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Decay of Fourier Transforms and Summability of Eigenfunction Expansions

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Abstract. Fourier coefficients $\int_0^1 f(x)\exp(-2\pi inx)dx$ of piecewise smooth functions are of the order of $|n|^{-1}$ and Fourier series $\sum_{n=-\infty}^{+\infty} \widehat{f}(n)\exp(2\pi inx)$ converge everywhere. Here we consider analogs of these results for eigenfunction expansions $f(x) = \sum_{\lambda} \mathcal{F}f(\lambda)\varphi_\lambda(x)$, where $\{\lambda^2\}$ and $\{\varphi_\lambda(x)\}$ are eigenvalues and an orthonormal complete system of eigenfunctions of a second order positive elliptic operator on a N-dimensional manifold. We prove that the norms of projections of piecewise smooth functions on subspaces generated by eigenfunctions with $\Lambda \le \lambda \le \Lambda+1$ satisfy the estimates $\left\{\sum_{\Lambda\le\lambda\le\Lambda+1} |\mathcal{F}f(\lambda)|^2\right\}^{1/2} \le c\Lambda^{-1}$. Then we give some sharp results on the Riesz summability of Fourier series. In particular we prove that the Riesz means $\sum_{\lambda<\Lambda}\left(1-(\lambda/\Lambda)^{2k}\right)^{\delta}\mathcal{F}f(\lambda)\varphi_\lambda(x)$ of order $\delta > (N-3)/2$ converge.

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Let $\mathbb{M}$ be a smooth manifold of dimension N and let Δ be a second order positive elliptic operator on $\mathbb{M}$, with smooth real coefficients. Assume that this differential operator with suitable boundary conditions is self adjoint with respect to some positive smooth density $d\mu$ and admits a sequence of eigenvalues $\{\lambda^2\}$ and a system of eigenfunctions $\{\varphi_\lambda(x)\}$ orthonormal and complete in $\mathbb{L}^2(\mathbb{M}, d\mu)$. Then to every function in $\mathbb{L}^2(\mathbb{M}, d\mu)$ one can associate a Fourier transform and a Fourier series,

$$\mathcal{F}f(\lambda) = \int_{\mathbb{M}} f(y)\overline{\varphi_\lambda(y)}d\mu(y), \qquad f(x) = \sum_{\lambda} \mathcal{F}f(\lambda)\varphi_\lambda(x).$$

These Fourier series converge in the metric of $\mathbb{L}^2(\mathbb{M}, d\mu)$ and more generally in the topology of distributions, but under appropriate conditions the

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convergence holds also pointwise. As a reference on localization, convergence and summability of eigenfunction expansions, see the survey by Alimov, Il'in, Nikishin [1]. See also the paper by Meaney [19], in which it is observed that eigenfunction expansions of functions in Sobolev spaces of positive index $\mathbb{W}^2_\varepsilon(\mathbb{M}, d\mu)$ converge almost everywhere. Indeed, by the Weyl estimates on the growth of eigenvalues, the k-th eigenvalue λ^2 is of the order of $k^{2/n}$ and if $\sum_\lambda \left(1+\lambda^2\right)^\varepsilon |\mathcal{F}f(\lambda)|^2 < +\infty$, then the assumptions of the Rademacher-Menchoff theorem on the almost everywhere convergence of orthonormal series are satisfied. If $\{\phi_k(x)\}_{k=1}^{+\infty}$ is an orthonormal system and $\sum_{k=1}^{+\infty} \log^2(k)\,|c(k)|^2 < +\infty$, then $\sum_{k=1}^{+\infty} c(k)\phi_k(x)$ converges almost everywhere. However if one is interested in convergence at given points, then the situation is different and in order to obtain some results one has to restrict the class of functions and possibly use some summation methods. In particular, here we mainly consider Riesz summability of eigenfunction expansions of piecewise smooth functions.

Let $g(x)\chi_\Omega(x)$ be the product of an everywhere smooth function and a characteristic function of an open domain in $\mathbb{M}$ with smooth boundary and compact closure at a positive distance from $\partial\mathbb{M}$. It is convenient to normalize this function on $\partial\Omega$ by putting it equal to $g(x)/2$. Linear combinations of these functions generate a space which we call $\mathbb{X}(\mathbb{M})$. This is an obvious extension of the definition of piecewise smooth functions in one variable, however we can also give a slightly more general definition. Let Ω be a domain in $\mathbb{M}$ with smooth boundary and compact closure at a positive distance from $\partial\mathbb{M}$. In a small neighborhood of $\partial\Omega$ one can introduce appropriate tangential and normal coordinates, writing $x = (\vartheta, t)$, with $\vartheta \in \partial\Omega$ and $d(x, \partial\Omega) = d(x, \vartheta) = t \in \mathbb{R}$, where $d(x, y)$ is a smooth Riemannian distance on $\mathbb{M}$. One can also identify each small piece of $\partial\Omega$ with an open set in $\mathbb{R}^{N-1}$, so that $(\vartheta, t) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Let $f(x)$ be a function on $\mathbb{M}$, which is smooth in Ω and vanishes outside. We assume that for some $-\infty < \alpha < +\infty$ and every j and β one has the estimates

$$\left| \frac{\partial^j}{\partial t^j} \frac{\partial^\beta}{\partial \vartheta^\beta} f(\vartheta, t) \right| \le c\, |t|^{\alpha - j},$$

with c independent on (ϑ, t). Linear combinations of such functions generate a space which we call $\mathbb{X}^\alpha(\mathbb{M})$. Roughly speaking, functions in this space have compact support and are piecewise smooth with singularities along smooth surfaces, derivatives $\partial^\beta/\partial\vartheta^\beta$ in directions tangential to these surfaces are bounded while normal derivatives $\partial^j/\partial t^j$ may grow as $|t|^{\alpha-j}$. In particular, functions in $\mathbb{X}^\alpha(\mathbb{M})$ with $\alpha > -1$ are integrable and if $\alpha > 0$ they also are continuous. The space $\mathbb{X}(\mathbb{M})$ is contained in $\mathbb{X}^0(\mathbb{M})$, but as a proper subspace. Functions in $\mathbb{X}(\mathbb{M})$ can have only jump discontinuities and derivatives do not deteriorate, on the contrary the definition of $\mathbb{X}^0(\mathbb{M})$ allows more complicated singularities. We emphasize that these piecewise smooth functions have compact support at a positive distance from $\partial\mathbb{M}$. Of course the emphasis is superfluous if $\mathbb{M}$ has no boundary.

Let us define the Sobolev spaces $\mathbb{W}^2_\varepsilon(\mathbb{M}, d\mu)$ and Besov spaces $\mathbb{B}^2_{\varepsilon,\infty}(\mathbb{M}, d\mu)$ by the norms

$$\|f\|_{\mathbb{W}^2_\varepsilon(\mathbb{M},d\mu)} = \left\{ \sum_\lambda \left(1+\lambda^2\right)^\varepsilon |\mathcal{F}f(\lambda)|^2 \right\}^{1/2} < +\infty,$$

$$\|f\|_{\mathbb{B}^2_{\varepsilon,\infty}(\mathbb{M},d\mu)} = \sup_\Lambda \left\{ \Lambda^{2\varepsilon} \sum_{\Lambda\le\lambda\le2\Lambda} |\mathcal{F}f(\lambda)|^2 \right\}^{1/2} < +\infty.$$

Functions in $\mathbb{X}^\alpha(\mathbb{M})$ have, roughly speaking, almost $\alpha+1/2$ derivatives in $\mathbb{L}^2(\mathbb{M}, d\mu)$, hence $\mathbb{X}^\alpha(\mathbb{M})$ is contained in $\mathbb{W}^2_\varepsilon(\mathbb{M}, d\mu)$ whenever $\varepsilon < \alpha+1/2$ and also in $\mathbb{B}^2_{\alpha+1/2,\infty}(\mathbb{M}, d\mu)$. In particular, the norms of projections on subspaces generated by eigenvectors with eigenvalues $\Lambda \le \lambda \le 2\Lambda$ when $\Lambda \to +\infty$ satisfy the estimates $\{\sum_{\Lambda\le\lambda\le2\Lambda} |\mathcal{F}f(\lambda)|^2\}^{1/2} \le c\Lambda^{-\alpha-1/2}$. However, motivated by the study of Riesz summability of eigenfunction expansions, we are interested in more precise estimates, not over dyadic intervals but over $\Lambda \le \lambda \le \Lambda+1$.

THEOREM. *If $f(x)$ is a piecewise smooth function in $\mathbb{X}^\alpha(\mathbb{M})$ with $\alpha > -1$, then as $\Lambda \to +\infty$,*

$$\left\{ \sum_{\Lambda\le\lambda\le\Lambda+1} |\mathcal{F}f(\lambda)|^2 \right\}^{1/2} \le c\Lambda^{-\alpha-1}.$$

The result is best possible, since it is possible to have

$$\limsup_{\Lambda\to+\infty} \Lambda^{2+2\alpha} \sum_{\Lambda\le\lambda\le\Lambda+1} |\mathcal{F}f(\lambda)|^2 > 0.$$

Observe that these estimates agree with the ones provided by Sobolev or Besov norms. Also observe that since piecewise smooth functions are exactly in $\mathbb{B}^2_{\alpha+1/2,\infty}(\mathbb{M}, d\mu)$, the estimates are essentially best possible. However, the information contained in this theorem is more precise than the one contained in the Sobolev and Besov norms and this is crucial in our study of the localization and convergence of the partial sums of the Fourier series $\sum_{\lambda<\Lambda} \mathcal{F}f(\lambda)\varphi_\lambda(x)$ and more generally of the Riesz means $\sum_{\lambda<\Lambda}(1-(\lambda/\Lambda)^{2k})^\delta \mathcal{F}f(\lambda)\varphi_\lambda(x)$.

THEOREM. *If $f(x)$ is a piecewise smooth function in $\mathbb{X}^\alpha(\mathbb{M})$ with $\alpha > -1$ and if $\alpha+\delta > (N-3)/2$, then at every point x where $f(x)$ is smooth,*

$$\lim_{\Lambda\to+\infty} \sum_{\lambda<\Lambda} \left(1-(\lambda/\Lambda)^{2k}\right)^\delta \mathcal{F}f(\lambda)\varphi_\lambda(x) = f(x).$$

The convergence is uniform in every compact set disjoint from the singularities of $f(x)$. The condition $\alpha+\delta > (N-3)/2$ is best possible and cannot be replaced by the equality.

In the definition of Riesz means, the index δ gives the degree of smoothness of the multiplier and hence the decay of the associated kernel. This index is allowed to be negative, but then $(1-(\lambda/\Lambda)^{2k})_+^{\delta}$ becomes arbitrarily large when λ approaches to $\Lambda-$. To avoid this problem, when $\delta<0$ it suffices to stop the sums that define the Riesz means at $\lambda<\Lambda-1$. The positive integer k measures the flatness of the multiplier near the origin and this affects the speed of convergence of the means. A natural question concerns the behavior of Riesz means at points where functions are not smooth. It turns out that under the assumptions of the theorem it is possible to exhibit examples with divergence and examples with convergence. On the other hand, it is possible to prove that Riesz means of order $\delta>(N-3)/2$ of functions in $\mathbb{X}(\mathbb{M})$, a proper subspace of $\mathbb{X}^0(\mathbb{M})$, converge everywhere. Also, in neighborhoods of discontinuities there is a Gibbs phenomenon.

