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Boundary Variation for a Neumann Problem

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Abstract. We study the stability of the solution of a two dimensional elliptic problem with Neumann boundary conditions, for geometric domain perturbations in the Hausdorff topology. We prove that the solution is stable if two conditions are satisfied: the number of the connected components of the complement of the variable domain is uniformly bounded and the Lebesgue measure is stable.

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1. - Introduction

Given a ball $B \subseteq \mathbb{R}^2$ and $f \in L^2(B)$, for every open set $\Omega \subseteq B$ we denote by u_Ω the weak variational solution of

$$(1) \qquad \begin{cases} -\Delta u_\Omega + u_\Omega = f & \text{in } \Omega \\ \dfrac{\partial u_\Omega}{\partial n} = 0 & \text{on } \partial\Omega . \end{cases}$$

Denoting $\tilde{u}_\Omega$ the extension by zero of u_Ω to an element of $L^2(B)$, we study the continuity of the mapping $\Omega \mapsto \tilde{u}_\Omega \in L^2(B)$, if the family of domains is endowed with the Hausdorff complementary topology (see [8] and Section 2 for the exact definition).

Even if this topology does not have much in common with problem (1), there are two main reasons to use it: the family of all open subsets of B is compact, and the family of admissible perturbations is quite large; in particular, it contains a large class of non-smooth perturbations which can change the topology of the domain. Of course, a simple counter example proves that without any constraint the continuity of the mapping $\Omega \mapsto \tilde{u}_\Omega \in L^2(B)$ does not hold in this topology (see for example [10]). The Hausdorff topology is also used to describe the behavior of the solution of a Dirichlet problem associated to the Laplace operator in terms of the geometric domain variation. We refer

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to the paper of Sverak [21] where the 2-dimensional case is treated, and to [8] for an extension in N-dimensions.

When dealing with moving boundaries, is more difficult to handle the Neumann boundary conditions than the Dirichlet conditions, since, in general, there do not exist extension operators if the boundaries are not smooth (as for example domains with cracks). This is one reason for which a suitable relaxation result for the Neumann problem is difficult to obtain in the general frame. We refer to the paper of Chenais [12], where the continuity of the mapping $\Omega \mapsto \tilde{u}_\Omega \in L^2(B)$ is obtained, provided that all Ω_n satisfy a uniform geometric condition (called uniform cone condition). The proof is based on the existence of extension operators with uniform norms, which is a consequence of the geometric constraints.

By means of representation theorems and Γ-convergence tools, in [13] is found the equation satisfied by the weak limit of a sequence of solutions of Neumann problems for domains having the boundary converging in the Hausdorff sense to some set contained in a smooth, a priori given, manifold. In [14], [18] the authors study the limit of a sequence of solutions for a a periodical structure by homogenization techniques. We also refer to [2], [3], [17] for several results of relaxation and for description of some non-local effects. We refer to [7] for a stability result obtained under some suitable capacitary constraints on the moving boundary and under the hypothesis that the moving boundary lies in a fixed smooth manifold.

The departure point for our paper is the result of Chambolle and Doveri [10]. There is proved the following two dimensional result: if Ω_n converges in the Hausdorff complementary topology (simply denoted H^c) to some Ω such that the number of connected components of $\partial\Omega_n$ and the length of the boundaries $\mathcal{H}^1(\partial\Omega_n)$ are uniformly bounded, then $\tilde{u}_{\Omega_n}$ converges to $\tilde{u}_\Omega$. The condition that the $\mathcal{H}^1$-measure of the boundary has to be finite, along with the connectivity assumption, might be seen as a regularity assumption on $\partial\Omega_n$ through the representation theorem of a continuum.

In this paper, we find a necessary and sufficient condition for the continuity of the mapping $\Omega \mapsto \tilde{u}_\Omega \in L^2(B)$, for a sequence converging in the Hausdorff complementary topology, and having the number of the connected components of the complementaries uniformly bounded. From this point of view, this result might be seen as a "dual" of the the result of Sverak. More precisely, we prove that if Ω_n converges in the Hausdorff complementary topology to Ω such that the number of connected components of the complementaries is uniformly bounded, then for every $f \in L^2(B)$ one has $\tilde{u}_{\Omega_n} \to \tilde{u}_\Omega$ if and only if $|\Omega_n| \to |\Omega|$, where $|\cdot|$ denotes the Lebesgue measure. The key argument is based on a topological property of the H^c-convergence in relation with the capacity. It can be easily seen that if the number of connected components of $\partial\Omega_n$ and the length of the boundaries $\mathcal{H}^1(\partial\Omega_n)$ are uniformly bounded, then the number of connected components of Ω_n^c is uniformly bounded and the Lebesgue measure is stable for the Hausdorff complementary convergence.

We apply this result to prove existence of an optimal domain which minimizes a shape functional depending on u_Ω. Using the density perimeter [9], we identify some compact classes for the H^c-topology with the property that if $\Omega_n \xrightarrow{H^c} \Omega$ then $|\Omega_n| \to |\Omega|$. Existence results were already been given for the minimization of energy type functionals like, for example, the Mumford-Shah functional. Using the results of this paper, we can prove existence of a solution for a larger class of functionals which are not necessarily of energy type, hence not "min-min" problems. For example, given $g \in L^2(B)$ we can prove existence of an optimal domain which minimizes the functional

$$\Omega \mapsto \int_\Omega (u_\Omega - g)^2 dx + \sharp(\Omega^c) + P(\partial\Omega),$$

where u_Ω is the solution of problem (1), $\sharp(\Omega^c)$ denotes the number of the connected components of $\mathbb{R}^2 \setminus \Omega$, and P denotes the density perimeter (see the exact definition in the last section).

2. – General considerations

In this section are recalled the main notations and tools used in the paper.

Let us denote by $B = B_{0,R}$ the ball centered in the origin of radius R in $\mathbb{R}^2$. The family of open subsets of B is denoted $\mathcal{O}(B)$ and is endowed with the Hausdorff complementary topology [8] given by the metric

$$d_{H^c}(\Omega_1, \Omega_2) = d_H(\overline{B} \setminus \Omega_1, \overline{B} \setminus \Omega_2),$$

where

$$d_H(K_1, K_2) = \max\left\{ \sup_{x\in K_1} \inf_{y\in K_2} |x-y|, \sup_{y\in K_2} \inf_{x\in K_1} |x-y| \right\}$$

is the Hausdorff distance between two closed sets. The classical distance between two sets of $\mathbb{R}^2$ is denoted by d and given by

$$d(A_1, A_2) = \inf_{(x_1,x_2)\in A_1\times A_2} d(x_1, x_2).$$

For every $l \in \mathbb{N}$, we set $\mathcal{O}_l(B) = \{\Omega \in \mathcal{O}(B) : \sharp(\Omega^c) \le l\}$, where $\sharp(\Omega^c)$ is the number of connected components of $\mathbb{R}^2 \setminus \Omega$.

