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Characterization of Homogeneous Gradient Young Measures in Case of Arbitrary Integrands

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Abstract. In the case of continuous integrands $L : \mathbf{R}^{m\times n} \to \mathbf{R}\cup\{\infty\}$ we indicate conditions both necessary and sufficient for a probability measure ν to be generated as a homogeneous Young measure by the gradients of piecewise affine (or Sobolev) functions $u_k \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with the property $L(Du_k) \rightharpoonup \langle L; \nu\rangle$ in L^1. Here A is the center of mass of ν and l_A is a linear function with the gradient equal to A everywhere and $L(v) \to \infty$ as $|v| \to \infty$. We also show that in the scalar case $m = 1$ any probability measure with finite action on L satisfies these conditions.

We discuss some applications of these results to various problems related to behavior of integral functionals on weakly convergent sequences.

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1. – Introduction

Recent results in the area of Young measure theory, see e.g. [Ba], [B1], [KP1]-[KP3], [Kr], [S1]-[S4], showed that this theory presents a powerful tool for studying classical problems of the Calculus of Variations related to behavior of integral functionals on weakly convergent sequences. In fact, the relaxation theorem was proved recently for those Caratheodory integrands which satisfy the p-growth condition using this theory in [S2]. This result completed the series of the relaxation results for integrands with p-growth, see e.g. [AF], [Bu], [D], [FM]. Note that those works relied on other methods.

The basic idea of the Young measure technique is to work directly with Young measures instead of sequences generating them, provided the action of a measure on an integrand is equal to the limit of values assumed by the integral

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functional at the sequence. In the case of integrands $L = L(Du)$ with p-growth

$$A_1|Du|^p + B_1 \le L(Du) \le A_2|Du|^p + B_2,\ p > 1,\ A_2 \ge A_1 > 0$$

the class of homogeneous Young measures generated by gradients and having the above property was characterized by Kinderlehrer and Pedregal in [KP3]. These measures were named *homogeneous gradient p-Young measures*.

In order to move analysis towards a wider class of integral functionals, including the realistic problems in Elasticity (see [B2], [B3], [BM], [C, Ch. 4]), one has to obtain a characterization of Young measures arising in an analogous way in the case of arbitrary integrands. Note that even the case of realistic homogeneous isotropic materials demands to deal with integrands $L = L(Du)$: $\mathbf{R}^{3\times 3} \to \mathbf{R} \cup \{\infty\}$ meeting the requirement

$$L(Du) \to 0 \text{ as } \det Du \to +0.$$

Therefore, in this paper the basic assumption on L will be

(H1) $L : \mathbf{R}^{m\times n} \to \bar{\mathbf{R}} := \mathbf{R} \cup \{+\infty\}$ is continuous and $L(v) \to \infty$ as $|v| \to \infty$

We adopt the following conventions: for a subset A of $\mathbf{R}^n$ the sets int$A$, reint$A$, co$A$, and extr$A$ are respectively the interior, the relative interior, the convex hull, and the set of extreme points of A. $B(a, \epsilon)$ denotes the ball of radius ϵ centered at the point $a \in \mathbf{R}^n$; l_a is a linear function with the gradient equal to a everywhere. Weak and strong convergences of sequences are denoted by $\rightharpoonup$ and $\to$ respectively. We will assume that Ω is a bounded open subset of $\mathbf{R}^n$ with meas $(\partial\Omega) = 0$.

We will use notation $\tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$ for the set of piecewise affine functions vanishing at the boundary, i.e. for those $u \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ for which there is an at most countable decomposition of Ω into Lipschitz domains, such that the restriction of the function u to the closure of each of these domains is an affine function, and a set of null measure. $C_0(\mathbf{R}^l)$ is the class of continuous functions vanishing at infinity. $C_c^\infty(\mathbf{R}^l)$ is the class of C^∞-functions $\Phi : \mathbf{R}^l \to \mathbf{R}$ with compact support. We use notation $\langle\cdot;\cdot\rangle$ to denote the action of a measure on a function.

We will use the following

DEFINITION 1.1. *For an integrand L, which satisfies the condition* (H1), *and a probability measure ν, which has finite action on L and is centered at a point* $A \in \mathbf{R}^{m\times n}$, *we call this measure a homogeneous gradient L-Young measure provided there exists a sequence* $u_k \in l_A + \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$ *such that* Du_k *generates ν as a Young measure:*

$$\Phi(Du_k) \overset{*}{\rightharpoonup} \langle\Phi; \nu\rangle \textit{ in } L^\infty(\Omega) \textit{ for all } \Phi \in C_c^\infty(\mathbf{R}^{m\times n}),$$

and $L(Du_k) \rightharpoonup \langle L; \nu\rangle$ *in* L^1 *as* $k \to \infty$.

REMARK 1. We do not associate ν with the set Ω since validity of this definition for Ω implies its validity for all bounded open sets, cf. Lemma 2.2.

The main result of this paper is

THEOREM 1.2. *Let L satisfy* (H1), *and let ν be a probability measure, which is supported in $\mathbf{R}^{m\times n}$, with finite action on L and center of mass at $A \in \mathbf{R}^{m\times n}$. Then ν is a gradient L-Young measure if and only if for each $\Phi \in C_c^\infty(\mathbf{R}^{m\times n})$ the inequality*

$$(1.1) \quad \inf_{\psi \in \tilde{C}_0^\infty(\Omega;\mathbf{R}^m)} \frac{1}{\text{meas } \Omega} \int_\Omega \{L(A+D\psi(x))+\Phi(A+D\psi(x))\}dx \le \langle L+\Phi; \nu\rangle$$

holds.

