , the intersection number  $AB$  is an element of  $\frac{1}{2}\mathbf{Z}$ , and it is an integer whenever  $A$  or  $B$  is Cartier.

Assume that  $Z$  is a union of lines and let  $R$  be a line contained in  $Z$ . Then  $C_0 = \phi^*(R) \in |L|$  is of the form  $C_0 = mA + B$ , with  $2 \leq m \leq 4$  and with  $A, B$  effective, Weil divisors such that  $A$  is irreducible and not contained in  $B$ . We set  $C'_0 = h^*C_0$ ,  $A' = h^*A$ ,  $B' = h^*B$ , so that  $C'_0 = mA' + B'$ . Notice that  $4 = L^2 = mL'A + L'B \geq 2L'A$  yields  $1 \leq L'A \leq 2$ . Assume first  $L'A = 2$ . Then one has  $B = 0$ ,  $m = 2$ , and thus  $C'_0 = 2A'$ . Recall that by Proposition 6.1 the abelian variety  $\text{Prym}(C', C)$  is naturally isomorphic to the Albanese variety of  $V$  and denote by  $\alpha : V \rightarrow \text{Prym}(C', C)$  the Albanese map. If  $\Xi$  is the principal polarization of  $\text{Prym}(C', C)$ , then by Welters criterion  $\alpha_*C'$  is homologically equivalent to  $\frac{2}{(g-2)!} \wedge^{g-2} \Xi$  for every  $C' \in |L|$ . Thus  $\alpha_*A'$  is homologically equivalent to  $\frac{1}{(g-2)!} \wedge^{g-2} \Xi$ , and it follows that  $A'$  is smooth and  $\text{Prym}(C', C)$  is isomorphic to the Jacobian of  $A'$ . This is impossible by



Step 1, and therefore  $LA \cong 1$ . The condition  $LA = 1$  implies that  $m \leq 3$ ,  $B$  is nonempty and  $A$  is smooth and irreducible. Assume that  $m = 3$ , and let  $R_1 \subset Z$  be another line; write  $\phi^*R_1 = m_1A_1 + B_1$ , with  $2 \leq m_1 \leq 3$ ,  $A_1$  irreducible and not contained in  $B_1$ . The equality  $1 = A_1L = 3A_1A + A_1B$  gives  $AA_1 = 0$ ,  $A_1B = 1$ , and thus  $1 = BL = m_1BA_1 + BB_1 \geq m_1 \geq 2$ , a contradiction. Thus, for every line  $R \subset Z$ , we have  $\phi^*R = 2A + B$ , with  $A$  irreducible and not contained in  $B$ . In particular,  $Z$  is reduced. Notice also that  $AB > 0$ , since the curves of  $|L'|$  are 2-connected by Step 2, and thus  $A$  and  $B$  have nonempty intersection. Let  $x_0 \in A \cap B$ , let  $y_0 = \phi(x_0)$ , and let  $C$  be the pull-back of a general line through  $y_0$ ; then  $C(2A + B) = L(2A + B) = 4$  and  $x_0$  accounts at least for 3 intersections. Thus  $\phi^{-1}(y_0)$  either consists of  $x_0$  only, or contains also a point  $x'_0$  that is not a branch point of  $\phi$ ; in either case  $x_0$  is not a simple ramification point of  $\phi$  and therefore  $Z$  is not smooth at  $x_0$ . Thus there is another line  $R_1 \subset Z$  that contains  $x_0$ . Write  $\phi^*R_1 = C_1 = 2A_1 + B_1$ . From  $1 = AL = 2AA_1 + AB_1$  we see that either  $AA_1 = \frac{1}{2}$  and  $AB_1 = 0$  or  $AA_1 = 0$  and  $AB_1 = 1$ . On the other hand,  $A_1$  contains  $x_0$  and thus we have  $A_1B > 0$  and  $A_1A > 0$ . Thus we have a contradiction, and  $Z$  is not a union of lines.

We can now consider an irreducible component  $Z'$  of  $Z$  that is not a line and a general tangent line  $R$  to  $Z'$ . The curve  $C_0 = \phi^*(R)$  is reduced, but singular at some point  $x$ . It moves in a base point free continuous system on  $W$ . Set  $h^{-1}(x) = \{x_1, x_2\}$  and let  $C'_0 = h^*C_0$ . Notice that the map  $h : C'_0 \rightarrow C_0$  is étale. Moreover  $C'_0$  is singular at  $x_1$  and  $x_2$ , and we can apply Theorem 3.2 from [ML]. Then we have  $C'_0 = A' + B'$  with  $A'$ ,  $B'$  reduced and with no common component, since  $C'_0$  is reduced as well as  $C_0$ , and  $A'B' = 2$ . Actually  $A' \cap B' = \{x_1, x_2\}$ , which proves that  $A'$  and  $B'$  are smooth at  $x_1$  and  $x_2$ .