It is said that Ω_n converges in the sense of the measure to Ω, and written $\Omega_n \xrightarrow{L^1} \Omega$, if the characteristic functions converge strongly in L^1, i.e. $\chi_{\Omega_n} \xrightarrow{L^1(B)} \chi_\Omega$. By $|\cdot|$ we denote the Lebesgue measure in $\mathbb{R}^2$.

Recall that $\mathcal{O}(B)$ and $\mathcal{O}_l(B)$ are compact in the H^c-topology. Moreover, if $\Omega_n \xrightarrow{H^c} \Omega$ then a.e. $\chi_\Omega \le \liminf_{n\to\infty} \chi_{\Omega_n}$ and $\lim_{n\to\infty} |\Omega \setminus \Omega_n| = 0$. If

$\Omega_n \xrightarrow{H^c} \Omega$, then for any $K \subset\subset \Omega$, there exists $n_K \in \mathbb{N}$ such that $\forall n \geq n_K$ we have $K \subseteq \Omega_n$, and for every $x \in \partial\Omega$, there exists $x_n \in \partial\Omega_n$ such that $x_n \to x$.

An open set $\Omega \in \mathcal{O}(B)$ is simply connected, if $\overline{B} \setminus \Omega$ is connected, i.e. $\Omega \in \mathcal{O}_1(B)$.

DEFINITION 2.1. Let $x_0 \in \Omega$ be fixed and $\{\Omega_n\}_{n\in\mathbb{N}}$, Ω open connected and simply connected subsets of B such that $\forall n \geq n(x_0)$, $x_0 \in \Omega_n$. It is said that Ω_n converges to Ω in the sense of the kernel with respect to x_0 if

- For every $x \in \Omega$, there exists a neighborhood U of x such that for every $n \geq n(U)$ we have $U \subseteq \Omega_n$.
- For every $x \in \partial\Omega$, there exists $x_n \in \partial\Omega_n$ such that $x_n \to x$.

Let $\mathbb{D}$ be the unit disk of $\mathbb{R}^2$ and $g_n : \mathbb{D} \to \Omega_n$, $g : \mathbb{D} \to \Omega_n$ be the conformal mappings such that $g_n(0) = g(0) = x_0$, $g_n'(0) > 0, g'(0) > 0$. Following [19], g_n converges to g locally uniformly on $\mathbb{D}$ if Ω_n converges in the sense of the kernel to Ω with respect to x_0.

If Ω_n and Ω are connected and simply connected and Ω_n converges in H^c to Ω, then Ω_n converges in the sense of kernel to Ω for every $x_0 \in \Omega$. The following property of the H^c convergence can be easily proven.

PROPOSITION 2.2. *If $\{\Omega_n^1\}_{n\in\mathbb{N}}$ $\{\Omega_n^2\}_{n\in\mathbb{N}}$ are two sequences in $\mathcal{O}(B)$ such that for every $n \in \mathbb{N}$, $\Omega_n^1 \cap \Omega_n^2 = \emptyset$ and $\Omega_n^1 \xrightarrow{H^c} \Omega^1$ and $\Omega_n^2 \xrightarrow{H^c} \Omega^2$, then $\Omega_n^1 \cup \Omega_n^2$ converges in H^c to $\Omega^1 \cup \Omega^2$.*

The weak solution of problem (1) is the unique function $u_\Omega \in H^1(\Omega)$ satisfying

$$\int_\Omega (\nabla u_\Omega \nabla v + u_\Omega v)dx = \int_\Omega f v dx \;\; \forall v \in H^1(\Omega). \tag{2}$$

In order to compare the solution of problem (1) on two different domains, all functions of $H^1(\Omega)$ are extended by zero to elements of $L^2(B)$ as well as their gradients. Hence, for every $u \in H^1(\Omega)$ we denote $\tilde{u}$ an element of $L^2(B)$, defined as $\tilde{u}(x) = u(x)$ if $x \in \Omega$ and $\tilde{u}(x) = 0$ if $x \in B \setminus \Omega$. The gradient of u is extended in the same way to an element of $L^2(B, \mathbb{R}^2)$. We write $\tilde{\nabla} u(x) = \nabla u(x)$ if $x \in \Omega$ and $\tilde{\nabla} u(x) = 0$ if $x \in B \setminus \Omega$. In this way, $H^1(\Omega)$ can be seen as a closed subspace in $L^2(B) \times L^2(B, \mathbb{R}^2)$.

Given a sequence of elements $\{\Omega_n\}_{n\in\mathbb{N}}$ in $\mathcal{O}(B)$, it is said that $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$ if

M_1) For all $\phi \in H^1(\Omega)$ there exists a sequence $\phi_n \in H^1(\Omega_n)$ such that $\tilde{\phi}_n$ converges strongly in $L^2(B)$ to $\tilde{\phi}$ and $\tilde{\nabla}\phi_n$ converges strongly in $L^2(B, \mathbb{R}^2)$ to $\tilde{\nabla}\phi$;

M_2) For every sequence $k \mapsto \phi_k \in H^1(\Omega_{n_k})$ such that $(\tilde{\phi}_k, \tilde{\nabla}\phi_k)$ is weakly convergent in $L^2(B) \times L^2(B, \mathbb{R}^2)$ to (u, v_1, v_2) we have that $(u, v_1, v_2) = 0$ a.e. $B \setminus \Omega$ and $\nabla u = (v_1, v_2)$ in Ω, this equality being understood in the sense of distributions.

In order to simplify notations, for $u_n \in H^1(\Omega_n)$ and $u \in H^1(\Omega)$, we write $u_n \rightsquigarrow u$ instead of $(\tilde{u}_n, \tilde{\nabla} u_n) \stackrel{L^2(B)\times L^2(B,\mathbb{R}^2)}{\longrightarrow} (\tilde{u}, \tilde{\nabla} u)$.

The capacity of a set $E \subseteq \mathbb{R}^2$ is defined by

$$\mathrm{Cap}(E) = \inf\left\{\int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2 dx, \quad u \in \mathcal{U}_E\right\}$$

where $\mathcal{U}_E$ is the class of all functions $u \in H^1(\mathbb{R}^2)$ such that $u \geq 1$ a.e. in a neighborhood of E. We say that a property $p(x)$ holds quasi everywhere on E (shortly q.e. on E) if the set of all points $x \in E$ for which $p(x)$ does not hold has capacity zero. We refer to [16] for further details concerning capacity.