REMARK 2. As it will follow from the proof, the analogous result holds if in the definition of gradient L-Young measures the class $\tilde{C}_0^\infty(\Omega;\mathbf{R}^m)$ is replaced by the Sobolev class $W_0^{1,p}(\Omega;\mathbf{R}^m)$, $p \in [1,\infty]$. In this case ψ in (1.1) should be taken in the same class. In fact instead of $\tilde{C}_0^\infty(\Omega;\mathbf{R}^m)$ or $W_0^{1,p}(\Omega;\mathbf{R}^m)$ one can even take *any class of functions* with the property that the construction from Lemma 2.2 keeps functions in the class. One of such classes is *generalized piecewise affine functions* which differ from the standard piecewise affine functions at the point that the requirement $u \in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ is replaced by $u \in W_0^{1,1}(\Omega;\mathbf{R}^m)$.

Note that classes of gradient Young measures generated by various classes of functions can be different. Moreover this issue plays essential role in problems arising in elasticity. We will discuss this in § 5.

REMARK 3. Note that in (1.1) the function L can be replaced by an equivalent integrand $\tilde{L}$, i.e.

$$C_1 L + B_1 \le \tilde{L} \le C_2 L + B_2, \; C_2 \ge C_1 > 0, \; B_2, B_1 \in \mathbf{R}.$$

In case of integrands L with p-growth this means that ν is a gradient L-Young measure if and only if

$$\inf_{\psi \in \tilde{C}_0^\infty(\Omega;\mathbf{R}^m)} \int_\Omega \{|A + D\psi(x)|^p + \Phi(A + D\psi(x))\}dx \le \langle |\cdot|^p + \Phi(\cdot); \nu\rangle$$

for each $\Phi \in C_c^\infty(\mathbf{R}^{m\times n})$ (here A is the center of mass of ν).

The original result of Kinderlehrer and Pedregal [KP3] says that ν is a gradient p-Young measure if and only if for each quasiconvex function G with p-growth the inequality $\langle G;\nu\rangle \ge G(A)$ holds. Recall that an integrand G is called *quasiconvex* if for each $A \in \mathbf{R}^{m\times n}$ we have

$$\int_\Omega G(A + D\phi(x))dx \ge G(A)\,\text{meas } \Omega, \;\; \forall \phi \in \tilde{C}_0^\infty(\Omega;\mathbf{R}^m).$$

Our result shows that one has to check this inequality only for those quasiconvex functions which are equal to $|\cdot|^p$ for all sufficiently large $|\cdot|$. Moreover, the analogous result still holds for convex integrands L with superlinear growth at infinity.

In the case $m = 1$ we can prove that any probability measure ν with finite action on L is a gradient L-Young measure.

THEOREM 1.3. *Let L satisfy the condition* (H1) *with $m = 1$ and let L have superlinear growth:*

$$L(v) \geq \theta(v), \text{ where } \theta(v)/|v| \to \infty \text{ as } |v| \to \infty.$$

Let also ν be a probability measure supported in $\mathbf{R}^n$ with $\langle L; \nu\rangle < \infty$.

Then ν is a homogeneous gradient L-Young measure. Moreover, ν can be generated as a homogeneous gradient L-Young measure by the gradients of a sequence ϕ_k with $D\phi_k$ lying in $1/k$-neighborhoods of $\operatorname{supp}\nu$, $k \in \mathbf{N}$.

This fact follows from a possibility to generate any convex combination of Dirac masses by the gradients of a sequence of piecewise affine functions, which is bounded in $W^{1,\infty}(\Omega)$, see Lemma 4.2. In the case $m > 1$ not each probability measure is a gradient L-Young measure. For different types of nontrivial restrictions these measures have to satisfy see e.g. [Sv1], [Sv2].

Note that approximation in energy of convex combinations of Dirac masses by the gradients of Sobolev functions is the main idea of the relaxation theorems, see e.g. [ET, Ch.10]. There the authors presented straightforward approximation arguments sufficient to prove Theorem 1.3 in the case when L is compatible with a convex function $F = F(|Du|)$, see Theorem 1.2 and Proposition 2.8 of Chapter 10 of [ET]. Theorem 1.3 shows that the approximation still exists for any continuous $L : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ with superlinear growth at infinity. Note that in case of integrands with p-growth this result follows also from the characterization result by Kinderlehrer and Pedregal [KP3] since in the scalar case each quasiconvex function is convex, cf. [D].

Recall that the standard approach to Young measures is to consider them as elements of the duals of certain Banach spaces. The idea to use duality arguments to show that one can approximate in energy a class of measures by a smaller one was fruitfully used by French School. In [BL], [T1], [T2] e.g. these arguments were applied to the case of all probability measures Y_A centered at A and the subset $\tilde{Y}_A$ of Y_A which consists of convex combinations of Dirac masses. Duality arguments and convexity of the set $\tilde{Y}_A$ allow to replace straightforward approximation by verification of the inequality

$$\langle L; \nu_1\rangle \geq \inf_{\nu_2 \in \tilde{Y}_A} \langle L; \nu_2\rangle, \quad \nu_1 \in Y_A,$$

for each element L of the Banach space.

More recently Kinderlehrer and Pedregal showed in [KP3] that duality arguments can be applied to characterize homogeneous gradient p-Young measures as those probability measures which satisfy the Jensen inequality with quasiconvex functions having p-growth at infinity. To be more specific consider the case when the smaller class (which replaces $\tilde{Y}_A$) consists of measures $\mathrm{Av}(D\phi)_\Omega$ with $\phi \in l_A + \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$, where

$$\langle L; \mathrm{Av}(D\phi)_\Omega \rangle := \frac{1}{\mathrm{meas}\ \Omega} \int_\Omega L(D\phi(x))dx.$$

Then the class of probability measures, which can be approximated in energy by the measures $\mathrm{Av}(D\phi)_\Omega$, consists of measures ν such that

$$\langle L; \nu \rangle \geq \frac{1}{\mathrm{meas}\ \Omega} \inf_{\psi \in \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)} \int_\Omega L(A + D\psi(x))dx \tag{1.2}$$

for every L in the space. In case of integrands with p-growth the later space is the set of all integrands L such that $\lim_{|v| \to \infty} L(v)/|v|^p$ exists and is finite. Since the formula in the right-hand side of (1.2) defines a function L^{qc} which is quasiconvex and stays in the same Banach space, cf. [D], we infer that (1.2) can be replaced by the inequality $\langle G; \nu \rangle \geq G(A)$ for quasiconvex G with p-growth.