**STEP 4.** *One has  $A^2 = B^2 = 2$  hence  $2A'$  and  $2B'$  are numerically equivalent to  $L'$ . Since  $A'$  and  $B'$  move without fixed components on  $V$ , we have  $A'^2 \geq 0$  and  $B'^2 \geq 0$ . Furthermore we have  $A'^2 + B'^2 = 4$  and  $L'A' = A'^2 + 2$  and  $L'B' = B'^2 + 2$ . Suppose  $A'^2 = 0$ , hence  $L'A' = A'B' = 2$ . We claim that in this case  $A'$  is irreducible: in fact, if  $A' = A_1 + A_2$ , then  $A_1^2, A_2^2 \geq 0$ , since  $A_1$  and  $A_2$  move, and thus  $A_1^2 = A_2^2 = 0$  and  $A_1 \sim_{num} A_2$ ,  $A_1B' = A_2B' = 1$ , contradicting the fact that the curves of  $|L'|$  are 2-connected. Thus the general curve  $A'$  is irreducible and moves in an irrational pencil  $\mathcal{A}'$  on  $V$ . The involution  $\iota$  determined by  $h : V \rightarrow W$  fixes  $C'_0$ , hence it maps  $A'$  to an irreducible curve  $A''$ . If  $A' = A''$ , then there exists  $A \subset C_0$  on  $W$  such that  $A' = h^*A$ ,  $A^2 = 0$ ,  $LA = 1$ . Thus  $A$  is smooth rational and, since  $A'$  is general,  $W$  is covered by rational curves, contradicting  $\kappa(W) \geq 0$ . So  $A''$  is contained in  $B'$ . The curve  $A''$  also moves in an irrational pencil  $\mathcal{A}''$ , and  $A'A'' \geq 2$ , since  $A'$  and  $A''$  both contain  $x_1$  and  $x_2$ . Write  $B = A'' + D$ ,  $C'_0 = A' + A'' + D$ ; since  $C'_0A' = L'A' = 2$ , we get  $A'A'' = 2$  and  $DA' = 0$ . Since  $D$  also moves on  $V$  without fixed components, it consists of curves of  $\mathcal{A}'$ , hence  $D^2 = 0$ . Since  $L'$  is fixed by  $\iota$ , we have  $L'A'' = L'A' = 2$ ,  $A''^2 = A^2 = 0$ , and therefore:  $2 = A''L' = A''(A' + A'' + D) = 2 + A''D$ ,  $A''D = 0$ . Thus  $D$  and  $A''$  are numerically equivalent, but this contradicts  $A'A'' = 2$ .*

Suppose that  $A^2 = 1$ . By Proposition 0.18 of [CCML] we deduce that  $A'$  is smooth irreducible and  $V$  is isomorphic to the symmetric product of  $A'$ . The canonical maps of symmetric products are well known. Thus, the fact that the canonical map of  $V$  is not birational, since it factors through  $h$ , tells us that either  $3 \geq p_a(A') = q(V) = g - 1$  or  $A'$  is hyperelliptic of genus  $p_a(A') \geq 4$ . The former case is impossible by Corollary 7.7. The latter case is also impossible because  $|L'|$  cuts out on  $A'$  a base point free  $g_3^1$ . Hence we are left with the only possibility  $A'^2 \geq 2$  and, similarly,  $B'^2 \geq 2$ , which implies the assertion.

STEP 5. *The divisors  $A'$  and  $B'$  are exchanged by  $\iota$ .* The divisor  $\iota(A') = A''$  is contained in  $C'_0$  and is numerically equivalent to  $A'$ , since  $2A'$  and  $2A''$  are both numerically equivalent to  $L'$ . If  $A' = A''$ , then there exists  $A$  on  $W$  such that  $h^*A = A'$ ,  $A^2 = 1$ . We apply Proposition 0.18 of [CCML] to the pull-back of  $A$  to the minimal desingularization  $\tilde{W}$  of  $W$  and deduce that  $\tilde{W}$  is birational to the symmetric product of  $A$ , contradicting  $q(W) = 0$ . If  $A'$  is irreducible, this is enough to prove that  $A'' = B'$ . So assume that  $A'$  is reducible and write  $A' = N + M$ , with  $N, M$  effective nonzero. Then  $2 = A'^2 = A'N + A'M$ , hence  $A'N = A'M = 1$  since  $A$ , as well as  $L'$ , is ample. This proves that  $N, M$  are both irreducible. Since they move on  $V$ , we have  $N^2 \geq 0$  and  $M^2 \geq 0$  and the index theorem yields  $N^2 = M^2 = 0, NM = 1$  and  $N$  and  $M$  both describe base point free pencils on  $V$ . Since  $A' \neq A'', B'$  and  $A''$  have at least a common component. Thus we may write  $B' = M' + N'$ , where  $M'$  is equal to, say,  $\iota(M)$ . We have  $M'B' = MA' = 1, M'^2 = M^2 = 0$ , hence  $B'N' = 1$  and  $N'$  is irreducible by the ampleness of  $B'$ . If  $\iota(N) = N'$ , then the claim is proven. So assume  $\iota(N) \neq N'$ . Then we have  $\iota(N) = N$  and there exists  $N_0 \subset W$  such that  $h^*N_0 = N$ . It follows that  $LN_0 = 1$ , and thus  $N_0$  is a rational curve. This is impossible, since otherwise  $W$  would be covered by rational curves. Thus  $\iota(N) = N'$  and  $\iota$  exchanges  $A'$  and  $B'$ .