LEMMA 2.3. *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ and $\{\tilde{\Omega}_n\}_{n\in\mathbb{N}}$ be two sequences of $\mathcal{O}(B)$ such that for all $n \in \mathbb{N}$, $\tilde{\Omega}_n \subset \Omega_n$ and such that $\lim_{n\to\infty} \mathrm{Cap}(\overline{\Omega_n \setminus \tilde{\Omega}_n}) = 0$. If the first Mosco condition holds for the Sobolev spaces associated to $\{\tilde{\Omega}_n\}_{n\in\mathbb{N}}$ and Ω, then it also holds for the Sobolev spaces associated to $\{\Omega_n\}_{n\in\mathbb{N}}$ and Ω.*

PROOF. Let us suppose in a first step that $u \in H^1(\Omega) \cap L^\infty(\Omega)$, $M = \|u\|_{L^\infty(\Omega)}$. Let $\phi_n \in H^1(\tilde{\Omega}_n)$ such that $\phi_n \rightsquigarrow u$. Using the sequence $\{\phi_n\}_{n\in\mathbb{N}}$, we will construct a sequence $\{u_n \in H^1(\Omega_n)\}_{n\in\mathbb{N}}$ such that $u_n \rightsquigarrow u$. Let $v_n = -(M \vee \phi_n) \wedge M$. Obviously, v_n belongs to $H^1(\tilde{\Omega}_n) \cap L^\infty(\tilde{\Omega}_n)$ and $v_n \rightsquigarrow u$.

There exists a function $w_n \in C_0^\infty(\mathbb{R}^2)$ such that $0 \leq w_n \leq 1$, $w_n = 1$ a.e. on a neighborhood of $\overline{\Omega_n \setminus \tilde{\Omega}_n}$ and $\int_{\mathbb{R}^2} |\nabla w_n|^2 dx + \int_{\mathbb{R}^2} w_n^2 dx \leq \mathrm{Cap}(\overline{\Omega_n \setminus \tilde{\Omega}_n}) + \frac{1}{n}$. We define $u_n(x) = (1 - w_n(x))v_n(x)$ for $x \in \tilde{\Omega}_n$ and $u_n(x) = 0$ for $x \in \Omega_n \setminus \tilde{\Omega}_n$. It is easy to verify that $u_n \in H^1(\Omega_n)$ and that $\nabla u_n(x) = \nabla[(1 - w_n(x))v_n](x)$ for $x \in \tilde{\Omega}_n$ and $\nabla u_n(x) = 0$ for a.e. $x \in \Omega_n \setminus \tilde{\Omega}_n$.

Let us prove that $u_n \rightsquigarrow u$. We have

$$\|u_n - v_n\|_{L^2(B)}^2 \leq M \int_B w_n^2 dx \to 0 \text{ for } n \to \infty .$$

On the other side

$$\int_B |\tilde{\nabla} u_n - \tilde{\nabla} v_n|^2 dx = \int_{\tilde{\Omega}_n} |\nabla(w_n v_n)|^2 dx$$
$$\leq 2 \int_{\tilde{\Omega}_n} w_n^2 |\nabla v_n|^2 dx + 2M \int_B |\nabla w_n|^2 dx \to 0 \text{ for } n \to \infty .$$

Since $H^1(\Omega) \cap L^\infty(\Omega)$ is dense in $H^1(\Omega)$, the first Mosco condition holds. □

3. – Continuity with respect to the geometric domain variation

The main result of the paper can be formulated as follows.

THEOREM 3.1. *Let $l \in \mathbb{N}$ be fixed, and $\Omega_n, \Omega \in \mathcal{O}_l(B)$ be such that $\Omega_n \xrightarrow{H^c} \Omega$. Then for every $f \in L^2(B)$ we have $u_{\Omega_n} \rightsquigarrow u_\Omega$ if and only if $|\Omega_n| \to |\Omega|$.*

The proof of this theorem is divided in three steps. In the first step we assume that all the open sets of the sequence Ω_n and the limit set Ω are connected and simply connected and we use conformal mappings for proving the convergence in the sense of Mosco of the associated Sobolev spaces. In the second step, we assume only Ω_n to be connected and simply connected, while Ω is arbitrary. In this step, by a geometrical lemma and several arguments relating the convergence of domains to the capacity, we reduce the problem to the first case. In the last step, by a localization procedure, we recover the theorem in the general case.

The convergence in the sense of Mosco of the Sobolev spaces is the main tool we use to obtain continuity. The relation between the continuity with respect to the geometric domain variation for problem (1) and the convergence in the sense of Mosco of the Sobolev spaces is given in the following proposition, which we prove only for the sake of completeness.

PROPOSITION 3.2. *Let $\Omega_n, \Omega \in \mathcal{O}(B)$ such that $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$. Then for every $f \in L^2(B)$ we have that $u_{\Omega_n} \rightsquigarrow u_\Omega$.*

PROOF. For the simplicity of the notation, we set $u_n = u_{\Omega_n}$ and $u = u_\Omega$. Taking u_n as test function in equation (1) on Ω_n, by the Cauchy-Schwartz inequality we obtain that $(\tilde{u}_n, \tilde{\nabla} u_n)$ are uniformly bounded in $L^2(B, \mathbb{R}^3)$. Hence, there exists a subsequence (still denoted with the same index) which weakly converges to (v, v_1, v_2) in $L^2(B, \mathbb{R}^3)$. From the second Mosco condition we get that $(v, v_1, v_2) = 0$ a.e. in Ω^c and that $\nabla v = (v_1, v_2)$ on Ω in the sense of distributions. Hence $v_{|\Omega} \in H^1(\Omega)$. In order to prove that $v = u$ it is sufficient to verify that v satisfies the equation on Ω. Let us consider an element $\phi \in H^1(\Omega)$. From the first Mosco condition, there exists $\phi_n \in H^1(\Omega_n)$ such that $\phi_n \rightsquigarrow \phi$. Writing (2) on Ω_n with ϕ_n as test function, and extending the integrals by zero on $B \setminus \Omega_n$ we have

$$\int_B (\tilde{\nabla} u_n \tilde{\nabla} \phi_n + \tilde{u}_n \tilde{\phi}_n) dx = \int_B f \tilde{\phi}_n dx .$$

Making $n \to \infty$ we get

$$\int_B (\tilde{\nabla} v \tilde{\nabla} \phi + \tilde{v} \tilde{\phi}) dx = \int_B f \tilde{\phi} dx ,$$

hence v is solution for (1) on Ω. Consequently we can write $v = \tilde{u}$.

In order to prove the strong convergence, we only remark that

$$\int_B (|\tilde{\nabla} u_n|^2 + |\tilde{u}_n|^2) dx = \int_B f \tilde{u}_n dx \to \int_B f \tilde{u} dx = \int_B (|\tilde{\nabla} u|^2 + |\tilde{u}|^2) dx .$$

From the uniqueness of the solution on Ω we get that the whole sequence $\{u_n\}_{n \in \mathbb{N}}$ converges strongly to u in the sense $u_n \rightsquigarrow u$. □

REMARK 3.3. Given $\{\Omega_n\}_{n\in\mathbb{N}}$ and Ω in $\mathcal{O}(B)$ such that for every $f \in L^2(B)$ we have $u_{\Omega_n} \rightsquigarrow u_\Omega$, it is not clear if $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$. If Ω_n converges in H^c to Ω, then the converse of Proposition 3.2 follows immediately.