The inequality (1.2) is a necessary condition for ν to be a gradient p-Young measure. On the other hand the measures $\mathrm{Av}(D\phi)_\Omega$ are homogeneous gradient p-Young measures, cf. [KP3] or Lemma 2.2 below, and automatically satisfy (1.2). Then all probability measures which can be approximated by $\mathrm{Av}(D\phi)_\Omega$ in energy are also homogeneous gradient p-Young measures. Therefore (1.2) is a characterization of the later class of measures.

The class of measures with finite action on an integrand L, which satisfies the requirement (H1) only, does not form a linear space. Therefore we can not apply the above scheme by Kinderlehrer and Pedregal to characterize gradient L-Young measures. However it is possible to follow the approach of the papers [S2], [S3] to consider Young measures as measurable functions $\nu : \Omega \to (M, \rho)$, where M is the space of measures with the metric

$$\rho(\mu_1, \mu_2) := \sum_{i=1}^{\infty} \frac{1}{2_i ||\Phi_i||_C} |\langle \Phi_i; \mu_1 \rangle - \langle \Phi_i; \mu_2 \rangle|, \tag{1.3}$$

and the sequence $\{\Phi_i\}$ is dense in the space $C_0(\mathbf{R}^{m \times n})$. This form of the metric turned out to be essential to prove some properties of Young measures, cf. [S2], [S3], and, as we will see, it can be also used to establish characterizations given by Theorems 1.2 and 1.3.

Note that to make gradient Young measure theory applicable to lower semicontinuity, relaxation, or similar problems in the Calculus of Variations one can e.g. follow the schemes suggested in [S2]. However for each particular class of integrands one has to show that the actions of gradient L-Young measures coincide with the actions of Young measures generated by the sequences Du_k

with $\int_\Omega |L(Du_k)| < c < \infty$, $k \in \mathbf{N}$. This property holds for integrands with p-growth at infinity, see [AF], [KP2], [KP3] or [Kr], as well as for some other classes, see [S3], [S4]. Establishing this property for new interesting classes of integrands is an open problem. However there are examples where the property fails, as it follows from [M]. Another subtle issue is a class of functions which is used to define gradient Young measures, as it follows from [BM, § 7], [JS].

We will discuss both these matters in § 5.

We will prove some auxiliary results about Young measures in § 2. In § 3 we give a proof to Theorem 1.2. Theorem 1.3 will be proved in § 4. § 5 will be devoted to comparison of various classes of gradient Young measures. There we will also discuss the issue of passing from homogeneous to nonhomogeneous cases, which is not trivial in the general case.

2. – Some facts from Young measure theory

In this section we recall some facts from Young measure theory which will be involved in the proof of Theorem 1.2.

Recall that a sequence $\xi_k : \Omega \to \mathbf{R}^l$ *generates a homogeneous Young measure* ν if ν is a probability measure and for each $\Phi \in C_0(\mathbf{R}^l)$ the convergence

$$\Phi(\xi_k) \stackrel{*}{\rightharpoonup} \langle \Phi; \nu\rangle \text{ in } L^\infty(\Omega) \tag{2.1}$$

holds .

Let $\mathrm{Av}(\xi_k)_\Omega$ (k is fixed) be a measure defined as

$$\langle \Phi; \mathrm{Av}(\xi_k)_\Omega\rangle := \frac{1}{\text{meas } \Omega}\int_\Omega \Phi(\xi_k(x))dx, \ \forall \Phi \in C_0(\mathbf{R}^l).$$

It is easy to prove that $\mathrm{Av}(\xi_k)_\Omega$ is a homogeneous Young measure generated by scaled copies of the function ξ_k. Therefore the convergence in (2.1) implies the convergence $\mathrm{Av}(\xi_k)_\Omega \stackrel{*}{\rightharpoonup} \nu$, i.e.

$$\langle \Phi; \mathrm{Av}(\xi_k)_\Omega\rangle \to \langle \Phi; \nu\rangle,\ k \to \infty,\ \forall \Phi \in C_0(\mathbf{R}^l).$$

In the proof of Theorem 1.2 we will use a similar construction showing that for a special sequence $\phi_k \in l_A + \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$ the convergence $\langle L; \mathrm{Av}(D\phi_k)_\Omega\rangle \to \langle L; \nu\rangle$ holds.

To make the proof complete we will need

1) to show that $\mathrm{Av}(D\phi_k)_\Omega$ are gradient L-Young measures provided $L(D\phi_k) \in L^1(\Omega)$;

2) to show that the convergences $\mathrm{Av}(D\phi_k)_\Omega \stackrel{*}{\rightharpoonup} \nu$, $\langle L; \mathrm{Av}(D\phi_k)_\Omega\rangle \to \langle L; \nu\rangle$ imply that ν is a gradient L-Young measure;

3) to establish a connection of these convergences with the inequality in the statement of Theorem 1.2.

To answer the third question we will use

LEMMA 2.1. *Let ν_k, $k = 0, 1, \ldots$, be a sequence of probability measures supported in $\mathbf{R}^l$. Then $\nu_k \overset{*}{\rightharpoonup} \nu_0$ if and only if $\rho(\nu_k, \nu_0) \to 0$, where*

$$\rho(\mu, \nu) = \sum_{i=1}^{\infty} \frac{1}{2^i ||\Phi_i||_C} |\langle \Phi_i; \mu\rangle - \langle \Phi_i; \nu\rangle|$$

and $\{\Phi_i\} \subset C_0(\mathbf{R}^l)$ is dense in $C_0(\mathbf{R}^l)$.

PROOF. is straightforward since the convergence $\nu_k \overset{*}{\rightharpoonup} \nu_0$ means convergence $\langle \Phi; \nu_k\rangle \to \langle \Phi; \nu_0\rangle$, $k \to \infty$, for all $\Phi \in C_0(\mathbf{R}^l)$. □

The following lemma answers the first question. Here we use arguments developing the ones presented first in [BM].