STEP 6. *Conclusion of the proof.* We use the notation introduced in Step 1. By Step 5, if the base point of the Albanese map  $\alpha : V \rightarrow \text{Prym}(C', C)$  is invariant for  $i$ , then  $\alpha_*B' = (-1)_*\alpha_*A'$ , since  $\alpha$  is equivariant with respect to  $\iota$  on  $V$  and multiplication by  $-1$  on  $\text{Prym}(C', C)$ . Thus  $\alpha_*(C'_0) = \alpha_*(A') + \alpha_*(B')$  and  $2\alpha_*A'$  represent the same cohomology class. By Welters criterion, this implies that  $\alpha_*A'$  is equivalent in cohomology to  $\frac{1}{(g-2)!} \wedge^{g-2} \Xi$ . By the criterion of Matsusaka-Ran,  $\text{Prym}(C/C)$  is isomorphic as a principally polarized abelian variety either to a Jacobian or to a product of Jacobians. This contradicts Step 1, and the proof is complete.  $\square$

REMARK 7.9. The same ideas we exploited in the proof of the previous proposition would also yield the following result: *in case (ii) of Proposition 7.5, the generating pair is either the pair of the Example 3.3 or it is obtained from Beauville's Example 3.1, with a blow-up procedure.* We will next prove the same theorem with a different technique, which also seems illuminating to us. Hence we give here only an idea of its proof with the present methods.

Assume for simplicity  $L^2 = 3$ . Then one considers the finite, degree 3 map  $\phi : W \rightarrow \mathbf{P}^2$  determined by  $|L|$ . First one shows that no line is in the branch divisor of  $\phi$ . Then one proves the existence of a 1-dimensional family of reduced curves  $C'_0 \in |L'|$  which split as  $C'_0 = A + B$ , with  $AB = 2$ . This implies that  $A^2 = B^2 = 1$ . At this point one uses Proposition 0.18 from [CCML] and proves that  $V$  is birational to the symmetric product of  $A = B$ . The fact that the canonical map of  $V$  is not birational tells us that either  $g \leq 3$ , which leads to the two cases which actually occur, or  $A$  is hyperelliptic of genus  $g \geq 4$ . But this not possible because, via  $h$ ,  $A$  is birational to the image  $C_0$  of  $C'_0$ , and  $C_0$  has a base point free  $g_3^1$ .

The rest of this section is devoted to the analysis of case (ii) of Proposition 7.5 under the hypothesis that  $L$  be ample. We prove the following result:

**THEOREM 7.10.** *Let  $(h : V \rightarrow W, L)$  be a good generating pair of genus  $g$  such that  $L$  is ample and  $h^0(W, L) = 3$ . Then  $L^2 = 3$  and:*

- (i) *either there exists a smooth plane quartic  $\Gamma$  such that  $(h : V \rightarrow W, L)$  is constructed from  $\Gamma$  as explained in Example 3.3;*
- (ii) *or  $(h : V \rightarrow W, L)$  is obtained from Beauville’s Example 3.1 via a simple blow-up of weight 1.*

By Propositions 7.5 and 7.8, a pair satisfying the assumption of Theorem 7.10 either has  $L^2 = 3$  and  $|L|$  is base point free or has  $L^2 = 4$  and  $|L|$  has a simple base point. So, up to a simple blow up of the pair, we may assume that  $L^2 = 3$  and  $|L|$  is base point free. Thus for the rest of the section we make the following assumption:

**ASSUMPTION 7.11.**  *$(h : V \rightarrow W, L)$  is a good generating pair of genus  $g$  such that  $L$  is ample,  $h^0(W, L) = 3$ ,  $L^2 = 3$  and  $|L|$  is base point free.*

If Assumption 7.11 holds, then  $|L|$  defines a finite morphism  $f : W \rightarrow \mathbf{P}^2$  of degree 3. The restriction of  $f$  to the general curve  $C \in |L|$  exhibits  $C$  as a triple cover of  $\mathbf{P}^1$  showing that  $C$  is trigonal. Given a curve  $C$  of genus  $g$ , a degree 3 map  $f : C \rightarrow \mathbf{P}^1$ , and an unramified double cover  $\pi : C' \rightarrow C$ , the *trigonal construction* ([Rec], cf. [B3]) yields a degree 4 map  $\phi : D \rightarrow \mathbf{P}^1$ , where  $D$  is a smooth curve of genus  $g - 1$  and  $\phi$  has no double fibre, such that the Jacobian of  $D$  and  $\text{Prym}(C', C)$  are isomorphic as principally polarized abelian varieties. We briefly recall the trigonal construction. One considers the induced morphism  $\pi^{(3)} : C'^{(3)} \rightarrow C^{(3)}$  between the symmetric products of  $C'$  and  $C$ . The curve  $\tilde{D} = \pi^{(3)-1}(g_3^1)$  has a natural morphism  $\tilde{D} \rightarrow \mathbf{P}^1$ ; it turns out that  $\tilde{D} \rightarrow \mathbf{P}^1$  splits as the disjoint union of two isomorphic smooth connected degree 4 covers  $\phi_i : D_i \rightarrow \mathbf{P}^1$ ,  $i = 1, 2$ , and one can set  $D = D_1$ ,  $\phi = \phi_1$ .