Remark also that if Ω_n converges in H^c to Ω, then Ω_n converges in measure to Ω if and only if $|\Omega_n| \to |\Omega|$. This is an easy consequence of the lower semi continuity of the measure in the H^c-topology. If Ω_n converges in both H^c and in the sense of measure to Ω, then the second Mosco condition is satisfied. Indeed, let $\phi_k \in H^1(\Omega_{n_k})$ such that $(\tilde{\phi}_k, \tilde{\nabla}\phi_k)$ is weakly convergent in $L^2(B) \times L^2(B, \mathbb{R}^2)$ to (u, v_1, v_2). From the convergence in measure of Ω_{n_k} to Ω we get that $(u, v_1, v_2) = 0$ a.e. $B \setminus \Omega$. On the other side from the H^c-convergence, for every $\varphi \in C_0^\infty(\Omega)$, there exists $k \in \mathbb{N}$ large enough such that $\varphi \in C_0^\infty(\Omega_{n_k})$. Then we can write

$$\int_{\Omega_{n_k}} \frac{\partial \phi_k}{\partial x_i} \varphi dx = -\int_{\Omega_{n_k}} \frac{\partial \varphi}{\partial x_i} \phi_k dx .$$

Making $k \to \infty$ and using the weak convergences, we find

$$\int_\Omega v_i \varphi dx = -\int_\Omega \frac{\partial \varphi}{\partial x_i} u dx ,$$

i.e. $\nabla u = (v_1, v_2)$ in Ω.

The first Mosco condition does not hold, in general, for a sequence of domains converging in H^c and in measure. This is the reason for which the constraint on the number of connected components has to be imposed in Theorem 3.1.

3.1. – Connected and simply connected sets

Let us suppose that every Ω_n and the limit set Ω are connected and simply connected. Since it is sufficient to prove the continuity for a subsequence, without restricting the generality we can suppose that $\chi_{\Omega_n}(x) \to \chi_\Omega(x)$ a.e. $x \in B$. Fixing a point $x_0 \in \Omega$, we have that Ω_n converges in the sense of the kernel to Ω with respect to x_0. Let us denote by g_n, g the conformal mappings $g_n : \mathbb{D} \longrightarrow \Omega_n, g : \mathbb{D} \longrightarrow \Omega$ such that $g_n(0) = g(0) = x_0$, $g_n'(0) > 0, g'(0) > 0$. In order to prove the continuity result of Theorem 3.1 in this particular case, it is sufficient to prove the first Mosco condition. This is contained in the following lemma.

LEMMA 3.4. *Under the previous hypotheses, for every $\phi \in H^1(\Omega)$ there exists a sequence $\phi_n \in H^1(\Omega_n)$ such that $\phi_n \rightsquigarrow \phi$.*

PROOF. Let us consider a function $\phi \in H^1(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega)$. We simply define on Ω_n the function $\phi_n(x) = \phi \circ g \circ g_n^{-1}(x)$ and prove in the sequel that $\phi_n \rightsquigarrow \phi$. Classical properties of the conformal mappings give

$\phi_n \in H^1(\Omega_n) \cap L^\infty(\Omega_n)$. Moreover, $\int_{\Omega_n} |\nabla\phi_n|^2 dx = \int_\Omega |\nabla\phi|^2 dx$ and $\|\phi_n\|_{L^\infty} = \|\phi\|_{L^\infty}$.

We have $\tilde{\phi}_n(x) \to \tilde{\phi}(x)$ a.e. $x \in B$. Indeed, for a.e. $x \in B \setminus \Omega$, we have $\chi_\Omega(x) = 0$ and for n large enough $\chi_{\Omega_n}(x) = 0$. Hence, $\tilde{\phi}(x) = 0$ and $\tilde{\phi}_n(x) = 0$, for n large enough. For $x \in \Omega$, we prove in the sequel that $\phi_n(x) \to \phi(x)$, i.e. $\phi \circ g \circ g_n^{-1}(x) \to \phi(x)$. Since ϕ is continuous in x, it is sufficient to prove that $g \circ g_n^{-1}(x) \to x$, or equivalently

$$|g \circ g_n^{-1}(x) - g_n \circ g_n^{-1}(x)| \to 0. \tag{3}$$

From the Hausdorff convergence, for n large enough we have $x \in \Omega_n$, hence $g_n^{-1}(x)$ is well defined. On the other side, g_n converges uniformly to g on compact sets in $\mathbb{D}$. Is therefore sufficient to prove the existence of some $1 > r > 0$ such that $g_n^{-1}(x) \in D_r$, where D_r is the disk centered in 0 with ray equal to r. Let us denote $z \in \mathbb{D}$ the point such that $g(z) = x$, and take $1 > r > 0$ such that $z \in D_r$. Then $g_n(\partial D_r) = \partial g_n(D_r)$ converges in the sense of Hausdorff to $g(\partial D_r) = \partial g(D_r)$. Therefore, there exists $\rho > 0$ such that for all $n \geq n_\rho$ we have $B(x,\rho) \cap \partial D_r = \emptyset$. On the other hand, $g_n(z) \to g(z) = x$, hence $B(x,\rho) \cap g_n(D_r) \neq \emptyset$ for n large enough, hence $B(x,\rho) \subseteq g_n(D_r)$, for n large enough. Finally, $g_n^{-1}(x) \in D_r$, hence (3) holds.

Since for a.e. $x \in B$ $\tilde{\phi}_n(x) \to \tilde{\phi}(x)$ and ϕ_n, ϕ are uniformly bounded in $L^\infty(B)$, the Lebesgue dominated convergence theorem gives that $\tilde{\phi}_n \stackrel{L^2(B)}{\longrightarrow} \tilde{\phi}$. Since $\int_{\Omega_n} |\nabla\phi_n|^2 dx = \int_\Omega |\nabla\phi|^2 dx$, we deduce immediately that $\phi_n \rightsquigarrow \phi$.

Since $H^1(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega)$ is dense in $H^1(\Omega)$, by a classical diagonal procedure the first Mosco condition follows. □

3.2. – Simply connected sets converging to an arbitrary set

In this subsection, we suppose that every open set Ω_n is connected and simply connected, but the limit set Ω is arbitrary. In fact, Ω should also be simply connected, as a consequence of the H^c-convergence, but may have more than one connected component. In this case, the argument used in Lemma 3.4 fails to work. It is exactly in this point, that Chambolle and Doveri [10] had to impose the constraints on the number of the connected components and on the $\mathcal{H}^1$-measure of the boundaries, in order to describe the structure of the limit of the Jacobians of the conformal mappings. In the sequel, we give a geometrical lemma which will allow us to reduce this case to the previous one. This is the key result of the paper.

A Lipschitz continuous mapping $\gamma : [\alpha, \beta] \subset \mathbb{R} \mapsto \mathbb{R}^2$ is called curve. If $\gamma(\alpha) = x, \gamma(\beta) = y$, we denote $\gamma_{x,y}$ the range of γ. A curve is called simple if γ is injective. For any $\delta > 0$ we denote by

$$C(x,\delta) = \left\{ y \in \mathbb{R}^2 : |y_1 - x_1| \leq \frac{\delta}{2},\ |y_2 - x_2| \leq \frac{\delta}{2} \right\}$$

a closed square centered in x of the side of length equal to δ.