LEMMA 2.2. *Let $\phi \in l_A + \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$ and let $\tilde{\Omega}$ be an open bounded subset of $\mathbf{R}^n$. Then, there exists a sequence $\phi_k \in l_A + \tilde{C}_0^\infty(\tilde{\Omega}; \mathbf{R}^m)$ such that $D\phi_k$ generates $\operatorname{Av}(D\phi)_\Omega$ as a homogeneous Young measure in $\tilde{\Omega}$ and for each $k \in \mathbf{N}$ the function $D\phi_k$ has the same distribution in $\tilde{\Omega}$ as $D\phi$ in Ω, that is $\operatorname{Av}(D\phi_k)_{\tilde{\Omega}} = \operatorname{Av}(D\phi)_\Omega$.*

Moreover, if L satisfy (H1) *and $\int_\Omega L(D\phi(x))dx < \infty$ then $D\phi_k$ generates $\operatorname{Av}(D\phi)_\Omega$ as a homogeneous gradient L-Young measure.*

In the proof we will use the following standard result: a family F of closed subsets of $\mathbf{R}^n$ is said to be a *Vitaly cover* of a bounded measurable set A if for a.a. $x \in A$ there exists a positive number $r(x) > 0$, a sequence of balls $B(x, \epsilon_k)$ with $\epsilon_k \to 0$, and a sequence $C_k \in F$ such that $x \in C_k$, $C_k \subset B(x, \epsilon_k)$, and $(\operatorname{meas} C_k / \operatorname{meas} B(x, \epsilon_k)) > r(x)$ for all $k \in \mathbf{N}$.

The version of Vitaly covering theorem from [Sa, p. 109] says that each Vitaly cover of A contains at most countable subfamily of disjoint sets C_k such that $\operatorname{meas}(A \setminus \cup_k C_k) = 0$.

PROOF OF LEMMA 2.2. Let $\tilde{\Omega}$ be a bounded open subset of $\mathbf{R}^n$.

By the above version of the Vitaly covering theorem for each $k \in \mathbf{N}$ we can find a decomposition of $\tilde{\Omega}$ into sets $\Omega_i^k := x_i^k + \epsilon_i^k \bar{\Omega} \subset \tilde{\Omega}$, where $\epsilon_i^k \leq 1/k$ for all $i \in \mathbf{N}$, and a set N_k of zero measure. We can assume also that for $k' \geq k$, $i, i' \in \mathbf{N}$ either $\Omega_{i'}^{k'} \subset \Omega_i^k$ or $\Omega_{i'}^{k'} \cap \Omega_i^k = \emptyset$.

Define ϕ_k as follows:

$$\phi_k(x) = \epsilon_i^k \phi\left(\frac{x - x_i^k}{\epsilon_i^k}\right) \text{ if } x \in \Omega_i^k,\ i \in \mathbf{N};\quad \phi_k(x) = l_A(x) - \text{ otherwise.}$$

Then $\phi_k \in l_A + \tilde{C}_0^\infty(\tilde{\Omega}; \mathbf{R}^m)$. Notice that for each Ω_i^j and all $k \geq j$ the identity

$$\int_{\Omega_i^j} \Phi(D\phi_k)dx = \langle \Phi; \operatorname{Av}(D\phi)_\Omega\rangle \operatorname{meas} \Omega_i^j \tag{2.2}$$

holds. To show that for each $\Phi \in C_0(\mathbf{R}^{m\times n})$

$$\Phi(D\phi_k) \overset{*}{\rightharpoonup} \langle \Phi; \operatorname{Av}(D\phi)_\Omega\rangle \text{ in } L^\infty(\tilde{\Omega})$$

we have to establish convergence

$$\frac{1}{\text{meas } U}\int_U \Phi(D\phi_k(x))dx \to \langle \Phi; \text{Av}(D\phi)_\Omega\rangle$$

for all open subsets U of $\tilde{\Omega}$.

Let U be fixed. Note that the sets Ω_i^j, $i, j \in \mathbf{N}$, form a Vitaly cover of U. Hence for each $\epsilon > 0$ there exists a finite collection of disjoint sets $\bar{\Omega}_{i(l)}^{j(l)} \subset U$, $l = 1, \dots, q$, such that

$$\text{meas}\left(U \setminus \cup_l \Omega_{i(l)}^{j(l)}\right) \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary and (2.2) holds for each pair $i, j \in \mathbf{N}$ with $k \in \mathbf{N}$ sufficiently large, we infer

$$\Phi\left(D\phi_k\big|_U\right) \to \langle \Phi; \text{Av}(D\phi)_\Omega\rangle \text{ meas } U, \ \forall \Phi \in C_0(\mathbf{R}^{m\times n}).$$

Since U is any open subset of $\tilde{\Omega}$ we obtain

$$\Phi(D\phi_k) \overset{*}{\rightharpoonup} \langle \Phi; \text{Av}(D\phi)_\Omega\rangle \text{ in } L^\infty(\tilde{\Omega}), \ \forall \Phi \in C_0(\mathbf{R}^{m\times n}). \tag{2.3}$$

By construction $D\phi_k$, $k \in \mathbf{N}$, has the same distribution in $\tilde{\Omega}$ as $D\phi$ in Ω, therefore $\text{Av}(D\phi_k)_{\tilde{\Omega}} = \text{Av}(D\phi)_\Omega$. This completes the proof of the first assertion of the lemma. To prove the second one notice that the convergence $L(D\phi_k) \rightharpoonup \langle L; \text{Av}(D\phi)_\Omega\rangle$ in $L^1(\tilde{\Omega})$ can be established by the same arguments as (2.3) since the sequence $L(D\phi_k)$ is equi-integrable. In fact it is easy to see that it has the modulus of integrability of the function $|L(D\phi)|$ multiplied by the factor (meas $\tilde{\Omega}$/ meas Ω). □

The claim 2) follows from

LEMMA 2.3. *Let $\nu_k, k \in \mathbf{N}$, be a sequence of homogeneous Young measures generated by the gradients of functions $\phi_i^k \in l_A + \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$ (as $i \to \infty$), respectively, and let $\nu_k \overset{*}{\rightharpoonup} \nu$.*

Then ν is generated as a homogeneous Young measure by the gradients of a sequence $\phi_{i(k)}^k$, $k \in \mathbf{N}$. Moreover, if ν_k are homogeneous gradient L-Young measures then ν is a homogeneous gradient L-Young measure provided $\langle L; \nu_k\rangle \to \langle L; \nu\rangle$.