The trigonal construction is a one-to-one correspondence, whose inverse is the Recillas’ construction ([Rec], cf. [BL1] page 391). Given a smooth genus  $g - 1$  curve  $D$  with a degree 4 morphism  $\varphi : D \rightarrow \mathbf{P}^1$  without double fibres, one defines a curve  $C' \subset D^{(2)}$  by setting:

$$C' = \{p_1 + p_2 \in D^{(2)} \mid p_1 + p_2 + p_3 + p_4 \text{ is a fibre of } \varphi \text{ for some } p_3, p_4 \in D\}.$$

The curve  $C'$  is smooth and connected of genus  $2g - 1$ , and has a natural free involution  $\sigma$ , which maps an element  $p_1 + p_2$  (in a fibre of  $\varphi$ ) to the complementary element  $p_3 + p_4$ . If  $\pi : C' \rightarrow C = C' / \langle \sigma \rangle$  denotes the natural projection, it is easy to check that  $C$  is trigonal.

Recillas' correspondence has been generalized in [Ca2], where the author introduces the *discriminant* of a degree 4 Gorenstein cover  $\varphi : Z \rightarrow Y$ , which is a degree 3 morphism  $f : \Delta(Z) \rightarrow Y$ . We recall that a cover  $\varphi : Z \rightarrow Y$  is said to be a Gorenstein cover if the scheme theoretic fibre  $\varphi^{-1}(y)$  is Gorenstein over  $k(y)$  for every  $y \in Y$  (cf. [Ca1]). If  $Y = \mathbf{P}^2$ , then the discriminant construction gives a one-to-one correspondence between the following objects:

- (A) normal Gorenstein covers  $f : W \rightarrow \mathbf{P}^2$  of degree 3 such that the singularities of  $W$  are at most RDP's and such that there exists a double cover  $h : V \rightarrow W$  branched exactly over the singularities of  $W$ ;
- (B) degree 4 Gorenstein covers  $\varphi : Z \rightarrow \mathbf{P}^2$  with  $Z$  smooth such that:
  - (i) for every  $y \in \mathbf{P}^2$  the Zariski tangent space to the fibre  $\varphi^{-1}(y)$  has dimension  $\leq 1$  at each point.
  - (ii) the set  $R_0 \subset \mathbf{P}^2$  of points  $y$  such that the fibre  $\varphi^{-1}(y)$  is isomorphic either to  $\text{spec } \mathbf{C}[t]/(t^4)$  or to  $\text{spec } \mathbf{C}[t, s]/(t^2 + 1, s^2)$  is finite.

The properties of this correspondence ensure that the branch loci of the associated covers  $\varphi$  and  $f = \Delta(\varphi)$  coincide as divisor of  $\mathbf{P}^2$ . Moreover, the singularities of  $W$  occur precisely over the points  $y \in R_0$ . Notice that, in the case we are interested in, the singular locus of  $W$  is not empty (see the proof of Theorem 6.1), and therefore  $R_0$  is not empty. Finally, fibrewise,  $Z_y$  is the base locus of a pencil of conics whose discriminant is  $W_y$ .

Assumption 7.11 allows us to apply the trigonal construction to the present case. Thus, given a good generating pair  $(h : V \rightarrow W, L)$  as in 7.11, there exists a unique degree 4 Gorenstein cover  $\varphi : Z \rightarrow \mathbf{P}^2$  as in (B) such that the morphism  $f : W \rightarrow \mathbf{P}^2$  associated to the system  $|L|$  is obtained from  $\varphi$  via the trigonal construction. We denote by  $|M|$  the pull-back to  $Z$  of the linear system of lines in  $\mathbf{P}^2$ .

LEMMA 7.12. *The smooth elements of  $|M|$  are isomorphic curves of genus  $g - 1 \geq 2$ .*

PROOF. Let  $H$  be a general line in  $\mathbf{P}^2$ , let  $D = \varphi^{-1}H$ , let  $C = f^{-1}H$  and let  $C' \rightarrow C$  be the unramified cover determined by  $h$ . By Theorem 6.1, the Prym variety  $P = \text{Prym}(C', C)$  is independent of  $H$ . On the other hand,  $C' \rightarrow C$  is obtained from  $D$  via the trigonal construction, and thus  $P$  and the Jacobian of  $D$  are isomorphic as p.p.a.v.'s. In particular, since the genus of  $|L|$  is at least 3, the genus of  $|M|$  is at least 2. By the global Torelli theorem for curves, the isomorphism class of  $D$  is also independent of  $H$ . This implies that the natural map from the open set of smooth curves of  $|M|$  to the moduli space of curves is constant. □

LEMMA 7.13. *Let  $y \in \mathbf{P}^2$  and let  $|M_y| \subset |M|$  be the pull-back on  $Z$  of the pencil of lines through  $y$ . Then the general curve of  $|M_y|$  is smooth.*

PROOF. The base scheme of  $|M_y|$  is  $\varphi^{-1}(y)$ . The statement follows by Bertini's theorem since  $\varphi : Z \rightarrow \mathbf{P}^2$  satisfies condition (i) of (B).  $\square$

LEMMA 7.14. *Z is a minimal geometrically ruled surface, and the smooth elements of  $|M|$  are sections of the ruling.*

PROOF. Denote by  $R \subset Z$  and by  $B \subset \mathbf{P}^2$  the ramification divisor and the branch divisor of  $\varphi$ . By condition (B), the ramification order of  $\varphi$  along each component of  $R$  is  $\leq 3$ , each component of  $R$  is mapped birationally onto its image and different components of  $R$  are mapped to different components of  $B$ . Let  $(M_t)_{t \in \mathbf{P}^1}$  be a general pencil contained in  $|M|$  and assume that  $M_0$  is singular. By applying stable reduction, one can replace  $M_0$  by a stable curve  $M'_0$ . Lemma 7.12 implies that  $M'_0$  is isomorphic to  $M_t$ , for  $t$  general.