For more on the Euclidean Fourier transform of characteristic functions see Hlawka [12], Herz [13], Podkorytov [23], Varchenko [32], and for eigenfunction expansions see Sogge [24], Torlaschi [31]. For more on the Riesz summability of Euclidean Fourier integrals, see Bochner [4], and for convergence and summability of eigenfunction expansions see Bérard [3], Kahane [17], Pinsky [20], [21], Pinsky-Taylor [22], Sogge [25], Taylor [27], [28], [29, [30]. For a discussion of the Gibbs phenomenon see Colzani-Vignati [7], DeMichele-Roux [8], [9], [10], and Weyl [34], [35].

A final remark. For simplicity in this paper we consider functions with singularities on smooth surfaces. However, we believe that similar results holds also on piecewise smooth domains, that is intersections of domains with smooth boundaries. Here it is a pseudo proof. Let A and B be domains with smooth boundaries. The theorems apply to $\chi_A(x)$ and $\chi_B(x)$, and to the sum $g(x)=\chi_A(x)+\chi_B(x)$. Observe that $g(x)=0$ outside $A\cup B$, $g(x)=1$ in $(A-B)\cup(B-A)$, and $g(x)=2$ in $A\cap B$. Let $f(x)$ be such that $f(x)=0$ outside $A\cup B$, $f(x)=1$ in $(A-B)\cup(B-A)$ and $f(x)=3$ in $A\cap B$. Since $f(x)$ and $g(x)$ have essentially the same discontinuities, the theorems should apply also to $f(x)$ and hence to $f(x)-g(x)=\chi_{A\cap B}(x)$.

The index of the paper is the following. In order to introduce problems and techniques in a simple model case, in the first section we consider the convergence and divergence of Riesz means of Fourier integrals in the Euclidean space $\mathbb{R}^N$. This section is independent of the others. In the second section we study operators associated to eigenfunction expansions of type $\mathcal{M}f(x)=\sum_\lambda m(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)$. The classical idea is to synthesize the kernels associated to these operators using the fundamental solution of the wave equation $\cos(t\sqrt{\Delta})=\sum_\lambda \cos(t\lambda)\varphi_\lambda(x)\overline{\varphi_\lambda(y)}$ and the Hadamard parametrix construction. In the third section we estimate $\left\{\sum_{\Lambda\leq\lambda\leq\Lambda+1}|\mathcal{F}f(\lambda)|^2\right\}^{1/2}$ by studying $\sum_\lambda m(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)$ where $m(\lambda)$ is a suitable bump function concentrated around Λ. In the fourth section we study the Riesz means $\sum_{\lambda<\Lambda}(1-(\lambda/\Lambda)^{2k})^\delta\mathcal{F}f(\lambda)\varphi_\lambda(x)$. In particular we decompose these means into a localized part which is essentially Euclidean and is good where the function is smooth, plus a remainder which can be controlled using estimates on the Fourier coefficients. It may be worthy to point out that the techniques used in the study of Riesz means are general enough to be applicable to other operators. In the last section we give a short proof of a generalization of a formula of Voronoi and Hardy on the number of integer points in a disc.

The content of this paper is somehow related to our previous paper [5], in which we studied the convergence of the partial sums $\sum_{\lambda<\Lambda} \mathcal{F}f(\lambda)\varphi_\lambda(x)$ on two dimensional compact manifolds. However the results in this paper are sharper and more complete. Some of our results have been extended in [30].

1. – Riesz means of Fourier integrals in Euclidean spaces

In this section we illustrate the theorems stated in the introduction in the context of the harmonic analysis on Euclidean spaces. The exponentials $\{\exp(2\pi i x \cdot \xi)\}$ are eigenfunctions of the Laplace operator $-\sum_{j=1}^{N} \partial^2/\partial x_j^2$ on $\mathbb{R}^N$ with eigenvalues $\{4\pi^2 |\xi|^2\}$. For functions in $\mathbb{L}^1 + \mathbb{L}^2(\mathbb{R}^N)$ one has the Fourier transform and Fourier expansion

$$\mathbb{F}f(\xi) = \int_{\mathbb{R}^N} f(y)\exp(-2\pi i \xi \cdot y)dy,\ f(x) = \int_{\mathbb{R}^N} \mathbb{F}f(\xi)\exp(2\pi i x \cdot \xi)d\xi\,.$$

The projection of a function on the subspace associated to eigenvalues between $4\pi^2\Lambda^2$ and $4\pi^2(\Lambda+1)^2$ is given by

$$\int_{\{\Lambda\le|\xi|\le\Lambda+1\}} \mathbb{F}f(\xi)\exp(2\pi i \xi \cdot x)d\xi,$$

with norm

$$\left\{\int_{\{\Lambda\le|\xi|\le\Lambda+1\}} |\mathbb{F}f(\xi)|^2\,d\xi\right\}^{1/2}.$$

We want to show that the theorems stated in the introduction are essentially best possible. In order to illustrate the first theorem let us consider the function $(1-|x|^2)_+^\alpha$. This function is in $\mathbb{X}^\alpha(\mathbb{R}^N)$ and it has Fourier transform

$$\begin{aligned}&\pi^{-\alpha}\Gamma(\alpha+1)\frac{J_{\alpha+N/2}(2\pi\,|\xi|)}{|\xi|^{\alpha+N/2}}\\ &\qquad\approx \pi^{-\alpha-1}\Gamma(\alpha+1)\,|\xi|^{-\alpha-(N+1)/2}\cos\left(2\pi\,|\xi| - (2\alpha+N+1)\pi/4\right).\end{aligned}$$

Hence

$$0 < \limsup_{\Lambda\to+\infty} \Lambda^{2\alpha+2}\int_{\{\Lambda\le|\xi|\le\Lambda+1\}} |\mathbb{F}f(\xi)|^2\,d\xi < +\infty.$$

Let us now consider the Riesz means

$$\int_{\mathbb{R}^N}\left(1-(\Lambda^{-1}|\xi|)^{2k}\right)_+^\delta \mathbb{F}f(\xi)\exp(2\pi i x \cdot \xi)d\xi.$$

Again as a test one can consider the means of the function $\left(1-|x|^2\right)_+^\alpha$ at $x=0$,

$$\begin{aligned}
&\pi^{-\alpha}\Gamma(\alpha+1)\int_{\{|\xi|<\Lambda\}}\left(1-\left(\Lambda^{-1}|\xi|\right)^{2k}\right)^{\delta}|\xi|^{-\alpha-N/2}J_{\alpha+N/2}(2\pi|\xi|)\,d\xi\\
&=\pi^{-\alpha}\Gamma(\alpha+1)\Lambda^{N/2-\alpha}\int_{\{|\sigma|=1\}}\int_0^1\left(1-t^{2k}\right)^{\delta}t^{N/2-1-\alpha}J_{\alpha+N/2}(2\pi\Lambda t)dtd\sigma\\
&\approx\pi^{-\alpha-1}\Gamma(\alpha+1)\,|\{|\sigma|=1\}|\,\Lambda^{(N-1)/2-\alpha}\\
&\times\int_0^1\left(1-t^{2k}\right)^{\delta}t^{(N-3)/2-\alpha}\cos\left(2\pi\Lambda t-(2\alpha+N+1)\pi/4\right)dt.
\end{aligned}$$

The singularity of $t^{(N-3)/2-\alpha}$ in $t=0+$ gives to the Fourier transform of $t^{(N-3)/2-\alpha}(1-t^{2k})_+^\delta$ a decay $\Lambda^{\alpha-(N-1)/2}$, while the singularity of $(1-t^{2k})_+^\delta$ in $t=1-$ gives a decay $\Lambda^{-\delta-1}$ with an oscillation. See also the next Lemma 1.2. Hence, the Riesz means with index $\delta\le(N-3)/2-\alpha$ do not converge at the origin. It turns out that this example also describes the speed of convergence of Riesz means $\Lambda^{(N-3)/2-\alpha-\delta}$.

We also want to present other examples, with $\alpha=\delta=0$ and $N=1$. In this case the Riesz means are nothing but the partial sums of the Fourier integrals,

$$\int_{-\Lambda/2\pi}^{+\Lambda/2\pi}\left(\int_{-\infty}^{+\infty}f(y)\exp(-2\pi i\xi y)dy\right)\exp(2\pi ix\xi)d\xi=\int_{-\infty}^{+\infty}\frac{\sin(\Lambda(x-y))}{\pi(x-y)}f(y)dy.$$

By the Riemann localization principle, the behavior of these partial sums at a given point depends only on values of the function in arbitrary small neighborhoods of this point, in particular these partial sums converge at all points where the function is smooth. Let us then restrict our attention to singular points. Let $\chi(x)$ be a smooth function with compact support and equal to one in a neighborhood of zero and let $g(x)=\begin{cases}\chi(x) & \text{if } x>0,\\ 0 & \text{if } x\le 0.\end{cases}$ This function is in $\mathbb{X}^0(\mathbb{R})$ with a jump discontinuity in $x=0$, and at this singular point

$$\int_0^{+\infty}\frac{\sin(\Lambda y)}{\pi y}\chi(y)dy=\frac{1}{2}-\int_0^{+\infty}\frac{1-\chi(y)}{\pi y}\sin(\Lambda y)dy.$$

Hence the partial sums converge to $\dfrac{g(0-)+g(0+)}{2}$.

Now let $b(x)=\begin{cases}\chi(x)\exp(i\lg(x)) & \text{if } x>0,\\ 0 & \text{if } x\le 0.\end{cases}$ Also this function is in $\mathbb{X}^0(\mathbb{R})$ with a discontinuity in $x=0$, but this singularity is worse than the previous one and

$$\begin{aligned}
&\int_0^{+\infty} \frac{\sin(\Lambda y)}{\pi y} \chi(y) \exp(i \lg(y))\, dy \\
&\quad = \exp(-i \lg(\Lambda)) \int_0^{+\infty} \frac{\sin(y)}{\pi y} \exp(i \lg(y))\, dy \\
&\quad + \exp(-i \lg(\Lambda)) \int_0^{+\infty} \frac{\chi(y/\Lambda) - 1}{\pi y} \sin(y) \exp(i \lg(y))\, dy.
\end{aligned}$$

Here the main term is the first one, and it does not converge. These one dimensional examples can be easily extended by radial symmetry to several dimensions. Hence, under the sole assumptions of the theorems in the introduction it is not possible to decide the convergence or divergence of the Riesz means at singular points and some extra assumptions seem necessary. Here we want to prove that Riesz means of order $\delta > (N-3)/2$ of functions in $\mathbb{X}(\mathbb{R}^N)$ converge everywhere. This result is already contained in [9], [10] and in [21], but our proof is different and it can be generalized to eigenfunction expansions.