PROOF. For each fixed k there exists a sequence $\phi_i^k \in l_A + \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$, $i \in \mathbf{N}$, such that its gradients generate ν_k as a homogeneous L-Young measure:

$$\Phi(D\phi_i^k) \overset{*}{\rightharpoonup} \langle \Phi; \nu_k\rangle \text{ in } L^\infty(\Omega), \ i \to \infty, \ \forall \Phi \in C_0(\mathbf{R}^{m\times n}).$$

Since $C_0(\mathbf{R}^{m\times n})$ is separable by standard diagonalization arguments we can find a sequence $\phi_{i(k)}^k$, $k \in \mathbf{N}$, with the properties

$$\Phi(D\phi_{i(k)}^k) \overset{*}{\rightharpoonup} \langle \Phi; \nu\rangle \text{ in } L^\infty(\Omega), \ k \to \infty, \forall \Phi \in C_0(\mathbf{R}^{m\times n}).$$

This means that the sequence $D\phi^k_{i(k)}$ generates ν as a homogeneous Young measure.

In the case when ν_k are homogeneous gradient L-Young measures we have also

$$L(D\phi^k_i) \rightharpoonup \langle L; \nu_k\rangle \text{ in } L^1 \text{ as } i \to \infty,\ k \in \mathbf{N}.$$

Because of the convergence $\langle L; \nu_k\rangle \to \langle L; \nu\rangle$, $k \to \infty$, the sequence $\phi^k_{i(k)}$ can be also chosen in such a way that

$$L(D\phi^k_{i(k)}) \rightharpoonup \langle L; \nu\rangle \text{ in } L^1(\Omega),\ k \to \infty.$$

Therefore ν is a homogeneous gradient L-Young measure. □

3. – Proof of Theorem 1.2

PROOF OF THEOREM 1.2. If ν is a gradient L-Young measure centered at A then there exists a sequence $\phi_k \in l_A + \tilde{C}^\infty_0(\Omega; \mathbf{R}^m)$ with the properties $L(D\phi_k) \rightharpoonup \langle L; \nu\rangle$ in L^1, $\Phi(D\phi_k) \overset{*}{\rightharpoonup} \langle \Phi; \nu\rangle$ in $L^\infty(\Omega)$ for all $\Phi \in C_0(\mathbf{R}^{m\times n})$. In this case

$$\int_\Omega \{L(D\phi_k) + \Phi(D\phi_k)\}dx \to \langle L + \Phi; \nu\rangle \operatorname{meas} \Omega,\ \ \forall \Phi \in C_0(\mathbf{R}^{m\times n}),$$

that implies validity of (1.1).

To prove the converse we will first show that the set

$$G := \left\{ \operatorname{Av}(D\phi)_\Omega : \phi \in l_A + \tilde{C}^\infty_0(\Omega; \mathbf{R}^m),\ \int_\Omega L(D\phi)dx < \infty \right\}$$

is convex. Let $\nu^1 := \operatorname{Av}(D\phi_1)_\Omega$, $\nu^2 := \operatorname{Av}(D\phi_2)_\Omega$, $\lambda \in]0, 1[$. Let Ω_1, Ω_2 be disjoint open subsets of Ω such that $\operatorname{meas}(\partial\Omega_1) = \operatorname{meas}(\partial\Omega_2) = 0$ and $\operatorname{meas} \Omega_1 = \lambda \operatorname{meas} \Omega$, $\operatorname{meas} \Omega_2 = (1-\lambda) \operatorname{meas} \Omega$. By Lemma 2.2 there exist functions $u^1 \in l_A + \tilde{C}^\infty_0(\Omega_1; \mathbf{R}^m)$, $u^2 \in l_A + \tilde{C}^\infty_0(\Omega_2; \mathbf{R}^m)$ such that $\operatorname{Av}(Du^1)_{\Omega_1} = \nu^1$, $\operatorname{Av}(Du^2)_{\Omega_2} = \nu^2$.

Let $u = u^1$ in Ω_1, $u = u^2$ in Ω_2. Then $\operatorname{Av}(Du)_\Omega \in G$ and $\operatorname{Av}(Du)_\Omega = \lambda\nu^1 + (1-\lambda)\nu^2$. This proves convexity of G.

The theorem will be proved if we show that ν belongs to the closure of the set G in the following sense:

$$\inf_{\mu\in G}\{\rho(\mu, \nu) + |\langle L; \mu\rangle - \langle L; \nu\rangle|\} = 0. \tag{3.1}$$

In fact (3.1) implies existence of a sequence $\nu_k \in G$ such that $\rho(\nu_k, \nu) + |\langle L; \nu_k\rangle - \langle L; \nu\rangle| \to 0$, $k \to \infty$. Convergence of the first term to zero means that $\nu_k \overset{*}{\rightharpoonup} \nu$. Then, by Lemmata 2.2, 2.3 convergence of the second term to zero implies that ν is a homogeneous gradient L-Young measure.

We will prove (3.1) by contradiction. Recall that

$$\rho(\mu, \nu) := \sum_{i=1}^{\infty} \frac{1}{2^i ||\Phi_i||_C} \Big| \langle \Phi_i; \mu \rangle - \langle \Phi_i; \nu \rangle \Big|,$$

where the sequence $\{\Phi_i\} \subset C_c^{\infty}(\mathbf{R}^{m\times n})$ is dense in $C_0(\mathbf{R}^{m\times n})$.