Assume that there exists a component  $\Theta$  of  $R$  such that  $\Delta = \varphi(\Theta)$  is not a line. If  $\varphi$  is ramified of order 3 along  $\Theta$ , then the inverse image  $M_0$  of a generic line tangent to  $\Delta$  has an ordinary cusp over the tangency point and it is smooth elsewhere. It follows that  $M_0$  is irreducible. Since  $p_a(M) > 1$ ,  $M_0$  is not rational; the special fibre of the stable reduction of a general pencil containing  $M_0$  is the union of the normalization  $M''_0$  of  $M_0$  and of a smooth elliptic curve meeting  $M'_0$  at one point, but this is impossible by the remark above. If  $\varphi$  is simply ramified along  $\Theta$ , take  $(M_t)$  to be the pull-back of a pencil of lines such that  $M_0$  is the pull-back of a line simply tangent to  $\Delta$  at a point  $y_0$  and meeting  $B$  transversely elsewhere. Then  $M_0$  has an ordinary node at a point  $x_0$  such that  $\varphi(x_0) = y_0$  and no other singularities. By the remarks above, the curve  $M_0$  is not semistable; therefore we have  $M_0 = M'_0 + F$ , where  $F$  is a smooth rational curve,  $M'_0$  is isomorphic to  $M_t$  for  $t$  general, and  $M'_0 F = 1$ . We have:  $4 = M^2 = M_t(M'_0 + F)$ ,  $M_t M'_0 \geq 3$  (since  $M'_0$  is not hyperelliptic), and thus  $M_t M'_0 = 3$  and  $F^2 = 0$ . Noticing that  $y_0$  is a general point of  $\Delta$ , it follows that  $Z$  is ruled. Since the system  $|M|$  is ample,  $M F = 1$  and, by Lemma 7.12, the curves of  $|M|$  are not rational,  $Z$  is geometrically ruled and minimal.

So we have proven that either  $Z$  is as claimed or all the components of  $B$  are lines. Let  $\Delta \subset B$  be a line; by condition (B), it is not possible that  $f^* \Delta = 2A$ . Thus  $\varphi^* \Delta = mA + B$ , with  $m \leq 3$ ,  $A$  irreducible,  $B$  nonempty and  $A$  not contained in  $B$ . Then one can argue as in Step 3 of the proof of Proposition 7.6 and prove that  $B$  is not a union of lines.  $\square$

PROPOSITION 7.15. *Let  $B$  be the base curve of the ruled surface  $Z$  of Lemma 7.14 and let  $p : Z \rightarrow B$  be the projection. Then there exists a birational morphism  $s : B \rightarrow \Gamma \subset \mathbf{P}^2$  such that:*

- i)  $\Gamma$  is either a smooth quartic or a quartic with a double point;
- ii)  $Z = \mathbf{P}(s^* T_{\mathbf{P}^2}(-1))$ , and  $p_* \mathcal{O}_Z(M) = s^* T_{\mathbf{P}^2}(-1)$ .

PROOF. According to Lemma 7.14 there exists a rank 2 bundle  $E$  on  $B$  such that  $Z = \mathbf{P}(E)$  and  $p_* \mathcal{O}_Z(M) = E$  (in particular,  $\deg(\det E) = 4$ ). Let  $D$  be a smooth curve in  $|M|$ , which we may identify with  $B$  via the map  $p|_D$ . Then  $M|_D$  is identified with  $\det E$ . By condition (B), (ii), if  $D$  is general, then

$\varphi|_D$  has no multiple fibre, while if  $\varphi(D)$  contains a point of  $R_0$  (which, as we know, is not empty) then  $\varphi|_D$  has at least one multiple fibre. So the restriction of  $|M|$  to  $D$  is not a complete system, i.e.  $h^0(B, \det E) = 3$ . Let  $s : B \rightarrow \mathbf{P}^2$  be the morphism given by the linear system  $|\det E|$  and let  $\Gamma = s(B)$ . If  $\Gamma$  were a conic, then the map  $\varphi|_D$  would have two multiple fibres for every smooth  $D$  of  $|M|$ , contradicting condition (B). So  $\Gamma$  is a quartic and  $s$  is birational. Since  $B$  has genus  $g - 1 \geq 2$ , it follows that  $\Gamma$  is either smooth or it has one double point and  $\det E = s^* \mathcal{O}_\Gamma(K_\Gamma)$ .