THEOREM 1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary and let $g(x)$ be a smooth function in $\mathbb{R}^N$. Define*

$$f(x) = \begin{cases} g(x) & \text{if } x \in \Omega, \\ \dfrac{1}{2} g(x) & \text{if } x \in \partial\Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then the Riesz means of order $\delta > (N-3)/2$ of this function converge at every point,

$$\lim_{\Lambda \to +\infty} \int_{\{|\xi| < \Lambda\}} \left(1 - \left(\Lambda^{-1} |\xi|\right)^{2k}\right)^{\delta} \mathbb{F}f(\xi) \exp(2\pi i x \cdot \xi) d\xi = f(x).$$

The convergence is uniform in every compact set disjoint from the discontinuities and in a neighborhood of the discontinuities there is a Gibbs phenomenon.

PROOF. Let $\Lambda^N K(\Lambda |x|)$ be the Fourier transform of $\left(1 - (\Lambda^{-1}|\xi|)^{2k}\right)_+^{\delta}$. Integrating in polar coordinates one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(1 - (\Lambda^{-1}|\xi|)^{2k}\right)_+^{\delta} \mathbb{F}f(\xi) \exp(2\pi i x \cdot \xi) d\xi \\
&\quad = \int_{\mathbb{R}^N} \Lambda^N K(\Lambda |y|) f(x-y) dy \\
&\quad = \Lambda^N \int_0^{+\infty} \left(\int_{\mathbb{S}^{N-1}} f(x - sy) dy \right) K(\Lambda s) s^{N-1} ds.
\end{aligned}$$

The analysis in Bochner [4] points out that the convergence of Riesz means is related to the decay and the regularity of the kernels $\Lambda^N K(\Lambda |x|)$ and of the spherical means $\int_{\mathbb{S}^{N-1}} f(x - sy) dy$. We first consider the behavior of the kernels.

LEMMA 1.2. *We have*

$$\int_{\mathbb{R}^N} \left(1-(\Lambda^{-1}|\xi|)^{2k}\right)_+^\delta \exp(2\pi i x\cdot\xi)d\xi = \Lambda^N K(\Lambda|x|).$$

The kernel $K(z)$ is an entire function with asymptotic expansion for $z\to+\infty$

$$K(z) = Az^{-\delta-(N+1)/2}\cos(2\pi z+\alpha) + Bz^{-\delta-(N+3)/2}\cos(2\pi z+\beta)+\dots$$

PROOF. Fourier transforms of radial functions are radial and Fourier transforms of functions with compact supports are entire. We now consider the asymptotic expansion. When $k=1$ there is a simple expression of the kernel in terms of Bessel functions, see Bochner [4] or Stein-Weiss [25, IV],

$$\begin{aligned}\int_{\mathbb{R}^N} \left(1-|\xi|^2\right)_+^\delta \exp(2\pi i x\cdot\xi)d\xi &= \pi^{-\delta}\Gamma(\delta+1)\,|x|^{-\delta-N/2} J_{\delta+N/2}(2\pi\,|x|)\\ &= \pi^{-\delta-1}\Gamma(\delta+1)\,|x|^{-\delta-(N+1)/2}\cos\left(2\pi\,|x|-(2\delta+N+1)\pi/4\right)+\dots\end{aligned}$$

The asymptotic expansion when $k=2,3,\dots$ follows from the one with $k=1$. Indeed one can write

$$\begin{aligned}\left(1-|\xi|^{2k}\right)_+^\delta &= \left(1+|\xi|^2+\dots+|\xi|^{2(k-1)}\right)^\delta \left(1-|\xi|^2\right)_+^\delta\\ &= k^\delta\left(1-|\xi|^2\right)_+^\delta + \frac{\left(1+|\xi|^2+\dots+|\xi|^{2(k-1)}\right)^\delta - k^\delta}{1-|\xi|^2}\left(1-|\xi|^2\right)_+^{\delta+1}.\end{aligned}$$

Let $\Psi(\xi)$ be a smooth radial function with compact support and such that $\Psi(\xi) = \frac{\left(1+|\xi|^2+\dots+|\xi|^{2(k-1)}\right)^\delta-k^\delta}{1-|\xi|^2}$ when $|\xi|\le 1$. Since the Fourier transform of a product is a convolution of Fourier transforms, we can write

$$\mathbb{F}\left(\left(1-|\xi|^{2k}\right)_+^\delta\right)(x) = k^\delta\mathbb{F}\left(\left(1-|\xi|^2\right)_+^\delta\right)(x) + \mathbb{F}\Psi * \mathbb{F}\left(\left(1-|\xi|^2\right)_+^{\delta+1}\right)(x).$$

Since $\mathbb{F}\Psi(\xi)$ is rapidly decreasing, we conclude that $\mathbb{F}((1-|\xi|^{2k})_+^\delta)(x)$ differs from $k^\delta\mathbb{F}((1-|\xi|^2)_+^\delta)(x)$ by a quantity of the order of $\mathbb{F}((1-|\xi|^2)_+^{\delta+1})(x)$. Iterating the argument one can obtain the complete asymptotic expansion. □

We now consider the behavior of $\int_{\mathbb{S}^{N-1}} f(x-sy)dy$. Spherical means of piecewise smooth functions are not necessarily smooth and can also be discontinuous. For example, if $f(y)$ has a discontinuity on a piece of sphere of center x and radius r, then the spherical means centered at x may have a jump when $s=r$. However, at least for small s these spherical means are smooth and this suffices for convergence. Of course the idea is that for piecewise smooth functions an analog of the Riemann localization principle holds.

LEMMA 1.3. *For every x there exists $\varepsilon > 0$ such that in the interval $0 < s < \varepsilon$ the spherical means $s \longmapsto \int_{\mathbb{S}^{N-1}} f(x-sy)dy$ are smooth and converge to $\left|\mathbb{S}^{N-1}\right| f(x)$ when $s \to 0+$.*

PROOF. When the function $f(y)$ is smooth at x the result is obvious, so that we only need to consider the case of x on a discontinuity. Let Ω be defined by the condition $\Phi(y) < 0$, with $\Phi(y)$ smooth and $\nabla\Phi(y) \neq 0$ when $\Phi(y) = 0$. The integral that defines the spherical means is over the set $\{|y| = 1,\ \Phi(x-sy) < 0\}$. Assume that $\Phi(x) = 0$ with $\nabla\Phi(x) = (0, ..., 0, 1)$, and introduce a system of polar coordinates $y = \cos(\vartheta)n + \sin(\vartheta)z$, with $n = (0, ..., 0, 1)$, $0 \leq \vartheta \leq \pi$, $z = (z_1, ..., z_{N-1}, 0)$ and $|z| = 1$. Then, if s is positive and small, $\Phi(x - sn) < 0$, $\Phi(x + sn) > 0$, and

$$\begin{aligned}
&\frac{\partial}{\partial\vartheta}\Phi\left(x - s\cos(\vartheta)n - s\sin(\vartheta)z\right)\\
&\quad= \nabla\Phi\left(x - s\cos(\vartheta)n - s\sin(\vartheta)z\right)\cdot\left(s\sin(\vartheta)n - s\cos(\vartheta)z\right)\\
&\quad= \left(\nabla\Phi(x) + O(s)\right)\cdot\left(s\sin(\vartheta)n - s\cos(\vartheta)z\right)\\
&\quad= s\sin(\vartheta) + O(s^2).
\end{aligned}$$

Hence, for every s small enough and every z there exists a unique ϑ such that $\Phi\left(x - s\cos(\vartheta)n - s\sin(\vartheta)z\right) = 0$ and this $\vartheta(s,z)$ is a smooth function of s and z. Thus we have

$$\begin{aligned}
&\int_{\{|y|=1,\ \Phi(x-sy)<0\}} f(x-sy)dy\\
&\quad= \int_{\{|z|=1\}}\int_0^{\vartheta(s,z)} f\left(x - s\cos(\vartheta)n - s\sin(\vartheta)z\right)\sin^{N-2}(\vartheta)d\vartheta dz.
\end{aligned}$$

It is then clear that this function is smooth in the variable s and converges to $\left|\mathbb{S}^{N-1}\right| f(x)$ when $s \to 0+$. □

Let $\phi(s)$ be a smooth even function on $\mathbb{R}$, with $\phi(s) = 1$ for $|s| \leq \varepsilon/2$ and $\phi(s) = 0$ for $|s| \geq \varepsilon$. Then one can decompose the Riesz means into

$$\begin{aligned}
&\Lambda^N\int_0^{+\infty}\left(\phi(s)\int_{\mathbb{S}^{N-1}} f(x-sy)dy\right)K(\Lambda s)s^{N-1}ds\\
&\quad+\Lambda^N\int_0^{+\infty}\left((1-\phi(s))\int_{\mathbb{S}^{N-1}} f(x-sy)dy\right)K(\Lambda s)s^{N-1}ds.
\end{aligned}$$

It turns out that the first term converges to $f(x)$ while the second vanishes.

LEMMA 1.4.

$$\lim_{\Lambda\to+\infty}\Lambda^N\int_0^{+\infty}\left(\phi(s)\int_{\mathbb{S}^{N-1}} f(x-sy)dy\right)K(\Lambda s)s^{N-1}ds = f(x).$$

PROOF. We can write

$$\begin{aligned}\Lambda^N &\int_0^{+\infty} \left(\phi(s) \int_{\mathbb{S}^{N-1}} f(x-sy)dy\right) K(\Lambda s)s^{N-1}ds \\ &= \int_0^{+\infty} \psi(s) \frac{d}{ds}\left(H(\Lambda s)\right) ds \\ &= -H(0)\psi(0) - \int_0^{+\infty} H(\Lambda s) \frac{d}{ds}\psi(s) ds,\end{aligned}$$

where

$$\psi(s) = \phi(s) \int_{\mathbb{S}^{N-1}} f(x-sy)dy, \qquad H(s) = -\int_s^{+\infty} u^{N-1} K(u)du.$$

Observe that $-H(0)\psi(0) = f(x)$. Also, by the previous lemmas, $\frac{d}{ds}\psi(s)$ is uniformly bounded and $H(s)$ is uniformly bounded and converges to zero if $s \to +\infty$. Hence, $\int_0^{+\infty} H(\Lambda s) \frac{d}{ds}\psi(s)ds \to 0$ as $\Lambda \to +\infty$. □

LEMMA 1.5.

$$\left\{ \int_{\mathbb{S}^{N-1}} |\mathbb{F}f(s\xi)|^2 d\xi \right\}^{1/2} \leq cs^{-(N+1)/2}.$$

PROOF. The result for characteristic functions is in [32], but the same techniques also apply to piecewise smooth functions. Anyhow, this result is also essentially contained in the third section. □

LEMMA 1.6.

$$\lim_{\Lambda \to +\infty} \Lambda^N \int_0^{+\infty} \left((1-\phi(s)) \int_{\mathbb{S}^{N-1}} f(x-sy)dy\right) K(\Lambda s)s^{N-1}ds = 0.$$