If (3.1) does not hold, then for a sufficiently large $l \in \mathbf{N}$ we have

$$\inf_{\mu\in G} \left\{ |\langle L; \mu \rangle - \langle L; \nu \rangle| + \sum_{i=1}^{l} \frac{1}{2^i ||\Phi_i||_C} \Big| \langle \Phi_i; \mu \rangle - \langle \Phi_i; \nu \rangle \Big| \right\} > \epsilon > 0.$$

Then, the subset of $\mathbf{R}^{l+1}$ given by the vectors

$$\left(\langle L; \mu \rangle, \frac{1}{2||\Phi_1||} \langle \Phi_1; \mu \rangle, \dots, \frac{1}{2^l ||\Phi_l||} \langle \Phi_l; \mu \rangle \right), \quad \mu \in G,$$

is convex since G is convex, and the vector generated by ν does not belong to its closure. Hence, there exists a vector $c \in \mathbf{R}^{l+1}$ such that

$$\inf_{\mu\in G} \left\{ c_0 \langle L; \mu \rangle + \sum_{i=1}^{l} c_i \langle \Phi_i; \mu \rangle \right\} > c_0 \langle L; \nu \rangle + \sum_{i=1}^{l} c_i \langle \Phi_i; \nu \rangle + \delta, \ \delta > 0.$$

Then

$$\inf_{\mu\in G} \langle \tilde{L}; \mu \rangle > \langle \tilde{L}; \nu \rangle + \delta, \ \text{with } \tilde{L} = c_0 L + \sum_{i=1}^{l} c_i \Phi_i. \tag{3.2}$$

Note that the coefficient c_0 cannot be negative, otherwise the value at the left-hand side is $-\infty$ since $L(v) \to \infty$ as $|v| \to \infty$. We can assume that $c_0 > 0$ since (3.2) still holds if we replace $\tilde{L}$ by $\tilde{L} + \eta L$ with $\eta > 0$ sufficiently small. Note that the integrand $\tilde{L}/c_0$ is of the type $L + \Phi$, $\Phi \in C_c^{\infty}(\mathbf{R}^{m\times n})$, and due to (3.2) the inequality (1.1) fails for this integrand.

This contradiction proves that (3.1) holds and that ν is a homogeneous gradient L-Young measure. This completes the proof of the theorem. □

4. – Proof of Theorem 1.3

To prove Theorem 1.3 we will need the following two lemmata.

LEMMA 4.1. *Assume that* $A \in \operatorname{int} \operatorname{co}\{v_1, \ldots, v_q\}$. *Then there is a function* $u_0 \in l_A + \tilde{C}_0^\infty(\Omega)$ *such that* $Du_0 \in \{v_1, \ldots, v_q\}$ *a.e. in* Ω.

PROOF. Without loss of generality we can assume that $v_1, \ldots, v_q$ are extreme points of a compact convex subset of $\mathbf{R}^n$.

To construct u_0 with desired properties consider the function

$$w_s(x) = \max_{1 \le i \le q} \langle v_i - A, x \rangle - s, \ s > 0.$$

It is easy to see that w_s is Lipschitz, $Dw_s(x) \in \{v_i - A : i = 1, \ldots, q\}$ a.e., and $w_s\big|_{\partial P_s} = 0$, where

$$P_s := \left\{ x : \max_{1 \le i \le q} \langle v_i - A, x \rangle \le s \right\}$$

is a compact set with Lipschitz boundary and nonempty interior.

Note also that $P_s = sP_1$.

Since Vitaly covering arguments let us decompose Ω into disjoint sets of the form $y_i + s_i P_1$, $i \in \mathbf{N}$, and a set of zero measure, we can define u_0 as

$$l_A(x) + w_{s_i}(x - y_i) \text{ for } x \in y_i + s_i P_1, \ i \in \mathbf{N}.$$

Then $u_0 \in l_A + \tilde{C}_0^\infty(\Omega)$ and $Du_0 \in \{v_1, \ldots, v_q\}$ a.e. in Ω. □

LEMMA 4.2. *Let* $v_i \in \mathbf{R}^n$, $c_i > 0$, $i = 1, \ldots, q$, *be such that* $\sum_i c_i = 1$, $\sum_i c_i v_i = A$. *Then there exists a sequence of piecewise affine functions* $\phi_k \in l_A + W_0^{1,\infty}(\Omega)$ *such that*

1) $D\phi_k(x) \in \cup_{i=1,\ldots,q} B(v_i, 1/k)$ *for a.a.* $x \in \Omega$;

2) $D\phi_k$ *generates the measure* $\sum_{i=1}^q c_i \delta_{v_i}$.

PROOF. will proceed by induction with respect to $q \in \mathbf{N}$.

Without loss of generality we can assume that $A = 0$.

1. Let $q = 2$. Then $c_1 v_1 + c_2 v_2 = 0$, where $c_i > 0$, $i = 1, 2$, and $c_1 + c_2 = 1$. We fix $k \in \mathbf{N}$ and take $u_3, \ldots, u_{n+2} \in B(v_1, 1/k)$ such that $u_1, \ldots, u_{n+2}$ are extreme points of a compact convex subset of $\mathbf{R}^n$ with $0 \in \operatorname{int} \operatorname{co}\{u_1, \ldots, u_{n+2}\}$, where $u_1 = v_1$, $u_2 = v_2$.