Let us make the following hypotheses. Suppose that $\Omega = \Omega_a \cup \Omega_b$ with Ω_a, Ω_b open and disjoint. Let ω_a and ω_b be smooth connected sets such that $\overline{\omega}_a \subset \Omega_a$ and $\overline{\omega}_b \subset \Omega_b$. Let $\varepsilon < \frac{1}{10}\min\{d(\omega_a, \partial\Omega_a), d(\omega_b, \partial\Omega_b)\}$, and U a simply connected set such that $d_{H^c}(U, \Omega) < \varepsilon$.

LEMMA 3.5. *Under the previous hypotheses, there exists a point $x \in \overline{B} \setminus \Omega$ such that $C(x, 9\varepsilon)$ intersects any curve γ contained in U and joining a point of ω_a to a point of ω_b.*

PROOF. Let ε and U satisfying the hypotheses. For every $x \in \overline{B} \setminus \Omega$ we have $C(x, 9\varepsilon) \cap (\omega_a \cup \omega_b) = \emptyset$ and $\omega_a \cup \omega_b \subset U$. If ω_a and ω_b lie in different connected components of U, the conclusion of the proposition follows directly since there are no curves in U joining ω_a to ω_b. Suppose that ω_a and ω_b belong to the same connected component of U. Let us fix $x_a \in \omega_a$ and $x_b \in \omega_b$. From the connectedness of ω_a and ω_b and the fact that $x \in \overline{B} \setminus \Omega$ and $\varepsilon < \frac{1}{10}\min\{d(\omega_a, \partial\Omega_a), d(\omega_b, \partial\Omega_b)\}$, it is sufficient to prove the assertion of the proposition only for curves joining x_a and x_b.

From the compactness of $\overline{B} \setminus \Omega$, there exists a finite family $\mathcal{F}$ of squares $C(x, 3\varepsilon)$ centered in points of $\overline{B} \setminus \Omega$ which covers $\overline{B} \setminus \Omega$. Since ω_a and ω_b are contained in different connected components of Ω, any curve joining x_a to x_b in U intersects at least one square of $\mathcal{F}$.

We construct an *essential family* of squares denoted $\mathcal{F}_{\text{ess}} \subseteq \mathcal{F}$ with the following properties: any curve γ_{x_a,x_b} contained in U intersects at least one element of $\mathcal{F}_{\text{ess}}$ and for any square of $\mathcal{F}_{\text{ess}}$ there exists at least one curve $\gamma_{x_a,x_b} \subseteq U$ which intersects this square and only this one from the family $\mathcal{F}_{\text{ess}}$.

In order to construct $\mathcal{F}_{\text{ess}}$ we define

$$\mathcal{F}_0 = \{C \in \mathcal{F} :\ \exists\, \gamma_{x_a,x_b} \subseteq U \quad \text{such that} \quad \gamma_{x_a,x_b} \cap C \neq \emptyset\}.$$

Since $\mathcal{F}_0$ is finite, we can write $\mathcal{F}_0 = \{C_j;\ j = 1, \dots, t\}$. We construct the sequence of sub families $\{\mathcal{F}_p\}_{p=0,\dots,t}$ in the following way. For $1 \le p \le t$, we set

$$\mathcal{F}_p = \mathcal{F}_{p-1} \setminus \{C_p\}$$

if for every γ_{x_a,x_b} such that $\gamma_{x_a,x_b} \cap C_p \neq \emptyset$, there exists $C_j \in \mathcal{F}_{p-1}$, $j \neq p$ such that $\gamma_{x_a,x_b} \cap C_j \neq \emptyset$. If not, we set $\mathcal{F}_p = \mathcal{F}_{p-1}$. Finally, we observe that $\mathcal{F}_t$ has the desired properties, and set $\mathcal{F}_{\text{ess}} = \mathcal{F}_t$.

We prove in the sequel that any two elements of $\mathcal{F}_{\text{ess}}$ have non empty intersection. Suppose for contradiction that $C_1, C_2 \in \mathcal{F}_{\text{ess}}$, and $C_1 \cap C_2 \neq \emptyset$. By the definition of $\mathcal{F}_{\text{ess}}$, there exist $\gamma_1 : [0, 1] \mapsto \mathbb{R}^2$ and $\gamma_2 : [0, 1] \mapsto \mathbb{R}^2$ two curves joining x_a to x_b in U such that γ_1 and γ_2, intersects only C_1 and C_2, respectively, from the family $\mathcal{F}_{\text{ess}}$. Since $\mathcal{F}_{\text{ess}}$ is finite, we can suppose that the curves are simple and consist of a union of segments parallels to the bisectrices and to the axes, respectively. Their intersection is therefore a finite set of points containing x_a and x_b. We can write

$$\gamma_1 \cap \gamma_2 = \{\gamma_1(0), \gamma_1(t_1), \dots, \gamma_1(t_{m-1}), \gamma_1(1)\} \text{ with } 0 = t_0 < \dots < t_i < \dots < t_m = 1.$$

For every $i = 0, 1, \dots, m-1$ we shall construct a simple curve γ_i joining $\gamma_1(t_i)$ to $\gamma_1(t_{i+1})$ such that

- γ_i is contained in U,
- γ_i does not intersect any square of $\mathcal{F}_{\text{ess}}$.

This contradicts the construction of $\mathcal{F}_{\text{ess}}$, since putting together the curves γ_i for $i = 0, 1, \dots, m-1$, we obtain a curve contained in U, joining x_a to x_b and which does not intersect any square of $\mathcal{F}_{\text{ess}}$.

Let i between 0 and $m-1$ be fixed. There exists $\alpha, \beta \in [0, 1]^2$ such that $\gamma_2(\alpha) = \gamma_1(t_i)$ and $\gamma_2(\beta) = \gamma_1(t_{i+1})$. Without restricting the generality, we can suppose that $\alpha < \beta$ (the case $\alpha > \beta$ is treated identically). Let $\Gamma = \{\gamma_1(t), : t \in [t_i, t_{i+1}]\} \cup \{\gamma_2(t), : t \in [\alpha, \beta]\}$, then Γ is a Jordan curve (without self intersections) consisting of a finite union of segments. Then $\mathbb{R}^2 \setminus \Gamma$ is the union of two connected components. We denote by Γ_{int} the bounded connected component, which has Γ as boundary [11]. Since U is simply connected, we have $\Gamma_{\text{int}} \subset U$.