PROOF. Since $\Lambda^{-2k\delta}\left(\Lambda^{2k} - |\xi|^{2k}\right)_+^\delta$ is the Fourier transform of $\Lambda^N K(\Lambda|x|)$ and since the Fourier transform of a product is the convolution of Fourier transforms, we have

$$\begin{aligned}&\int_0^{+\infty} \left(\int_{\mathbb{S}^{N-1}} f(x-sy)dy\right) \left(\Lambda^N K(\Lambda s) - \Lambda^N K(\Lambda s)\phi(s)\right) s^{N-1}ds \\ &= \Lambda^{-2k\delta} \int_{\mathbb{R}^N} \left(\left(\Lambda^{2k} - |\xi|^{2k}\right)_+^\delta - \left(\Lambda^{2k} - |\xi|^{2k}\right)_+^\delta * \Psi(\xi)\right) \mathbb{F}f(\xi) \exp(2\pi i x\cdot\xi)d\xi,\end{aligned}$$

where $\Psi(\xi)$ denotes the Fourier transform of $\phi(|y|)$. By the assumptions this Fourier transform is rapidly decreasing, with $\int_{\mathbb{R}^N} \Psi(\xi)d\xi = 1$ and $\int_{\mathbb{R}^N} \xi^\alpha \Psi(\xi)d\xi = 0$ for every multiindex $\alpha \neq 0$. Hence $\left(\Lambda^{2k} - |\xi|^{2k}\right)_+^\delta * \Psi(\xi)$

is a good approximation of $\left(\Lambda^{2k}-|\xi|^{2k}\right)_+^\delta$ where this function is smooth. If $\delta \geq 0$, then it is not difficult to see that for every j

$$\begin{aligned}
&\left|\left(\Lambda^{2k}-|\xi|^{2k}\right)_+^\delta * \Psi(\xi)-\left(\Lambda^{2k}-|\xi|^{2k}\right)_+^\delta\right| \\
&=\left|\int_{\mathbb{R}^N}\left(\left(\Lambda^{2k}-|\xi-\eta|^{2k}\right)_+^\delta-\sum_{|\alpha|<n}\frac{\partial^\alpha}{\partial\xi^\alpha}\left(\Lambda^{2k}-|\xi|^{2k}\right)_+^\delta\frac{(-\eta)^\alpha}{\alpha!}\right)\Psi(\eta)d\eta\right| \\
&\leq c\Lambda^{(2k-1)\delta}\left(1+|\Lambda-|\xi||\right)^{-j},
\end{aligned}$$

while if $-1<\delta<0$, then

$$\begin{aligned}
&\left|\left(\Lambda^{2k}-|\xi|^{2k}\right)_+^\delta-\left(\Lambda^{2k}-|\xi|^{2k}\right)_+^\delta * \Psi(\xi)\right| \\
&\qquad\leq\begin{cases} c\Lambda^{(2k-1)\delta}\,|\Lambda-|\xi||^\delta & \text{if } |\Lambda-|\xi||\leq 1, \\ c\Lambda^{(2k-1)\delta}\,|\Lambda-|\xi||^{-j} & \text{if } |\Lambda-|\xi||\geq 1. \end{cases}
\end{aligned}$$

Using these estimates, Lemma 1.5, and the inequality

$$\int_{\mathbb{S}^{N-1}}|\mathbb{F}f(s\xi)|\,d\xi\leq\left|\mathbb{S}^{N-1}\right|^{1/2}\left\{\int_{\mathbb{S}^{N-1}}|\mathbb{F}f(s\xi)|^2\,d\xi\right\}^{1/2}\leq cs^{-(N+1)/2},$$

we conclude that

$$\begin{aligned}
&\left|\Lambda^N\int_0^{+\infty}\left((1-\phi(s))\int_{\mathbb{S}^{N-1}}f(x-sy)dy\right)K(\Lambda s)s^{N-1}ds\right| \\
&\leq\Lambda^{-2k\delta}\int_{\mathbb{R}^N}\left|\left(\Lambda^{2k}-|\xi|^{2k}\right)_+^\delta-\left(\Lambda^{2k}-|\xi|^{2k}\right)_+^\delta * \Psi(\xi)\right||\mathbb{F}f(\xi)|\,d\xi \\
&\leq c\Lambda^{-\delta}\int_0^{\Lambda-1}\left(\int_{\mathbb{S}^{N-1}}|\mathbb{F}f(s\xi)|\,d\xi\right)(\Lambda-s)^{-j}\,s^{N-1}ds \\
&\quad+c\Lambda^{-\delta}\int_{\Lambda-1}^{\Lambda+1}\left(\int_{\mathbb{S}^{N-1}}|\mathbb{F}f(s\xi)|\,d\xi\right)\left(1+|\Lambda-s|^\delta\right)s^{N-1}ds \\
&\quad+c\Lambda^{-\delta}\int_{\Lambda+1}^{+\infty}\left(\int_{\mathbb{S}^{N-1}}|\mathbb{F}f(s\xi)|\,d\xi\right)(s-\Lambda)^{-j}\,s^{N-1}ds \\
&\leq c\Lambda^{(N-3)/2-\delta}. \qquad\square
\end{aligned}$$

This concludes the proof of the convergence of Bochner-Riesz means. For a discussion of the Gibbs phenomenon see Colzani-Vignati [7] and DeMichele-Roux [8], [9], [10]. □

We want to end this section with a remark. Lemma 1.5 gives an estimate for $\{\int_{\mathbb{S}^{N-1}} |\mathbb{F}f(s\xi)|^2 d\xi\}^{1/2}$, but in the proof of Theorem 1.1 one actually uses $\int_{\mathbb{S}^{N-1}} |\mathbb{F}f(s\xi)| d\xi$. In general, this $\mathbb{L}^1$ norm is not much better than the $\mathbb{L}^2$ norm, however there are interesting exceptions. For example, in [6] it is proved that for characteristic functions of polyhedra in $\mathbb{R}^N$,

$$\int_{\mathbb{S}^{N-1}} \left|\mathbb{F}\chi_P(s\xi)\right| d\xi \leq cs^{-N} \log^N(2+s).$$

Using this estimate one can prove convergence of Riesz means with index $\delta > -1$ for piecewise smooth functions with discontinuities along hyperplanes. This index do not depend on the dimension and is better than $(N-3)/2$.

2. – Operators associated to eigenfunctions expansions

In this section we consider operators of the form

$$\mathcal{M}f(x) = \sum_{\lambda} m(\lambda)\, \mathcal{F}f(\lambda)\varphi_\lambda(x).$$

If the multiplier $m(s)$ is bounded, then the operator $\mathcal{M}$ is bounded on $\mathbb{L}^2(\mathbb{M})$. If $m(s)$ has sufficiently rapid decay at infinity, then the series that defines $\mathcal{M}f(x)$ converges pointwise and defines a smooth function. Our purpose is to synthesize the kernels associated to these operators using the fundamental solution of the wave equation $\cos\left(t\sqrt{\Delta}\right)$ and to relate these operators to corresponding operators on $\mathbb{L}^2\left(\mathbb{R}^N\right)$. The following lemmas are well known, but they are crucial in what follows.

Lemma 2.1. *Let $m(s)$ be an even test function on $-\infty < s < +\infty$ with cosine Fourier transform $\widehat{m}(t) = \frac{2}{\pi}\int_0^{+\infty} m(s)\cos(ts)ds$. Also let $\cos(t\sqrt{\Delta})f(x)$ be the solution of the Cauchy problem for the wave equation in $\mathbb{R}\times\mathbb{M}$,*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) + \Delta u(t,x) = 0 \\ u(0,x) = f(x),\ \dfrac{\partial}{\partial t}u(0,x) = 0. \end{cases}$$

Then in the distribution sense we have the equality

$$\sum_{\lambda} m(\lambda)\, \mathcal{F}f(\lambda)\varphi_\lambda(x) = \int_0^{+\infty} \widehat{m}(t)\cos\left(t\sqrt{\Delta}\right) f(x)dt.$$

PROOF. Solving the wave equation by separation of variables we obtain

$$\cos\left(t\sqrt{\Delta}\right) f(x) = \sum_{\lambda} \cos(t\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x).$$

Hence,

$$\begin{aligned}
\sum_{\lambda} m(\lambda)\, \mathcal{F}f(\lambda)\varphi_\lambda(x) &= \sum_{\lambda}\left(\int_0^{+\infty} \widehat{m}(t)\cos(t\lambda)dt\right)\mathcal{F}f(\lambda)\varphi_\lambda(x) \\
&= \int_0^{+\infty} \widehat{m}(t)\left(\sum_{\lambda}\cos(t\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)dt\right)dt \\
&= \int_0^{+\infty} \widehat{m}(t)\cos\left(t\sqrt{\Delta}\right) f(x)dt.
\end{aligned}$$

The interchange in the order of summation and integration can be justified as in the proof of Fourier inversion formula. □

The above lemma suggests to study the fundamental solution of the wave equation and this is done via the Hadamard parametrix construction. We recall that by taking the trigonometric Fourier transform of $\cos(2\pi t\,|\xi|)$ one obtains the fundamental solution of the wave equation on $\mathbb{R}^N$,

$$\pi^{(1-N)/2} t \frac{\left(t^2 - |x-y|^2\right)_+^{-(N+1)/2}}{\Gamma((1-N)/2)},$$

where the tempered distributions $t_+^{-\alpha}/\Gamma(1-\alpha)$ are defined for every α recursively via an integration by parts,

$$\int_0^{+\infty} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(t)dt = -\int_0^{+\infty} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \frac{d}{dt} f(t)dt.$$

We also recall that the cosine Fourier transform of $t\frac{\left(t^2-\delta^2\right)_+^{\alpha}}{\Gamma(\alpha+1)}$ is

$$\frac{2}{\pi}\int_0^{+\infty} t\frac{\left(t^2-\delta^2\right)_+^{\alpha}}{\Gamma(\alpha+1)}\cos(st)dt = 2^{\alpha+1/2}\pi^{-1/2}s^{-2\alpha-2}\frac{J_{-\alpha-3/2}(\delta s)}{(\delta s)^{-\alpha-3/2}}.$$

Observe that when $\alpha \geq -1/2$ the term $s^{-2\alpha-2}$ has a non integrable singularity in $s = 0$, but one can define this distribution by analytic continuation. Substituting s with $s + i\varepsilon$ and taking the limit for $\varepsilon \to 0+$, one obtains a distribution of order at most $[2\alpha+2]+1$ in $s = 0$. See for example Hörmander [16, 3.1.11 and 7.1.17].

Since waves propagate with finite speed and since a manifold is locally Euclidean, it is natural to conjecture a relation between the fundamental solutions of the wave equations on the manifold $\mathbb{M}$ and on the Euclidean space $\mathbb{R}^N$, at least for small times.

LEMMA 2.2. *The kernel* $\cos\left(t\sqrt{\Delta}\right)(x,y)$ *has support in* $\{d(x,y)\le t\}$, *where* $d(x,y)$ *is the distance between* x *and* y *in the Riemannian metric associated to the principal part of the differential operator* Δ. *Also for* t *small,* $|t|\le\varepsilon$, *there exist smooth functions* $\{U_k(x,y)\}_{k=0}^n$ *such that*

$$\cos\left(t\sqrt{\Delta}\right)(x,y)=\sum_{k=0}^{n}U_k(x,y)t\frac{\left(t^2-d(x,y)^2\right)_+^{k-(N+1)/2}}{\Gamma\left(k-(N-1)/2\right)}+V_n(t,x,y),$$

where $V_n(t,x,y)$ *is a solution of*

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}+\Delta_x\right)V_n(t,x,y)=-\Delta_x\left(U_n(x,y)t\dfrac{\left(t^2-d(x,y)^2\right)_+^{n-(N+1)/2}}{\Gamma\left(n-(N-1)/2\right)}\right).\\[2ex] V_n(0,x,y)=0,\ \dfrac{\partial}{\partial t}V_n(0,x,y)=0.\end{cases}$$

PROOF. This is the Hadamard construction of a parametrix for the wave operator. See for example Hörmander [16, 17.4], or Bérard [2]. □

Using the above lemmas we easily obtain the following.