By Lemma 4.1 there exists a piece-wise affine function $\phi_k \in l_A + W_0^{1,\infty}(\Omega)$ such that

$$D\phi_k \in \{u_1, \ldots, u_{n+2}\} \subset (B(v_1, 1/k) \cup B(v_2, 1/k)), \tag{4.1}$$

Note that (4.1) implies that if for a subsequence of the sequence ϕ_k (not relabeled) we have

$$c_i^k := \frac{\operatorname{meas}\{x \in \Omega : D\phi_k(x) \in B(v_i, 1/k)\}}{\operatorname{meas}\Omega} \to \tilde{c}_i, \ k \to \infty, \ i \in \{1,2\},$$

then $\sum_{i=1}^2 \tilde{c}_i = 1$, $\sum_{i=1}^2 \tilde{c}_i v_i = 0$, and because of uniqueness of the representation of 0 in the form of a convex combination of v_i, $i = 1, 2$, we infer that $\tilde{c}_i = c_i$. Hence, $c_i^k \to c_i$ as $k \to \infty$ for the original sequences c_i^k, $i \in \{1, 2\}$.

Therefore the property 1) holds for the sequence $D\phi_k$ and, moreover,

$$\frac{\operatorname{meas}\{x \in \Omega : D\phi_k(x) \in B(v_i, 1/k)\}}{\operatorname{meas}\Omega} \to c_i, \ k \to \infty, \ i \in \{1,2\}.$$

Now we can apply Lemmata 2.2 and 2.3 to complete the proof.

2. Now we show that validity of the lemma for $q \geq 2$ implies its validity for $q+1$. The induction step will be reduced to the situation discussed in the first part of the proof, i.e. when $q = 2$.

Consider an auxiliary point $\hat{v} = (c_1 v_1 + c_2 v_2)/(c_1 + c_2)$. Then $0 = (c_1 + c_2)\hat{v} + \sum_{i=3}^{q+1} c_i v_i$. By the induction assumption we can find a sequence of piecewise affine functions $\psi_j \in W_0^{1,\infty}(\Omega)$ with the properties

1) $D\psi_j(x) \in (\cup_{i=3,\dots,q+1} B(v_i, 1/j)) \cup B(\hat{v}, 1/j)$ for a.a. $x \in \Omega$;

2) $D\psi_j$ generates a Young measure $(c_1 + c_2)\delta_{\hat{v}} + \sum_{i=3}^{q+1} c_i \delta_{v_i}$.

In each open set $\tilde{\Omega}$ where ψ_j is affine and $D\psi_j \in B(\hat{v}, 1/j)$ we can apply the arguments of the case $q = 2$ to perturb ψ_j by $\eta_j \in W_0^{1,\infty}(\tilde{\Omega})$ in such a way that

$$D\eta_j + D\psi_j \in (B(v_1, 1/j) \cup B(v_2, 1/j))$$

and

$$\left| \frac{\operatorname{meas}\{x \in \tilde{\Omega} : D\psi_j(x) + D\eta_j(x) \in B(v_i, 1/j)\}}{\operatorname{meas}\tilde{\Omega}} - c_i \right| \leq 1/j, \quad i \in \{1,2\}.$$

We define a sequence ϕ_j as the perturbations of ψ_j described above (i.e. in the sets where ψ_j is affine and $D\psi_j \in B(\hat{v}, 1/j)$ the function ψ_j satisfies the above requirements).

Applying Lemmata 2.2 and 2.3 to the sequence ϕ_j we prove the lemma. □

PROOF OF THEOREM 1.3. Let ν be a probability measure with finite action on L. Let A be the center of mass of ν. Without loss of generality we can assume that $A = 0$.

To prove the theorem it is enough to establish the inequality

$$\langle L + \Phi; \nu \rangle \geq \inf_{\phi \in \tilde{C}_0^\infty(\Omega)} \langle L + \Phi; \operatorname{Av}(D\phi)_\Omega \rangle, \tag{4.2}$$

for each $\Phi \in C_c^\infty(\mathbf{R}^n)$, cf. Theorem 1.2.

Fix $\Phi \in C_c^\infty(\mathbf{R}^n)$. By Lemma 4.2 the right-hand side of the inequality is equal to $(L+\Phi)^{**}(A)$, where $(L+\Phi)^{**}$ is convexification of $(L+\Phi)$ and, consequently, is a convex continuous function $\mathbf{R}^n \to \mathbf{R}\cup\{\infty\}$, cf. [ET].

Since the inequality $\langle G; \mu\rangle \geq G(A)$ holds for each convex function $G: \mathbf{R}^n \to \mathbf{R}\cup\{\infty\}$ and each probability measure μ centered at A, cf. [ET], we infer that

$$\langle L+\Phi; \nu\rangle \geq \langle (L+\Phi)^{**}; \nu\rangle \geq (L+\Phi)^{**}(A)$$

and the inequality (4.2) follows.

To complete the proof we have to show that ν can be generated as a gradient L-Young measure by the gradients $D\phi_k$ of a sequence $\phi_k \in \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$ with $||\text{dist}(D\phi_k, \text{supp}\,\nu)||_{L^\infty} \to 0$. To do this it is enough to notice that by the proved above ν is a gradient L_j-Young measure for each continuous integrand $L_j : \mathbf{R}^n \to \mathbf{R}\cup\{\infty\}$, $j \in \mathbf{N}$, such that $L_j \geq L$ a.e. and L_j is bounded in $1/j$-neighborhood of supp ν, $L_j = \infty$ away of $2/j$-neighborhood of supp ν. Standard diagonalization arguments complete the proof. □

5. – On some applications of Young measure theory

In this section we will discuss how different can be various classes of gradient Young measures and how this issue is connected with applications to lower semicontinuity and relaxation of variational functionals. We will also touch the topic of passing from homogeneous to nonhomogeneous cases.

