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Optimal Stability for Inverse Elliptic Boundary Value Problems with Unknown Boundaries

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Abstract. In this paper we study a class of inverse problems associated to elliptic boundary value problems. More precisely, those inverse problems in which the role of the unknown is played by an inaccessible part of the boundary and the role of the data is played by overdetermined boundary data for the elliptic equation assigned on the remaining, accessible, part of the boundary. We treat the case of arbitrary space dimension $n \geq 2$. Such problems arise in applied contexts of nondestructive testing of materials for either electric or thermal conductors, and are known to be ill-posed. In this paper we obtain essentially best possible stability estimates. Here, in the context of ill-posed problems, stability means the continuous dependence of the unknown upon the data when additional a priori information on the unknown boundary (such as its regularity) is available.

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1. – Introduction

In this paper we shall deal with two inverse boundary value problems.

Suppose Ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, a part of which, say I (perhaps some interior connected component of $\partial\Omega$ or some inaccessible portion of the exterior component of $\partial\Omega$), is not known. This could be the case of an electrically conducting specimen, which is possibly defective due to the presence of interior cavities or of corroded parts, which are not accessible to direct inspection. See for instance [K-S-V]. The aim is to detect the presence of such defects by nondestructive methods collecting current and voltage measurements on the accessible part A of the boundary $\partial\Omega$.

If we assume that the inaccessible part I of $\partial\Omega$ is electrically insulated, then, given a nontrivial function ψ on A, having zero average (which represents

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the assigned current density on the accessible part A of $\partial\Omega$), we have that the voltage potential u inside Ω satisfies the following Neumann type boundary value problem

$$\operatorname{div}(\sigma\nabla u) = 0, \quad \text{in } \Omega, \tag{1.1a}$$

$$\sigma\nabla u \cdot \nu = \psi, \quad \text{on } A, \tag{1.1b}$$

$$\sigma\nabla u \cdot \nu = 0, \quad \text{on } I. \tag{1.1c}$$

Here, ν is the exterior unit normal to $\partial\Omega$ and $\sigma = \{\sigma_{ij}(x)\}_{i,j=1}^{n}$ denotes the known symmetric conductivity tensor ant it is assumed to satisfy a hypothesis of uniform ellipticity. Let us remark that the solution to (1.1*a*)-(1.1*c*) is unique up to an undetermined additive constant. In order to specify a single solution, we shall assume, from now on, the following normalization condition

$$\int_{\Omega} u = 0. \tag{1.1d}$$

Suppose, now, that Σ is an open subset of $\partial\Omega$, which is contained in A, and on which the voltage potential can be measured. Then, the inverse problem consists of determining I provided $u_{|\Sigma}$ is known. This is the first object of our study and we shall refer to it as the *Inverse Neumann Problem* (*Neumann case*, for short).

An allied problem is the one associated to the direct Dirichlet problem

$$\operatorname{div}(\sigma\nabla u) = 0, \quad \text{in } \Omega, \tag{1.2a}$$

$$u = g, \quad \text{on } A, \tag{1.2b}$$

$$u = 0, \quad \text{on } I. \tag{1.2c}$$

Here, as above, I, A are the inaccessible, respectively, accessible, parts of $\partial\Omega$, and σ is the conductivity tensor satisfying the same hypotheses. Our second object of study is the inverse problem consisting in the determination of I from the knowledge of $\sigma\nabla u \cdot \nu_{|\Sigma}$, where $\Sigma \subset A$ is as above. We shall refer to it as the *Inverse Dirichlet Problem* (*Dirichlet case*, for short). We believe that also this problem may be of interest for concrete applications of nondestructing testing, for instance in thermal imaging. In this case, the inaccessible boundary I could represent a priviledged isothermal surface, such as a solidification front. Of course, it should be kept in mind that, dealing with thermal processes, the evolutionary model based on parabolic, rather than elliptic, equations is in general more appropriate, for related issues see, for instance, [B-K-W], [Bi], [V1]. However, we trust that also a preliminary study of a stationary model may be instructive.

Such two problems, the Neumann and Dirichlet cases, are known to be ill-posed. Indeed there are examples that show that, under a priori assumptions on the unknown boundary I regarding its regularity (up to any finite order of differentiability), the continuous dependence (stability) of I from the measured data ($u_{|\Sigma}$ in the Neumann case, $\sigma\nabla u \cdot \nu_{|\Sigma}$ in the Dirichlet case) is, at best, of logarithmic type. See [Al2] and also [Al-R].

The main purpose of this paper is to prove stability estimates of logarithmic type (hence, best possible) for both the Neumann and Dirichlet cases, (Theorems 2.1, 2.2), when the space dimension $n \geq 2$ is arbitrary. We recall that, for the case $n = 2$, results comparable to ours have been found in [Be-V] when σ is homogeneous and in [R], [Al-R] when σ can be inhomogeneous and also discontinuous. Other related results for the case of dimension two can be found in [Bu-C-Y1], [Bu-C-Y2], [Bu-C-Y3], [Bu-C-Y4], [An-B-J]. Let us also recall that, typically, the above mentioned results are based on arguments related, in various ways, to complex analytic methods, which do not carry over the higher dimensional case.

In the sequel of this Introduction, we shall illustrate the new tools we found necessary to develop and exploit when $n > 2$. But first, let us comment briefly on the connection with another inverse problem which has become quite popular in the last ten years, namely the inverse problem of cracks. On one hand there are similarities, in fact a crack can be viewed as a collapsed cavity, that is a portion of surface inside the conductor, such that a homogeneous Neumann condition like (1.1c) holds on the two sides of the surface. On the other hand there are differences, for the uniqueness in the crack problem at least two appropriate distinct measurements are necessary [F-V], whereas for our problems, either the Neumann or the Dirichlet case, any single nontrivial measurement suffices for uniqueness, see for instance [Be-V] for a discussion of the uniqueness issue. Let us also recall that for the crack problem in dimensions bigger than two, various basic problems regarding uniqueness are still unanswered. See, for the available results and for references [Al-DiB]. It is therefore clear that a study of the stability for the crack problem in dimensions higher than two shall require new ideas. Nonetheless, the authors believe that the techniques developed here might be useful also in the treatment of the crack problem.

The methods we use in this paper are based essentially on a single unifying theme: *Quantitative Estimates of Unique Continuation*, and we shall exploit it under various different facets, namely the following ones.

(a) *Stability Estimates of Continuation from Cauchy Data.* Since we are given the Cauchy data on Σ for a solution u to (1.1a), we shall need to evaluate how much a possible error on such Cauchy data can affect the interior values of u. Such stability estimates for Cauchy problems associated to elliptic equations have been a central topic of ill-posed problems since the beginning of their modern theory, [H], [Pu1], [Pu2]. Here, since one of our underlying aims will be to treat our problems under possibly minimal regularity assumptions, we shall assume the conductivity σ to be Lipschitz continuous (this is indeed the minimal regularity ensuring the uniqueness for the Cauchy problem, [Pl], [M]). Our present stability estimates (Propositions 3.1, 3.2, 4.1, 4.2) will elaborate on inequalities due to Trytten [T] who developed a method first introduced by Payne [Pa1], [Pa2]. The additional difficulty encountered here will be that we shall need to compare solutions u_1, u_2 which are defined on possibly different domains Ω_1, Ω_2

whose boundaries are known to agree on the accessible part A only. Let us recall that a similar approach, but restricted to the topologically simpler two-dimensional setting, has already been used in [Al1], [Be-V]. We shall obtain that, if the error on the measurement on the Cauchy data is small, then for the Neumann case, also $|\nabla u_1|$ is small, in an L^2 average sense, on $\Omega_1 \setminus \overline{\Omega_2}$, the part of Ω_1 which exceeds Ω_2. And the same holds for $|\nabla u_2|$ on $\Omega_2 \setminus \overline{\Omega_1}$ (Propositions 3.1, 3.2). In the Dirichlet case instead we shall prove that u_1 itself is small in $\Omega_1 \setminus \overline{\Omega_2}$, and the same holds for u_2 on $\Omega_2 \setminus \overline{\Omega_1}$ (Propositions 4.1, 4.2).

(*b*) *Estimates of Continuation from the Interior.* We shall also need interior average lower bounds on u and on its gradient (Propositions 3.3, 4.3), on small balls contained inside Ω. Bounds of this type have been introduced in [Al-Ros-S, Lemma 2.2] in the context of a different inverse boundary value problem. The tools here involve another form of quantitative unique continuation, namely the following.

(*c*) *Three Spheres Inequalities.* Also this one is a rather classical theme in connection with unique continuation. Aside from the classical Hadamard's three circles theorem, in the context of elliptic equations we recall Landis [La] and Agmon [Ag]. Under our assumptions of Lipschitz continuity on σ, our estimates (see (5.47) below) shall refer to differential inequalities on integral norms originally due to Garofalo and Lin [G-L], later developed by Brummelhuis [Br] and Kukavica [Ku].

(*d*) *Doubling Inequalities in the Interior.* This rather recent tool has been introduced by Garofalo and Lin in the above mentioned paper [G-L]. It provides an efficient method of estimating the local average vanishing rate of a solution to (1.1*a*). Let us recall that it also provides a remarkable bridge to the powerful theory of Muckenhoupt weights [C-F] and that this last connection has been crucially used in [Al-Ros-S] and also in [V2].

The last, fundamental, appearance of quantitative estimates of unique continuation is the following.

(*e*) *Doubling Inequalities at the Boundary.* For our purposes it will be crucial to evaluate the vanishing rate of ∇u (in the Neumann case) or of u (in the Dirichlet case) near the inaccessible boundary I. In particular, the fact that such a rate is not worse than polynomial (Propositions 3.5, 4.5) is an essential ingredient in proving that the stability for our inverse problems are not worse than logarithmic (see the proof of Theorem 2.1). Such evaluations on vanishing rates near I, where an homogeneous boundary condition applies (either (1.1*c*) or (1.2.*c*)), can be obtained by the so called Doubling Inequalities at the Boundary. The study of such inequalities has been initiated by Adolfsson, Escauriaza and Kenig [A-E-K] and later developed by Kukavica and Nystrom [Ku-N] and Adolfsson and Escauriaza [A-E]. In particular, in [A-E] such inequalities are proven, for the Neumann problem, when the boundary is $C^{1,1}$ smooth, and, for the Dirichlet problem, when the boundary is $C^{1,\omega}$ smooth, where the modulus of continuity ω is of

Dini type. We shall use essentially such results, with the only difference that, mainly for simplicity of exposition, we shall assume, in the Dirichlet case, that $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$, that is of Hölder type (Propositions 3.5, 4.5). Let us also recall that the conjecture which has been left open by the above mentioned papers is that the Doubling Inequality at the Boundary should hold true when the boundary is Lipschitz. Hopefully, if such conjecture were proven, then our stability results, Theorems 2.1, 2.2, might be generalized as follows. If I is a priori known to be Lipschitz with a sufficiently small Lipschitz constant, then the logarithmic stability estimates of Theorems 2.1, 2.2 should apply also to this case. If instead the Lipschitz constant of I is large, then the best possible stability estimate might be worse than logarithmic. This last expectation is motivated by the fact that two Lipschitz surfaces with large Lipschitz constant may be arbitrarily close in the sense of the Hausdorff distance, but locally they need not to be representable as graphs in a common reference system (see Rondi [R] for an example). If it happens that this is the case for the unknown boundaries I_1, I_2, then it might also happen that estimates on the smallness of $|\nabla u_1|$ in $\Omega_1 \setminus \overline{\Omega_2}$ (in the Neumann case, for instance) are worse than logarithmic. In fact from the proofs of Propositions 3.2, 4.2, the importance of proving that I_1, I_2 are locally represented as graphs in a common reference system will become evident. This property of I_1, I_2 will be referred to by saying that I_1, I_2 are *Relative Graphs*. Sufficient conditions guaranteeing that the boundaries of the two domains Ω_1, Ω_2 are Relative Graphs will be examined in Proposition 3.6. As we already mentioned, in this paper we intend to strive after optimal results under possibly minimal a priori assumptions of regularity (see *i*) and *iii*) in Section 2). Moreover, very general assumptions on the unknown boundary I are made. It may have a finite, but undetermined, number of connected components, and no restriction is placed on their topology. Furthermore we use a single measurement corresponding to one boundary data, either ψ or g, that can be prescribed arbitrarily. Concerning their regularity, the assumptions (2.7*a*), (2.8*a*) are quite loose and essentially correspond to the natural ones in the treatment of the direct problems (1.1), (1.2) respectively. In addition, we shall require a bound on the oscillation character (frequency) of ψ or of g. This is expressed as a bound on a ratio of two norms: either

$$\frac{\|\psi\|_{L^2(A)}}{\|\psi\|_{H^{-1/2}(A)}} \leq F, \quad \text{in the Neumann case},$$

or

$$\frac{\|g\|_{H^{1/2}(A)}}{\|g\|_{L^2(A)}} \leq F, \quad \text{in the Dirichlet case}.$$

Such control will be necessary in order to dominate the vanishing rates of the solutions in terms of quantities which depend only on the prescribed data. Notice that F may be arbitrarily large, but it is expected that the constants in the estimates of Theorems 2.1, 2.2 may deteriorate as $F \to \infty$.

The plan of the paper is as follows.

In Section 2 we shall state the main Theorems 2.1, 2.2, we also state Corollary 2.3 which provides a finer interpretation of the stability estimates in the previous theorems. Here, instead of estimating the Hausdorff distance of the domains Ω_1, Ω_2, we shall estimate their distance locally, in terms of the graph representation of their boundaries, and also globally, by viewing them as imbedded differentiable manifolds with boundary.

Sections 3 and 4 contain the proofs of Theorem 2.1 and Theorem 2.2, respectively. The proofs are preceded by the statements of various auxiliary propositions (Propositions 3.1-3.6, Propositions 4.1-4.5). Section 4 contains also the proof of Corollary 2.3.

Section 5 contains the proof of the propositions regarding the estimates of continuation for Cauchy problems, and namely Propositions 3.1, 3.2, 4.1 and 4.2. Such proofs are accompanied by some intermediate lemmas. Lemma 5.1 collects some regularity results for the direct Neumann problem. Lemmas 5.2, 5.3 deal with the technical notion of regularized distance as introduced by Lieberman.

Section 6 contains the proofs of Propositions 3.3, 4.3 concerning estimates of continuation from the interior.

Section 7 contains all the proofs concerning doubling inequalities. Namely, the proofs of Propositions 3.4, 4.4, dealing with the interior doubling inequalities, the proofs of Propositions 3.5, 4.5, where the results of Adolfsson and Escauriaza are adapted to the present purposes. Their result for the Dirichlet problem is summarized in Lemma 7.1.

Section 8 deals with Relative Graphs, first in Lemma 8.1 we treat the general case of Lipschitz boundaries, and we conclude with the proof of Proposition 3.6.

2. – The main results

When representing locally a boundary as a graph, it will be convenient to use the following notation. For every $x \in \mathbb{R}^n$ we shall set $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$.

Definition 2.1. Let Ω be a bounded domain in $\mathbb{R}^n$. Given α, $0 < \alpha \leq 1$, we shall say that a portion S of $\partial\Omega$ is of *class* $C^{1,\alpha}$ *with constants* ρ_0, $E > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{x \in B_{\rho_0}(0) \text{ s.t. } x_n > \varphi(x')\},$$

where φ is a $C^{1,\alpha}$ function on $B_{\rho_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$\|\varphi\|_{C^{1,\alpha}(B_{\rho_0}(0))} \leq E\rho_0 .$$

REMARK 2.1. To the purpose of simplifying the expressions in the various estimates throughout the paper, we have found it convenient to scale all norms in such a way that they are dimensionally equivalent to their argument and coincide with the standard definition when the dimensional parameter ρ_0 equals 1. For instance, for any $\varphi \in C^{1,\alpha}\left(B_{\rho_0}(0)\right)$ we set

$$\|\varphi\|_{C^{1,\alpha}(B_{\rho_0}(0))} = \|\varphi\|_{L^\infty(B_{\rho_0}(0))} + \rho_0\|\nabla\varphi\|_{L^\infty(B_{\rho_0}(0))} + \rho_0^{1+\alpha}|\nabla\varphi|_{\alpha,B_{\rho_0}(0)},$$

where

$$|\nabla\varphi|_{\alpha,B_{\rho_0}(0)} = \sup_{\substack{x',y'\in B_{\rho_0}(0)\\ x'\neq y'}} \frac{|\nabla\varphi(x') - \nabla\varphi(y')|}{|x'-y'|^\alpha}.$$

Similarly, we shall set

$$\|u\|_{L^2(\Omega)} = \rho_0^{-n/2}\left(\int_\Omega u^2\right)^{1/2},$$

$$\|u\|_{H^1(\Omega)} = \rho_0^{-n/2}\left(\int_\Omega u^2 + \rho_0^2\int_\Omega |\nabla u|^2\right)^{1/2},$$

and so on for boundary and trace norms such as $\|\cdot\|_{L^2(\partial\Omega)}$, $\|\cdot\|_{H^{1/2}(\partial\Omega)}$, $\|\cdot\|_{H^{-1/2}(\partial\Omega)}$.

i) A priori information on the domain.

Our main Theorems 2.1, 2.2 will be based on the following assumptions on the domain. Given ρ_0, $M > 0$, we assume:

$$|\Omega| \leq M\rho_0^n. \tag{2.1}$$

Here, and in the sequel, $|\Omega|$ denotes the Lebesgue measure of Ω. We shall distinguish two nonempty parts, A, I in $\partial\Omega$ and we assume

$$I \cup A = \partial\Omega, \qquad \overset{\circ}{I} \cap \overset{\circ}{A} = \emptyset, \qquad I \cap A = \partial A = \partial I. \tag{2.2}$$

Here, interiors and boundaries are intended in the relative topology in $\partial\Omega$. Moreover we assume that we can select a portion Σ within A satisfying for some $P_1 \in \Sigma$

$$\partial\Omega \cap B_{\rho_0}(P_1) \subset \Sigma, \tag{2.3}$$

and also, denoting by I^{ρ_0} the portion of $\partial\Omega$ of all $x \in \partial\Omega$ such that $\text{dist}(x, I) < \rho_0$,

$$\Sigma \cap I^{\rho_0} = \emptyset. \tag{2.4}$$

Regarding the regularity of $\partial\Omega$, given α, E, $0 < \alpha \leq 1$, $E > 0$, we assume that

$$\partial\Omega \text{ is of class } C^{1,\alpha} \text{ with constants } \rho_0, E. \tag{2.5}$$

In addition, in Theorem 2.1, we shall also assume the following

$$I \text{ is of class } C^{1,1} \text{ with constants } \rho_0, E. \tag{2.6}$$

REMARK 2.2. Observe that (2.5) automatically implies a lower bound on the diameter of every connected component of $\partial\Omega$. Moreover, by combining (2.1) with (2.5), an upper bound on the diameter of Ω can also be obtained. Note also that (2.1), (2.5) implicitly comprise an a priori upper bound on the number of connected components of $\partial\Omega$.

ii) Assumptions about the boundary data.