THEOREM 2.3. *Let* $m(s)$ *be an even test function on* $-\infty<s<+\infty$ *with cosine Fourier transform* $\widehat{m}(t)$ *vanishing for* $t\ge\varepsilon$, *with* ε *suitably small. Then the kernel associated to the operator* $\mathcal{M}f(x)=\sum_\lambda m(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)$ *has support in* $\{d(x,y)\le\varepsilon\}$ *and there exist* $\{A_k(s,x,y)\}_{k=0}^n$ *and* $R_n(t,x,y)$ *such that*

$$\begin{aligned}&\sum_\lambda m(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)\\&\quad=\int_{\mathbb{M}}\left(\int_0^{+\infty}\left(\sum_{k=0}^{n}A_k(s,x,y)+R_n(s,x,y)\right)m(s)ds\right)f(y)d\mu(y).\end{aligned}$$

The kernels $\{A_k(s,x,y)\}_{k=0}^n$ *are*

$$A_k(s,x,y)=2^{k-N/2}\pi^{-1/2}s^{N-2k-1}\frac{J_{(N-2)/2-k}\left(d(x,y)s\right)}{\left(d(x,y)s\right)^{(N-2)/2-k}}U_k(x,y).$$

Moreover, given h *there exist* n *and a constant* c *independent of* ε, x, y, *such that*

$$|R_n(s,x,y)|\le c\varepsilon(1+s)^{-h}.$$

PROOF. By the previous lemmas,

$$\sum_{\lambda} m(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x) = \int_0^{+\infty} \widehat{m}(t)\cos\left(t\sqrt{\Delta}\right) f(x)dt$$

$$= \sum_{k=0}^{n} \int_0^{+\infty} \left(\int_{\mathrm{M}} t \frac{\left(t^2 - d(x,y)^2\right)_+^{k-(N+1)/2}}{\Gamma(k-(N-1)/2)} U_k(x,y) f(y)d\mu(y) \right) \widehat{m}(t)dt$$

$$+ \int_0^{+\infty} \left(\int_{\mathrm{M}} V_n(t,x,y) f(y)d\mu(y) \right) \widehat{m}(t)dt$$

$$= \sum_{k=0}^{n} \int_{\mathrm{M}} \left(U_k(x,y) \int_0^{+\infty} t \frac{\left(t^2 - d(x,y)^2\right)_+^{k-(N+1)/2}}{\Gamma(k-(N-1)/2)} \widehat{m}(t)dt \right) f(y)d\mu(y)$$

$$+ \int_{\mathrm{M}} \left(\int_0^{+\infty} V_n(t,x,y)\widehat{m}(t)dt \right) f(y)d\mu(y).$$

By the definition of Fourier transform of tempered distributions, $\int_0^{+\infty} \psi(t)\widehat{\phi}(t)dt = \int_0^{+\infty} \widehat{\psi}(s)\phi(s)ds$, one obtain

$$\int_0^{+\infty} \left(t \frac{\left(t^2 - d(x,y)^2\right)_+^{k-(N+1)/2}}{\Gamma(k-(N-1)/2)} \right) \widehat{m}(t)dt$$
$$= \int_0^{+\infty} \left(2^{k-N/2}\pi^{-1/2}s^{N-2k-1} \frac{J_{(N-2)/2-k}\left(d(x,y)s\right)}{\left(d(x,y)s\right)^{(N-2)/2-k}} \right) m(s)ds.$$

Let $\chi(t)$ be a smooth even function with compact support and such that $\widehat{m}(t) = \widehat{m}(t)\chi(t)$. Then

$$\int_0^{+\infty} V_n(t,x,y)\widehat{m}(t)dt = \int_0^{+\infty} \left(\frac{2}{\pi} \int_0^{+\infty} V_n(t,x,y)\chi(t)\cos(st)dt \right) m(s)ds.$$

The kernel $V_n(t,x,y)$ is a solution of an hyperbolic equation and, when measured in appropriate Sobolev norm, it has at least the regularity of

$$\Delta_x \left(U_n(x,y) t \frac{\left(t^2 - d(x,y)^2\right)_+^{n-(N+1)/2}}{\Gamma(n-(N-1)/2)} \right),$$

which is the only non smooth term in the equation. Also observe that derivatives of order α of this term are dominated by $ct^{2n-N-|\alpha|}$. Thus similar estimates hold for $V_n(t,x,y)$ and a certain number of derivatives $\partial^j/\partial t^j V_n(t,x,y)$. A repeated integration by parts gives the desired decay in s for $R_n(t,x,y)$,

$$\begin{aligned} R_n(s,x,y) &= \frac{2}{\pi}\int_0^{+\infty} \cos(st)\chi(t)V_n(t,x,y)dt \\ &= -\frac{2}{\pi}\int_0^{+\infty} \frac{\sin(st)}{s}\frac{\partial}{dt}\left(\chi(t)V_n(t,x,y)\right)dt \\ &= -\frac{2}{\pi}\int_0^{+\infty} \frac{\cos(st)}{s^2}\frac{\partial^2}{dt^2}\left(\chi(t)V_n(t,x,y)\right)dt \ldots \qquad \square \end{aligned}$$

Let us explain the meaning of the above theorem. The exponentials $\{\exp(2\pi i x\cdot\xi)\}$ are eigenfunctions of the Laplace operator $-\sum_{j=1}^N \partial^2/\partial x_j^2$ on $\mathbb{R}^N$ with eigenvalues $\{4\pi^2|\xi|^2\}$. Using the Euclidean Fourier transform one can define the operator

$$\mathcal{T}f(x) = \int_{\mathbb{R}^N} m(2\pi|\xi|)\mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi,$$

which is analogous to the operator $\mathcal{M}$. Arguing exactly as in the above proof one can synthesized this operator using the fundamental solution of the wave equation on $\mathbb{R}^N$, obtaining

$$\begin{aligned} &\int_{\mathbb{R}^N} m(2\pi|\xi|)\mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi \\ &\quad = \int_{\mathbb{R}^N} f(y)\left(2^{-N/2}\pi^{-N/2}\int_0^{+\infty} \frac{J_{(N-2)/2}(|x-y|s)}{(|x-y|s)^{(N-2)/2}}s^{N-1}m(s)ds\right)dy. \end{aligned}$$

Of course, this last formula is also an immediate consequence of the Fourier inversion formula for radial functions. See Stein-Weiss [26, IV.3]. Thus in suitable local coordinates the principal part of the operator $\mathcal{M}$ is similar to the corresponding Euclidean operators $\mathcal{T}$. We assumed that the cosine Fourier transform of $m(s)$ has a small support, but in $\mathbb{R}^N$ this assumption is not necessary and even in a manifold the restriction can be relaxed.

3. – Fourier transforms of piecewise smooth functions

This section is devoted to estimate the size of Fourier coefficients of piecewise smooth functions, but we start with an easy lemma on some integrals involving Bessel functions.

LEMMA 3.1. *If $\alpha+\beta>-1$, there exist c and k such that for every $\psi(t)$ smooth with compact support and for every $\varepsilon>0$,*

$$\left|\int_0^{+\infty}\psi(\varepsilon t)\,t^{\beta}J_\alpha(t)\,dt\right|\le c\sum_{j=0}^{k}\sup_{t\ge 0}\left|\frac{\partial^j}{\partial t^j}\psi(t)\right|.$$

Moreover,

$$\lim_{\varepsilon\to 0+}\int_0^{+\infty}\psi(\varepsilon t)\,t^{\beta}J_\alpha(t)\,dt=2^{\beta}\frac{\Gamma((\alpha+\beta+1)/2)}{\Gamma((\alpha-\beta+1)/2)}\psi(0).$$

PROOF. Since $\frac{\partial}{\partial t}(t^{\alpha+1}J_{\alpha+1}(t))=t^{\alpha+1}J_\alpha(t)$, an integration by parts reduces an integral with parameters (α,β) for the function $\psi(\varepsilon t)$ to one with $(\alpha+1,\beta-1)$ for an associated function $\phi(\varepsilon t)$ with $\phi(0)=(1+\alpha-\beta)\psi(0)$,

$$\begin{aligned}
&\int_0^{+\infty}\psi(\varepsilon t)\,t^{\beta}J_\alpha(t)\,dt\\
&=\int_0^{+\infty}\psi(\varepsilon t)\,t^{\beta-\alpha-1}\frac{\partial}{\partial t}\left(t^{\alpha+1}J_{\alpha+1}(t)\right)dt\\
&=\psi(\varepsilon t)\,t^{\beta}J_{\alpha+1}(t)\Big|_0^{+\infty}-\int_0^{+\infty}t^{\alpha+1}J_{\alpha+1}(t)\frac{\partial}{\partial t}\left(\psi(\varepsilon t)\,t^{\beta-\alpha-1}\right)dt\\
&=\int_0^{+\infty}\left((1+\alpha-\beta)\psi(\varepsilon t)-\varepsilon t\dot{\psi}(\varepsilon t)\right)t^{\beta-1}J_{\alpha+1}(t)\,dt\\
&=\int_0^{+\infty}\phi(\varepsilon t)\,t^{\beta-1}J_{\alpha+1}(t)\,dt.
\end{aligned}$$

One can iterate until $t^{\beta-n}J_{\alpha+n}(t)$ becomes absolutely integrable, then the dominated convergence theorem applies. This show that $\int_0^{+\infty}\psi(\varepsilon t)t^{\beta}J_\alpha(t)dt$ is uniformly bounded in $\varepsilon>0$ by $c\sum_{j=0}^{k}\sup_{t\ge 0}|\frac{\partial^j}{\partial t^j}\psi(t)|$ and that $\lim_{\varepsilon\to 0+}\int_0^{+\infty}\psi(\varepsilon t)\,t^{\beta}J_\alpha(t)\,dt=c(\alpha,\beta)\psi(0)$. To determine the constant $c(\alpha,\beta)$ it is enough to test the distribution on a particular function. See [33, 13.24]. □

THEOREM 3.2. *If $f(x)$ is in $\mathbb{X}^\alpha(\mathbb{M})$ with $\alpha>-1$, then as $\Lambda\to+\infty$,*

$$\left\{\sum_{\Lambda\le\lambda\le\Lambda+1}|\mathcal{F}f(\lambda)|^2\right\}^{1/2}\le c\Lambda^{-\alpha-1}.$$

PROOF. We may assume that our piecewise smooth function has support in a domain Ω and it is smooth inside this domain. We may also assume that this support is small and concentrated around $\partial\Omega$. Indeed with a smooth partition of unity one can cut the function into several pieces, those pieces which are far from $\partial\Omega$ are smooth, and smooth functions have rapidly decreasing

Fourier coefficients. Let us introduce coordinates $x = (\vartheta, t)$, with $\vartheta \in \partial\Omega$ and $d(x, \partial\Omega) = d(x, \vartheta) = t \in \mathbb{R}$, also let us identify $\partial\Omega$ with an open set in $\mathbb{R}^{N-1}$ and write $d\mu(x) = \mu(\vartheta, t) d\vartheta dt$. Now, one can decompose $f(x)$ into thin layers of width comparable to 2^{-j},

$$f(x) = \sum_{j=0}^{+\infty} f(\vartheta, t)\chi\left(2^j t\right),$$

where $\chi(t)$ is positive, smooth, with support in $1/2 \leq t \leq 2$, and such that $\sum_{j=0}^{+\infty} \chi\left(2^j t\right) = 1$ if $0 < t < 1$. Write $f(\vartheta, t)\chi\left(2^j t\right) = 2^{-\alpha j} g_j(\vartheta, t)$ and observe that these normalized functions $\{g_j(x)\}$ are essentially the building blocks for the spaces $\mathbb{X}^\alpha(\mathbb{M})$. Since

$$\left\{\sum_{\Lambda \leq \lambda \leq \Lambda+1} |\mathcal{F}f(\lambda)|^2\right\}^{1/2} \leq \sum_{j=0}^{+\infty} 2^{-\alpha j} \left\{\sum_{\Lambda \leq \lambda \leq \Lambda+1} \left|\mathcal{F}g_j(\lambda)\right|^2\right\}^{1/2},$$

the theorem is a consequence of the following lemma.