Two situations may occur. If for $t \in [t_i, t_{i+1}]$, $\gamma_1(t)$ does not intersect the square C_1, we define γ_i as γ_1 restricted to $[t_i, t_{i+1}]$. If there exists $t \in [t_i, t_{i+1}]$ such that $\gamma_1(t) \in C_1$, we construct γ_i as follows. Let $d = \min\{d(C_1, C_2), d(\gamma_1, C_2), d(\gamma_2, C_1)\} > 0$. Let C_1' be the square of size $3\varepsilon + \frac{d}{2}$ having the same center as C_1. Then C_1' does not intersect neither C_2 nor γ_2. Then $\Gamma_{\text{int}} \setminus C_1'$ is a simply connected set with the boundary contained in $\Gamma \cup \partial C_1'$. Moreover $\gamma_2([\alpha, \beta])$ is contained in the boundary of one connected component having nonempty intersection with Γ_{int}. Let us denote this connected component Γ_{int}' and take for γ_i, $\partial \Gamma_{\text{int}}' \setminus \{\gamma_2([\alpha, \beta])\}$. Then, γ_i joins $\gamma_1(t_i)$ to $\gamma_1(t_{i+1})$, does not intersect C_1 and C_2, and is contained in $\Gamma_{\text{int}} \cup \Gamma$. Moreover, it does not intersect any other square of $\mathcal{F}_{\text{ess}}$, if not this square would be contained in Γ_{int}, hence in U, which is impossible from the fact that $d_{H^c}(U, \Omega) < \varepsilon$.

To finish the proof, we notice that since any two squares of $\mathcal{F}_{\text{ess}}$ have a nonempty intersection, there exists a square centered in a point of $\overline{B} \setminus \Omega$ and of side of length 9ε which contains all the squares of $\mathcal{F}_{\text{ess}}$. □

Let us suppose that $\{\Omega_n\}_{n \in \mathbb{N}}$ is a sequence of simply connected sets converging in H^c and in measure to Ω. Suppose that Ω is decomposed in its connected components $\Omega = \bigcup_{i=1}^{\infty} C_i$ (from a rank on, all components might be empty). We prove the following lemma.

LEMMA 3.6. *There exists a sequence $\{\tilde{\Omega}_n\}_{n \in \mathbb{N}}$ of simply connected sets, such that for every $n \in \mathbb{N}$ we have $\tilde{\Omega}_n \subset \Omega_n$ and $\tilde{\Omega}_n = \Omega_n^1 \cup R_n$, where Ω_n^1 and R_n are disjoint open sets such that Ω_n^1 is connected and*

1. $\lim_{n \to \infty} \text{Cap}(\overline{\Omega_n \setminus \tilde{\Omega}_n}) = 0$;
2. $\tilde{\Omega}_n \overset{H^c, L^1}{\longrightarrow} \Omega$ *for* $n \to \infty$;
3. $\Omega_n^1 \overset{H^c, L^1}{\longrightarrow} C_1$ *for* $n \to \infty$.

PROOF. Let us construct the sequence $\{\tilde{\Omega}_n\}_{n \in \mathbb{N}}$. For every $i \in \mathbb{N}$, let us consider the sequence $\{U_k^i\}_{k \in \mathbb{N}}$, such that $C_i = \cup_{k \in \mathbb{N}} U_k^i$, U_k^i are connected and smooth and $U_k^i \subset U_{k+1}^i$. We set $n_0 = 1$, and for $k \geq 1$ we note

$$\delta_k = \min_{1 \leq i \leq k} \{d(U_k^i, \partial C_i)\}.$$

We consider $\varepsilon_k > 0$ such that $\varepsilon_k \leq \frac{\delta_k}{10}$ and $\mathrm{Cap}(C(0, 9\varepsilon_k)) \leq \frac{1}{k^2}$. Let n_k the smallest index of the sequence $\{\Omega_n\}_{n \in \mathbb{N}}$ greater or equal than n_{k-1} such that for every $n \geq n_k$, we have $d_{H^c}(\Omega_n, \Omega) \leq \varepsilon_k$. For any couple (U_k^1, U_k^i), $i = 2, \dots, k$ we apply Lemma 3.5 and find the squares $C(x_i^k, 9\varepsilon_k)$. We construct then the sequence $\{\tilde{\Omega}_n\}_{n \in \mathbb{N}}$ as follows:

- for $n < n_1$, $\tilde{\Omega}_n = \Omega_n$
- for $n_k \leq n < n_{k+1}$, $\tilde{\Omega}_n = \Omega_n \setminus (\overline{\bigcup_{i=2}^k C(x_i^k, 9\varepsilon_k)})$.

We obviously have $\mathrm{Cap}(\bigcup_{i=2}^k C(x_i^k, 9\varepsilon_k)) \leq \frac{1}{k}$, hence $\mathrm{Cap}(\Omega_n \setminus \tilde{\Omega}_n) \leq \frac{1}{k}$. Moreover $\tilde{\Omega}_n \xrightarrow{H^c} \Omega$. Indeed, let Ω' be the H^c-limit of a subsequence of $\{\tilde{\Omega}_n\}_{n \in \mathbb{N}}$. Then $\Omega' \subset \Omega$, and for all $x \in \Omega$, there exists $\rho > 0$ such that $\overline{B}(x, \rho) \subset \Omega_n$ for n large enough. For k large enough such that $9\varepsilon_k < \frac{\rho}{2}$, the inclusion $\overline{B}(x, \frac{\rho}{3}) \subset \tilde{\Omega}_n$ holds, hence $x \in \Omega'$. Finally $\Omega' = \Omega$. Since the measure is lower semi-continuous for the H^c convergence, we also get that $\tilde{\Omega}_n$ converges in measure to Ω.

For $n_k \leq n < n_{k+1}$ let us denote by Ω_n^1 the connected component of $\tilde{\Omega}_n$ which contains U_k^1, and write $\tilde{\Omega}_n = \Omega_n^1 \cup R_n$ with $\Omega_n^1 \cap R_n = \emptyset$. Following Proposition 2.2, for a subsequence (still denoted with the same index) we have $\Omega_n^1 \cup R_n \xrightarrow{H^c} \Omega^1 \cup R$ with $\Omega^1 \cap R = \emptyset$ and $\Omega^1 \cup R \subseteq \Omega$. Let $x \in C_1$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq C_1$. Then, $B(x, \varepsilon) \subseteq U_k^1$ for k large enough, hence $x \in \Omega^1$. Consequently we get $C_1 \subseteq \Omega^1$. Conversely, let $x \in \Omega^1$, and suppose that $x \notin C_1$. Then, there exists $i \geq 2$ such that $x \in C_i$, hence for n large enough we have $\Omega_n^1 \cap C_i \neq \emptyset$ i.e. $\Omega_n^1 \cup C_i$ is a connected set. This is impossible for $n \geq n_i$ by the construction of Ω_n^1. Consequently we get $\Omega^1 = C_1$, hence $\Omega_n^1 \xrightarrow{H^c} C_1$ and $R_n \xrightarrow{H^c} \cup_{i \geq 2} C_i$ for the whole sequence. The convergence in measure of Ω_n^1 and R_n to C_1 and $\cup_{i \geq 2} C_i$, respectively, is immediate by the fact that

$$\chi_\Omega \leq \liminf_{n \to \infty} \chi_{\Omega_n^1} + \liminf_{n \to \infty} \chi_{R_n} \leq \liminf_{n \to \infty} \chi_{\Omega_n^1 \cup R_n} = \chi_\Omega . \tag{4}$$

□

We are able to give now the continuity result for simply connected sets.