Let  $U \subset H^0(Z, M)$  be the subspace such that  $\mathbf{P}(U) = \varphi^*|\mathcal{O}_{\mathbf{P}^2}(1)|$ . If we identify  $U$  with a subspace of  $H^0(B, E)$ , then i) the natural sheaf map  $U \otimes \mathcal{O}_Z \rightarrow E$  is surjective ( $|M|$  is base-point free). Moreover, ii) the map  $\wedge^2 U \rightarrow H^0(B, s^*K_\Gamma)$  is an isomorphism (this follows from the discussion above, since we have shown that  $|M|$  does not restrict to the same  $g_4^1$  on all the curves of  $|M|$ ).

If we choose a basis for  $U$ , then we have a short exact sequence:

$$(5) \quad 0 \rightarrow \mathcal{O}_B(-s^*K_\Gamma) \rightarrow \mathcal{O}_B^3 \rightarrow E \rightarrow 0.$$

Let the inclusion  $\mathcal{O}_B(-s^*K_\Gamma) \rightarrow \mathcal{O}_B^3$  be given by  $(s_0, s_1, s_2)$ , where  $s_i \in H^0(\Gamma, K_\Gamma)$ ,  $i = 0, 1, 2$ , and let  $S$  be the subspace of  $H^0(\Gamma, K_\Gamma)$  spanned by  $s_0, s_1, s_2$ . Notice that  $\dim S \geq 2$ , since  $E$  is torsion free. If  $\dim S = 2$ , then it is clear that  $E = \mathcal{O}_B \oplus \mathcal{O}_B(s^*K_\Gamma)$  and condition ii) above is not satisfied. Thus  $s_0, s_1, s_2$  are independent and sequence (5) is the pull-back via the map  $s$  of the twisted Euler sequence:

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbf{P}^2}^3 \rightarrow T_{\mathbf{P}^2}(-1) \rightarrow 0. \quad \square$$

Now we are ready to finish the proof of Theorem 7.10:

**PROPOSITION 7.16.** *Notation as in Proposition 7.15. The surface  $Z$  is the normalization of the incidence surface  $Y = \{(p, l) \in \Gamma \times (\mathbf{P}^2)^* | p \in l\}$ , and the maps  $p : Z \rightarrow B$  and  $\varphi : Z \rightarrow \mathbf{P}^2$  are induced by the projections of  $Y$  onto  $\Gamma$  and  $(\mathbf{P}^2)^*$  respectively.*

*Let  $f : W \rightarrow \mathbf{P}^2$  be the triple cover obtained from  $\varphi : Z \rightarrow \mathbf{P}^2$  via the discriminant construction,  $h : V \rightarrow W$  the corresponding double cover and  $L = f^* \mathcal{O}_{\mathbf{P}^2}(1)$ . Then:*

- (i) *if  $\Gamma$  is smooth, then  $V = \text{Sym}^2(\Gamma)$ , and  $(h : V \rightarrow W, L)$  is as in Example 3.3;*
- (ii) *if  $\Gamma$  has a double point, write  $p + q$  for the only effective divisor linearly equivalent to  $s^*K_\Gamma \otimes K_B^*$ . Then:*
  - (a)  *$V$  is the blow-up of  $\text{Sym}^2(B)$  at  $p + q$ , namely it is the blow-up of the Jacobian  $J = J(B) = \text{Pic}^2(B)$  of  $B$  at the points corresponding to  $K_B$  and  $p + q$ ;*
  - (b)  *$W$  is obtained as the quotient of  $V$  by the involution which is induced on  $V$  by the birational involution on  $\text{Sym}^2(B)$  which associates to the general divisor  $x + y$  the divisor  $|s^*K_\Gamma - x - y|$ . Notice that  $W$  is the blow-up of the Kummer surface  $\text{Kum}(J)$  at a smooth point;*
  - (c) *the generating pair  $(h : V \rightarrow W, L)$  is obtained from Example 3.1 by a simple blow-up of weight 1.*

PROOF. We keep the notation of the proof of Proposition 7.15. Assume first  $\Gamma$  is smooth. Then the first assertion immediately follows by the well known fact that  $\mathbf{P}(T_{\mathbf{P}^2}(-1))$  is the incidence correspondence inside  $\mathbf{P}^2 \times (\mathbf{P}^2)^*$ . Having in mind Recillas' construction described at the beginning of this section, also part i) immediately follows. The case  $\Gamma$  singular is completely analogous and can be dealt with in the same way. We leave the details to the reader.  $\square$

**8. – The other cases**

In this section we collect some information on pairs that are not good or not of Kodaira dimension 2. We start by classifying non good degree 2 pairs with  $L^2 = 4$ . (We recall that by Propositions 5.5 and 7.3 such a pair always has  $L^2 \leq 4$ .)

PROPOSITION 8.1. *Let  $(h : V \rightarrow W, L)$  be a non good generating pair of degree 2 with  $L^2 = 4$ ; then there exist smooth curves  $C_i, i = 1, 2$ , of genus  $g_i > 0$  and double covers  $\phi_i : C_i \rightarrow \mathbf{P}^1$  such that  $(h : V \rightarrow W, L)$  is obtained by a sequence of inessential blow-ups from a generating pair constructed from  $\phi_i : C_i \rightarrow \mathbf{P}^1$  as in Example 3.5.*

PROOF. By Proposition 6.3 we can assume that the pair is minimal.