LEMMA 3.3. *For every non negative integer h,*

$$\left\{\sum_{\Lambda \leq \lambda \leq \Lambda+1} \left|\mathcal{F}g_j(\lambda)\right|^2\right\}^{1/2} \leq c \begin{cases} \Lambda^{-2h} 2^{(2h-1)j} & \textit{if } 2^j \leq \Lambda, \\ 2^{-j} & \textit{if } 2^j \geq \Lambda. \end{cases}$$

PROOF. It suffices to estimate $\sum_\lambda m(\lambda) \left|\mathcal{F}g_j(\lambda)\right|^2$ for some function $m(s)$ with a bump around Λ, and we can assume that Theorem 2.3 applies. Let $m(s)$ be a non negative even test function with $m(s) \geq 1$ if $|s \pm \Lambda| \leq 1$ and with rapid decay away of $\pm\Lambda$, that is $|\frac{d^j}{ds^j} m(s)| \leq c(1 + |s \pm \Lambda|)^{-h}$ for every j and h. Also assume that the cosine Fourier transform $\widehat{m}(t)$ vanishes for $t \geq \varepsilon$. We have

$$\begin{aligned}
\left\{\sum_\lambda m(\lambda) \left|\mathcal{F}g_j(\lambda)\right|^2\right\}^{1/2} &= \left\{\sum_\lambda m(\lambda) \left|\int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} g_j(\vartheta, t)\overline{\varphi_\lambda(\vartheta, t)}\mu(\vartheta, t) d\vartheta dt\right|^2\right\}^{1/2} \\
&\leq \int_{2^{-1-j}}^{2^{1-j}} \left\{\sum_\lambda m(\lambda) \left|\int_{\mathbb{R}^{N-1}} g_j(\vartheta, t)\overline{\varphi_\lambda(\vartheta, t)}\mu(\vartheta, t) d\vartheta\right|^2\right\}^{1/2} dt \\
&= \int_{2^{-1-j}}^{2^{1-j}} \left\{\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} g_j(\vartheta, t)\overline{g_j(\zeta, t)}\mu(\vartheta, t)\mu(\zeta, t)\right. \\
&\quad \times \left.\left(\sum_\lambda m(\lambda)\varphi_\lambda(\zeta, t)\overline{\varphi_\lambda(\vartheta, t)}\right) d\vartheta d\zeta\right\}^{1/2} dt.
\end{aligned}$$

By Theorem 2.3, with the notation $d(t, \zeta, \vartheta)$ for the distance between the points of coordinates (ϑ, t) and (ζ, t),

$$\sum_{\lambda} m(\lambda)\varphi_\lambda(\zeta, t)\overline{\varphi_\lambda(\vartheta, t)}$$

$$= \sum_{k=0}^{n} \left(2^{k-N/2}\pi^{-1/2}U_k(t, \zeta, \vartheta) \int_0^{+\infty} \frac{J_{(N-2)/2-k}\,(d(t, \zeta, \vartheta)s)}{(d(t, \zeta, \vartheta)s)^{(N-2)/2-k}} s^{N-2k-1} m(s)ds\right)$$

$$+ \int_0^{+\infty} R_n(s, t, \zeta, \vartheta)m(s)ds.$$

Thus we are led to estimate integrals of the type

$$\int\limits_{2^{-1-j}}^{2^{1-j}} \left| \int\limits_{0}^{+\infty} \left(\int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}^{N-1}} G(t, \zeta, \vartheta) \frac{J_{(N-2)/2-k}\,(d(t, \zeta, \vartheta)s)}{(d(t, \zeta, \vartheta)s)^{(N-2)/2-k}} d\vartheta d\zeta \right) s^{N-2k-1} m(s)ds \right| dt,$$

and

$$\int_{2^{-1-j}}^{2^{1-j}} \left| \int_0^{+\infty} \left(\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} G(t, \zeta, \vartheta)R_n(s, t, \zeta, \vartheta)d\vartheta d\zeta \right) m(s)ds \right| dt,$$

where $G(t, \zeta, \vartheta)$ are suitable smooth compactly supported functions of the variables (t, ζ, ϑ). Observe that derivatives of these functions in tangential directions ζ and ϑ are bounded uniformly in j. We first consider the second integral. Since $|m(s)| \leq c\,(1 + |s \pm \Lambda|)^{-h}$ and $|R_n(s, t, \zeta, \vartheta)| \leq c\varepsilon(1 + s)^{-h}$, and since the integral over $\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ reduces to an integral over a compact set,

$$\int_{2^{-1-j}}^{2^{1-j}} \left| \int_0^{+\infty} \left(\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} G(t, \zeta, \vartheta)R_n(s, t, \zeta, \vartheta)d\zeta d\vartheta \right) m(s)ds \right| dt$$

$$\leq c\varepsilon \int_{2^{-1-j}}^{2^{1-j}} \int_0^{+\infty} (1 + |s - \Lambda|)^{-h}\,(1 + s)^{-h} ds\, dt$$

$$\leq c\varepsilon 2^{-j}\Lambda^{-h}.$$

We now estimate

$$\int_{\mathbb{R}^{N-1}} G(t, \zeta, \vartheta) \frac{J_{(N-2)/2-k}\,(d(t, \zeta, \vartheta)s)}{(d(t, \zeta, \vartheta)s)^{(N-2)/2-k}} d\vartheta.$$

This integral is a smooth function of s and we want to show that when $s \to +\infty$ it is of the order of s^{1-N}. Introducing a sort of polar coordinates centered at $\vartheta = \zeta$, we integrate first over $\{d(t, \zeta, \vartheta) = r\}$, which has surface measure of the order of r^{N-2}, and then over $r > 0$. Hence if $s \to +\infty$, by Lemma 3.1,

$$\int_{\mathbb{R}^{N-1}} G(t, \zeta, \vartheta) \frac{J_{(N-2)/2-k}(d(t, \zeta, \vartheta)s)}{(d(t, \zeta, \vartheta)s)^{(N-2)/2-k}} d\vartheta$$

$$= \int_0^\infty F(t, \zeta, r) r^{N-2} \frac{J_{(N-2)/2-k}(sr)}{(sr)^{(N-2)/2-k}} dr$$

$$= s^{1-N} \int_0^\infty F(t, \zeta, s^{-1}r) r^{k+(N-2)/2} J_{(N-2)/2-k}(r)\, dr$$

$$\approx 2^{k+(N-2)/2} \frac{\Gamma((N-1)/2)}{\Gamma((1-2k)/2)} F(t, \zeta, 0) s^{1-N}.$$

We conclude that

$$\int_{2^{-1-j}}^{2^{1-j}} \left| \int_0^{+\infty} \left(\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} G(t, \zeta, \vartheta) \frac{J_{(N-2)/2-k}(d(t, \zeta, \vartheta)s)}{(d(t, \zeta, \vartheta)s)^{(N-2)/2-k}} d\vartheta d\zeta \right) s^{N-2k-1} m(s) ds \right| dt$$

$$\leq c 2^{-j} \Lambda^{-2k}.$$

In conclusion we have proved the estimate $\leq c2^{-j}$. The other estimate $\leq c\Lambda^{-2h} 2^{(2h-1)j}$ is an immediate consequence. Since $g_j(x) = 2^{\alpha j} f(\vartheta, t) \chi(2^j t)$, the iterated Laplacian $\Delta^h g_j(x)$ is comparable to $2^{2hj} g_j(x)$ and

$$\left\{ \sum_{\Lambda \leq \lambda \leq \Lambda+1} \left| \mathcal{F} g_j(\lambda) \right|^2 \right\}^{1/2} = \left\{ \sum_{\Lambda \leq \lambda \leq \Lambda+1} \left| \lambda^{-2h} \mathcal{F} \Delta^h g_j(\lambda) \right|^2 \right\}^{1/2}$$

$$\leq \Lambda^{-2h} \left\{ \sum_{\Lambda \leq \lambda \leq \Lambda+1} \left| \mathcal{F} \Delta^h g_j(\lambda) \right|^2 \right\}^{1/2} \leq c \Lambda^{-2h} 2^{(2h-1)j}. \quad \square$$

We end this section observing that, when applied to the Euclidean Fourier transform, the techniques in the proof of the above theorem give a proof of Lemma 1.3 which is different from the one in [32].

4. – Localization and convergence of Riesz means

In this section we want to study the convergence of Riesz means of eigenfunction expansions and we start with an analog of the Riemann localization principle.

THEOREM 4.1. *Let $\varepsilon > 0$, $\delta > -1$, and assume that for some ϑ,*

$$\lim_{\Lambda\to+\infty} \Lambda^{\vartheta} \left\{ \sum_{\Lambda\leq\lambda\leq\Lambda+1} |\mathcal{F}f(\lambda)|^2 \right\}^{1/2} = 0.$$

Then if $\delta + \vartheta \geq (N-1)/2$, the behavior as $\Lambda \to +\infty$ of the Riesz means

$$\sum_{\lambda<\Lambda} \left(1 - (\lambda/\Lambda)^{2k}\right)^{\delta} \mathcal{F}f(\lambda)\varphi_\lambda(x)$$

at a point x depends only on the values of $f(y)$ at points $d(y,x) \leq \varepsilon$. In particular the Riesz means converge to the function in every open set where the function is smooth and the convergence is uniform in compact subsets.

PROOF. Let $m_\Lambda(s) = \left(1 - (s/\Lambda)^{2k}\right)_+^{\delta}$. The cosine Fourier transform of this function has not compact support, so that the arguments in Section 2 do not immediately apply. Nevertheless this Fourier transform concentrates around zero when $\Lambda \to +\infty$, and the idea is to decompose $m_\Lambda(s)$ into a term with compactly supported Fourier transform plus an error which can be controlled using estimates on $\{\mathcal{F}f(\lambda)\}$. Let $\psi(s)$ be an even test function with $\widehat{\psi}(t) = 0$ if $|t| \geq \varepsilon$, and with $\int_{-\infty}^{+\infty} \psi(s)ds = 1$ and $\int_{-\infty}^{+\infty} \psi(s)s^j ds = 0$ for $j = 1, 2, \ldots$. The convolution $m_\Lambda * \psi(s) = \int_{-\infty}^{+\infty} m_\Lambda(s-t)\psi(t)dt$ is an approximation of $m_\Lambda(s)$ and we can write

$$\sum_\lambda m(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)$$

$$= \sum_\lambda m_\Lambda * \psi(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x) + \sum_\lambda \left(m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)\right)\mathcal{F}f(\lambda)\varphi_\lambda(x).$$

The theorem is an immediate consequence of the above decomposition and the following lemmas.