Let us set

$$A_{\rho_0} = \{x \in A \text{ s.t. } \operatorname{dist}(x, I) \geq \rho_0\},$$

(that is: $A_{\rho_0} = \partial\Omega \setminus I^{\rho_0}$). We shall assume the following on the Neumann data ψ appearing in problem (1.1)

$$\psi \in L^2(A), \quad \psi \not\equiv 0, \tag{2.7a}$$

$$\int_A \psi = 0, \tag{2.7b}$$

$$\operatorname{supp}\psi \subset A_{\rho_0}, \tag{2.7c}$$

and, for a given constant $F > 0$,

$$\frac{\|\psi\|_{L^2(A)}}{\|\psi\|_{H^{-1/2}(A)}} \leq F. \tag{2.7d}$$

Concerning the Dirichlet data g appearing in (1.2), we assume

$$g \in H^{1/2}(A), \quad g \not\equiv 0, \tag{2.8a}$$

$$\operatorname{supp} g \subset A_{\rho_0}, \tag{2.8b}$$

$$\frac{\|g\|_{H^{1/2}(A)}}{\|g\|_{L^2(A)}} \leq F. \tag{2.8c}$$

As noted already in Remark 2.1, norms are suitably scaled so to be dimensionally equivalent to their argument.

iii) Assumptions about the conductivity.

The conductivity σ is assumed to be a given function from $\mathbb{R}^n$ with values $n \times n$ symmetric matrices satisfying the following conditions for given constants λ, Λ, $0 < \lambda \leq 1$, $\Lambda \geq 0$,

$$\lambda|\xi|^2 \leq \sigma(x)\xi \cdot \xi \leq \lambda^{-1}|\xi|^2, \text{ for every } x, \xi \in \mathbb{R}^n, \text{ (ellipticity)} \tag{2.9a}$$

$$|\sigma(x) - \sigma(y)| \leq \Lambda \frac{|x-y|}{\rho_0}, \text{ for every } x, y \in \mathbb{R}^n. \text{(Lipschitz continuity)} \tag{2.9b}$$

In the sequel, we shall refer to the set of constants E, α, M, F, λ, Λ as to the *a priori data.*

THEOREM 2.1. *Let Ω_1, Ω_2 be two domains satisfying* (2.1), (2.5). *Let A_i, I_i, $i = 1, 2$, be the corresponding accessible and inaccessible parts of their boundaries. Let us assume that $A_1 = A_2 = A$, Ω_1, Ω_2 lie on the same side of A and that* (2.2)-(2.4) *are satisfied by both pairs A_i, I_i. Let I_1, I_2 satisfy* (2.6). *Let $u_i \in H^1(\Omega_i)$ be the solution to* (1.1) *when $\Omega = \Omega_i$, $i = 1, 2$, and let* (2.7), (2.9) *be satisfied. If, given $\epsilon > 0$, we have*

$$\|u_1 - u_2\|_{L^2(\Sigma)} \leq \epsilon \,, \tag{2.10}$$

then we have

$$d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}) \leq \rho_0 \omega \left(\frac{\epsilon}{\rho_0 \|\psi\|_{H^{-1/2}(A)}} \right) , \tag{2.11}$$

where ω is an increasing continuous function on $[0, \infty)$ which satisfies

$$\omega(t) \leq C |\log t|^{-\eta}, \quad \text{for every } t < 1 \,, \tag{2.12}$$

and C, η, $C > 0$, $0 < \eta \leq 1$ are constants only depending on the a priori data.

Here $d_{\mathcal{H}}$ denotes the Hausdorff distance between bounded closed sets of $\mathbb{R}^n$

THEOREM 2.2. *Let Ω_1, Ω_2 and A_i, I_i, $i = 1, 2$, be as in Theorem* 2.1. *Let* (2.1)-(2.5) *be satisfied. Let $u_i \in H^1(\Omega_i)$ be the solution to* (1.2) *when $\Omega = \Omega_i$, $i = 1, 2$, and let* (2.8), (2.9) *be satisfied. If, given $\epsilon > 0$, we have*

$$\rho_0 \|\sigma \nabla u_1 \cdot \nu - \sigma \nabla u_2 \cdot \nu\|_{L^2(\Sigma)} \leq \epsilon \,, \tag{2.13}$$

then we have

$$d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}) \leq \rho_0 \omega \left(\frac{\epsilon}{\|g\|_{H^{1/2}(A)}} \right) , \tag{2.14}$$

where ω is as in (2.12) *and the constants C, η, $C > 0$, $0 < \eta \leq 1$ only depend on the a priori data.*

COROLLARY 2.3. *Let the hypotheses of either Theorem* 2.1 *or Theorem* 2.2 *be satisfied. There exist r_0, $0 < r_0 \leq \rho_0$, only depending on ρ_0, E, α, and $\epsilon_0 > 0$, only depending on the a priori data, such that if $\epsilon \leq \epsilon_0$ then for every $P \in \partial\Omega_1 \cup \partial\Omega_2$ there exists a rigid transformation of coordinates under which $P = 0$ and*

$$\Omega_i \cap B_{r_0}(0) = \{x \in B_{r_0}(0) \text{ s.t. } x_n > \varphi_i(x')\}, \quad i = 1, 2 \,, \tag{2.15}$$

where φ_1, φ_2 are $C^{1,\alpha}$ functions on $B_{r_0}(0) \subset \mathbb{R}^{n-1}$ which satisfy, for every β, $0 < \beta < \alpha$,

$$\|\varphi_1 - \varphi_2\|_{C^{1,\beta}(B_{r_0}(0))} \leq \rho_0 K \omega(\tilde{\epsilon})^{\frac{\alpha-\beta}{1+\alpha}} \,, \tag{2.16}$$

where

$$\tilde{\epsilon} = \begin{cases} \dfrac{\epsilon}{\rho_0 \|\psi\|_{H^{-1/2}(A)}}, & \textit{when Theorem 2.1 applies}, \\ \dfrac{\epsilon}{\|g\|_{H^{1/2}(A)}}, & \textit{when Theorem 2.2 applies}, \end{cases}$$

ω is as in (2.12) *and $K > 0$ only depends on E, α and β. Furthermore, there exists a $C^{1,\alpha}$ diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(\Omega_2) = \Omega_1$ and for every β, $0 < \beta < \alpha$,*

$$\|F - Id\|_{C^{1,\beta}(\mathbb{R}^n)} \leq \rho_0 K \omega(\tilde{\epsilon})^{\frac{\alpha-\beta}{1+\alpha}}, \tag{2.17}$$

with K, ω as above. Here $Id : \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity mapping.

3. - Proof of Theorem 2.1

Here and in the sequel we shall denote by G the connected component of $\Omega_1 \cap \Omega_2$ such that $\Sigma \subset \bar{G}$.

The proof of Theorem 2.1 is obtained from the following sequence of propositions.

PROPOSITION 3.1 (Stability Estimate of Continuation from Cauchy Data). *Let the hypotheses of Theorem* 2.1, *except* (2.6), *be satisfied. We have*

$$\int_{\Omega_i \setminus G} |\nabla u_i|^2 \leq \rho_0^n \|\psi\|^2_{H^{-1/2}(A)} \omega\left(\frac{\epsilon}{\rho_0 \|\psi\|_{H^{-1/2}(A)}}\right), \quad i = 1, 2, \tag{3.1}$$

where ω is an increasing continuous function on $[0, \infty)$ *which satisfies*

$$\omega(t) \leq C(\log|\log t|)^{-\alpha/n}, \qquad \textit{for every } t < e^{-1}, \tag{3.2}$$

and $C > 0$ depends on λ, Λ, E, α and M only.

DEFINITION 3.1. Let Ω be a bounded domain in $\mathbb{R}^n$. We shall say that a portion S of $\partial\Omega$ is of *Lipschitz class with constants* ρ_0, $E > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{x \in B_{\rho_0}(0) \text{ s.t. } x_n > \varphi(x')\},$$

where φ is a Lipschitz continuous function on $B_{\rho_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = 0$$

and

$$\|\varphi\|_{C^{0,1}(B_{\rho_0}(0))} \leq E\rho_0 .$$

Here the $C^{0,1}$ norm is scaled according to the principles stated in Remark 2.1, that is

$$\|\varphi\|_{C^{0,1}\left(B_{\rho_0}(0)\right)} = \|\varphi\|_{L^\infty\left(B_{\rho_0}(0)\right)} + \rho_0|\varphi|_{1,B_{\rho_0}(0)}\,.$$

PROPOSITION 3.2 (Improved Stability Estimate of Continuation from Cauchy Data). *Let the hypotheses of Proposition* 3.1 *hold and, in addition, let us assume that there exist* $L > 0$ *and* r_0, $0 < r_0 \le \rho_0$, *such that* ∂G *is of Lipschitz class with constants* r_0, L. *Then* (3.1) *holds with* ω *given by*

$$\omega(t) \le C|\log t|^{-\gamma}, \qquad \text{for every } t < 1\,, \tag{3.3}$$

where $\gamma > 0$ *and* $C > 0$ *only depend on* λ, Λ, E, α, M, L *and* ρ_0/r_0.

We shall denote

$$\Omega_r = \{x \in \Omega \text{ s.t. } \operatorname{dist}(x, \partial\Omega) > r\}\,.$$

PROPOSITION 3.3 (Lipschitz Stability Estimate of Continuation from the Interior). *Let* Ω *be a domain satisfying* (2.1), *such that* $\partial\Omega$ *is of Lipschitz class with constants* ρ_0, E. *Let* $u \in H^1(\Omega)$ *be the solution to* (1.1), *where* ψ *satisfies*

$$\psi \in L^2(\partial\Omega)\,, \quad \psi \not\equiv 0\,, \tag{3.4a}$$

$$\int_{\partial\Omega} \psi = 0\,, \tag{3.4b}$$

and, for a given constant $F > 0$,

$$\frac{\|\psi\|_{L^2(\partial\Omega)}}{\|\psi\|_{H^{-1/2}(\partial\Omega)}} \le F\,. \tag{3.4c}$$

and σ *satisfies* (2.9). *For every* $\rho > 0$ *and every* $x_0 \in \Omega_{4\rho}$, *we have*

$$\int_{B_\rho(x_0)} |\nabla u|^2 \ge C\rho_0^n \|\psi\|^2_{H^{-1/2}(\partial\Omega)}\,, \tag{3.5}$$

where $C > 0$ *depends on* λ, Λ, E, M, F *and* ρ/ρ_0 *only.*

REMARK 3.1. Let us notice that if ψ satisfies (2.7*a*)-(2.7*d*), then it also satisfies (3.4*a*)-(3.4*c*) up to possibly replacing F with a multiple cF, where c only depends on E. In fact, for functions ψ satisfying (2.7*c*) the following equivalence relations can be obtained

$$\frac{1}{c}\|\psi\|_{H^{-1/2}(A)} \le \|\psi\|_{H^{-1/2}(\partial\Omega)} \le \|\psi\|_{H^{-1/2}(A)}\,. \tag{3.6}$$

PROPOSITION 3.4 (Interior Doubling Inequality). *Let* Ω *be a domain satisfying* (2.1), *such that* $\partial\Omega$ *is of Lipschitz class with constants* ρ_0, E. *Let* $u \in H^1(\Omega)$ *be the solution to* (1.1), *where* ψ *satisfies* (3.4) *and* σ *satisfies* (2.9). *For every* $\rho > 0$ *and every* $x_0 \in \Omega_\rho$, *we have*

$$\int_{B_{\beta r}(x_0)} |\nabla u|^2 \le C\beta^K \int_{B_r(x_0)} |\nabla u|^2\,, \quad \text{for every } r, \beta \text{ s.t. } 1 \le \beta \text{ and } 0 < \beta r \le \rho\,, \tag{3.7}$$

where $C > 0$ *and* $K > 0$ *depend on* λ, Λ, E, M, F *and* ρ/ρ_0 *only.*

PROPOSITION 3.5 (Doubling Inequality at the Boundary). *Let Ω be a domain satisfying* (2.1) *and* (2.5). *Let us assume that the accessible and inaccessible parts A, I of its boundary satisfy* (2.2)-(2.4) *and* (2.6). *Let $u \in H^1(\Omega)$ be the solution to* (1.1) *and let* (2.7) *and* (2.9) *be satisfied. Let $x_0 \in I$. For any $r > 0$ and any $\beta \geq 1$ we have*

$$\int_{\Omega\cap B_{\beta r}(x_0)} |\nabla u|^2 \leq C\beta^K \int_{\Omega\cap B_r(x_0)} |\nabla u|^2 , \tag{3.8}$$

where $C > 0$ and $K > 0$ depend on λ, Λ, E, M and F only.

In the sequel, it will be expedient to introduce a quantity which is a slight variation of the Hausdorff distance between $\bar{\Omega}_1$ and $\bar{\Omega}_2$.

DEFINITION 3.2. We call *modified distance* between Ω_1 and Ω_2 the number

$$d_m(\Omega_1, \Omega_2) = \max\left\{ \sup_{x\in\partial\Omega_1} \operatorname{dist}(x, \Omega_2),\ \sup_{x\in\partial\Omega_2} \operatorname{dist}(x, \Omega_1) \right\} . \tag{3.9}$$

Notice that we obviously have

$$d_m(\Omega_1, \Omega_2) \leq d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}) , \tag{3.10}$$

but, in general, d_m does not dominate the Hausdorff distance, and indeed it does not satisfy the axioms of a distance function. This is made clear by the following example: $\Omega_1 = B_1(0)$, $\Omega_2 = B_1(0) \setminus \overline{B_{1/2}(0)}$. In this case $d_m(\Omega_1, \Omega_2) = 0$, whereas $d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}) = 1/2$.

PROPOSITION 3.6 (Relative Graphs). *Let Ω_1, Ω_2 be bounded domains satisfying* (2.5). *There exist numbers d_0, r_0, $d_0 > 0$, $0 < r_0 \leq \rho_0$, for which the ratios $\frac{d_0}{\rho_0}$, $\frac{r_0}{\rho_0}$ only depend on α and E, such that if we have*

$$d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}) \leq d_0 , \tag{3.11}$$

then the following facts hold:

i) For every $P \in \partial\Omega_1$, up to a rigid transformation of coordinates which maps P into the origin, we have

$$\Omega_i \cap B_{r_0}(P) = \{x \in B_{r_0}(0) \text{ s.t. } x_n > \varphi_i(x')\} , \quad i = 1, 2 ,$$

where φ_1, φ_2 are $C^{1,\alpha}$ functions on $B_{r_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\|\varphi_1 - \varphi_2\|_{C^{1,\beta}(B_{r_0}(0))} \leq C\rho_0^{\frac{1+\beta}{1+\alpha}} (d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}))^{\frac{\alpha-\beta}{1+\alpha}} , \tag{3.12}$$

for every β, $0 < \beta < \alpha$,

where $C > 0$ only depends on α, β and E.

ii) There exists an absolute constant $C > 0$ such that

$$d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}) \leq C d_m(\Omega_1, \Omega_2) . \tag{3.13}$$

iii) Any connected component G of $\Omega_1 \cap \Omega_2$ has boundary of Lipschitz class with constants r_0, L, where r_0 is as above and $L > 0$ only depends on α and E.

PROOF OF THEOREM 2.1. Let us denote, for simplicity, $d = d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2})$. Let $\eta > 0$ be such that

$$\max_{i=1,2} \int_{\Omega_i \setminus G} |\nabla u_i|^2 \leq \eta . \tag{3.14}$$

Our first goal is the proof of the following inequality

$$d \leq C\rho_0 \left(\frac{\eta}{\rho_0^n \|\psi\|^2_{H^{-1/2}(A)}} \right)^{1/K} , \tag{3.15}$$

where $C > 0$ and $K > 0$ depend on λ, Λ, E, M and F only. As a preliminary step, let us show that (3.15) holds true when d is replaced with $d_m = d_m(\Omega_1, \Omega_2)$, the quantity introduced in Definition 3.2. Let us assume, with no loss of generality, that there exists $x_0 \in I_1 \subset \partial\Omega_1$ such that $\text{dist}(x_0, \Omega_2) = d_m$. From (3.14) we obviously have

$$\int_{\Omega_1 \cap B_{d_m}(x_0)} |\nabla u_1|^2 \leq \eta . \tag{3.16}$$

Suppose now $d_m < \rho_0$. By Proposition 3.5, picking $r = d_m$, $\beta = \frac{\rho_0}{d_m}$, we have

$$\int_{\Omega_1 \cap B_{d_m}(x_0)} |\nabla u_1|^2 \geq C \left(\frac{d_m}{\rho_0} \right)^K \int_{\Omega_1 \cap B_{\rho_0}(x_0)} |\nabla u_1|^2 , \tag{3.17}$$

where $C > 0$ and $K > 0$ depend on λ, Λ, E, M and F only. From (2.5) we can find a ball $B_r(w_0)$ of radius $r = \frac{\rho_0}{2\sqrt{1+E^2}}$ compactly contained in $\Omega_1 \cap B_{\rho_0}(x_0)$. Hence, applying Proposition 3.3 with $\rho = r/4$, we have

$$\int_{\Omega_1 \cap B_{\rho_0}(x_0)} |\nabla u_1|^2 \geq \int_{B_\rho(w_0)} |\nabla u_1|^2 \geq C\rho_0^n \|\psi\|^2_{H^{-1/2}(A)} , \tag{3.18}$$

where $C > 0$ depends on λ, Λ, E, M and F only. From (3.16)-(3.18) we derive

$$\eta \geq C\rho_0^n \|\psi\|^2_{H^{-1/2}(A)} \left(\frac{d_m}{\rho_0} \right)^K . \tag{3.19}$$

On the other hand, when $d_m \geq \rho_0$, (3.19) follows from (3.18) and from the trivial estimate

$$\frac{d_m}{\rho_0} \leq \frac{\text{diam}(\Omega_1) + \text{diam}(\Omega_2)}{\rho_0} \leq C , \tag{3.20}$$

with C only depending on E and M. Hence we have proved that

$$d_m \leq C\rho_0 \left(\frac{\eta}{\rho_0^n \|\psi\|^2_{H^{-1/2}(A)}} \right)^{1/K}, \tag{3.21}$$

where $C > 0$ and $K > 0$ depend on λ, Λ, E, M and F only.

With no loss of generality, let $y_0 \in \overline{\Omega_1} \setminus \overline{\Omega_2}$ be such that $\text{dist}(y_0, \overline{\Omega_2}) = d$. Let us notice that in general y_0 needs not to belong to $\partial\Omega_1$, see the example below Definition 3.2. For this reason it is necessary to analyse various different cases separately. Denoting by $h = \text{dist}(y_0, \partial\Omega_1)$, let us distinguish the following three cases:

i) $h \leq \frac{d}{2}$,
ii) $h > \frac{d}{2}$, $h > \frac{d_0}{2}$,
iii) $h > \frac{d}{2}$, $h \leq \frac{d_0}{2}$,
where d_0 is the number introduced in Proposition 3.6.