Let  $C \in |L|$  be general and let  $C' = h^*C$ . By [Mu], p. 346, we see that the Galois group  $G$  of the composition of  $C' \rightarrow C$  with the hyperelliptic involution on  $C$  can be identified with  $\mathbf{Z}_2 \times \mathbf{Z}_2$ . As in the proof of Lemma 6.8, denote by  $\sigma$  the element of  $G$  such that  $C'/\langle \sigma \rangle = C$  and by  $\sigma_i (i = 1, 2)$  the remaining non trivial elements. For  $i = 1, 2$ , set  $p_i : C' \rightarrow C_i = C'/\langle \sigma_i \rangle$  the corresponding projection and notice that  $C_i$  is a smooth curve of genus  $g_i$ , where  $g_1 + g_2 = g - 1$  and there exists a cartesian diagram:

$$(6) \quad \begin{array}{ccc} C' & \xrightarrow{\pi_2} & C_2 \\ \pi_1 \downarrow & & \downarrow \phi_2 \\ C_1 & \xrightarrow{\phi_1} & \mathbf{P}^1 \end{array}$$

where, for  $i = 1, 2$ ,  $\phi_i : C_i \rightarrow \mathbf{P}^1$  is a double cover. In the present case there is an isomorphism  $\text{Prym}(C', C) = J(C_1) \times J(C_2)$  as principally polarized abelian varieties, and  $A = \text{Alb}(V)$  is also isomorphic to  $\text{Prym}(C', C)$  by Theorem 6.1. We can assume that  $g_1 \leq g_2$  and the condition that  $C'$  is not hyperelliptic ensures that  $g_1 > 0$ . Notice the existence of commutative diagrams:

$$(7) \quad \begin{array}{ccc} C' & \longrightarrow & \text{Prym}(C', C) \\ \pi_i \downarrow & & \downarrow p_i \\ C_i & \longrightarrow & J(C_i) \end{array}$$

where  $C' \rightarrow Prym(C', C)$  is the Abel-Prym map,  $C_i \rightarrow J(C_i)$  is the Abel-Jacobi map and  $p_i : Prym(C', C) = J(C_1) \times J(C_2) \rightarrow J(C_i)$  is the  $i$ -th projection,  $i = 1, 2$ .

As in the proof of Lemma 6.8, one shows that there exist involutions  $\tau_1, \tau_2$  on  $V$  that act on  $C'$  as  $\sigma_1$ , respectively,  $\sigma_2$ . Clearly, the involution  $\iota$  associated to  $h$  is equal to  $\tau_1 \circ \tau_2$ . We denote by  $S_i$  the quotient surface  $V / \langle \tau_i \rangle$ , by  $h_i : V \rightarrow S_i$  the projection onto the quotient and by  $C_i$  the image in  $S_i$  of a general  $C'$ . The singularities of  $S_1$  and  $S_2$ , if any, are  $A_1$  points and  $q(S_i) = g_i$ . More precisely, we claim that  $J(C_i)$  is the Albanese variety of  $S_i$ . Indeed, the map  $V \rightarrow J(C_i)$  obtained by composing the Albanese map of  $V$  with the projection onto  $C_i$  is equivariant with respect to  $\tau_i$ , provided that the base point of the Albanese map is invariant for  $\tau_i$ . Thus we have an induced map  $S_i \rightarrow J(C_i)$  and thus the Albanese variety  $A_i$  of  $S_i$  is isogenous to  $J(C_i)$ . To show that this isogeny is actually an isomorphism, it is enough to remark that the map  $H_1(V, \mathbf{Z}) \rightarrow H_1(J(C_i), \mathbf{Z})$  is surjective, since it is the composition of  $H_1(V, \mathbf{Z}) \rightarrow H_1(A, \mathbf{Z})$ , that is an isomorphism up to torsion, and of  $H_1(A, \mathbf{Z}) \rightarrow H_1(J(C_i), \mathbf{Z})$  which is surjective. On the other hand,  $H_1(V, \mathbf{Z}) \rightarrow H_1(J(C_i), \mathbf{Z})$  is also the composition of  $H_1(V, \mathbf{Z}) \rightarrow H_1(S_i, \mathbf{Z})$  and  $H_1(S_i, \mathbf{Z}) \rightarrow H_1(J(C_i), \mathbf{Z})$ , hence the latter map is surjective and  $A_i$  is isomorphic to  $J(C_i)$ .

We claim that  $S_i$  is birational to  $\mathbf{P}^1 \times C_i$ . Indeed, by Proposition 4.3 the curve  $C_i$  does not vary in moduli and the Albanese image of  $S_i$  is a curve. By Proposition 4.4, this concludes the proof of the claim. In particular the Albanese image of  $S_i$  is the curve  $C_i$ .