LEMMA 4.2. *If the point x varies in a compact set at a positive distance from $\partial\mathbb{M}$, then as $\Lambda \to +\infty$,*

$$\sum_{\Lambda\leq\lambda\leq\Lambda+1} |\varphi_\lambda(x)|^2 \leq c\Lambda^{N-1}.$$

PROOF. This estimate on the spectral function of an elliptic operator is well known, see Hörmander [15], [16, 17.5], but we include the proof since it is an easy corollary of Theorem 2.3. As in Lemma 3.3 it suffices to estimate $\sum_\lambda m(\lambda)\,|\varphi_\lambda(x)|^2$ for some function $m(s)$ with a bump around Λ and with $\widehat{m}(t) = 0$ if $|t| \geq \varepsilon$. By Theorem 2.3,

$$\sum_\lambda m(\lambda)\varphi_\lambda(x)\overline{\varphi_\lambda(y)} = \int_0^{+\infty} \left(\sum_{k=0}^{n} A_k(s,x,y) + R_n(s,x,y)\right) m(s)ds.$$

Since $|A_k(s,x,y)| \leq cs^{N-2k-1}$ and $|R_n(s,x,y)| \leq c(1+s)^{-h}$, we have

$$\left| \int_0^{+\infty} A_k(s,x,y)m(s)ds \right| \leq c\Lambda^{N-2k-1},$$
$$\left| \int_0^{+\infty} R_n(s,x,y)m(s)ds \right| \leq c\Lambda^{-h}. \qquad \square$$

LEMMA 4.3. *The values of the means* $\sum_\lambda m_\Lambda * \psi(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)$ *at a point* x *depend only on values of* $f(y)$ *at points* $d(y,x) \leq \varepsilon$.

PROOF. Since $\widehat{m_\Lambda * \psi}(t) = \pi\Lambda\widehat{m}(\Lambda t)\widehat{\psi}(t) = 0$ if $|t| \geq \varepsilon$, Theorem 2.3 applies. $\square$

LEMMA 4.4. *If* $\left\{\sum_{\Lambda\leq\lambda\leq\Lambda+1} |\mathcal{F}f(\lambda)|^2\right\}^{1/2} \leq c\Lambda^{-\vartheta}$, *then*

$$\left|\sum_\lambda \left(m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)\right) \mathcal{F}f(\lambda)\varphi_\lambda(x)\right| \leq c\Lambda^{(N-1)/2-\delta-\vartheta}.$$

Moreover if $\lim_{\Lambda\to+\infty} \Lambda^\vartheta \left\{\sum_{\Lambda\leq\lambda\leq\Lambda+1} |\mathcal{F}f(\lambda)|^2\right\}^{1/2} = 0$, *then*

$$\lim_{\Lambda\to+\infty} \Lambda^{\vartheta+\delta-(N-1)/2} \left|\sum_\lambda \left(m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)\right) \mathcal{F}f(\lambda)\varphi_\lambda(x)\right| = 0.$$

PROOF. It can be easily proved that $m_\Lambda * \psi(\lambda)$ is a good approximation of $m_\Lambda(\lambda)$ where this function is smooth, that is away from $\pm\Lambda$, and in a neighborhood of $\pm\Lambda$ the approximation is of the order of $c\Lambda^{-\delta}$. In particular if $\delta \geq 0$, then for every k we have

$$|m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)| \leq c\Lambda^{-\delta}\left(1 + |\lambda \pm \Lambda|\right)^{-k}.$$

This estimate holds also when $-1 < \delta < 0$ if the terms $m_\Lambda(\lambda)$ with $|\lambda \pm \Lambda| \leq 1$ are omitted. Hence

$$\left|\sum_\lambda \left(m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)\right) \mathcal{F}f(\lambda)\varphi_\lambda(x)\right|$$

$$\leq \left\{\sum_\lambda |m_\Lambda(\lambda)-m_\Lambda * \psi(\lambda)|\, |\mathcal{F}f(\lambda)|^2\right\}^{1/2} \left\{\sum_\lambda |m_\Lambda(\lambda)-m_\Lambda * \psi(\lambda)|\, |\varphi_\lambda(x)|^2\right\}^{1/2}$$

$$\leq c\Lambda^{-\delta} \left\{\sum_{n=1}^{+\infty} (1+|\Lambda-n|)^{-k} \sum_{n-1\leq\lambda<n} |\varphi_\lambda(x)|^2\right\}^{1/2} \left\{\sum_{n=1}^{+\infty} (1+|\Lambda-n|)^{-k} \sum_{n-1\leq\lambda<n} |\mathcal{F}f(\lambda)|^2\right\}^{1/2}.$$

By Lemma 4.2, $\sum_{n-1\leq\lambda\leq n} |\varphi_\lambda(x)|^2 \leq cn^{N-1}$ for every $n = 1, 2, \ldots$, so that

$$\left\{\sum_{n=1}^{+\infty} (1+|\Lambda-n|)^{-k} \sum_{n-1\leq\lambda<n} |\varphi_\lambda(x)|^2\right\}^{1/2} \leq c\Lambda^{(N-1)/2}.$$

Also, assuming that $\sum_{n-1\leq\lambda\leq n} |\mathcal{F}f(\lambda)|^2 \leq cn^{-2\vartheta}$,

$$\left\{\sum_{n=1}^{+\infty} (1+|\Lambda-n|)^{-k} \sum_{n-1\leq\lambda<n} |\mathcal{F}f(\lambda)|^2\right\}^{1/2} \leq c\Lambda^{-\vartheta}. \qquad \square$$

If $f(x)$ is an integrable function with support in a compact set A disjoint from $\partial\mathbb{M}$, then, by Lemma 4.2,

$$\begin{aligned}
&\left\{\sum_{\Lambda\leq\lambda\leq\Lambda+1} |\mathcal{F}f(\lambda)|^2\right\}^{1/2} \\
&= \left\{\int_{\mathbb{M}}\int_{\mathbb{M}} \left(\sum_{\Lambda\leq\lambda\leq\Lambda+1} \varphi_\lambda(x)\overline{\varphi_\lambda(y)}\right) \overline{f(x)} f(y)\,d\mu(x)\,d\mu(y)\right\}^{1/2} \\
&\leq \left\{\sup_{x\in A} \sum_{\Lambda\leq\lambda\leq\Lambda+1} |\varphi_\lambda(x)|^2 \int_{\mathbb{M}} |f(x)|\,d\mu(x) \int_{\mathbb{M}} |f(y)|\,d\mu(y)\right\}^{1/2} \\
&\leq c\Lambda^{(N-1)/2} \int_{\mathbb{M}} |f(x)|\,d\mu(x).
\end{aligned}$$

This a priory estimate an a density argument also yield

$$\lim_{\Lambda\to\infty} \Lambda^{(1-N)/2} \left\{\sum_{\Lambda\leq\lambda\leq\Lambda+1} |\mathcal{F}f(\lambda)|^2\right\}^{1/2} = 0.$$

It then follows that in $\mathbb{L}^1(\mathbb{M}, d\mu)$ localization holds if $\delta \geq N-1$. If $f(x)$ is square integrable, then $\{\sum_{\Lambda\leq\lambda\leq\Lambda+1} |\mathcal{F}f(\lambda)|^2\}^{1/2} \to 0$. Hence in $\mathbb{L}^2(\mathbb{M}, d\mu)$ localization holds if $\delta \geq (N-1)/2$. These results are also in [14], [15]. For estimates on $\{\sum_{\Lambda\leq\lambda\leq\Lambda+1} |\mathcal{F}f(\lambda)|^2\}^{1/2}$ in the spaces $\mathbb{L}^p(\mathbb{M}, d\mu)$, $1 < p < 2$, see Sogge [24]. These estimates are related to the problem of restriction of Fourier transforms to spheres and imply localization results in $\mathbb{L}^p(\mathbb{M}, d\mu)$. But now let's come back to piecewise smooth functions. Combining Theorem 4.1 with Theorem 3.2, one obtains the following.

THEOREM 4.4. *If $f(x)$ is a piecewise smooth function in $\mathbb{X}^{\alpha}(\mathbb{M})$ with $\alpha > -1$ and if $\gamma = min\{2k, \quad \alpha + \delta - (N-3)/2\}$, then at every point x where $f(x)$ is smooth,*

$$\left| f(x) - \sum_{\lambda<\Lambda} \left(1 - (\lambda/\Lambda)^{2k}\right)^{\delta} \mathcal{F}f(\lambda)\varphi_{\lambda}(x) \right| \leq c(1+\Lambda)^{-\gamma}.$$

In particular, if $\alpha+\delta > (N-3)/2$ then the Riesz means converge to the function at every point where this function is smooth and the convergence is uniform in every compact disjoint from the singularities.

PROOF. Assume $\delta \geq 0$, the case $-1 < \delta < 0$ only requires minor changes. Let x be a point where $f(y)$ is smooth and let $g(y)$ be an everywhere smooth function that coincides with $f(y)$ in a neighborhood of x. Then if the support of $\widehat{\psi}(t)$ is suitably small, the operator associated to the multiplier $\{m_{\Lambda} * \psi(\lambda)\}$ is local and at the point x we have

$$\begin{aligned}
&\left| f(x) - \sum_{\lambda} m_{\Lambda}(\lambda)\mathcal{F}f(\lambda)\varphi_{\lambda}(x) \right| \\
&\leq \left| f(x) - \sum_{\lambda} m_{\Lambda}*\psi(\lambda)\mathcal{F}f(\lambda)\varphi_{\lambda}(x) \right| + \left| \sum_{\lambda} (m_{\Lambda}(\lambda) - m_{\Lambda}*\psi(\lambda))\,\mathcal{F}f(\lambda)\varphi_{\lambda}(x) \right| \\
&= \left| \sum_{\lambda} (1 - m_{\Lambda}*\psi(\lambda))\,\mathcal{F}g(\lambda)\varphi_{\lambda}(x) \right| + \left| \sum_{\lambda} (m_{\Lambda}(\lambda) - m_{\Lambda}*\psi(\lambda))\,\mathcal{F}f(\lambda)\varphi_{\lambda}(x) \right|.
\end{aligned}$$

By Theorem 3.2 and Lemma 4.3,

$$\sum_{\lambda} |m_{\Lambda}(\lambda) - m_{\Lambda} * \psi(\lambda)|\, |\mathcal{F}f(\lambda)|\, |\varphi_{\lambda}(x)| \leq c\Lambda^{(N-3)/2-\alpha-\delta}.$$