Lemma 3.7. *Let $\{\Omega_n\}_{n \in \mathbb{N}} \subseteq \mathcal{O}_1(B)$, such that $\Omega_n \xrightarrow{H^c, L^1} \Omega$. Then the first Mosco condition is satisfied.*

Proof. It is clear that it is sufficient to prove the first Mosco condition for a set of functions which has its span dense in $H^1(\Omega)$. Splitting Ω into its connected components $\Omega = \bigcup_{i=1}^\infty C_i$, it is sufficient to prove the first Mosco condition for a function $u \in H^1(\Omega) \cap L^\infty(\Omega)$ which vanishes on $\bigcup_{i=2}^\infty C_i$.

We apply Lemma 3.6 to construct the sequence $\{\tilde{\Omega}_n\}_{n\in\mathbb{N}}$. Since $\Omega_n^1 \overset{H^c,L^1}{\longrightarrow} C_1$, from Lemma 3.4 there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ such that $\phi_n \in H^1(\Omega_n^1)$ and $\phi_n \rightsquigarrow u_{|C_1}$. Defining $\phi_n'(x) = \phi_n(x)$ for $x \in \Omega_n^1$ and $\phi_n'(x) = 0$ on R_n we get $\phi_n' \in H^1(\tilde{\Omega}_n)$ and $\phi_n' \rightsquigarrow u$. Using Lemma 2.3 we conclude the proof. □

3.3. – Proof of Theorem 3.1

In this subsection, we prove Theorem 3.1 using a localization procedure and Lemma 3.7.

PROOF OF THEOREM 3.1.

Necessity. Let us suppose that $\Omega_n, \Omega \in \mathcal{O}_l(B)$, Ω_n converges in H^c to Ω and for every $f \in L^2(B)$ we have $u_{\Omega_n} \rightsquigarrow u_\Omega$. If we choose $f \equiv 1_B$, then $\tilde{u}_{\Omega_n} = \chi_{\Omega_n}$ and $\tilde{u}_\Omega = \chi_\Omega$, hence $|\Omega_n| \to |\Omega|$.

Sufficiency. Let us suppose that $\Omega_n, \Omega \in \mathcal{O}_l(B)$ and that Ω_n converges in H^c to Ω. Suppose moreover that $|\Omega_n| \to |\Omega|$. Then Ω_n converges also in measure to Ω. Since it is sufficient to prove the continuity for a subsequence, we can suppose $\chi_{\Omega_n}(x) \to \chi_\Omega(x)$ for a.e. $x \in B$.

Let

$$\overline{B} \setminus \Omega_n = K_1^n \cup \ldots \cup K_l^n$$

be the decomposition of $\overline{B}\setminus\Omega_n$ in l connected components (compact and disjoint, eventually empty). For every index $i = 1, \ldots, l$ there exists a subsequence (still denoted with the same index) such that

$$K_i^n \overset{H}{\longrightarrow} K_i$$

where K_i are compact connected sets, not necessarily disjoint and nonempty. Since the solution of problem (1) on Ω is unique and the Hausdorff topology is compact, it is sufficient to prove the continuity with respect to the domain only for this subsequence.

We obviously have $\overline{B}\setminus\Omega = K_1 \cup \ldots \cup K_l$. Three possibilities may occur for the sets K_i. They may have a strictly positive diameter, they may contain only one point, or they are empty (see [4] for a similar analysis for the Dirichlet problem). We construct a new open set Ω^+ such that $\Omega \subset \Omega^+ \subset B$ and $\mathrm{Cap}(\Omega^+ \setminus \Omega) = 0$, by eliminating those K_i which have zero diameter. After a renotation of the indices, we can write

$$B \setminus \Omega^+ = K_1 \cup \ldots \cup K_{l'}$$

where $l' \leq l$, and $\mathrm{diam}(K_i) \geq \delta > 0$. We then consider the sequence $\{\Omega_n^+\}_{n\in\mathbb{N}}$ defined by

$$\overline{B} \setminus \Omega_n^+ = K_1^n \cup \ldots \cup K_{l'}^n$$

for every $i = 1, \ldots, l'$. We have $K_i^n \overset{H}{\longrightarrow} K_i$ and K_i^n connected. From the construction of Ω_n^+ we have

$$\Omega_n^+ \overset{L^1,H^c}{\longrightarrow} \Omega^+ .$$

The Hausdorff convergence of K_i^n to K_i gives the existence of $n = n_\delta \in \mathbb{N}$ such that

$$\forall n \geq n_\delta \ \forall i = 1, \dots, l' \quad \operatorname{diam}(K_i^n) > \frac{2\delta}{3}. \tag{5}$$

Consequently, for every $x \in \overline{B}$ and for every $n \geq n_\delta$, the set $\Omega_n^+ \cap B(x, \frac{\delta}{3})$ is simply connected. Moreover

$$\Omega_n^+ \cap B\left(x, \frac{\delta}{3}\right) \xrightarrow{H^c, L^1} \Omega^+ \cap B\left(x, \frac{\delta}{3}\right).$$

We can then apply Lemma 3.7 for the previous sequences. Using an argument based on the partition of unity as in [10, Appendix A], we obtain that for every $\phi \in H^1(\Omega^+)$ there exists $\phi_n^+ \in H^1(\Omega_n^+)$ such that $\phi_n^+ \rightsquigarrow \phi$. Since $\operatorname{Cap}(\Omega^+ \setminus \Omega) = 0$, taking the restriction $\phi_n = \phi_n^+{}_{|\Omega_n} \in H^1(\Omega_n)$ we obtain that for every $\phi \in H^1(\Omega)$ there exists $\phi_n \in H^1(\Omega_n)$ such that $\phi_n \rightsquigarrow u$, hence the first Mosco condition is satisfied.

Using now Proposition 3.2 we have that $u_n \rightsquigarrow u$. □

REMARK 3.8. In the result of Sverak, the continuity of the solution of an elliptic problem with homogeneous Dirichlet boundary conditions in two dimensions is a consequence of the convergence in H^c and of the uniform bound on the number of the connected components of the complementaries. From this point of view, it might be surprising that the Lebesgue measure can control the continuity for the Neumann problem. In fact, the constraint on the measure can be dropped, if instead of problem (1), a different Neumann problem is considered in Dirichlet spaces (see [6]).

4. – Further remarks and applications

Let $H : [0, \infty) \to \mathbb{R}$ be continuous with $H(0) = 0$ and $\gamma > 0$ a fixed number. We recall from [9] the definition of the density perimeter. The function H plays a "corrector" role for the perimeter while γ acts like a scale.