If case *i)* occurs, taking $z_0 \in \partial\Omega_1$ such that $|y_0 - z_0| = h$, we have that $\text{dist}(z_0, \overline{\Omega_2}) \geq d - h \geq \frac{d}{2}$, so that $d \leq 2d_m$ and (3.15) follows from (3.21).

If case *ii)* occurs, let us set

$$d_1 = \min\left\{\frac{d}{2}, \frac{d_0}{2}\right\}. \tag{3.22}$$

We have that

$$B_{d_1}(y_0) \subset \Omega_1 \setminus \Omega_2 . \tag{3.23}$$

By applying Proposition 3.4 with $r = d_1$, $\beta = \frac{d_0}{2d_1}$, we have

$$\eta \geq \int_{\Omega_1\setminus\Omega_2} |\nabla u_1|^2 \geq \int_{B_{d_1}(y_0)} |\nabla u_1|^2 \geq C\left(\frac{2d_1}{d_0}\right)^K \int_{B_{d_0/2}(y_0)} |\nabla u_1|^2, \tag{3.24}$$

where $C > 0$ and $K > 0$ depend on λ, Λ, E, α, M and F only. Since $h > \frac{d_0}{2}$, we can apply Proposition 3.3 with $\rho = \frac{d_0}{8}$, obtaining

$$\int_{B_{d_0/2}(y_0)} |\nabla u_1|^2 \geq C\rho_0^n \|\psi\|^2_{H^{-1/2}(A)}, \tag{3.25}$$

where $C > 0$ depends on λ, Λ, E, α, M and F only. From (3.24) and (3.25) we have

$$d_1 \leq \tilde{C}\rho_0 \left(\frac{\eta}{\rho_0^n \|\psi\|^2_{H^{-1/2}(A)}} \right)^{1/K}. \tag{3.26}$$

Let $\bar{\eta} = (\frac{d_0}{2\rho_0 \bar{C}})^K \rho_0^n \|\psi\|^2_{H^{-1/2}(A)}$. If $\eta < \bar{\eta}$, then $d_1 < \frac{d_0}{2}$, so that $d = 2d_1$ and (3.15) follows from (3.26). If, otherwise, $\eta \geq \bar{\eta}$, then (3.15) follows trivially, likewise we did in (3.20).

If case *iii*) occurs, then $d < d_0$ and Proposition 3.6 applies, so that by (3.13) and (3.19) we again obtain (3.15).

Hence, by Proposition 3.1, we obtain

$$d \leq C\rho_0 \left(\left| \log \left| \log \left(\frac{\epsilon}{\rho_0 \|\psi\|_{H^{-1/2}(A)}} \right) \right| \right| \right)^{-K}, \tag{3.27}$$

where $C > 0$ depends on λ, Λ, E, M and F only, whereas $K > 0$ depends on the same quantities and in addition on α. Thus we have obtained a stability estimate of log-log type. Next, by (3.27), we can find $\epsilon_0 > 0$, only depending on λ, Λ, E, α, M and F, such that if $\epsilon \leq \epsilon_0$ then $d \leq d_0$. Therefore, by Proposition 3.6, G satisfies the hypotheses of Proposition 3.2. Hence in (3.15) we may replace η with $\rho_0^n |\psi\|^2_{H^{-1/2}(A)} \omega(\frac{\epsilon}{\rho_0 \|\psi\|_{H^{-1/2}(A)}})$, where ω is as in Proposition 3.2 (a modulus of continuity of log type) and obtain (2.11), (2.12). □

4. – Proof of Theorem 2.2 and of Corollary 2.3

Here and in the sequel we shall denote by G the connected component of $\Omega_1 \cap \Omega_2$ such that $\Sigma \subset \bar{G}$.

The proof of Theorem 2.2 is obtained from the following sequence of propositions, which closely parallel Propositions 3.1-3.5.

PROPOSITION 4.1 (Stability Estimate of Continuation from Cauchy Data). *Let the hypotheses of Theorem 2.2 be satisfied. We have*

$$\max_{\Omega_i \setminus G} |u_i| \leq \|g\|^2_{H^{1/2}(A)} \omega \left(\frac{\epsilon}{\|g\|_{H^{1/2}(A)}} \right), \quad i = 1, 2, \tag{4.1}$$

where ω is an increasing continuous function on $[0, \infty)$ *which satisfies*

$$\omega(t) \leq C(\log|\log t|)^{-1/n}, \qquad \textit{for every } t < e^{-1}, \tag{4.2}$$

where $C > 0$ depends on λ, Λ, E, α and M only.

PROPOSITION 4.2 (Improved Stability Estimate of Continuation from Cauchy Data). *Let the hypotheses of Proposition* 4.1 *hold and, in addition, let us assume that there exist $L > 0$ and r_0, $0 < r_0 \leq \rho_0$, such that ∂G is of Lipschitz class with constants r_0, L. Then* (4.1) *holds with ω given by*

$$\omega(t) \leq C|\log t|^{-\gamma}, \qquad \text{for every } t < 1, \tag{4.3}$$

where $\gamma > 0$ and $C > 0$ only depend on λ, Λ, E, α, M, L and ρ_0/r_0.

PROPOSITION 4.3 (Lipschitz Stability Estimate of Continuation from the Interior). *Let Ω be a domain satisfying* (2.1), *such that $\partial\Omega$ is of Lipschitz class with constants ρ_0, E. Let $u \in H^1(\Omega)$ be the solution to* (1.2), *where g satisfies*

$$g \in H^{1/2}(\partial\Omega), \quad g \not\equiv 0, \tag{4.4a}$$

and, for a given constant $F > 0$,

$$\frac{\|g\|_{H^{1/2}(\partial\Omega)}}{\|g\|_{L^2(\partial\Omega)}} \leq F. \tag{4.4b}$$

and σ satisfies (2.9). *For every $\rho > 0$ and every $x_0 \in \Omega_{2\rho}$, we have*

$$\int_{B_\rho(x_0)} u^2 \geq C\rho_0^n \|g\|^2_{H^{1/2}(\partial\Omega)}, \tag{4.5}$$

where $C > 0$ depends on λ, Λ, E, M, F and ρ/ρ_0 only.

REMARK 4.1. Let us notice that if g satisfies (2.8a)-(2.8c), then it also satisfies (4.4a)-(4.4b) up to possibly replacing F with a multiple cF, where c only depends on E. In fact, for functions g satisfying (2.8b) the following equivalence relations can be obtained

$$\|g\|_{H^{1/2}(A)} \leq \|g\|_{H^{1/2}(\partial\Omega)} \leq c\|g\|_{H^{1/2}(A)}. \tag{4.6}$$

PROPOSITION 4.4 (Interior Doubling Inequality). *Let Ω be a domain satisfying* (2.1), *such that $\partial\Omega$ is of Lipschitz class with constants ρ_0, E. Let $u \in H^1(\Omega)$ be the solution to* (1.2), *where g satisfies* (4.4) *and σ satisfies* (2.9). *For every $\rho > 0$ and every $x_0 \in \Omega_\rho$, we have*

$$\int_{B_{\beta r}(x_0)} u^2 \leq C\beta^K \int_{B_r(x_0)} u^2, \quad \text{for every } r, \beta \text{ s.t. } 1 \leq \beta \text{ and } 0 < \beta r \leq \rho, \tag{4.7}$$

where $C > 0$ and $K > 0$ depend on λ, Λ, E, M, F and ρ/ρ_0 only.

PROPOSITION 4.5 (Doubling Inequality at the Boundary). *Let Ω be a domain satisfying* (2.1) *and* (2.5). *Let us assume that the accessible and inaccessible parts A, I of its boundary satisfy* (2.2)-(2.4). *Let $u \in H^1(\Omega)$ be the solution to* (1.2) *and let* (2.8) *and* (2.9) *be satisfied. Let $x_0 \in I$. For any $r > 0$ and any $\beta \geq 1$ we have*

$$\int_{\Omega\cap B_{\beta r}(x_0)} u^2 \leq C\beta^K \int_{\Omega\cap B_r(x_0)} u^2, \tag{4.8}$$

where $C > 0$ and $K > 0$ depend on λ, Λ, E, α, M and F only.

PROOF OF THEOREM 2.2. By using the trivial estimate

$$\int_{\Omega_i \setminus G} u_i^2 \leq M\rho_0^n \left(\max_{\Omega_i \setminus G} |u_i| \right)^2, \quad i = 1, 2,$$

the proof is obtained similarly to the proof of Theorem 2.1, up to obvious changes. □

PROOF OF COROLLARY 2.3. We have that (2.15)-(2.16) follow immediately from (3.12), (2.12) and either (2.11) (when Theorem 2.1 applies) or (2.14) (when Theorem 2.2 applies).

Next, let us prove (2.17). We can find r_1, h, $0 < r_1 < r_0$, $0 < h < r_0$, only depending on α, E, ρ_0, and a number N only depending on α, E, M, such that there exist points $P_l \in \partial\Omega_1$ and cylinders C^l, $l = 1, \ldots, N$, centered at P_l, having height $2h$ and basis a $(n-1)$-dimensional disk of radius r_1, such that $\cup_{l=1}^N C^l$ covers both $\partial\Omega_1$ and $\partial\Omega_2$, and each C^l has axis along the direction labeled by x_n in the local representation (2.15) when $P = P_l$. Moreover we assume $2C^l \subset B_{r_0}(P_l)$ for every l. Here $2C^l$ denotes the cylinder with double sizes and the same center. Notice that, possibly replacing ϵ_0 by a smaller number, we may assume that the functions φ_i in (2.15) satisfy

$$|\varphi_1(x')|, |\varphi_2(x')| \leq \frac{h}{2}, \quad \text{for every } x', |x'| \leq 2r_1 .$$

Let us fix $l = 1$ and let us define $F_1 : \mathbb{R}^n \to \mathbb{R}^n$ as follows. Letting $x = (x', x_n)$ suitable coordinates near $P = P_1$,

$$F_1(x', x_n) = (x', z(x', x_n)),$$

where

$$z(x', x_n) = \eta(x')\tau(\varphi_1(x'), \varphi_2(x'), x_n) + (1 - \eta(x'))x_n .$$

Here: η, $0 \leq \eta \leq 1$, is a smooth function such that $\eta(x') \equiv 1$ when $|x'| \leq r_1$, $\eta(x') \equiv 0$ when $|x'| \geq 2r_1$, and $\tau(a, b; \cdot) : \mathbb{R} \to \mathbb{R}$ is an uniformly smooth function for every a, $b \in [-h/2, h/2]$ satisfying

$$\frac{\partial \tau}{\partial s}(a, b; s) \geq c > 0, \quad \text{for every } s \in \mathbb{R},$$
$$\tau(a, b; s) = s, \quad \text{for every } s, |s| \geq h,$$

and also

$$\tau(a, b; b) = a .$$

Here $c < 1$ is an absolute constant. For instance we can choose τ as a suitable smoothing of the piecewise linear function whose graph joins $(-h, -h)$, (b, a),

(h, h), within the square $(-h, h) \times (-h, h)$ and coincides with the bisector of the first and third quadrant outside. Now we have

$$\frac{\partial F_1}{\partial(x', x_n)} = \begin{pmatrix} Id & 0 \\ (\nabla_{x'} z)^T & \frac{\partial z}{\partial x_n} \end{pmatrix},$$

and hence

$$\left|\det\left(\frac{\partial F_1}{\partial(x', x_n)}\right)\right| = \frac{\partial z}{\partial x_n} \geq c > 0\,.$$

One can verify that $F_1(\Omega_2 \cap C^1) = \Omega_1 \cap C^1$, $F_1(x) = x$ for every $x \in \partial\Omega_1 \cap \partial\Omega_2$, $F = F_1$ satisfies (2.17) and also that if Ω_2 is replaced with $F_1(\Omega_2)$, then (2.16) continues to hold. We may iterate this procedure defining inductively analogous maps F_l which deform coordinates within the cylinder $2C^l$ and replacing at each stage Ω_2 with $F_l(\Omega_2)$. In the end we set $F = F_N \circ \ldots \circ F_1$. F is an orientation preserving $C^{1,\alpha}$ diffeomorphism satisfying (2.17) such that $F(\partial\Omega_2) = \partial\Omega_1$ and also $F = Id$ outside of the fixed small neighbourhood of $\partial\Omega_1$ given by $\cup_{l=1}^{N} 2C^l$. Therefore $F(\Omega_2) = \Omega_1$. □

5. – Proofs of the estimates of continuation for Cauchy problems

Throughout this section, let Ω_1, Ω_2 be two domains satisfying (2.1), (2.5). Let A_i, I_i, $i = 1, 2$, be the corresponding accessible and inaccessible parts of their boundaries. Let us assume that $A_1 = A_2 = A$, Ω_1, Ω_2 lie on the same side of A and that (2.2)-(2.4) are satisfied by both pairs A_i, I_i.

We shall denote

$$\mathcal{U}_i^\rho = \{x \in \overline{\Omega_i} \text{ s.t. } \operatorname{dist}(x, A_{\rho_0}) \leq \rho\}\,.$$

It is clear that

$$\mathcal{U}_1^\rho = \mathcal{U}_2^\rho = \mathcal{U}^\rho, \qquad \text{for every } \rho < \rho_0\,.$$

LEMMA 5.1. *The following Schauder type estimates hold*

$$\|u_i\|_{C^{1,\alpha}\left(\overline{\Omega_i \setminus \mathcal{U}^{\rho_0/8}}\right)} \leq C\rho_0 \|\psi\|_{H^{-1/2}(A)}, \quad \text{for } i = 1, 2, \tag{5.1}$$

$$\|u_1 - u_2\|_{C^{1,\alpha}(\overline{\Omega_1 \cap \Omega_2})} \leq C\rho_0 \|\psi\|_{H^{-1/2}(A)}, \tag{5.2}$$

where $C > 0$ depends on λ, Λ, E, α and M only.

PROOF OF LEMMA 5.1. Set $\Omega = \Omega_i$, $u = u_i$, $i = 1, 2$. Since $\sigma\nabla u \cdot \nu = 0$ on $\partial\Omega \setminus A_{\rho_0}$, by standard elliptic estimates we have

$$\|u\|_{C^{1,\alpha}\left(\overline{\Omega\setminus\mathcal{U}^{\rho_0/8}}\right)} \leq C\|u\|_{L^\infty\left(\overline{\Omega\setminus\mathcal{U}^{\rho_0/16}}\right)} \leq C\|u\|_{L^2(\Omega)}, \tag{5.3}$$

where C depends on λ, E, α and M only. Moreover we have

$$\lambda\|\psi\|_{H^{-1/2}(\partial\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} \leq \lambda^{-1}\|\psi\|_{H^{-1/2}(\partial\Omega)}. \tag{5.4}$$

From (5.3), (5.4), (3.6) and Poincaré inequality, (5.1) follows. Since $\sigma\nabla(u_1 - u_2)\cdot\nu = 0$ on A, we derive similarly a bound for the $C^{1,\alpha}$ norm of $u_1 - u_2$ in $\mathcal{U}^{\rho_0/2}$ and, in view of (5.1), (5.2) follows. □

In the proof of Proposition 3.1 we shall need to approximate the domains Ω_r with *regularized domains*, say $\tilde{\Omega}_r$, $r > 0$. We shall define $\tilde{\Omega}_r$, roughly speaking, as the level set of a *regularized distance* $\tilde{d}$, approximating $d(\cdot, \partial\Omega)$, which was constructed by Lieberman (see [Li]). To this aim, let us state the following Lemma 5.2 (about *regularized distance*) and Lemma 5.3 (about *regularized domains*).

LEMMA 5.2 (Lieberman). *For any bounded domain Ω satisfying* (2.5), *one can construct a function $\tilde{d} \in C^2(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ (called regularized distance) such that the following facts hold.*

$$\gamma_0 \leq \frac{\operatorname{dist}(x, \partial\Omega)}{\tilde{d}(x)} \leq \gamma_1, \tag{5.5}$$

$$|\nabla\tilde{d}(y)| \geq C_1, \qquad \textit{for every } y \in \bar{\Omega}\setminus\Omega_{b\rho_0}, \tag{5.6}$$

$$\|\tilde{d}\|_{1,\alpha} \leq C_2\rho_0, \tag{5.7}$$

where γ_0, γ_1, b, C_1 and C_2 are positive constants only depending on E, α. Furthermore, let Ω_1, Ω_2 be two domains as above, with A_i, I_i being the accessible and inaccessible parts of $\partial\Omega_i$, $i = 1, 2$, as in (2.2), *and such that, as in Theorem 2.1, $A_1 = A_2 = A$ and Ω_1, Ω_2 lie on the same side of A. If $\tilde{d}_i$ are the regularized distances associated to Ω_i, $i = 1, 2$, then we have*

$$\tilde{d}_1 = \tilde{d}_2, \qquad \textit{in } \mathcal{U}^{\rho_0/4}. \tag{5.8}$$

PROOF OF LEMMA 5.2. The proof follows from [Li, Theorem 1.3, Theorem 2.1, Theorem 2.3]. □

LEMMA 5.3. *For any bounded domain Ω satisfying* (2.5), *one can construct a family of regularized domains $\tilde{\Omega}_h \subset \Omega$, for $0 < h \leq a\rho_0$, having C^1 boundary such that*

$$\tilde{\Omega}_{h_2} \subset \tilde{\Omega}_{h_1}, \qquad 0 < h_1 \leq h_2, \tag{5.9}$$

$$\gamma_0 h \leq \operatorname{dist}(x, \partial\Omega) \leq \gamma_1 h, \qquad \textit{for every } x \in \partial\tilde{\Omega}_h, \tag{5.10}$$

$$|\Omega\setminus\tilde{\Omega}_h| \leq \gamma_2 M\rho_0^{n-1}h, \tag{5.11}$$

$$|\partial\tilde{\Omega}_h|_{n-1} \leq \gamma_3 M\rho_0^{n-1}, \tag{5.12}$$

for every $x \in \partial\tilde{\Omega}_h$, there exists $y \in \partial\Omega$ s.t.

$$(5.13) \qquad |y - x| = \operatorname{dist}(x, \partial\Omega), \quad |\nu(x) - \nu(y)| \le \gamma_4 \frac{h^\alpha}{\rho_0^\alpha},$$

where $\nu(x)$, $\nu(y)$ denote the outer unit normal to $\tilde{\Omega}_h$ at x and to Ω at y respectively, and a, γ_j, $j = 0, 1, \dots, 4$, are positive constants depending on E and α only. Here $|\cdot|_{n-1}$ denotes the surface measure. Furthermore, let Ω_1, Ω_2 be as in Lemma 5.2. If $\tilde{\Omega}_{i,h}$ are the families of regularized domains associated to Ω_i, $i = 1, 2$, then we have

$$(5.14) \qquad \tilde{\Omega}_{1,h} \cap \mathcal{U}^{\rho_0/4} = \tilde{\Omega}_{2,h} \cap \mathcal{U}^{\rho_0/4}.$$