Composing  $h_i$  with the Albanese map  $S_i \rightarrow C_i$ , we get morphisms  $f_i : V \rightarrow C_i$ ,  $i = 1, 2$ . Denote by  $F_i$  a fibre of  $f_i$ . The Index theorem applied to  $F_1 + F_2$  and  $L'$  gives  $2(F_1 F_2) L'^2 \leq [L'(F_1 + F_2)]^2 = 16$ , namely  $L'^2 \leq 8$ ,  $L^2 \leq 4$ . If  $L^2 = 4$ , then  $F_1 F_2 = 1$  and  $L'$  is numerically equivalent to  $2F_1 + 2F_2$ . Thus  $f = f_1 \times f_2 : V \rightarrow C_1 \times C_2$  is birational, and therefore it is an isomorphism since  $V$  is minimal. One has:  $\tau_1 = \sigma_1 \times Id$ ,  $\tau_2 = Id \times \sigma_2$ ,  $\iota = \sigma_1 \times \sigma_2$  and the curves of  $|L'|$  are invariant for  $\tau_1, \tau_2$  and it is easy to see that  $(h : V \rightarrow W, L)$  is precisely as in Example 3.5. □

Next we classify pairs of degree 2 and Kodaira dimension 0.

**PROPOSITION 8.2.** *Let  $(h : V \rightarrow W, L)$  be a generating pair of degree 2 and genus  $g$ ; if the Kodaira dimension of the pair is 0, then it can be obtained from Example 3.1 (Beauville's example) by a sequence of simple blow-ups, only three of which at most essential, of weight 1.*

**PROOF.** Assume that the pair has Kodaira dimension 0 and is minimal. By Proposition 5.4 and Corollary 6.4, we see that  $g = 3$  and the irregularity of  $V$  is 2, hence  $V$  is an abelian surface. Since  $q(W) = 0$  by Proposition 5.4,  $W$  is the Kummer surface of  $V$ . By Theorem 6.1, if  $C \in |L|$  is general then  $V$  is isomorphic to  $Prym(C', C)$  and thus, in particular, it is principally polarized. In addition, by Welters criterion,  $C'$  is a divisor of type  $(2, 2)$  and thus we have precisely Example 3.1. By Corollary 6.3, this implies that if the pair is



not minimal, then it is obtained from Example 3.1 by a sequence of blow-ups of weight 0 or 1. Since  $L$  is big by assumption and Example 3.1 has  $L^2 = 4$ , there are at most 3 blow-ups of weight 1 in the sequence.  $\square$

The next result is an almost complete classification of pairs of degree 2 and Kodaira dimension 1.

**PROPOSITION 8.3.** *Let  $(h : V \rightarrow W, L)$  be a generating pair of degree 2 and genus  $g$  with Kodaira dimension 1. Then there exist an elliptic curve  $E$  and an hyperelliptic curve  $B$  of genus  $g - 2 \geq 2$  such that  $(h : V \rightarrow W, L)$  is obtained by a sequence of simple blow-ups of degree 0 or 1 from one of the following:*

- (a) *the pair constructed from  $E$  and  $B$  as in Example 3.5. In this case the pair is not good;*
- (b) *a pair  $(h_0 : V_0 \rightarrow W_0, L_0)$  such that  $g = 4$  (and thus  $B$  has genus 2). In this case  $L^2 = 2$ , and, if the pair is good, then  $h^0(W, L) = 2$ .*

**PROOF.** By Corollary 6.3 we may assume that  $(h : V \rightarrow W, L)$  is minimal. Let  $p : V \rightarrow B$  be the elliptic fibration. The involution  $\iota$  determined by  $h$  on  $V$  preserves the fibration  $V \rightarrow B$ , and, since the quotient of  $V$  by  $\iota$  is regular, it acts on  $B$  as an involution  $j$  with rational quotient. Since the Albanese image of  $V$  is a surface,  $E$  is isogenous to the kernel of  $\text{Alb}(V) \rightarrow J(B)$ , hence  $p$  has constant moduli. By [Se], Section 1,  $V$  is a free quotient of a product  $E \times C$  and, in particular,  $|K_V| = p^*|K_B|$ . Notice that  $B$  has genus  $g - 2 \geq 2$ , since  $\kappa(V) = 1$ .

Denote by  $F$  the general fibre of the pencil  $p_1 : W \rightarrow \mathbf{P}^1 = B / \langle j \rangle$ ;  $F$  is isomorphic to  $E$  and  $p_1$  has  $2g - 2$  double fibres, each containing 4 nodes of  $W$ . Now, with the usual notation, we take  $C \in |L|$  a general curve and  $C' = h^*C$ . Note that  $CF$  is even, since  $p_1$  has double fibres and the general  $C$  contains no singular point of  $W$ . So we set  $CF = 2l$ . The system  $|K_W|$  is equal to  $|(g - 3)F|$ , and thus the adjunction formula on  $V$  gives:

$$4(g - 1) = C'^2 + C'K_V = C'^2 + 4l(g - 3).$$

If  $l = 1$ , then we have  $C'^2 = 8$ , namely  $L^2 = 4$ , and  $p_1$  restricts to a  $g_2^1$  on  $C$ , so that  $C$  is hyperelliptic and the pair is not good. Thus Proposition 8.1 implies that we are in case (a). Then we have  $0 < C'^2 = 4[g(1 - l) + 3l - 1]$ , which leaves us with the only possibility  $l = 2$ ,  $g = 4$ ,  $C'^2 = 4$ , and therefore  $C^2 = 2$  and this is case (b). If the pair is good, then  $h^0(W, L) = 2$  by Proposition 7.5.  $\square$

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