Hence the last summand satisfies the required estimates. It remain the first summand. Since $1 - \left(1 - s^{2k}\right)^{\delta} \approx \delta s^{2k}$ for s small, we have

$$|1 - m_{\Lambda} * \psi(\lambda)| \leq c \begin{cases} \Lambda^{-2k}\left(1 + \lambda^{2k}\right) & \text{if } \lambda \leq \Lambda, \\ 1 & \text{if } \lambda \geq \Lambda. \end{cases}$$

Also, by the sharp estimates on the remainder of the spectral function in Lemma 4.2,

$$\left\{ \sum_{n-1 \leq \lambda \leq n} |\varphi_{\lambda}(x)|^2 \right\}^{1/2} \leq cn^{(N-1)/2}.$$

Finally, since smooth functions have rapidly decreasing Fourier coefficients, for every $h > 0$,

$$\left\{ \sum_{n-1\le\lambda\le n} |\mathcal{F}g(\lambda)|^2 \right\}^{1/2} \le cn^{-h}.$$

Collecting all these estimates, we thus obtain

$$\sum_{\lambda} |1 - m_\Lambda * \psi(\lambda)|\, |\mathcal{F}g(\lambda)|\, |\varphi_\lambda(x)|$$

$$\le \sum_{n=1}^{+\infty} \left\{ \sup_{n-1\le\lambda<n} |1 - m_\Lambda * \psi(\lambda)| \right\} \left\{ \sum_{n-1\le\lambda<n} |\varphi_\lambda(x)|^2 \right\}^{1/2} \left\{ \sum_{n-1\le\lambda<n} |\mathcal{F}g(\lambda)|^2 \right\}^{1/2}$$

$$\le c\Lambda^{-2k}. \qquad \square$$

It may be of some interest to compare the speed of pointwise convergence with the speed of mean square convergence. Observe that for every λ,

$$\left\{ \int_{\mathbb{M}} \left| f(x) - \sum_{\lambda<\Lambda} \left(1 - (\lambda/\Lambda)^{2k}\right)^{\delta} \mathcal{F}f(\lambda)\varphi_\lambda(x) \right|^2 d\mu(x) \right\}^{1/2}$$

$$\ge \left| \int_{\mathbb{M}} \left(f(x) - \sum_{\lambda<\Lambda} \left(1 - (\lambda/\Lambda)^{2k}\right)^{\delta} \mathcal{F}f(\lambda)\varphi_\lambda(x) \right) \overline{\varphi_\lambda(x)} d\mu(x) \right|$$

$$= \left| 1 - \left(1 - (\lambda/\Lambda)^{2k}\right)_+^{\delta} \right| |\mathcal{F}f(\lambda)|.$$

Hence, the speed of convergence is not better than $\left| 1 - \left(1 - (\lambda/\Lambda)^{2k}\right)_+^{\delta} \right| \approx \Lambda^{-2k}$. On the other hand, if $f(x)$ is in $\mathbb{B}^2_{\varepsilon,\infty}(\mathbb{M}, d\mu)$ and $\varepsilon < 2k$, then

$$= \left\{ \int_{\mathbb{M}} \left| f(x) - \sum_{\lambda<\Lambda} \left(1 - (\lambda/\Lambda)^{2k}\right)^{\delta} \mathcal{F}f(\lambda)\varphi_\lambda(x) \right|^2 d\mu(x) \right\}^{1/2}$$

$$= \left\{ \sum_{\lambda} \left| 1 - \left(1 - (\lambda/\Lambda)^{2k}\right)_+^{\delta} \right|^2 |\mathcal{F}f(\lambda)|^2 \right\}^{1/2} \le c \, \|f\|_{\mathbb{B}^2_{\varepsilon,\infty}(\mathbb{M},d\mu)} \Lambda^{-\varepsilon}.$$

Observe that when $f(x)$ is in $\mathbb{X}^{\alpha}(\mathbb{M})$ then $\varepsilon = \alpha + 1/2$, however the pointwise and the mean square results are independent and do not overlap.

We have seen in the first section that under the assumptions of the above theorem it is not possible to decide the convergence of the Riesz means at singular points, however we have the following.

THEOREM 4.5. *Riesz means of order $\delta > (N-3)/2$ of functions in $\mathbb{X}(\mathbb{M})$ converge everywhere.*

PROOF. The idea is that, by Theorem 4.1, localization holds. Moreover, by Theorem 2.3, the local behavior of Riesz means on the manifold is similar to the one on the Euclidean space. Finally, one can apply Theorem 1.1 to the Riesz means on this Euclidean space. We skip the details. □

5. - An Application

The Gauss circle problem is the estimate of the number of integer points in a large disc in the plane. More in general, one can try to estimate the number of integer points in large balls in $\mathbb{R}^N$. The number of integer points in a ball $B(x,r)$ of center x and radius r is a periodic function of x with Fourier expansion

$$\sum_{k\in\mathbb{Z}^N} \chi_{B(x,r)}(k) = \sum_{k\in\mathbb{Z}^N} r^{N/2}\,|k|^{-N/2}\,J_{N/2}\left(2\pi r\,|k|\right)\exp\left(2\pi i k\cdot x\right).$$

This function is piecewise constant, with discontinuities on piecewise smooth surfaces, and since $\left|J_{N/2}(t)\right| \le ct^{-1/2}$ one immediately obtain

$$\left\{\sum_{n-1\le 2\pi|k|\le n} \left|r^{N/2}\,|k|^{-N/2}\,J_{N/2}\left(2\pi r\,|k|\right)\right|^2\right\}^{1/2} \le cr^{(N-1)/2}n^{-1}.$$

This is of course the thesis of Theorem 3.2, one can apply Theorem 4.1 and conclude that the series is Riesz summable with index $\delta > (N-3)/2$. The convergence of the series for $N=2$, $\delta=0$ and $x=0$ was first conjectured by Voronoi and then proved in [11], but the connection between this problem in analytic number theory and Fourier series was first stated in [18].

REFERENCES

[1] S. A. ALIMOV – V. A. IL'IN – E. M. NIKISHIN, *Convergence problems of multiple trigonometric series and spectral decompositions, I, II*, Russian Math. Surveys **31** (1976), 29-86, **32** (1977), 115-139.

[2] P. BÉRARD, *On the wave equation on a manifold without conjugate points*, Math. Z. **155** (1977), 249-276.

[3] P. BÉRARD, *Riesz means on Riemannian manifolds*, Amer. Math. Soc. Proc. Symp. Pure Math. **XXXVI** (1980), 1-12.

[4] S. BOCHNER, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. **40** (1936), 175-207.

[5] L. BRANDOLINI – L. COLZANI, *Localization and convergence of eigenfunction expansions*, J. Fourier Anal. Appl. **5** (1999), 431-447.

[6] L. BRANDOLINI – L.COLZANI – G. TRAVAGLINI, *Average decay of Fourier transforms and integer points in polyhedra*, Ark. Mat. **35** (1997), 253-275.

[7] L. COLZANI – M.VIGNATI, *The Gibbs phenomenon for multiple Fourier integrals*, J. Approx. Th. **80** (1995), 119-131.

[8] L. DE MICHELE – D. ROUX, *Approximate units and Gibbs phenomenon*, Boll. Un. Mat. Ital. A (7) (1997), 739-746.

[9] L. DE MICHELE – D. ROUX, *The Gibbs phenomenon for L^1_{loc} kernels*, J. Approx. Th. **100** (1999), 144-156.

[10] L. DE MICHELE – D. ROUX, *The Gibbs phenomenon for multiple Fourier integrals and series: restriction theorems*, Atti Sem. Mat. Fis. Univ. Modena **46** (1998), 351-360.

[11] G. H. HARDY, *On the expression of a number as a sum of two squares*, Quart. J. Math. **46** (1915), 263-283.

[12] E. HLAWKA, *Uber Integrale auf convexen Korpen, I & II*, Monats. Math. **54** (1950), 1-36, 81-99.

[13] C. HERZ, *Fourier transform related to convex sets*, Ann. of Math. **75** (1962), 81-92.

[14] L. HÖRMANDER, *On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators*, In: "Some recent advances in the basic sciences", Yeshiva University 1966, pp. 155-202.

[15] L. HÖRMANDER, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193-218.

[16] L. HÖRMANDER, "The analysis of linear partial differential operators", I, II, III, IV, Springer Verlag, 1985-1985.

[17] J. P. KAHANE, *Le phénomène de Pinsky et la géométrie des surfaces*, C. R. Acad. Sci. Paris **321** (1995), 1027-1029.

[18] D. G. KENDALL, *On the number of lattice points inside a random oval*, Quart. J. Math. Oxford **19** (1948), 1-26.

[19] C. MEANEY, *On almost-everywhere convergent eigenfunction expansions of the Laplace-Beltrami operator*, Math. Proc. Cambridge Philos. Soc. **92** (1982), 129-131.

[20] M. A. PINSKY, *Pointwise Fourier inversion and related eigenfunction expansions*, Comm. Pure Appl. Math. **47** (1994), 653-681.

[21] M. A. PINSKY, *Fourier inversion in the piecewise smooth category*, In: "Fourier Analysis, analytic and geometric aspects", W. O. Bray – P. S. Milojevic' – C. V. Stanojevic' (eds.), Marcel Dekker (1994).

[22] M. A. PINSKY – M.TAYLOR, *Pointwise Fourier inversion: a wave equation approach*, J. Fourier Anal. Appl. **3** (1997), 647-703.

[23] A. N. PODKORYTOV, *The asymptotic of Fourier transform of a convex curve*, Vestnik Leningr. Univ. Mat. **24** (1991), 57-65.

[24] C. D. SOGGE, *Concerning the L^p norm of spectral clusters for second order elliptic differential operators on compact manifolds*, J. Funct. Anal. **77** (1988), 123-134.

[25] C. D. SOGGE, *On the convergence of Riesz means on compact manifolds*, Ann. of Math. **126** (1987), 439-447.

[26] E. M. STEIN – G. WEISS, "Introduction to Fourier analysis on Euclidean spaces", Princeton University Press, 1971.

[27] M. E. TAYLOR, *Pointwise Fourier inversion on tori and other compact manifolds*, J. Fourier Anal. Appl. **5** (1999), 449-463.

[28] M. E. TAYLOR, *Pointwise Fourier inversion - an addendum*, Proc. Amer. Math. Soc., to appear.

[29] M. E. TAYLOR, *The Dirichlet-Jordan test and multidimensional extensions*, Proc. Amer. Math. Soc., to appear.

[30] M. E. TAYLOR, *Eigenfunction expansions and the Pinsky phenomenon on compact manifolds*, preprint.

[31] A. TORLASCHI, *Sviluppi in armoniche sferiche di funzioni regolari a tratti*, Tesi di Laurea, Università degli Studi di Milano (1998).

[32] A. N. VARCHENKO, *Number of lattice points in families of homethetic domains in* $\mathbb{R}^n$, Funktional An. **17** (1983), 1-6.

[33] G. N. WATSON, "A treatise on the theory of Bessel functions", Cambridge University Press, 1944.

[34] H. WEYL, *Die Gibbsche Erscheinung in der Theorie der Kugelfunktionen*, Rendiconti Circ. Mat. Palermo **29** (1910), 308-323.

[35] H. WEYL, *Über die Gibbsche Erscheinung und verwandte Konvergenzphänomene*, Rendiconti Circ. Mat. Palermo **30** (1910), 377-407.

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