PROOF OF LEMMA 5.3. Let $\tilde{d}$ be the Lieberman's *regularized distance* introduced in Lemma 5.2. Let

$$a = \min\left\{ \frac{b}{\gamma_1}, \frac{1}{\gamma_0} \left(\frac{C_1}{2\sqrt{n}C_2} \right)^{1/\alpha} \right\}.$$

For $h \le a\rho_0$, let $\tilde{\Omega}^h$ be the connected component of the set $\{x \in \Omega \text{ s.t. } \tilde{d}(x) < h\}$ whose closure contains $\partial\Omega$. Let us define $\tilde{\Omega}_h = \bar{\Omega} \setminus \overline{\tilde{\Omega}^h}$. We have that (5.9) is trivial and (5.10) follows from (5.5). Since, by (5.5), $\tilde{\Omega}^h \subset \Omega^{\gamma_1 h}$, (5.11) follows from the following estimate

$$(5.15) \qquad |\Omega \setminus \Omega_r| \le C|\Omega| \frac{r}{\rho_0},$$

where C depends on E only. It is nearly evident that, locally, the width of $\Omega \setminus \Omega_r$ is of the order of r, as $r \to 0$. A complete proof of (5.15) requires somewhat lengthy but not difficult estimates, details can be found in [Al-Ros, Lemma 2.8]. By (5.10) and by the choice of a, we have that $\operatorname{dist}(x, \partial\Omega) \le b\rho_0$, for any $x \in \partial\tilde{\Omega}_h$. Applying the implicit function theorem, from (5.6) and (5.7) we have that $\partial\tilde{\Omega}_h$ is a surface of class C^1 with constants ρ_1, E_1, where

$$\rho_1 = h\gamma_0, \qquad E_1 = \frac{2\sqrt{n(n-1)}C_2}{C_1}.$$

In order to prove (5.12), let us tessellate $\mathbb{R}^n$ with internally nonoverlapping closed cubes of diameter $\gamma_0 h$. Let $\{Q_1, \dots, Q_N\}$ be the collection of those cubes having nonempty intersection with $\partial\tilde{\Omega}_h$. By (5.10), each Q_i is contained in $\Omega \setminus \Omega_{(\gamma_0+\gamma_1)h}$, so that

$$(5.16) \qquad \frac{N\gamma_0^n h^n}{n^{n/2}} \le |\Omega \setminus \Omega_{(\gamma_0+\gamma_1)h}|.$$

From (5.15) and (5.16) we have that

$$|\partial\tilde{\Omega}_h|_{n-1} = \sum_{j=1}^{N} |\partial\tilde{\Omega}_h \cap Q_j|_{n-1} \le \gamma_3 M \rho_0^{n-1},$$

where γ_3 only depends on E and α. For any $x \in \partial\tilde{\Omega}_h$ and for any $y \in \partial\Omega$ such that $|y-x| = \operatorname{dist}(x, \partial\Omega)$, we have

$$\nu(x) = -\frac{\nabla\tilde{d}(x)}{|\nabla\tilde{d}(x)|}, \qquad \nu(y) = -\frac{\nabla\tilde{d}(y)}{|\nabla\tilde{d}(y)|},$$

so that (5.13) follows easily from (5.5)-(5.7). Finally, (5.14) follows from (5.8). □

In order to derive the Cauchy estimates for the difference of the solutions u_1 and u_2, first of all we need to dominate in terms of ϵ the L^2 norm of $u_1 - u_2$ and of $\nabla(u_1 - u_2)$ in a neighbourhood in Σ of the point $P_1 \in \Sigma$ appearing in (2.3).

According to (2.5), there exists a cartesian coordinate system under which $P_1 = 0$ and

$$\Omega_i \cap B_{\rho_0}(0) = \{x \in B_{\rho_0}(0) \text{ s.t. } x_n > \varphi(x')\}, \quad i = 1, 2,$$

where φ is a $C^{1,\alpha}$ function on $B_{\rho_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$\|\varphi\|_{C^{1,\alpha}(B_{\rho_0}(0))} \le E\rho_0.$$

Let

$$r_1 = \frac{\rho_0}{\sqrt{1+E^2}}, \tag{5.17}$$

$$\Sigma_0 = \{(x', x_n) \text{ s.t. } |x'| < r_1, x_n = \varphi(x')\}.$$

By the choice of r_1, we have $\Sigma_0 \subset \Sigma$.

Proof of Proposition 3.1 (Preparation). We premise the proof with two auxiliary steps.

Step 1. Let $\epsilon < \rho_0 \|\psi\|_{H^{-1/2}(A)}$. We have

$$\begin{aligned} &\int_{\Sigma_0} (u_1 - u_2)^2 + \rho_0^2 \int_{\Sigma_0} |\nabla(u_1 - u_2)|^2 \\ &\le C\rho_0^{n-1} (\rho_0 \|\psi\|_{H^{-1/2}(A)})^2 \left(\frac{\epsilon}{\rho_0 \|\psi\|_{H^{-1/2}(A)}} \right)^{\frac{4\alpha}{2\alpha+n+1}}, \end{aligned} \tag{5.18}$$

where $C > 0$ depends on λ, Λ, E, α and M only.

STEP 2. Let $\epsilon < \rho_0\|\psi\|_{H^{-1/2}(A)}$. There exist $C>0$ only depending on λ, Λ and E, and δ, $0<\delta<1$, only depending on λ, Λ, E and α, such that

$$(5.19)\qquad \int_{B_{\bar\rho}(z_0)} |\nabla(u_1-u_2)|^2 \le C\rho_0^n\|\psi\|^2_{H^{-1/2}(A)}\left(\frac{\epsilon}{\rho_0\|\psi\|_{H^{-1/2}(A)}}\right)^{2\delta},$$

where $z_0 = P_1 - \frac{r_1}{16}\nu$, $\bar\rho = \frac{\rho_0}{16(1+E^2)}$ and ν denotes the outer unit normal to Ω at P_1.

PROOF OF STEP 1. Let us denote

$$(5.20)\qquad w = u_1 - u_2,$$

and let v defined in $B_{r_1}(0)\subset\mathbb{R}^{n-1}$ by

$$v(x') = w(x',\varphi(x')).$$

Given $x=(x',\varphi(x'))\in\Sigma_0$, we have

$$(5.21)\qquad v_{x_i}(x') = w_{x_i}(x) + w_{x_n}(x)\varphi_{x_i}(x'),\quad i=1,\dots,n-1,$$

$$(5.22)\qquad \nu(x) = \frac{(-\nabla\varphi(x'),1)}{\sqrt{1+|\nabla\varphi(x')|^2}}.$$

Denoting by $\nabla_s w$ the tangential gradient of w in Σ_0, from (5.21) and (5.22) we have

$$(5.23)\qquad |\nabla_s w(x)| \le |\nabla v(x')|.$$

Since $(\sigma\nabla w\cdot\nu)(x)=0$, we have

$$(5.24)\qquad \lambda\left|\frac{\partial w}{\partial\nu}\right| \le C|\nabla_s w|,$$

with C only depending on λ. Hence

$$(5.25)\qquad |\nabla w| \le C|\nabla_s w|,\qquad \text{on } \Sigma_0,$$

where C only depends on λ.

Let us recall the following interpolation inequalities

$$(5.26)\qquad \|\nabla v\|_\infty \le C\left(|\nabla v|_\alpha^{1/(1+\alpha)}\|v\|_\infty^{\alpha/(1+\alpha)} + \frac{1}{\rho_0}\|v\|_\infty\right),$$

where C only depends on E and α,

$$(5.27)\qquad \|v\|_\infty \le C\left(\|\nabla v\|_\infty^{(n-1)/(n+1)}\|v\|_{L^2}^{2/(n+1)}\rho_0^{(n-1)/(n+1)} + \|v\|_{L^2}\right),$$

where C only depends on E. From (5.26) we have

$$\|v\|_{C^1} \leq C\|v\|_{\infty}^{\alpha/(1+\alpha)}\|v\|_{C^{1,\alpha}}^{1/(1+\alpha)}, \tag{5.28}$$

where C only depends on E and α, and from (5.27) we have

$$\|v\|_{\infty} \leq C\|v\|_{L^2}^{2/(n+1)}\|v\|_{C^1}^{(n-1)/(n+1)}, \tag{5.29}$$

where C only depends on E. From (5.28) and (5.29) it follows that

$$\|v\|_{C^1} \leq C\|v\|_{C^{1,\alpha}}^{\frac{n+1}{2\alpha+n+1}}\|v\|_{L^2}^{\frac{2\alpha}{2\alpha+n+1}}, \tag{5.30}$$

where C only depends on E and α. By using (5.21), we can estimate the norms of v in terms of those of w:

$$\|v\|_{\infty} \leq \|w\|_{\infty,\Sigma_0}, \tag{5.31}$$

$$\|\nabla v\|_{\infty} \leq (1+E)\|\nabla w\|_{\infty,\Sigma_0}, \tag{5.32}$$

$$\|v\|_{L^2(B_{r_1}(0))} \leq \|w\|_{L^2(\Sigma_0)}. \tag{5.33}$$

Since $|x-y| \leq \sqrt{1+E^2}|x'-y'|$, we have

$$|\nabla v|_{\alpha} \leq (1+E)(1+E^2)^{\alpha/2}|\nabla w|_{\alpha,\mathcal{U}^{\rho_0/2}} + \frac{E}{\rho_0^{\alpha}}\|\nabla w\|_{\infty}. \tag{5.34}$$

By using (5.31), (5.32) and (5.34), we have that

$$\|v\|_{C^{1,\alpha}} \leq C\|w\|_{C^{1,\alpha}(\mathcal{U}^{\rho_0/2})}, \tag{5.35}$$

where C only depends on E. From (5.17), (5.23), (5.25), (5.30), (5.33) and (5.35) we have

$$\rho_0\|\nabla w\|_{\infty,\Sigma_0} \leq C\|w\|_{C^{1,\alpha}(\mathcal{U}^{\rho_0/2})}^{\frac{n+1}{2\alpha+n+1}}\|w\|_{L^2(\Sigma_0)}^{\frac{2\alpha}{2\alpha+n+1}},$$

where C depends on E, α and λ only. From (5.2) we have

$$\|\nabla w\|_{\infty,\Sigma_0} \leq C\|\psi\|_{H^{-1/2}(A)}\left(\frac{\epsilon}{\rho_0\|\psi\|_{H^{-1/2}(A)}}\right)^{\frac{2\alpha}{2\alpha+n+1}},$$

where C depends on λ, Λ, E and α only. Notice also that the surface measure $|\Sigma_0|$ of Σ_0 satisfies

$$|\Sigma_0| \leq \frac{\omega_{n-1}\rho_0^{n-1}}{(1+E^2)^{(n-2)/2}},$$

hence we have

$$\left(\int_{\Sigma_0} |\nabla w|^2\right)^{1/2} \leq C\rho_0^{(n-1)/2} \|\psi\|_{H^{-1/2}(A)} \left(\frac{\epsilon}{\rho_0 \|\psi\|_{H^{-1/2}(A)}}\right)^{\frac{2\alpha}{2\alpha+n+1}},$$

where C depends on λ, Λ, E and α only. Therefore (5.18) follows. □

PROOF OF STEP 2. We shall make use of the stability estimates for the Cauchy problem for elliptic equations in divergence form with Lipschitz coefficients established by Trytten ([T]).

Let $P_0 = P_1 + \frac{r_1}{4}\nu$ and let $w = u_1 - u_2$. For appropriately chosen $p > 1$, $K > 0$, only depending on λ, Λ and E, we can derive from [T] the following estimate

$$\begin{aligned} \mathcal{F}\left(\frac{r_1}{2}\right) \leq \frac{C}{\rho_0^p} & \left(\int_{\Sigma_0} w^2 + \rho_0^2 \int_{\Sigma_0} |\nabla w|^2\right)^{\delta_1} \\ & \times \left(\int_{\Sigma_0} w^2 + \rho_0^2 \int_{\Sigma_0} |\nabla w|^2 + \rho_0 \int_G \sigma \nabla w \cdot \nabla w\right)^{1-\delta_1}, \end{aligned} \tag{5.36}$$

where

$$\mathcal{F}(r) = \int_{r_2}^{r} s^{-p} \int_{G\cap B_s(P_0)} \sigma \nabla w \cdot \nabla w + \frac{K}{\rho_0^p}\left(\int_{\Sigma_0} w^2 + \rho_0^2 \int_{\Sigma_0} |\nabla w|^2\right), \tag{5.37}$$

$$r_2 = \frac{1}{4\sqrt{1+E^2}} r_1, \tag{5.38}$$

with δ_1, $0 < \delta_1 < 1$ and $C > 0$ only depending on λ, Λ and E. By (2.9a), (3.6) and (5.4) we have

$$\begin{aligned} \int_G \sigma \nabla w \cdot \nabla w & \leq 2\left(\int_{\Omega_1} \sigma \nabla u_1 \cdot \nabla u_1 + \int_{\Omega_2} \sigma \nabla u_2 \cdot \nabla u_2\right) \\ & \leq 4\lambda^{-3} \rho_0^n \|\psi\|^2_{H^{-1/2}(A)}. \end{aligned} \tag{5.39}$$

On the other hand,

$$\mathcal{F}\left(\frac{r_1}{2}\right) \geq \int_{\frac{3}{8}r_1}^{\frac{1}{2}r_1} s^{-p} \int_{G\cap B_s(P_0)} \sigma \nabla w \cdot \nabla w \geq C\rho_0^{1-p} \int_{G\cap B_{\frac{3}{8}r_1}(P_0)} |\nabla w|^2, \tag{5.40}$$

where C depends on λ, Λ and E only. Recalling that $\epsilon < \rho_0 \|\psi\|_{H^{-1/2}(A)}$, from (5.18), (5.36), (5.39) and (5.40) we have that

$$\int_{G\cap B_{\frac{3}{8}r_1}(P_0)} |\nabla w|^2 \leq C\rho_0^n \|\psi\|^2_{H^{-1/2}(A)} \left(\frac{\epsilon}{\rho_0 \|\psi\|_{H^{-1/2}(A)}}\right)^{2\delta},$$

where $\delta = \frac{2\alpha\delta_1}{2\alpha+n+1}$. Finally, it is evident that $B_{\bar{\rho}}(z_0) \subset G \cap B_{\frac{3}{8}r_1}(P_0) \subset G$, so that the thesis follows. □

PROOF OF PROPOSITION 3.1 (Conclusion). With no loss of generality we can assume that $\epsilon \leq \rho_0 \|\psi\|_{H^{-1/2}(A)} \tilde{\mu}$, where $\tilde{\mu}$, $0 < \tilde{\mu} < e^{-1}$, is a constant only depending on λ, Λ, E, α and M, which will be chosen later on, since, otherwise, (3.1) becomes trivial.

Let $\theta = \min\{a, \frac{1}{16(1+E^2)\gamma_1}\}$, where a, γ_1 have been introduced in Lemma 5.3. We have that θ depends on E and α only. Let $\bar{r} = \rho_0\theta$ and let

$$\Sigma^i_{\gamma_1\bar{r}} = \{x \in \Omega_i \text{ s.t. } \operatorname{dist}(x, \Sigma) = \operatorname{dist}(x, \partial\Omega_i) = \gamma_1\bar{r}\}, \qquad i = 1, 2.$$

Since $\gamma_1\bar{r} \leq \frac{\rho_0}{16}$, we have

$$\Sigma^1_{\gamma_1\bar{r}} = \Sigma^2_{\gamma_1\bar{r}} = \Sigma_{\gamma_1\bar{r}}.$$

Let $r \leq \bar{r}$, that is $\frac{r}{\rho_0} \leq \theta$. By (5.10) we have $\Sigma_{\gamma_1\bar{r}} \subset \overline{\tilde{\Omega}_{1,r} \cap \tilde{\Omega}_{2,r}}$. Let $\tilde{V}_r$ be the connected component of $\tilde{\Omega}_{1,r} \cap \tilde{\Omega}_{2,r}$ whose closure contains $\Sigma_{\gamma_1\bar{r}}$. Let us notice that, by (5.14), $\operatorname{dist}(x, \Sigma) \geq \rho_0/4$ for any $x \in \overline{\tilde{\Omega}_{i,r} \setminus \tilde{V}_r}$. Let us prove (3.1)-(3.2) when $i = 1$, the case $i = 2$ being analogous. We have

$$\begin{aligned} &\Omega_1 \setminus G \subset [(\Omega_1 \setminus \tilde{\Omega}_{1,r}) \setminus G] \cup [\tilde{\Omega}_{1,r} \setminus \tilde{V}_r], \\ &\partial(\tilde{\Omega}_{1,r} \setminus \tilde{V}_r) = \tilde{\Gamma}_{1,r} \cup \tilde{\Gamma}_{2,r}, \end{aligned}$$

where $\tilde{\Gamma}_{1,r}$ is the part of boundary contained in $\partial\tilde{\Omega}_{1,r}$ and $\tilde{\Gamma}_{2,r}$ is the part contained in $\partial\tilde{\Omega}_{2,r} \cap \partial\tilde{V}_r$. Therefore we have

$$\int_{\Omega_1 \setminus G} |\nabla u_1|^2 \leq \int_{(\Omega_1 \setminus \tilde{\Omega}_{1,r}) \setminus G} |\nabla u_1|^2 + \int_{\tilde{\Omega}_{1,r} \setminus \tilde{V}_r} |\nabla u_1|^2. \tag{5.41}$$

By (5.1) and (5.11) we have

$$\int_{(\Omega_1 \setminus \tilde{\Omega}_{1,r}) \setminus G} |\nabla u_1|^2 \leq C\rho_0^{n-1} \|\psi\|^2_{H^{-1/2}(A)} r \leq C\rho_0^{n-\alpha} \|\psi\|^2_{H^{-1/2}(A)} r^\alpha, \tag{5.42}$$

with C only depending on λ, Λ, E, α and M. From the divergence theorem we have

$$\int_{\tilde{\Omega}_{1,r} \setminus \tilde{V}_r} |\nabla u_1|^2 \leq \lambda^{-1} \left(\int_{\tilde{\Gamma}_{1,r}} |(\sigma\nabla u_1 \cdot \nu) u_1| + \int_{\tilde{\Gamma}_{2,r}} |(\sigma\nabla u_1 \cdot \nu) u_1| \right). \tag{5.43}$$

Let $x \in \tilde{\Gamma}_{1,r}$. By (5.10), $\operatorname{dist}(x, \partial\Omega_1) \leq \gamma_1 r$. On the other hand, $x \in \overline{\tilde{\Omega}_{1,r} \setminus \tilde{V}_r}$, so that, as noticed above, $\operatorname{dist}(x, \Sigma) \geq \rho_0/4 > \gamma_1 r$. Hence, there exists $y \in$

$\partial\Omega_1 \setminus \Sigma$ such that $|y-x| = \text{dist}(x, \partial\Omega_1) \leq \gamma_1 r$. Since $(\sigma\nabla u_1 \cdot \nu)(y) = 0$, from (5.1), (5.10) and (5.13) we have

$$|(\sigma\nabla u_1 \cdot \nu)(x)| \leq C\|\psi\|_{H^{-1/2}(A)} \left(\frac{r}{\rho_0}\right)^{\alpha}, \tag{5.44}$$

where C only depends on λ, E, α and M.