DEFINITION 4.1. The (γ, H)-density perimeter of a set $A \subseteq \mathbb{R}^2$ is

$$P_{\gamma,H}(A) = \sup_{\varepsilon \in (0,\gamma)} \left[\frac{|A^\varepsilon|}{2\varepsilon} + H(\varepsilon)\right], \tag{6}$$

where $A^\varepsilon = \bigcup_{x \in A} B(x, \varepsilon)$.

The following results hold (see [9]). If $\Omega_n \xrightarrow{H^c} \Omega$ and there exist $M > 0$ such that for every $n \in \mathbb{N}$ we have $P_{\gamma,H}(\partial\Omega_n) \leq M$, then $P_{\gamma,H}(\partial\Omega) \leq M$ and $\Omega_n \xrightarrow{L^1} \Omega$. For every $\Omega \in \mathcal{O}(B)$ such that $\sharp\partial\Omega \leq l$, we have $P_{1,-\frac{\pi l \varepsilon}{2}}(\partial\Omega) = \mathcal{H}^1(\partial\Omega)$.

We give the following compactness result, which in view of Theorem 3.1 can be applied for proving existence for a class of shape optimization problems.

THEOREM 4.2. *Let $l \in \mathbb{N}$, $M > 0$. Then the set*

$$\mathcal{O}_{l,M}(B) = \{\Omega \in \mathcal{O}_l(B) : P_{\gamma,H}(\partial\Omega) \leq M\}$$

is compact in the H^c-topology and for every sequence $\{\Omega_n\}_{n\in\mathbb{N}} \subseteq \mathcal{O}_{l,M}(B)$ such that $\Omega_n \xrightarrow{H^c} \Omega$ we have $\Omega_n \xrightarrow{L^1} \Omega$.

PROOF. It is a direct consequence of the properties of the Hausdorff topology and of the density perimeter (see [9]). □

The continuity result of [10] is a particular case of Theorems 3.1 and 4.2. Indeed, if Ω_n is such that $\sharp(\partial\Omega_n) \leq l$, then

$$\mathcal{H}^1(\partial\Omega_n) = P_{1,-\frac{\pi\varepsilon}{2}.l}(\partial\Omega_n),$$

hence the condition $\mathcal{H}^1(\partial\Omega_n) \leq M$ becomes in fact $P_{1,-\frac{\pi\varepsilon}{2}.l}(\partial\Omega_n) \leq M$. Consequently, Theorem 4.2 can be applied.

Remark also that the number of the connected components of the boundary is not lower semi-continuous for the H^c-convergence. It appears that Theorem 4.2 gives existence results for minimization problems of a large class of shape functionals defined on $\mathcal{O}_{l,M}(B)$. The direct method of calculus of variations can be applied using the compactness-continuity result given by Theorem 4.2.

Let $F : B \times [0, 1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a positive lower semi-continuous function and $f \in L^2(B)$. Then, the following problem

$$\min_{\Omega\in\mathcal{O}_{l,M}(B)} \int_B F(x, \chi_\Omega(x), \tilde{u}_\Omega(x), \tilde{\nabla} u_\Omega(x))dx$$

has at least one solution. For fixed $g \in L^2(B)$, we consider

$$J(\Omega) = \int_\Omega |u_\Omega - g|^2 dx.$$

Using the density perimeter as a penalty term, the direct method of the calculus of variations gives the existence of a solution for the following minimization problem

$$\min_{\Omega\in\mathcal{O}(B)} \int_\Omega |u_\Omega - g|^2 dx + \sharp(\Omega^c) + P_{\gamma,H}(\partial\Omega).$$

REFERENCES

[1] J. BAXTER —G. DAL MASO —U. MOSCO, *Stopping times and Γ-convergence*, Trans. American Math. Soc. **303**, 1, September (1987), 1-38.

[2] M. BELLIEUD —G. BOUCHITTÉ, *Homogénéisation de problèmes eliptiques dégénérés*, C. R. Acad. Sci. Paris, Sér. I Math. **327** (1988), 787-792.

[3] M. Briane, *Homogenization of a nonperiodic material*, J. Math. Pures Appl. (9) **73** (1994), no. 1, 47-66.

[4] D. Bucur, *Shape analysis for Nonsmooth Elliptic Operators*, Appl. Math. Lett. **9** No. 3 (1996), 11-16.

[5] D. Bucur —P. Trebeschi, *Shape optimization problem governed by nonlinear state equation,*, Proc. Roy. Soc. Edinburgh **128 A** (1998), p. 945-963.

[6] D. Bucur —N. Varchon, *A duality approach for the boundary variation for the Neumann problem*, Preprint Université de Besançon, **14/2000**.

[7] D. Bucur —J. P. Zolésio, *Continuité par rapport au domaine dans le problème de Neumann*, C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), p. 57-60.

[8] D. Bucur —J. P. Zolésio, *N-Dimensional Shape Optimization under Capacitary Constraints*, J. Differential Equations **123** (1995), No. 2, 504-522.

[9] D. Bucur —J. P. Zolésio, *Boundary Optimization under Pseudo Curvature Constraint*, Annali Scuola Norm. Sup. Pisa Cl. Sci. (4), **XXIII**, Fasc. 4, (1996), p. 681-699.

[10] A. Chambolle —F. Doveri, *Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets.*, Comm. Partial Differential Equations **22**, No. 5-6 (1997), 811-840.

[11] A. Chambert-loir —S. Fermigier —V. Maillot, "Exercices de mathématiques pour l'agrégation", Analyse 1, 2^e édition, Masson, 1997.

[12] D. Chenais, *On the existence of a solution in a domain identification problem*, J. Math. Anal. Appl., **52** (1975), 189-289.

[13] G. Cortesani, *Asymptotic behaviour of a sequence of Neumann problems*, Comm. Partial Differential Equations **22**, No.9-10 (1997), 1691-1729.

[14] T. Del Vecchio, *The thick Neumann's sieve*, Ann. Mat. Pura Appl. (4) **147** (1987), 363-402.

[15] L. C. Evans —R. F. Gariepy, "Measure Theory and Fine Properties of Functions", CRC Press, Inc. 1992.

[16] J. Heinonen —T. Kilpelainen —O. Martio, "Nonlinear Potential Theory of Degenerate Elliptic Equations", Clarendon Press, Oxford, New York, Tokyo, 1993.

[17] E. Ya. Khruslov —G. I. Vorobceva, *Averaging of the Neumann problem for higher-order elliptic operators*, Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki 1997, no. 4, 43-47.

[18] F. Murat, *On the Neumann sieve*, in A. Marino et al., editor, *Nonlinear Variational Problems*, **127** in Res. Notes in Math., Pitman 1985, 24-32.

[19] Ch. Pommerenke, "Boundary Behaviour of Conformal Maps", Springer-Verlag, Berlin, 1992.

[20] W. Rudin, "Analyse réelle et complexe", Masson, Paris, 1975.

[21] V. Sverak, *On optimal shape design*, J. Math. Pures Appl. **72** (1993), 537-551.

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