Similarly, given $x \in \tilde{\Gamma}_{2,r}$, there exists $y \in \partial\Omega_2 \setminus \Sigma$ such that $|y-x| = \text{dist}(x, \partial\Omega_2) \leq \gamma_1 r$. Since $(\sigma\nabla u_2 \cdot \nu)(y) = 0$, we have

$$|(\sigma\nabla u_1 \cdot \nu)(x)| \leq C\left(\|\psi\|_{H^{-1/2}(A)} \left(\frac{r}{\rho_0}\right)^{\alpha} + |\nabla w(x)|\right), \tag{5.45}$$

where w is given by (5.20) and C only depends on λ, E, α and M. From (5.1), (5.12), (5.41)-(5.45), we have

$$\int_{\Omega_1\setminus G} |\nabla u_1|^2 \leq C\rho_0^n \left(\|\psi\|^2_{H^{-1/2}(A)} \left(\frac{r}{\rho_0}\right)^{\alpha} + \|\psi\|_{H^{-1/2}(A)} \max_{\overline{\tilde{V}_r}} |\nabla w|\right), \tag{5.46}$$

where C only depends on λ, E, α and M.

In order to estimate $\max_{\overline{\tilde{V}_r}} |\nabla w|$, we shall make use of Step 2.

Since $\min\{\text{dist}(z_0, \partial\Omega_1), \text{dist}(z_0, \partial\Omega_2)\} \geq \frac{\rho_0}{16(1+E^2)} \geq \gamma_1 r$, we have that $z_0 \in \overline{\tilde{V}_r}$, where z_0 has been introduced in Step 2. Let x be any other point in $\overline{\tilde{V}_r}$. Since $\min\{\text{dist}(x, \partial\Omega_1), \text{dist}(x, \partial\Omega_2)\} \geq \gamma_0 r$, it follows that $x \in \overline{G_{\gamma_0 r}}$. Let γ be an arc in $\overline{\tilde{V}_r}$ joining x to z_0. Let us define $\{x_i\}$, $i = 1, \dots, s$, as follows: $x_1 = z_0$, $x_{i+1} = \gamma(t_i)$, where $t_i = \max\{t \text{ s. t. } |\gamma(t) - x_i| = \frac{\gamma_0 r}{2}\}$ if $|x_i - x| > \frac{\gamma_0 r}{2}$, otherwise let $i = s$ and stop the process. By construction, the balls $B_{\frac{\gamma_0 r}{4}}(x_i)$ are pairwise disjoint, $|x_{i+1} - x_i| = \frac{\gamma_0 r}{2}$, for $i = 1, \dots, s-1$, $|x_s - x| \leq \frac{\gamma_0 r}{2}$. Hence we have $s \leq S\left(\frac{\rho_0}{r}\right)^n$, with S only depending on E, α and M.

At this stage we shall make use of a three spheres inequality for solutions v to (1.1a), where σ satisfies (2.9), more precisely: for every β_1, β_2, $1 < \beta_1 < \beta_2$, there exist τ, $0 < \tau < 1$, $C \geq 1$, only depending on λ, Λ, β_1 and β_2, such that for every $x \in \Omega_{\beta_2 r}$ we have

$$\int_{B_{\beta_1 r}(x)} v^2 \leq C\left(\int_{B_r(x)} v^2\right)^{\tau} \left(\int_{B_{\beta_2 r}(x)} v^2\right)^{1-\tau}. \tag{5.47}$$

This result can be derived, through minor adaptations, from the estimates found by Garofalo and Lin in their proof of the unique continuation properties for this type of equations, [G-L]. See also [Ku,Theorem 4.1]. In particular, writing (5.47) for $v-c$, with $c = (\frac{1}{\omega_n r^n})\int_{B_r(x)} v$ and with $\beta_1 = 7r/2$, $\beta_2 = 4r$, and applying Caccioppoli and Poincaré inequalities, we have

$$\int_{B_{3r}(x)} |\nabla v|^2 \leq C\left(\int_{B_r(x)} |\nabla v|^2\right)^{\tau}\left(\int_{B_{4r}(x)} |\nabla v|^2\right)^{1-\tau}. \tag{5.48}$$

An iterated application of the three spheres inequality (5.48) for w gives that there exist τ, $0<\tau<1$, $C \geq 1$, only depending on λ and Λ such that for any r, $0 < r \leq \bar{r}$,

$$\int_{B_{\frac{\gamma_0 r}{4}}(x)} |\nabla w|^2 \leq C\left(\int_G |\nabla w|^2\right)^{1-\tau^s}\left(\int_{B_{\frac{\gamma_0 r}{4}}(z_0)} |\nabla w|^2\right)^{\tau^s}. \tag{5.49}$$

From now on, let us denote

$$\tilde{\epsilon} = \frac{\epsilon}{\rho_0\|\psi\|_{H^{-1/2}(A)}}.$$

Since $\frac{\gamma_0 r}{4} \leq \bar{\rho}$, we can estimate the right hand side of (5.49) by Step 2 and by (5.39) and obtain

$$\int_{B_{\frac{\gamma_0 r}{4}}(x)} |\nabla w|^2 \leq C\rho_0^n\|\psi\|^2_{H^{-1/2}(A)}\tilde{\epsilon}^{2\delta\tau^s}, \tag{5.50}$$

where δ, $0<\delta<1$, and $C\geq 1$ depend on λ, Λ, E and α only. Let us recall now the following interpolation inequality

$$\|v\|_\infty \leq C\left(\left(\int_{B_\rho} v^2\right)^{\frac{\alpha}{2\alpha+n}}|v|_\alpha^{\frac{n}{2\alpha+n}} + \frac{1}{\rho^{n/2}}\left(\int_{B_\rho} v^2\right)^{1/2}\right), \tag{5.51}$$

which holds for any function v defined in the ball $B_\rho \subset \mathbb{R}^n$ and for any α, $0<\alpha\leq 1$. By applying (5.51) to ∇w in $B_{\frac{\gamma_0 r}{4}}(x)$, we have, by (5.1) and (5.50),

$$\|\nabla w\|_{\infty, B_{\frac{\gamma_0 r}{4}}(x)} \leq C\left(\frac{\rho_0}{r}\right)^{n/2}\|\psi\|_{H^{-1/2}(A)}\tilde{\epsilon}^{\gamma\tau^s}, \tag{5.52}$$

where $\gamma = \frac{2\alpha\delta}{2\alpha+n}$, $0<\gamma<1$, and C depends on λ, Λ, E α and M only. From (5.46) and (5.52) we have that for any $r \leq \bar{r}$

$$\int_{\Omega_1\setminus G} |\nabla u_1|^2 \leq C\rho_0^n\|\psi\|^2_{H^{-1/2}(A)}\left(\left(\frac{r}{\rho_0}\right)^{\alpha} + \left(\frac{\rho_0}{r}\right)^{n/2}\tilde{\epsilon}^{\gamma\tau^s}\right), \tag{5.53}$$

with C only depending on λ, Λ, E α and M.

Let us set $\bar{\mu} = \exp\{-\frac{1}{\gamma}\exp(\frac{2S|\log\tau|}{\theta^n})\}$, $\tilde{\mu} = \min\{\bar{\mu}, \exp(-\gamma^{-2})\}$. We have that $\tilde{\mu} < e^{-1}$ and it depends on λ, Λ, E α and M only. Let $\tilde{\epsilon} \le \tilde{\mu}$ and let

$$r(\tilde{\epsilon}) = \rho_0 \left(\frac{2S|\log\tau|}{\log|\log\tilde{\epsilon}^{\gamma}|} \right)^{1/n} .$$

Since $r(\tilde{\epsilon})$ is increasing in $(0, e^{-1})$ and since $r(\tilde{\mu}) \le r(\bar{\mu}) = \rho_0\theta = \bar{r}$, inequality (5.53) is applicable when $r = r(\tilde{\epsilon})$ and we obtain

$$\int_{\Omega_1\setminus G} |\nabla u_1|^2 \le C\rho_0^n \|\psi\|^2_{H^{-1/2}(A)} \left(\log\left|\log\tilde{\epsilon}^{\gamma}\right|\right)^{-\alpha/n} , \tag{5.54}$$

where C depends on λ, Λ, E α and M only. On the other hand, since $\tilde{\epsilon} \le \exp(-\gamma^{-2})$, we have that $\log\gamma \ge -\frac{1}{2}\log|\log\tilde{\epsilon}|$, so that

$$\log\left|\log\tilde{\epsilon}^{\gamma}\right| \ge \frac{1}{2}\log|\log\tilde{\epsilon}| . \tag{5.55}$$

Therefore (3.1)-(3.2) follow. □

PROOF OF PROPOSITION 3.2. Also in this case, it is not restrictive to assume $\tilde{\epsilon} = \frac{\epsilon}{\rho_0\|\psi\|_{H^{-1/2}(A)}} \le \tilde{\mu}$, where $\tilde{\mu}$, $0 < \tilde{\mu} < 1$, is a constant only depending on λ, Λ, E, α, L and M, which will be chosen later on. Let us prove (3.1) and (3.3) when $i = 1$, the case $i = 2$ being analogous. We have

$$\int_{\Omega_1\setminus G} |\nabla u_1|^2 \le \lambda^{-1} \int_{\partial(\Omega_1\setminus G)} u_1(\sigma\nabla u_1 \cdot \nu) , \tag{5.56}$$

$$\partial(\Omega_1 \setminus G) \subset (\partial\Omega_1 \setminus A) \cup (\partial\Omega_2 \cap \partial G \setminus \mathcal{U}^{\rho_0/2}) .$$

Since $\sigma\nabla u_1 \cdot \nu = 0$ on $\partial\Omega_1 \setminus A_{\rho_0}$ and $\sigma\nabla u_2 \cdot \nu = 0$ on $\partial\Omega_2 \setminus A_{\rho_0}$, denoting $w = u_1 - u_2$, by (5.1) we have that

$$\begin{aligned} &\int_{\Omega_1\setminus G} |\nabla u_1|^2 \\ &\le \lambda^{-1} \int_{\partial\Omega_2\cap\partial G\setminus\mathcal{U}^{\rho_0/2}} |u_1(\sigma\nabla w \cdot \nu)| \le C\rho_0^n \|\psi\|_{H^{-1/2}(A)} \max_{\partial G} |\nabla w| , \end{aligned} \tag{5.57}$$

where C depends on λ, Λ, E, α and M only. Let us introduce the following notation.

Given $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $|\xi| = 1$, $\theta > 0$, $r > 0$, we shall denote by

$$C(z, \xi, \theta, r) = \left\{ x \in \mathbb{R}^n \text{ s. t. } \frac{(x - z)\cdot\xi}{|x - z|} > \cos\theta, |x - z| < r \right\} ,$$

the intersection of the ball $B_r(z)$ with the open cone having vertex z, axis in the direction ξ and width 2θ. Since ∂G is of Lipschitz class with constants r_0,

L, for any $z \in \partial G$ there exists $\xi \in \mathbb{R}^n$, $|\xi| = 1$, such that $C(z, \xi, \theta, r_0) \subset G$, where $\theta = \arctan \frac{1}{L}$.

Let us notice that G_{4r} is connected for $r \leq \frac{r_0}{12}$. Let us fix $z \in \partial G$, and let

$$\begin{aligned} \lambda_1 &= \min\left\{\frac{r_0}{1+\sin\theta}, \frac{r_0}{3\sin\theta}, \frac{\rho_0}{16(1+E^2)\sin\theta}\right\}, \\ \theta_1 &= \arcsin\left(\frac{\sin\theta}{4}\right), \\ w_1 &= z + \lambda_1 \xi, \\ \rho_1 &= \lambda_1 \sin\theta_1 . \end{aligned}$$

We have that $B_{\rho_1}(w_1) \subset C(z, \xi, \theta_1, r_0)$, $B_{4\rho_1}(w_1) \subset C(z, \xi, \theta, r_0) \subset G$, so that $w_1 \in \overline{G_{4\rho_1}}$, and $\overline{G_{4\rho_1}}$ is connected since $\rho_1 \leq \frac{r_0}{12}$. Moreover $4\rho_1 \leq \bar{\rho}$, so that $B_{4\rho_1}(z_0) \subset G$, where z_0, $\bar{\rho}$ have been introduced in Step 2 of the proof of Proposition 3.1. Arguing as in the proof of Proposition 3.1, we obtain, by an iterated application of (5.48),

$$\int_{B_{\rho_1}(w_1)} |\nabla w|^2 \leq C \left(\int_G |\nabla w|^2\right)^{1-\tau^s} \left(\int_{B_{\rho_1}(z_0)} |\nabla w|^2\right)^{\tau^s} . \tag{5.58}$$

where τ, $0 < \tau < 1$, and $C \geq 1$ depend on λ and Λ only, and $s \leq \frac{M\rho_0^n}{\omega_n \rho_1^n}$. By (5.19) and (5.39) we have

$$\int_{B_{\rho_1}(w_1)} |\nabla w|^2 \leq C\rho_0^n \|\psi\|^2_{H^{-1/2}(A)} \tilde{\epsilon}^{2\beta_1} , \tag{5.59}$$

where β_1, $0 < \beta_1 < 1$, depends on λ, Λ, E, α, L, M and $\frac{\rho_0}{r_0}$ only, and $C \geq 1$ depends on λ, Λ, E, L and $\frac{\rho_0}{r_0}$ only. Let us approach $z \in \partial G$, by constructing a sequence of balls contained in $C(z, \xi, \theta_1, r_0)$. We define, for $k \geq 2$,

$$\begin{aligned} w_k &= z + \lambda_k \xi , \\ \lambda_k &= \chi \lambda_{k-1} , \\ \rho_k &= \chi \rho_{k-1} , \end{aligned}$$

with

$$\chi = \frac{1-\sin\theta_1}{1+\sin\theta_1} .$$

Hence $\rho_k = \chi^{k-1}\rho_1$, $\lambda_k = \chi^{k-1}\lambda_1$, $B_{\rho_{k+1}}(w_{k+1}) \subset B_{3\rho_k}(w_k) \subset B_{4\rho_k}(w_k) \subset C(z, \xi, \theta, r_0) \subset G$. Denoting

$$d(k) = |w_k - z| - \rho_k ,$$

we have

$$d(k) = \chi^{k-1} d(1),$$

with

$$d(1) = \lambda_1(1 - \sin\theta_1).$$

For any r, $0 < r \leq d(1)$, let $k(r)$ be the smallest positive integer such that $d(k) \leq r$, that is

$$\frac{\left|\log \dfrac{r}{d(1)}\right|}{|\log \chi|} \leq k(r) - 1 \leq \frac{\left|\log \dfrac{r}{d(1)}\right|}{|\log \chi|} + 1. \tag{5.60}$$

By an iterated application of the three spheres inequality (5.48) over the chain of balls $B_{\rho_1}(w_1), \dots, B_{\rho_{k(r)}}(w_{k(r)})$, we have

$$\int_{B_{\rho_{k(r)}}(w_{k(r)})} |\nabla w|^2 \leq C\rho_0^n \|\psi\|^2_{H^{-1/2}(A)} \tilde{\epsilon}^{2\beta_1 \tau^{k(r)-1}}, \tag{5.61}$$

where C depends on λ, Λ, E, L and $\frac{\rho_0}{r_0}$ only. From the interpolation inequality (5.51) and from (5.2) we have

$$\|\nabla w\|_{\infty, B_{\rho_{k(r)}}(w_{k(r)})} \leq C\|\psi\|_{H^{-1/2}(A)} \frac{\tilde{\epsilon}^{\beta_2 \tau^{k(r)-1}}}{\chi^{\frac{n}{2}(k(r)-1)}}, \tag{5.62}$$

where $\beta_2 = \frac{2\alpha\beta_1}{2\alpha+n}$ depends on λ, Λ, E, α, L, M and $\frac{\rho_0}{r_0}$ only. Let us consider the point $z_r = z + r\xi$. We have that $z_r \in B_{\rho_{k(r)}}(w_{k(r)})$. From (5.62) and (5.2), we have that for any r, $0 < r \leq d(1)$,

$$|\nabla w(z)| \leq C\|\psi\|_{H^{-1/2}(A)} \left(\left(\frac{r}{\rho_0}\right)^\alpha + \frac{\tilde{\epsilon}^{\beta_2 \tau^{k(r)-1}}}{\chi^{\frac{n}{2}(k(r)-1)}} \right). \tag{5.63}$$

Let

$$r(\tilde{\epsilon}) = d(1) \left|\log \tilde{\epsilon}^{\beta_2}\right|^{-B},$$

with

$$B = \frac{|\log \chi|}{2|\log \tau|}.$$

Let $\tilde{\mu} = \exp(-\beta_2^{-1})$. We have that $r(\tilde{\mu}) = d(1)$ and $r(\tilde{\epsilon}) \leq d(1)$ for any $\tilde{\epsilon}$, $0 < \tilde{\epsilon} \leq \tilde{\mu}$. Choosing $r = r(\tilde{\epsilon})$ in (5.63) and recalling (5.57) and (5.60), we have

$$\int_{\Omega_1 \setminus G} |\nabla u_1|^2 \leq C\rho_0^n \|\psi\|^2_{H^{-1/2}(A)} \left|\log \tilde{\epsilon}^{\beta_2}\right|^{-\alpha B},$$

where C depends on λ, Λ, E, α, M, L and $\frac{\rho_0}{r_0}$ only. Therefore (3.1) and (3.3) follow with $\gamma = \alpha B$. $\square$

PROOF OF PROPOSITION 4.1. Similarly to the previous propositions, it is not restrictive to assume $\epsilon \leq \|g\|_{H^{1/2}(A)}\tilde{\mu}$, where $\tilde{\mu}$, $0 < \tilde{\mu} < e^{-1}$, is a constant only depending on λ, Λ, E, α and M, which will be chosen later on. Since $u_i = 0$ on $\partial\Omega_i \setminus A_{\rho_0}$ and $u_1 - u_2 = 0$ on A, arguing similarly to the proof of Lemma 5.1, and recalling (4.6), we derive the following Schauder type estimates

$$\|u_i\|_{C^{1,\alpha}\left(\overline{\Omega_i \setminus \mathcal{U}^{\rho_0/8}}\right)} \leq C\|g\|_{H^{1/2}(A)}, \quad \text{for } i = 1, 2, \tag{5.64}$$

$$\|u_1 - u_2\|_{C^{1,\alpha}(\overline{\Omega_1 \cap \Omega_2})} \leq C\|g\|_{H^{1/2}(A)}, \tag{5.65}$$

where $C > 0$ depends on λ, Λ, E, α and M only. Let us prove (4.1)-(4.2) when $i = 1$, the case $i = 2$ being analogous. Let us denote $w = u_1 - u_2$. Since $w = 0$ on Σ, we have that $\nabla w = \frac{\partial w}{\partial \nu}\nu$ on Σ and therefore

$$\left|\frac{\partial w}{\partial \nu}\right| \leq \lambda^{-1} |\sigma \nabla w \cdot \nu| , \qquad \text{on } \Sigma,$$

so that

$$\int_\Sigma w^2 + \rho_0^2 \int_\Sigma |\nabla w|^2 = \rho_0^2 \int_\Sigma |\nabla w|^2 \leq \lambda^{-2}\epsilon^2\rho_0^{n-1}. \tag{5.66}$$

From now on, let us denote

$$\tilde{\epsilon} = \frac{\epsilon}{\|g\|_{H^{1/2}(A)}}.$$

From (5.36), (5.65) and (5.66) we have

$$\mathcal{F}\left(\frac{r_1}{2}\right) \leq C\rho_0^{n-p-1}\|g\|^2_{H^{1/2}(A)}\tilde{\epsilon}^{2\delta_1}, \tag{5.67}$$

where δ_1, $0 < \delta_1 < 1$, $C > 0$ and $p > 1$ depend on λ, Λ and E only and we refer to the notations introduced in the proof of Step 2 of Proposition 3.1. On the other hand, we have (see [T])

$$\int_{G \cap B_r(P_0)} w^2 \leq C r^{p+1}\mathcal{F}(r), \tag{5.68}$$

where C depends on λ, Λ and E only. From (5.67) and (5.68) we have

$$\int_{B_{\bar{\rho}}(z_0)} w^2 \leq C\rho_0^n\|g\|^2_{H^{1/2}(A)}\tilde{\epsilon}^{2\delta_1}, \tag{5.69}$$

where C depends on λ, Λ and E only and z_0 and $\bar{\rho}$ have been introduced in Step 2 of the proof of Proposition 3.1.

Let $r \leq \bar{\rho} = \frac{\rho_0}{16\sqrt{1+E^2}}$ and let

$$\Sigma_r^i = \{x \in \Omega_i \text{ s.t. } \operatorname{dist}(x, \Sigma) = \operatorname{dist}(x, \partial\Omega_i) = r\}, \qquad i = 1, 2.$$

Notice that $\Sigma_r^1 = \Sigma_r^2 = \Sigma_r$, for any $r \leq \bar{\rho}$. Let V_r be the connected component of $\Omega_{1,r} \cap \Omega_{2,r}$ whose closure contains Σ_r. Since $\min\{\operatorname{dist}(z_0, \partial\Omega_1), \operatorname{dist}(z_0, \partial\Omega_2)\} \geq \bar{\rho} \geq r$, we have that $z_0 \in \bar{V}_r$. Let x be any other point in $\overline{V_r}$. By repeating arguments in the proof of Proposition 3.1, with the obvious changes, and by using (5.65) and (5.69), we obtain that there exists τ, $0 < \tau < 1$, such that for any r, $0 < r \leq \bar{\rho}$, we have

$$\int_{B_{\frac{r}{4}}(x)} w^2 \leq C\rho_0^n \|g\|^2_{H^{1/2}(A)} \tilde{\epsilon}^{2\delta_1 \tau^s}, \tag{5.70}$$

where $s \leq S(\rho_0/r)^n$, with S only depending on M and $C > 0$ only depending on λ, Λ, E, α and M. By the local boundedness of solutions to elliptic equations in divergence form (see, for instance, [G-T]), we have

$$\|w\|_{\infty, B_{\frac{r}{8}}(x)} \leq C\left(\frac{\rho_0}{r}\right)^{n/2} \|g\|_{H^{1/2}(A)} \tilde{\epsilon}^{\delta_1 \tau^s}, \tag{5.71}$$

where C depends on λ, Λ, E, α and M only. Let W_r be the connected component of the set $\{x \in G \text{ s.t. } \operatorname{dist}(x, \partial G \backslash A) > r\}$ whose closure contains Σ. Since $\Omega_1 \setminus G \subset \Omega_1 \setminus W_r$ and by using the maximum principle, we have

$$\max_{\Omega_1 \backslash G} |u_1| \leq \max_{\partial(\Omega_1 \backslash W_r)} |u_1|. \tag{5.72}$$

Hence it is sufficient to estimate $\sup_{\partial W_r \backslash A} |u_1|$.

Let $x \in \partial W_r \setminus A$, hence $x \in G$ and $\operatorname{dist}(x, \partial G \setminus A) = r$. Let us distinguish two cases:

i) $\operatorname{dist}(x, \partial G) = r$,

ii) $\operatorname{dist}(x, \partial G) < r$.

Let us assume that *i*) occurs and let us again distinguish two cases:

I) there exists $y \in I_1$ such that $|x - y| = \operatorname{dist}(x, \partial G \setminus A) = r$.

II) there exists $y \in I_2$ such that $|x - y| = \operatorname{dist}(x, \partial G \setminus A) = r$.

If *I*) occurs, from (5.64) we have

$$|u_1(x)| \leq C\|g\|_{H^{1/2}(A)} \frac{r}{\rho_0},$$

where C only depends on λ, E, α and M.

If II) occurs, by (5.71) and (5.64), we have

$$|u_1(x)| \leq |w(x)| + |u_2(x) - u_2(y)| \leq C\left(\frac{\rho_0}{r}\right)^{n/2} \|g\|_{H^{1/2}(A)} \tilde{\epsilon}^{\delta_1 \tau^s} + C\|g\|_{H^{1/2}(A)} \frac{r}{\rho_0},$$

where C only depends on λ, Λ, E, α and M.

If case ii) occurs, then there exists $z \in A$ such that $|x-z| = \mathrm{dist}(x, \partial G) < r$. Since

$$\mathrm{dist}(x, \Sigma) \geq \mathrm{dist}(\Sigma, \partial G \setminus A) - \mathrm{dist}(x, \partial G \setminus A) \geq \rho_0 - r > r > |x-z|,$$

it follows that $z \notin \Sigma$, so that $u_1(z) = 0$. Hence

$$|u_1(x)| \leq C\|g\|_{H^{1/2}(A)} \frac{r}{\rho_0},$$

where C only depends on λ, E, α and M.

Therefore, for any r, $0 < r \leq \bar{\rho}$, we have

$$\max_{\Omega_1 \setminus G} |u_1| \leq C\|g\|_{H^{1/2}(A)} \left(\frac{r}{\rho_0} + \left(\frac{\rho_0}{r}\right)^{n/2} \tilde{\epsilon}^{\delta_1 \tau^s}\right), \tag{5.73}$$

where C only depends on λ, Λ, E, α and M.

Let us set $\bar{\mu} = \exp\{-\frac{1}{\delta_1} \exp(2S|\log \tau|(\frac{\rho_0}{\bar{\rho}})^n)\}$, $\tilde{\mu} = \min\{\bar{\mu}, \exp(-\delta_1^{-2})\}$. We have that $\tilde{\mu} < e^{-1}$ and it depends on λ, Λ, E, α and M only. Let $\tilde{\epsilon} \leq \tilde{\mu}$ and let

$$r(\tilde{\epsilon}) = \rho_0 \left(\frac{2S|\log \tau|}{\log|\log \tilde{\epsilon}^{\delta_1}|}\right)^{1/n}.$$

Since $r(\tilde{\epsilon})$ is increasing in $(0, e^{-1})$ and since $r(\tilde{\mu}) \leq r(\bar{\mu}) = \bar{\rho}$, inequality (5.73) is applicable when $r = r(\tilde{\epsilon})$ and we obtain

$$\max_{\Omega_1 \setminus G} |u_1| \leq C\|g\|_{H^{1/2}(A)} (\log|\log \tilde{\epsilon}|)^{-1/n}, \tag{5.74}$$

where C depends on λ, Λ, E, α and M only. Therefore (4.1)-(4.2) follow. □

Proof of Proposition 4.2. The same arguments involved in the proof of Proposition 3.2, with the obvious changes, give the desired estimate. □

6. – Proofs of the estimates of continuation from the interior

Proof of Proposition 3.3. This estimate was obtained in [Al-Ros-S, Lemma 2.2] and we refer to it for a proof. Let us just recall that the main ingredients are: a repeated use of the three spheres inequality (5.48) and the following estimate

$$\int_{\Omega\setminus\Omega_{5r}} |\nabla u|^2 \leq C\rho_0^{n-1} r \|\psi\|^2_{L^2(\partial\Omega)}, \quad \text{for every } r > 0, \tag{6.1}$$

where $C > 0$ only depends on λ, Λ, M and E, see the final remark in [Al-Ros-S]. □

Proof of Proposition 4.3. Let $P \in \partial\Omega$. There exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{x \in B_{\rho_0}(0) \text{ s.t. } x_n > \varphi(x')\},$$

where φ is a Lipschitz continuous function on $B_{\rho_0}(0) \subset \mathbb{R}^{n-1}$ satisfying $\varphi(0) = 0$ and $\|\varphi\|_{C^{0,1}(B_{\rho_0}(0))} \leq E\rho_0$. Let $r > 0$, with $2r \leq \rho_0/4$, and let $\rho_0' = \frac{\sqrt{9E^2+8}-E}{4(1+E^2)}\rho_0$. If $x' \in B_{\rho_0'} \subset \mathbb{R}^{n-1}$ and $s \in [0, 2r]$, then $(x', \varphi(x') + s) \in \Omega \cap B_{\rho_0}(0)$ and $\text{dist}((x', \varphi(x')+s), \partial B_{\rho_0}(0)) \geq \text{dist}((x', \varphi(x')+s), G_\varphi)$, where G_φ denotes the intersection of the graph of φ with $B_{\rho_0}(0)$. Hence $\text{dist}((x', \varphi(x')+s), \partial\Omega) = \text{dist}((x', \varphi(x') + s), G_\varphi)$, for every $x' \in B_{\rho_0'}$ and every $s \in [0, 2r]$. Let us denote by $S_r = \{(x', x_n) \text{ s.t. } x' \in B_{\rho_0'}, \varphi(x') + r \leq x_n \leq \varphi(x') + 2r\}$ and let $s \in [r, 2r]$, with $2r \leq \rho_0/4$. For every $x' \in B_{\rho_0'}$ we have

$$|u(x', \varphi(x'))|^2 \leq 2\left(s\int_{\varphi(x')}^{\varphi(x')+s} |u_{x_n}(x', x_n)|^2 dx_n + |u(x', \varphi(x') + s)|^2\right), \tag{6.2}$$

and integrating with respect to s in $[r, 2r]$ one gets

$$\begin{aligned} &r|u(x', \varphi(x'))|^2 \\ &\leq 2\left(2r^2 \int_{\varphi(x')}^{\varphi(x')+2r} |u_{x_n}(x', x_n)|^2 dx_n + \int_r^{2r} |u(x', \varphi(x') + s)|^2 ds\right). \end{aligned} \tag{6.3}$$

Integrating (6.3) over $B_{\rho_0'}$, we derive

$$\int_{B_{\rho_0'}} |u(x', \varphi(x'))|^2 dx' \leq 2\left(2r \int_{\Omega\cap B_{\rho_0}(0)} |\nabla u|^2 dx + \frac{1}{r}\int_{S_r} u^2 dx\right). \tag{6.4}$$

Denoting by γ the intersection of the graph of φ with $B_{\rho_0'} \times \mathbb{R}$, we obtain

$$(6.5)\qquad \int_\gamma u^2 ds \le 2\sqrt{1+E^2}\left(2r\int_{\Omega\cap B_{\rho_0}(0)} |\nabla u|^2 dx + \frac{1}{r}\int_{S_r} u^2 dx\right).$$

Let $\bar{d}$ denote the diameter of Ω and let Q be a cube of side $2\bar{d}$ containing Ω. We have that $\bar{d} \le C\rho_0$, where $C > 0$ depends on E and M only. Let us divide the sides of Q in m equal parts where

$$(6.6)\qquad m = \left[\frac{2\sqrt{n}\bar{d}}{\rho_0'}\right] + 1.$$

In this way we divide Q in m^n subcubes with sides of size $\tau \le \frac{\rho_0'}{\sqrt{n}}$. Let us denote by $\{Q_\tau^j\}_{j=1,\dots,J}$ the family of the subcubes of this partition such that $\partial\Omega \cap Q_\tau^j \ne \emptyset$. For any $j \in \{1,\dots,J\}$ we choose $x_j \in \partial\Omega\cap Q_\tau^j$ and we denote by C_j, up to a suitable rotation which brings x_j in the origin, the set $B_{\rho_0}\cap(B_{\rho_0'}\times\mathbb{R})$. We have that $Q_\tau^j \subset B_{\rho_0'}(x_j) \subset C_j$ and $\partial\Omega = \cup_{j=1}^J \partial\Omega\cap Q_\tau^j \subset \cup_{j=1}^J \partial\Omega \cap C_j$. Let $d = \frac{r}{4\sqrt{1+E^2}}$. Then $S_r \subset \Omega_{4d}$. In fact let $\tau_s = \mathrm{dist}((x', \varphi(x')+s), \partial\Omega) = \mathrm{dist}((x',\varphi(x')+s), G_\varphi)$, with $s \in [r, 2r]$. It is easy to see that $\tau_s \le s \le \sqrt{1+E^2}\tau_s$. Hence $4d \le \tau_s$, which implies that if $(x',\varphi(x')+s) \in S_r$ then $(x',\varphi(x')+s) \in \overline{\Omega_{4d}}$. Since inequality (6.5) holds for $\gamma_j = \partial\Omega\cap C_j$ for every $j = 1,\dots,J$, summing up over j and using the fact that $S_r \subset \Omega_{4d}$, we have

$$(6.7)\qquad \int_{\partial\Omega} u^2 ds \le 2\sqrt{1+E^2}J\left(2r\int_\Omega |\nabla u|^2 dx + \frac{1}{r}\int_{\Omega_{4d}} u^2 dx\right),$$

where $J \le m^n$, with m^n depending on M and E only. Let $d_0 = \min\{\frac{\rho}{2}, \frac{\rho_0/8}{4\sqrt{1+E^2}}, \frac{\rho_0}{12}\}$ and let $d \le d_0$. Let us fix $x_0 \in \Omega_{2\rho}$ and let us denote

$$\epsilon = \left(\int_{B_\rho(x_0)} u^2\right)^{1/2}.$$

Given $x \in \Omega_{4d}$, repeating arguments in the proof of Proposition 3.1, we can consider a chain of pairwise disjoint spheres of radius d with center at points of a path l connecting x to x_0 in Ω_{4d}, for $d \le d_0$. An iterated application of the three spheres inequality (5.47) gives

$$(6.8)\qquad \int_{B_d(x)} u^2 dx \le C\left(\int_\Omega u^2\right)^{1-\delta^{N_d}} \epsilon^{2\delta^{N_d}},$$

where δ, $0 < \delta < 1$, and C depend on λ and Λ only and $N_d \leq \frac{M\rho_0^n}{\omega_n d^n}$. Covering Ω_{4d} with internally nonoverlapping closed cubes $\{Q_i\}_{i=1,\dots N_d'}$, of side $d/\sqrt{n}$, we have

$$
\tag{6.9}
\begin{aligned}
\int_{\Omega_{4d}} u^2 dx &\leq \sum_{i=1}^{N_d'} \int_{Q_i} u^2 dx \\
&\leq \sum_{i=1}^{N_d'} \int_{B_d(x^i)} u^2 dx \leq C N_d' \left(\int_\Omega u^2 \right)^{1-\delta^{N_d}} \epsilon^{2\delta^{N_d}} ,
\end{aligned}
$$

where $x^i \in \Omega_{4d} \cap Q_i$ and $N_d' \leq \frac{n^{n/2} M\rho_0^n}{d^n}$. From (6.7) and (6.9) we get

$$
\tag{6.10}
\int_{\partial\Omega} u^2 ds \leq C \left(d \int_\Omega |\nabla u|^2 dx + \frac{\rho_0^n}{d^{n+1}} \left(\int_\Omega u^2 \right)^{1-\delta^{N_d}} \epsilon^{2\delta^{N_d}} \right)
$$

where C depends on λ, Λ, E and M only. Letting $\alpha = \frac{\epsilon^2}{e^2 \rho_0^n \|u\|_{H^1}^2}$, we rewrite (6.10) as follows

$$
\tag{6.11}
\int_{\partial\Omega} u^2 ds \leq C \rho_0^{n-1} \|u\|_{H^1}^2 \left(\frac{d}{\rho_0} + \left(\frac{\rho_0}{d} \right)^{n+1} \alpha^{\delta^{N_d}} \right) ,
$$

where C depends on λ, Λ, E and M only. Let

$$
\alpha_0 = \exp\left(-\exp\left(\frac{2M|\log\delta|}{\omega_n} \left(\frac{\rho_0}{d_0} \right)^n \right) \right) < 1 .
$$

Let us notice that $\frac{\rho_0}{d_0}$ depends on E and $\frac{\rho}{\rho_0}$ only, so that α_0 depends on λ, Λ, E, M and $\frac{\rho}{\rho_0}$ only. If $\alpha \leq \alpha_0$ we choose $d = d_\alpha$, where

$$
d_\alpha = \rho_0 \left(\frac{2M|\log\delta|}{\omega_n \log|\log\alpha|} \right)^{1/n} .
$$

Then $d_\alpha \leq d(\alpha_0) = d_0$ and from (6.11) we have

$$
\tag{6.12}
\int_{\partial\Omega} u^2 ds \leq C \rho_0^{n-1} \|u\|_{H^1}^2 (\log|\log\alpha|)^{-1/n} ,
$$

where C depends on λ, Λ, E and M only. If $\alpha > \alpha_0$, from (6.11), using the fact that $\alpha \leq 1$, and substituting d with d_0, we derive

$$
\tag{6.13}
\int_{\partial\Omega} u^2 ds \leq C \rho_0^{n-1} \|u\|_{H^1}^2 \left(\frac{d_0}{\rho_0} + \left(\frac{\rho_0}{d_0} \right)^{n+1} \right) \frac{\alpha}{\alpha_0} ,
$$

where C depends on λ, Λ, E and M only. Since $\alpha \leq e^{-2}(\log 2)^{1/n}(\log|\log\alpha|)^{-1/n}$, from (6.12) and (6.13) we have

$$\int_{\partial\Omega} u^2 ds \leq \frac{C}{\alpha_0}\rho_0^{n-1}\left(\frac{\rho_0}{d_0}\right)^{n+1} \|u\|_{H^1}^2 (\log|\log\alpha|)^{-1/n}, \tag{6.14}$$

where C depends on λ, Λ, E and M only. From (6.14) and from

$$\|g\|_{H^{1/2}(\partial\Omega)}^2 \leq \|u\|_{H^1}^2 \leq C\|g\|_{H^{1/2}(\partial\Omega)}^2, \tag{6.15}$$

where C depends on λ, E and M only, we have

$$\epsilon^2 \geq e^2\rho_0^n \exp\left(-\exp\left(\frac{C}{\alpha_0}\left(\frac{\rho_0}{d_0}\right)^{n+1} F^2\right)^n\right)\|g\|_{H^{1/2}(\partial\Omega)}^2, \tag{6.16}$$

where C depends on λ, Λ, E and M only. Therefore (4.5) follows with the stated dependence. □

7. – Proofs of the Doubling Inequalities

PROOF OF PROPOSITION 3.4. The starting point here is the doubling inequality of Garofalo and Lin [G-L]. Given a solution v to

$$\operatorname{div}(\sigma\nabla v) = 0, \quad \text{in } B_R(x),$$

where σ satisfies (2.9), and denoting by $N = N(r)$ (this is a slight variant of the so called Almgren's frequency function)

$$N(r) = \frac{r^2\int_{B_r(x)} |\nabla v|^2}{\int_{B_r(x)} v^2}, \tag{7.1}$$

we have that there exists τ, $0 < \tau \leq 1$, only depending on Λ, such that

$$\int_{B_{\gamma r}(x)} v^2 \leq C\gamma^K \int_{B_r(x)} v^2, \tag{7.2}$$
$$\text{for every } r, \gamma \text{ s.t. } 1 \leq \gamma \text{ and } 0 < \gamma r \leq \tau R.$$

Here $C > 0$ only depends on λ and Λ, whereas $K > 0$ only depends on λ, Λ and, increasingly, on $N(\tau R)$. See [G-L, Theorem 1.3] and also, for a more recent proof, Kukavica [Ku, Theorem 3.1], to which we refer for notation and details.

The matter here is to translate (7.2) into a doubling inequality for ∇u and to evaluate $N(\tau R)$ in terms of F, the frequency of the boundary data ψ.

Let $x_0 \in \Omega_\rho$, let us fix $\beta \geq 1$ and pick $v = u - c$ and $\gamma = 2\beta$, where $c = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u$. By the use of Caccioppoli and Poincaré inequalities, the doubling inequality (7.2) leads to

$$\begin{gathered}\int_{B_{\beta r}(x_0)} |\nabla u|^2 \leq C(2\beta)^K \int_{B_r(x_0)} |\nabla u|^2 , \\ \text{for every } r, \beta \text{ s.t. } 1 \leq \beta \text{ and } 0 < \beta r \leq \frac{\tau\rho}{2} .\end{gathered} \tag{7.3}$$

Here $C > 0$ only depends on λ and Λ, whereas $K > 0$ only depends on λ, Λ and, increasingly, on

$$N(\tau\rho) = \frac{(\tau\rho)^2 \int_{B_{\tau\rho}(x_0)} |\nabla u|^2}{\int_{B_{\tau\rho}(x_0)} |u - c|^2} .$$

Using once more Caccioppoli inequality and by Proposition 3.3, (3.6) and (5.4), we can majorize $N(\tau\rho)$ by a constant $C > 0$ only depending on λ, Λ, E, M, $\frac{\rho}{\rho_0}$ and F. Thus (3.7) follows, provided $\beta r \leq \frac{\tau\rho}{2}$. The case when $\frac{\tau\rho}{2} < \beta r \leq \rho$ can be easily treated, again by the use of Proposition 3.3. □

Proof of Proposition 4.4. The arguments here are analogous to those used for the proof of Proposition 3.4, the key difference being in the evaluation of $N(\tau\rho)$ in terms of the frequency of the Dirichlet data g, instead of the Neumann data. Such an evaluation is obtained by the use of Proposition 4.3 instead of Proposition 3.3. □

Proof of Proposition 3.5. First, let us assume that $\sigma(x_0) = Id$. We fix coordinates (x', x_n) suitable for the local representation of the boundary as a graph as in Definition 2.1. Namely we have $x_0 = 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{x \in B_{\rho_0}(0) \text{ s.t. } x_n > \varphi(x')\} ,$$

where φ is a $C^{1,1}$ function on $B_{\rho_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$\|\varphi\|_{C^{1,1}\left(B_{\rho_0}(0)\right)} \leq E\rho_0 .$$

Following ideas in [A-E], we can costruct a map $\Phi \in C^{1,1}(B_{\rho_2}(0), \mathbb{R}^n)$ such that

$$\Phi(B_{\rho_2}(0)) \subset B_{\rho_1}(0), \tag{7.4a}$$

$$\Phi(y', 0) = (y', \varphi(y')), \quad \text{for every } y' \in B_{\rho_2}(0) \subset \mathbb{R}^{n-1}, \tag{7.4b}$$

$$\Phi(B^+_{\rho_2}(0)) \subset \Omega \cap B_{\rho_1}(0), \tag{7.4c}$$

$$\frac{1}{2}|y - z| \leq |\Phi(y) - \Phi(z)| \leq C_1|y - z|, \quad \text{for every } y, z \in B_{\rho_2}(0), \tag{7.4d}$$

$$\frac{1}{2^n} \leq |\det D\Phi| \leq C_2, \tag{7.4e}$$

where $\rho_i = \theta_i \rho_0$, $0 < \theta_i < 1$, $i = 1, 2$, and C_1, C_2, θ_1, θ_2 only depend on λ, Λ and E.

Denoting

$$\bar{\sigma}(y) = |\det D\Phi(y)|(D\Phi^{-1})(\Phi(y))\sigma(\Phi(y))(D\Phi^{-1})^T(\Phi(y)),$$
$$v(y) = u(\Phi(y)),$$

we have

$$\bar{\sigma}(0) = Id, \tag{7.5a}$$

$$\bar{\sigma}_{nk}(y', 0) = 0, \quad \text{for } k = 1, \dots, n-1, \tag{7.5b}$$

$$\frac{\lambda}{2^{n+2}}|\xi|^2 \leq \bar{\sigma}(y)\xi \cdot \xi \leq C_3|\xi|^2, \text{ for every } y \in B^+_{\rho_2}(0) \text{ and every } \xi \in \mathbb{R}^n, \tag{7.5c}$$

$$|\bar{\sigma}(y) - \bar{\sigma}(z)| \leq \frac{C_4}{\rho_0}|y - z|, \quad \text{for every } y, z \in B^+_{\rho_2}(0), \tag{7.5d}$$

where C_3 and C_4 only depend on λ, Λ and E. We have that $v \in H^1(B^+_{\rho_2}(0))$ is a weak solution to

$$\operatorname{div}(\bar{\sigma}(y)\nabla v(y)) = 0, \quad \text{in } B^+_{\rho_2}(0), \tag{7.6a}$$

$$\frac{\partial v}{\partial y_n}(y', 0) = 0, \quad \text{for } |y'| \leq \rho_2. \tag{7.6b}$$

For every $y \in B_{\rho_2}(0)$, let us denote by $\sigma'(y)$ the symmetric matrix whose entries are given by

$$\sigma'_{ij}(y', y_n) = \bar{\sigma}_{ij}(y', |y_n|), \quad \text{if either } 1 \leq i, j \leq n-1, \text{ or } i = j = n,$$
$$\sigma'_{nj}(y', y_n) = \sigma'_{jn}(y', y_n) = \operatorname{sgn}(y_n)\bar{\sigma}_{jn}(y', |y_n|), \quad \text{if } 1 \leq j \leq n-1.$$

We have that σ' satisfies the same ellipticity and Lipschitz continuity conditions as $\bar{\sigma}$.

Denoting

$$w(y) = v(y', |y_n|), \quad \text{for } y \in B_{\rho_2}(0),$$

we have that $w \in H^1(B_{\rho_2}(0))$ is a weak solution to

$$\operatorname{div}(\sigma'(y)\nabla w(y)) = 0, \quad \text{in } B_{\rho_2}(0). \tag{7.7}$$

Moreover, from (7.4d) we have that

$$\Omega \cap B_{\rho/2}(0) \subset \Phi(B_\rho^+(0)) \subset \Omega \cap B_{C_1\rho}(0), \quad \text{for every } \rho \leq \rho_2. \tag{7.8}$$

Choosing $\rho = \rho_2/2$ in (7.3), we have

$$\int_{B_{\beta r}(0)} |\nabla w|^2 \leq C\beta^K \int_{B_r(0)} |\nabla w|^2, \tag{7.9}$$
$$\text{for every } r, \beta \text{ s.t. } 1 \leq \beta \text{ and } 0 < \beta r \leq \rho_3,$$

where $\rho_3 = \delta\rho_2$, $0 < \delta \leq 1/4$, with δ only depending on λ, Λ and E and where C and K depend on λ, Λ, E, and $N(\rho_3)$ only, with

$$N(\rho_3) = \frac{\rho_3^2 \int_{B_{\rho_3}(0)} |\nabla w|^2}{\int_{B_{\rho_3}(0)} |w - c|^2}, \tag{7.10}$$

where $c = \frac{1}{\omega_n r^n} \int_{B_r(0)} w$.

Let $0 < r \leq \beta r \leq \rho_3/2$. From (7.8) we have that

$$\int_{\Omega \cap B_{\beta r}(0)} |\nabla u|^2 \leq C \int_{B^+_{2\beta r}(0)} |\nabla v|^2,$$

where C depends on λ, Λ and E only. From (7.9) we have

$$\int_{B^+_{2\beta r}(0)} |\nabla v|^2 \leq C(2\beta C_1)^K \int_{B^+_{r/C_1}(0)} |\nabla v|^2 \leq C(2\beta C_1)^K \int_{\Omega \cap B_r(0)} |\nabla u|^2,$$

where C and K depend on λ, Λ, E and $N(\rho_3)$ only. From the two last inequalities we have

$$\int_{\Omega \cap B_{\beta r}(0)} |\nabla u|^2 \leq C\beta^K \int_{\Omega \cap B_r(0)} |\nabla u|^2, \tag{7.11}$$

where C and K depend on λ, Λ, E and $N(\rho_3)$ only. From Caccioppoli inequality and (7.8), we can estimate

$$\begin{aligned}\int_{B_{\rho_3}(0)} |w-c|^2 &\geq C\rho_3^2 \int_{B_{\rho_3/2}(0)} |\nabla w|^2 \\ &\geq C\rho_3^2 \int_{B^+_{\rho_3/2}} |\nabla v|^2 \geq C\rho_3^2 \int_{\Omega\cap B_{\rho_3/4}} |\nabla u|^2 ,\end{aligned} \tag{7.12}$$

where C depends on λ, Λ and E only. Setting $\rho_4 = \frac{\rho_3}{64\sqrt{1+E^2}}$, $P = (0, \rho_3/8)$, we have that $B_{4\rho_4}(P) \subset \Omega \cap B_{\rho_3/4}(0)$. From (7.12), (5.4) and Proposition 3.3 we have

$$\int_{B_{\rho_3}(0)} |w-c|^2 \geq C\rho_3^2 \int_{\Omega} |\nabla u|^2 , \tag{7.13}$$

where C depends on λ, Λ, E M and F only.

On the other hand,

$$\int_{B_{\rho_3}(0)} |\nabla w|^2 \leq 2 \int_{B^+_{\rho_3}(0)} |\nabla v|^2 \leq C \int_{\Omega} |\nabla u|^2 , \tag{7.14}$$

where C only depends on λ, Λ and E. Therefore we can majorize $N(\rho_3)$ by a constant $C > 0$ only depending on λ, Λ, E, M and F. Thus (3.8) follows, provided $\beta r \leq \rho_3/2$. The case when $\rho_3/2 < \beta r$ can be easily treated, again by the use of Proposition 3.3.

In the general case $\sigma(x_0) \neq Id$, we can consider a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that, setting $\tilde{\sigma}(Sx) = \frac{S\sigma(x)S^T}{|\det S|}$, we have $\tilde{\sigma}(x_0) = Id$ (here, as above, we identify $x_0 = 0$). We have that, under such a transformation, the modified coefficient $\tilde{\sigma}$, the transformed domain $S(\Omega)$ and the boundary data satisfy a priori assumptions analogous to (2.1)-(2.7) and (2.9) with constants which are dominated by the a priori constants ρ_0, M, E, α, F, λ, Λ, up to multiplicative factors which only depend on λ. We also have that the ellipsoids $S(B_\rho(x_0))$ satisfy

$$B_{\sqrt{\lambda}\rho}(x_0) \subset S(B_\rho(x_0)) \subset B_{\frac{\rho}{\sqrt{\lambda}}}(x_0), \quad \text{for every } \rho > 0 .$$

Therefore, by a change of variables, using the result just proved when $\sigma(x_0) = Id$, we obtain (3.8). □

We premise the proof of Proposition 4.5 with an auxiliary lemma which, in a more general form, is due to Adolfsson and Escauriaza [A-E, Theorem 1.1].

LEMMA 7.1 (Adolfsson-Escauriaza). *Let Ω' be a domain such that $\partial\Omega'$ is of Lipschitz class with constants r_0, $L > 0$ and $0 \in \partial\Omega'$. Let w be a nonconstant solution to*

$$\text{(7.15a)} \qquad div(\sigma'\nabla w) = 0, \quad in\ \Omega' \cap B_{R_0}(0),$$

$$\text{(7.15b)} \qquad w = 0 \quad on\ \partial\Omega' \cap B_{R_0}(0),$$

for some $R_0 > 0$, where σ' is a function from $\mathbb{R}^n$ with values $n \times n$ symmetric matrices satisfying the following assumptions, for given constants λ_0 and $\tilde{C}$:
i)

$$\text{(7.16a)} \qquad \lambda_0|\xi|^2 \leq \sigma'(x)\xi \cdot \xi \leq \lambda_0^{-1}|\xi|^2 \quad for\ every\ x, \xi \in \mathbb{R}^n,$$

ii)

$$\text{(7.16b)} \qquad \sigma'(0) = Id,$$

iii)

$$\text{(7.16c)} \qquad \sigma'(x)x \cdot \nu \geq 0 \ for\ a.e.\ x \in \partial\Omega' \cap B_{R_0}(0),$$

iv)

$$\text{(7.16d)} \quad |\nabla\sigma'(x)| \leq \frac{\tilde{C}}{r_0^\alpha}|x|^{\alpha-1}, \ |\sigma'(x) - \sigma'(0)| \leq \frac{\tilde{C}}{r_0^\alpha}|x|^\alpha, \ for\ every\ x \in B_{R_0}(0).$$

Then there exists R, $0 < R \leq R_0$, with R/R_0 only depending on λ_0 and L, such that

$$\text{(7.17)} \qquad \int_{\Omega' \cap B_{\beta r}(0)} w^2 \leq C\beta^K \int_{\Omega' \cap B_r(0)} w^2, \quad for\ every\ r, \beta\ s.t.\ 1 \leq \beta\ and\ 0 < \beta r < R,$$

where $C > 0$ only depends on λ_0, α, $\tilde{C}$ and $\frac{R_0}{r_0}$, whereas $K > 0$ only depends on λ_0, α, $\tilde{C}$, $\frac{R_0}{r_0}$ and, increasingly, on $\tilde{N}(R_0)$, where

$$\text{(7.18)} \qquad \tilde{N}(r) = \frac{r^2 \int_{\Omega' \cap B_r(0)} |\nabla w|^2}{\int_{\Omega' \cap B_r(0)} w^2}.$$

PROOF OF LEMMA 7.1. The proof is contained in [A-E, Proof of Theorem 1.1], the only differences being in a more explicit evaluation of the constants C and K in terms of the a priori data, and a slight modification of the expression of $\tilde{N}$. □

PROOF OF PROPOSITION 4.5. Up to a rigid motion, we can set $x_0 = 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{x \in B_{\rho_0}(0) \text{ s.t. } x_n > \varphi(x')\},$$

where φ is a $C^{1,\alpha}$ function on $B_{\rho_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$\|\varphi\|_{C^{1,\alpha}\left(B_{\rho_0}(0)\right)} \leq E\rho_0 .$$

We shall follow the main lines of the proof of Theorem 0.4 of [A-E]. The idea is to construct a $C^{1,\alpha}$ diffeomorphism Φ from $\Omega' \cap B_{\rho_2}(0)$ to $\Omega \cap B_{\rho_1}(0)$, where $\Omega' \cap B_{\rho_2}(0) = \{y \in B_{\rho_2}(0) \text{ s.t. } y_n > \psi(y')\}$ and $\psi \in C^{1,\alpha}\left(B_{\rho_2}(0) \subset \mathbb{R}^{n-1}\right)$, for some $\rho_1 > 0$, $\rho_2 > 0$, showing that $w(y) = u(\Phi(y))$ satisfies the assumptions of Lemma 7.1 and hence inequality (7.17).

Let $C_1 = \frac{3(2^\alpha-1)^2 E}{(\alpha \log 2)^2}$, and let

$$\Phi(y', y_n) = \left(y', y_n + \frac{C_1}{\rho_0^\alpha}|y|^{\alpha+1}\right), \tag{7.19}$$

$$\Omega' = \left\{y \in B_{\rho_0}(0) \text{ s.t. } y_n > \varphi(y') - \frac{C_1}{\rho_0^\alpha}|y|^{\alpha+1}\right\}. \tag{7.20}$$

Following the computations in [A-E], we have that there exist ρ_1, ρ_2, $0 < \rho_1 \leq \rho_0$, $0 < \rho_2 \leq \rho_0$, with ρ_1/ρ_0, ρ_2/ρ_0 only depending on E and α, such that $\Phi \in C^{1,\alpha}\left(B_{\rho_2}(0), \mathbb{R}^n\right)$ satisfies

$$\Phi\left(y', \varphi(y') - \frac{C_1}{\rho_0^\alpha}|y|^{\alpha+1}\right) = (y', \varphi(y')), \tag{7.21a}$$

$$\Phi(\Omega' \cap B_{\rho_2}(0)) \subset \Omega \cap B_{\rho_1}(0), \tag{7.21b}$$

$$\frac{1}{2}|y-z| \leq |\Phi(y) - \Phi(z)| \leq C_2|y-z|, \quad \text{for every } y, z \in B_{\rho_2}(0), \tag{7.21c}$$

$$\frac{1}{2} \leq |\det D\Phi(y)| \leq 2, \quad \text{for every } y \in B_{\rho_2}(0), \tag{7.21d}$$

(7.21*e*) $\partial\Omega' \cap B_{\rho_2}(0)$ is of *Lipschitz class with constants* ρ_2, L,

where $C_2 > 0$ and $L > 0$ only depend on E and α.

Denoting by

$$\sigma'(y) = |\det D\Phi(y)|(D\Phi^{-1})(\Phi(y))\sigma(\Phi(y))(D\Phi^{-1})^T(\Phi(y)),$$
$$w(y) = u(\Phi(y)),$$

we have

(7.22a) $\frac{\lambda}{8}|\xi|^2 \leq \sigma'(y)\xi\cdot\xi \leq \frac{8}{\lambda}|\xi|^2$ for every $y\in\Omega'\cap B_{\rho_2}(0)$ and every $\xi\in\mathbb{R}^n$,

$$\sigma'(0) = Id, \tag{7.22b}$$

$$\sigma'(y)y\cdot\nu \geq 0 \text{ for every } y\in\partial\Omega'\cap B_{\rho_2}(0), \tag{7.22c}$$

$$|\nabla\sigma'(y)| \leq \frac{C_3}{\rho_0^\alpha}|y|^{\alpha-1} \quad \text{for every } y\in\Omega'\cap B_{\rho_2}(0), \tag{7.22d}$$

$$|\sigma'(y)-\sigma'(z)| \leq \frac{C_3}{\rho_0^\alpha}|y-z|^\alpha \quad \text{for every } y,z\in\Omega'\cap B_{\rho_2}(0), \tag{7.22e}$$

where $C_3 > 0$ only depends on λ, Λ, α and E. Moreover, we have that $w\in H^1(\Omega'\cap B_{\rho_2}(0))$ is a weak solution to

$$\operatorname{div}(\sigma'\nabla w) = 0, \quad \text{in } \Omega'\cap B_{\rho_2}(0), \tag{7.23a}$$

$$w = 0 \quad \text{on } \partial\Omega'\cap B_{\rho_2}(0). \tag{7.23b}$$

All the above properties (7.21)-(7.23) follow extending the arguments of the proof of Theorem 0.4 of [A-E] to operators in divergence form. In particular, (7.21e) and (7.22c) are obtained by observing that $\partial\Omega'\cap B_{\rho_2}(0)$ is the C^1 graph defined implicitly by a function ψ satisfying

$$\psi(y') = \varphi(y') - \frac{C_1}{\rho_0^\alpha}|y|^{\alpha+1}, \tag{7.24a}$$

$$\psi(0) = 0, \tag{7.24b}$$

and by differentiating implicitly (7.24a).

Hence we can apply Lemma 7.1 to the solution w to (7.23), with $r_0 = R_0 = \rho_2$, $\lambda_0 = \lambda/8$, $\tilde{C} = C_3(\frac{\rho_2}{\rho_0})^\alpha$, so that $\tilde{C}$ only depends on λ, Λ, E and α. Hence we obtain

$$\int_{\Omega'\cap B_{\beta r}(0)} w^2 \leq C\beta^K \int_{\Omega'\cap B_r(0)} w^2, \tag{7.25}$$
$$\text{for every } r,\beta \text{ s.t. } 1\leq\beta \text{ and } 0<\beta r<\rho_3,$$

where $0<\rho_3\leq\rho_2$, with ρ_3/ρ_0, $C>0$ only depending on λ, Λ, E and α, whereas $K>0$ only depends on λ, Λ, E, α and $\tilde{N}(\rho_3)$.

Moreover, from (7.21c) we have that

$$\Omega\cap B_{\rho/2}(0) \subset \Phi(\Omega'\cap B_\rho(0)) \subset \Omega\cap B_{C_2\rho}(0) \quad \text{for every } \rho\leq\rho_2. \tag{7.26}$$

Let $0 < r \leq \beta r < \rho_3/2$. From (7.25) and (7.26) we have, arguing similarly to the proof of Proposition 3.5, that

$$\int_{\Omega\cap B_{\beta r}(0)} u^2 \leq C\beta^K \int_{\Omega\cap B_r(0)} u^2\,, \tag{7.27}$$

where C and K only depend on λ, Λ, E, α and $\tilde{N}(\rho_3)$. From (7.26) and (6.15) we can estimate

$$\int_{\Omega'\cap B_{\rho_3}(0)} |\nabla w|^2 \leq C\rho_0^{n-2}\|g\|^2_{H^{1/2}(\partial\Omega)}\,, \tag{7.28}$$

where $C > 0$ only depends on λ, E, α and M. On the other hand

$$\int_{\Omega'\cap B_{\rho_3}(0)} w^2 \geq C \int_{\Omega\cap B_{\rho_3/2}(0)} u^2\,, \tag{7.29}$$

where C only depends on λ, E and α.

Setting $\rho_4 = \frac{\rho_3}{32\sqrt{1+E^2}}$, $P = (0, \rho_3/4)$, we have that $B_{4\rho_4}(P) \subset \Omega\cap B_{\rho_3/2}(0)$. From (7.29) and Proposition 4.3 we have

$$\int_{\Omega'\cap B_{\rho_3}(0)} w^2 \geq C\rho_0^n\|g\|^2_{H^{1/2}(\partial\Omega)}\,, \tag{7.30}$$

where C only depends on λ, Λ, E, M and F. Therefore we can majorize $\tilde{N}(\rho_3)$ by a constant $C > 0$ only depending on λ, Λ, E, α, M and F. Thus (4.8) follows, provided $\beta r < \rho_3/2$. The case when $\rho_3/2 \leq \beta r$ can be easily treated, again by the use of Proposition 4.3.

In the general case $\sigma(x_0) \neq Id$, (4.8) is derived by repeating arguments in the proof of Proposition 3.5. □

8. – Relative Graphs

We premise the proof of Proposition 3.6 with one lemma.

LEMMA 8.1. *Let Ω_1, Ω_2 be two bounded domains. There exist absolute constants L_0, δ_0, $L_0 > 0$, $0 < \delta_0 < 1$, such that if*

(8.1*a*) *the boundaries of Ω_1, Ω_2 are of Lipschitz class with constants ρ_0, L,*

and also

$$L \leq L_0\,, \tag{8.1b}$$

$$d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}) \leq \delta_0\rho_0\,, \tag{8.1c}$$

then the following facts hold true:

i) for every $P \in \partial\Omega_1$, up to a rigid transformation of coordinates which maps P into the origin, we have

$$\Omega_i \cap B_{r_0}(P) = \{x \in B_{r_0}(0) \text{ s.t. } x_n > \varphi_i(x')\}, \quad i = 1, 2, \tag{8.2a}$$

where $r_0 = \delta_1 \rho_0$, with a suitable absolute constant δ_1, $\delta_0 < \delta_1 \leq 1$, and φ_1, φ_2 are Lipschitz functions on $B_{r_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\|\varphi_i\|_{C^{0,1}\left(B_{r_0}(0)\right)} \leq \rho_0 L_1, \quad i = 1, 2, \tag{8.2b}$$

$$\|\varphi_1 - \varphi_2\|_{L^\infty\left(B_{r_0}(0)\right)} \leq K_1 d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}). \tag{8.2c}$$

ii) we have

$$d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}) \leq K_2 d_m(\Omega_1, \Omega_2), \tag{8.3}$$

where d_m is the modified distance introduced in Definition 3.2. Here, the quantities L_1, K_1, K_2 are also positive absolute constants.

REMARK 8.1. It may be interesting to note that the above lemma may fail if the smallness hypothesis (8.1*b*) on the Lipschitz constant is dropped. In fact, a two dimensional example by Rondi [R, Remark 2.3] shows that *i*) may not hold when $L = 1$.

PROOF OF LEMMA 8.1. Up to a dilation of the coordinate system, we can assume, without loss of generality, that $\rho_0 = 1$. Let us denote, for simplicity,

$$d = d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}), \tag{8.4}$$

and let us set

$$\alpha = \arctan L, \quad \alpha_0 = \arctan L_0. \tag{8.5}$$

Let us fix $P \in \partial\Omega_1$, and let (x', x_n) be the coordinate system centered at P appearing in the local graph representation of $\partial\Omega_1$ given in Definition 3.1. Let us define the following truncated conical regions

$$T_h^- = \{(x', x_n) \in B_1(0) \text{ s.t. } x_n \geq -L|x'| - h\}, \tag{8.6$-$}$$

$$T_h^+ = \{(x', x_n) \in B_1(0) \text{ s.t. } x_n > L|x'| + h\}, \tag{8.6$+$}$$

for every $h > 0$. By (8.1*c*) and (8.4) we have

$$\overline{\Omega_2} \cap B_{1-\delta_0}(0) \subset T^-_{d/\cos\alpha}, \tag{8.7}$$

and, again by (8.4), there exists $Q \in \overline{\Omega_2} \cap B_1(0)$ such that

$$|Q - P| \leq d.$$

Let $Q = (y', y_n)$ and let $l(Q)$ be the following vertical segment

$$l(Q) = \{(x', x_n) \in B_{1-\delta_0}(0) \text{ s.t. } x' = y', x_n \leq y_n\}, \tag{8.8}$$

Now, provided L_0 and δ_0 are chosen sufficiently small, we have that the top endpoint of $l(Q)$ is $Q \in \overline{\Omega_2}$, whereas the bottom endpoint of $l(Q)$ is outside of $T^-_{d/\cos\alpha}$ and hence, by (8.7), outside of $\overline{\Omega_2}$. Therefore there exists $Q' \in \partial\Omega_2 \cap l(Q) \cap T^-_{d/\cos\alpha}$, thus

$$|Q' - P| \leq |Q' - Q| + |Q - P| \leq \left(\frac{2}{\cos\alpha} + 1\right) d \leq \left(\frac{2}{\cos\alpha_0} + 1\right) d. \tag{8.9}$$

Observe also that, by the same reasoning, we can prove that for every $Q \in \partial\Omega_2$ there exists $P' \in \partial\Omega_1$ such that

$$|Q - P'| \leq Cd, \tag{8.10}$$

where $C = \frac{2}{\cos\alpha_0} + 1$. In other words, we have proven that

$$d_{\mathcal{H}}(\partial\Omega_1, \partial\Omega_2) \leq Cd. \tag{8.11}$$

Now, by the Lipschitz character of $\partial\Omega_2$ and by (8.11), we have that, given Q' as above,

$$\overline{\Omega_2} \supset RT_0^+ + Q',$$

where $R : \mathbb{R}^n \to \mathbb{R}^n$ is a suitable rotation such that $\Omega_2 \cap B_1(Q')$ has the local graph representation of Definition 3.1 in the rotated coordinates Rx, $x = (x', x_n)$. Therefore, by (8.7),

$$(RT_0^+ + Q') \cap B_{1-\delta_0}(0) \subset T^-_{d/\cos\alpha}.$$

This condition poses a constraint on the angle θ between the unit vectors e_n, Re_n. Indeed, by some trigonometry, we can obtain

$$\theta \leq \Theta(d, L),$$

where the function $\Theta(d, L) \geq 0$ tends to zero as $(d, L) \to (0, 0)$. Therefore, if δ_0, L_0 are chosen sufficiently small, then we can find constants δ_1, L_1, $\delta_0 < \delta_1 \leq 1$, $L_0 \leq L_1$, such that $\partial\Omega_2 \cap B_{\delta_1}(0)$ is a Lipschitz graph also with respect to the coordinates (x', x_n), with Lipschitz constant L_1. Therefore (8.2*a*), (8.2*b*) hold. Next we prove (8.2*c*). For every $x' \in B_{\delta_1}(0) \subset \mathbb{R}^{n-1}$, in view of (8.11), let $(y', \varphi_1(y')) \in \partial\Omega_1$ be such that

$$(|x' - y'|^2 + |\varphi_2(x') - \varphi_1(y')|^2)^{1/2} \leq Cd.$$

Therefore

$$|\varphi_2(x') - \varphi_1(x')| \leq |\varphi_2(x') - \varphi_1(y')| + |\varphi_1(x') - \varphi_1(y')| \leq Cd + LCd.$$

Hence we may pick $K_1 = (1 + L_0)C$, where C is as in (8.10), and (8.2c) follows. Finally, we prove (8.3). Suppose, without loss of generality, that $Q \in \Omega_2 \setminus \overline{\Omega_1}$, $P \in \partial\Omega_1$ are such that $|P - Q| = d = \mathrm{dist}(Q, \overline{\Omega_1})$. With the local representation of $\partial\Omega_1$ within $B_{\delta_1}(P)$, as introduced above, let $Q = (y', y_n)$. By (8.2a), we must have $y_n \geq \varphi_2(y')$ and, posing $P' = (y', \varphi_1(y'))$, we have $\varphi_1(y') > y_n$. Therefore

$$d \leq |Q - P'| = \varphi_1(y') - y_n \leq \varphi_1(y') - \varphi_2(y') . \tag{8.12}$$

Possibly choosing a smaller δ_0, we can find a point $(x', \varphi_1(x')) \in B_{\delta_1}(0)$ such that

$$|(y', \varphi_2(y')) - (x', \varphi_1(x'))| = \mathrm{dist}((y', \varphi_2(y')), \overline{\Omega_1}) \leq d_m(\Omega_1, \Omega_2) .$$

Therefore

$$\varphi_1(y') - \varphi_1(x') \leq L_1|x' - y'| \leq L_1 d_m(\Omega_1, \Omega_2) ,$$

and also, by (8.12),

$$\begin{aligned} d &\leq \varphi_1(y') - \varphi_2(y') \leq \varphi_1(y') - \varphi_1(x') + \varphi_1(x') - \varphi_2(y') \\ &\leq L_1 d_m(\Omega_1, \Omega_2) + d_m(\Omega_1, \Omega_2) . \end{aligned}$$

Therefore (8.3) holds with $K_2 = L_1 + 1$. □

Proof of Proposition 3.6. Notice that, if $\partial\Omega_i$, $i = 1, 2$, are of class $C^{1,\alpha}$ with constants ρ_0, E, then they are also of Lipschitz class with constants ρ, $E(\frac{\rho}{\rho_0})^\alpha$ for every ρ, $0 < \rho \leq \rho_0$. Therefore, if we fix $\rho_1 \leq \rho_0$ such that $E(\frac{\rho_1}{\rho_0})^\alpha \leq L_0$, and d_0 is chosen in such a way that $d_0 \leq \delta_0\rho_1$, then the hypotheses of Lemma 8.1 are met, provided ρ_0 is replaced with ρ_1. Notice that now we shall have $r_0 = \delta_1\rho_1$, hence r_0/ρ_0 only depends on α and E. Let $P \in \partial\Omega_1$ and let (x', x_n) the coordinates used in the local representation of $\partial\Omega_1$ as a $C^{1,\alpha}$ graph given by Definition 2.1.We already know that also $\partial\Omega_2$ is represented near P by the graph of a Lipschitz function φ_2. We need to show that also φ_2 is $C^{1,\alpha}$. We know that there exists a coordinate system for which $\partial\Omega_2 \cap B_{r_0}(0)$ is represented as a graph of a $C^{1,\alpha}$ function. Let (ξ', ξ_n), $\xi' \in \mathbb{R}^{n-1}$, $\xi_n \in \mathbb{R}$, be such coordinate system in $B_{r_0}(0)$, and let $\psi \in C^{1,\alpha}(B_{r_0}(0))$, $B_{r_0}(0) \subset \mathbb{R}^{n-1}$, the function such that

$$\partial\Omega_2 \cap B_{r_0}(0) = \{(\xi', \xi_n) \in B_{r_0}(0) \text{ s.t. } \xi_n = \psi(\xi')\} ,$$

and also $|\nabla\psi|_\alpha \leq \rho_0^{-\alpha}E$. Denoting by ν the exterior unit normal to $\partial\Omega_2$ within $B_{r_0}(0)$, we have

$$\nu = \frac{(\nabla_{\xi'}\psi, -1)}{\sqrt{1 + |\nabla_{\xi'}\psi|^2}} , \tag{8.13}$$

and therefore we can easily compute a constant $K \geq 1$ such that

$$|\nu(\xi_1', \psi(\xi_1')) - \nu(\xi_2', \psi(\xi_2'))| \leq K E \rho_0^{-\alpha} |\xi_1' - \xi_2'|^\alpha ,$$
$$\text{for every } \xi_1', \xi_2' \in B_{r_0}(0) \subset \mathbb{R}^{n-1} .$$

Setting $Q_i = (\xi_i', \psi(\xi_i'))$, $i = 1, 2$, we obtain

$$|\nu(Q_1) - \nu(Q_2)| \leq K E \rho_0^{-\alpha} |Q_1 - Q_2|^\alpha, \quad \text{for every } Q_1, Q_2 \in \partial\Omega_2 \cap B_{r_0}(0) .$$

Turning back to the (x', x_n) coordinates, if $Q_i = (x_i', \varphi_2(x_i'))$, $i = 1, 2$, we obtain

$$|\nu(Q_1) - \nu(Q_2)| \leq K E \rho_0^{-\alpha} (1 + L_0^2)^{\alpha/2} |x_1' - x_2'| .$$

Rephrasing now (8.13) in terms of the coordinates (x', x_n):

$$\nu = \frac{(\nabla_{x'}\varphi_2, -1)}{\sqrt{1 + |\nabla_{x'}\varphi_2|^2}} ,$$

we easily derive

$$|\nabla_{x'}\varphi_2|_\alpha \leq C E \rho_0^{-\alpha} ,$$

where $C > 0$ is an absolute constant. Now, for any β, $0 < \beta < \alpha$, we can use the interpolation inequalities

$$|\nabla_{x'}\varphi|_\beta \leq C_1 \left[|\nabla_{x'}\varphi|_\alpha^{\frac{1+\beta}{1+\alpha}} \|\varphi\|_\infty^{\frac{\alpha-\beta}{1+\alpha}} + \frac{1}{r_0^{1+\beta}} \|\varphi\|_\infty \right] , \tag{8.14}$$

$$\|\nabla\varphi\|_\infty \leq C_2 \left[|\nabla_{x'}\varphi|_\alpha^{\frac{1}{1+\alpha}} \|\varphi\|_\infty^{\frac{\alpha}{1+\alpha}} + \frac{1}{r_0} \|\varphi\|_\infty \right] , \tag{8.15}$$

where $C_1 > 0$ only depends on α and β, $C_2 > 0$ only depends on α, and norms and seminorms are taken on $B_{r_0}(0) \subset \mathbb{R}^{n-1}$. We obtain, by (8.2$c$),

$$|\nabla(\varphi_1 - \varphi_2)|_\beta \leq C_1 \left[E^{\frac{1+\beta}{1+\alpha}} \rho_0^{\frac{-\alpha(1+\beta)}{1+\alpha}} d^{\frac{\alpha-\beta}{1+\alpha}} + r_0^{-(1+\beta)} d \right] ,$$

$$\|\nabla(\varphi_1 - \varphi_2)\|_\infty \leq C_2 \left[E^{\frac{1}{1+\alpha}} \rho_0^{\frac{-\alpha}{1+\alpha}} d^{\frac{\alpha}{1+\alpha}} + r_0^{-1} d \right] ,$$

where, as above, $C_1 > 0$ only depends on α and β, $C_2 > 0$ only depends on α. Again by (8.2c) we deduce

$$\|\varphi_1 - \varphi_2\|_{C^{1,\beta}(B_{r_0}(0))} \leq C \Big[d + E^{\frac{1}{1+\alpha}} \rho_0^{1-\frac{\alpha}{1+\alpha}} d^{\frac{\alpha}{1+\alpha}} + \rho_0 r_0^{-1} d$$
$$+ E^{\frac{1+\beta}{1+\alpha}} \rho_0^{1+\beta-\frac{\alpha(1+\beta)}{1+\alpha}} d^{\frac{\alpha-\beta}{1+\alpha}} + \rho_0^{1+\beta} r_0^{-1-\beta} d \Big] ,$$

where $C > 0$ only depends on α and β. Now, using $d \leq d_0 \leq \delta_0\rho_1 \leq \rho_0$, we obtain (3.12). Finally $ii)$ and $iii)$ follow from Lemma 8.1, more precisely from (8.3) and (8.2) respectively. □

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