A standard example of a functional coming from Elasticity is

$$u \to J(u) := \int_\Omega |\text{det}Du(x)|dx. \tag{5.1}$$

The functional has a number of interesting features. The function $|\det(\cdot)|$ is quasiconvex (see Introduction or (5.5) below for the definition of quasiconvexity) and, since

$$|\text{det}Du| \leq |Du|^n,$$

it is also sequentially weak lower semicontinuous in $W^{1,n}(\Omega; \mathbf{R}^n)$ (see [AF]), i.e. the weak convergence $u_k \rightharpoonup u_0$ in $W^{1,n}(\Omega; \mathbf{R}^n)$ implies the inequality

$$\liminf_{k\to\infty} J(u_k) \geq J(u_0). \tag{5.2}$$

In fact one can prove stronger facts involving Young measures. Note that, if $(\nu_x)_{x\in\Omega}$ is a Young measure generated by Du_k, then

$$\liminf_{k\to\infty} J(u_k) \geq \int_\Omega \langle |\det(\cdot)|; \nu_x\rangle dx, \tag{5.3}$$

see e.g. [Ba], where the result is proved for arbitrary Caratheodory integrands. Moreover, due to Kinderlehrer and Pedregal, cf. [KP2], [KP3], ν_x is a homogeneous gradient n-Young measure centered at $Du_0(x)$ for a.a. $x \in \Omega$. Therefore quasiconvexity of the function $A \to |\det A|$ implies

$$\langle|\det(\cdot)|; \nu_x\rangle \geq |\det(\langle\cdot; \nu_x\rangle)| = |\det Du_0(x)| \text{ a.e. in } \Omega. \tag{5.4}$$

The inequalities (5.3) and (5.4) imply the inequality (5.2). In fact a general way to use Young measures to prove the lower semicontinuity is to apply these inequalites, of course with $|\det(\cdot)|$ replaced by an integrand of interest.

However properties of the functional J change if we consider a wider space, i.e. $W^{1,p}(\Omega; \mathbf{R}^n)$ with $p < n$. Ball & Murat [BM, Ch.7] constructed examples of sequences $u_k \in Id + W_0^{1,p}(\Omega; \mathbf{R}^n)$ such that both $u_k \rightharpoonup Id$ in $W^{1,p}$ for any $p < n$ and $J(u_k) \to 0$. Without loss of these properties one can assume that Du_k generates a homogeneous Young measure ν (just use Lemmata 2.2 and 2.3). It is clear that ν can not be a $|\det|$-gradient Young measure since in this case we could apply (5.3) and (5.4) to show the lower semicontinuity, i.e. that

$$\liminf_{k\to\infty} J(u_k) \geq J(Id) = \text{meas } \Omega > 0.$$

This shows that in general classes of gradient Young measures associated with a given integrand (here and later on "associated with L" means that the property of convergence in energy holds, i.e. $L(Du_k) \rightharpoonup \langle L; \nu\rangle$ in L^1) and generated by the gradients of sequences from different Sobolev spaces are different.

In case we take integral functionals with the integrands having the form

$$L(Du) = \mu|Du|^p + |\det Du|$$

the example above also says that if $\mu > 0$ is sufficiently small then gradient Young measures associated with L and generated by Sobolev functions $u \in Id + W_0^{1,p}(\Omega; \mathbf{R}^n)$ are different when $p \geq 3$ and $p < 3$. As we see in case $p < 3$ the inequality

$$\langle L(\cdot); \nu\rangle \geq L(\langle\cdot; \nu\rangle)$$

fails for gradient Young measures generated by Sobolev functions in spite the function $L : A \to |\det(A)|$ is quasiconvex, i.e.

$$\int_\Omega L(A + D\phi(x))dx \geq L(A) \text{ meas } \Omega, \ \forall\phi \in \tilde{C}_0^\infty(\Omega; \mathbf{R}^n). \tag{5.5}$$

The reason is that we can not extend the class $\phi \in \tilde{C}_0^\infty(\Omega; \mathbf{R}^n)$ to the whole $W_0^{1,p}(\Omega; \mathbf{R}^n)$ in the last inequality.

Note also that the general scheme to apply Young measure theory to show that quasiconvexity implies the lower semicontinuity, i.e. to apply (5.3) and (5.4), depends on the fact whether the actions of those Young measures which are generated by the gradients of functions u_k bounded in energy, i.e. with

$J(u_k) < c, \forall k \in \mathbf{N}$, are the same as actions of L-gradient Young measures. In the later case we would immediately derive that quasiconvexity of L is both necessary and sufficient requirement for the lower semicontinuity of the integral functional. However this property can fail even for very regular u_k. To construct counterexamples one can again use the integrand $A \to |\det A|$. A result of the paper [M] says that for each $p < n-1$ there exists a sequence of dipheomorphisms u_k such that $u_k \rightharpoonup Id$ in $W^{1,p}(\Omega; \mathbf{R}^n)$, but

$$\lim_{k\to\infty} \int_\Omega |\det(Du_k(x))| dx < \text{meas } \Omega.$$

This shows that in case of functionals with the integrands

$$L(Du) = \mu |Du|^p + |\det Du|$$

with sufficiently small $\mu > 0$ and $p < n-1$ the class of Young measures generated by the gradients of sequences bounded in energy is wider then L-gradient Young measures. This does not allow to apply the inequality (5.4). However a number of interesting classes, when this phenomena does not occur and the Young measure techniques can be applied both for deriving results on lower semicontinuity and relaxation, were indicated in [KP1-3], [S1-S4]: see [KP2-3] and [S1-2] for the case of integrands with p-growth, see [S3] for the case of integrands compatible with some convex functions having sufficiently fast growth at infinity, and see [S4] for integrands L with $L \geq \alpha |\cdot|^p + \gamma$, where $\alpha > 0$, $p > n-1$, in the scalar case. To clarify which other classes of integrands have this property is an interesting open problem. One of the most interesting classes to study seems to be isotropic problems from Elasticity, i.e. when $L : \mathbf{R}^{n\times n} \to \mathbf{R}$ is such that

$$L(A) = L(QAR), \ \forall Q, R \in SO(n),$$

$$L(A) \to \infty \text{ as } \det(A) \to 0,$$

$$L(A) = \infty \text{ if } \det(A) \leq 0,$$

and L has certain growth from below to prevent occurrence of essential discontinuites in deformations (in order the bulk term to be equal to the total energy), see e.g. [MQY], [Sv3].

Note also that all the issues discussed above play a similar role in relaxation. In fact, we need to define the integrand of the relaxed energy in such a way that its value on a linear function is exactly the infimum of limits of the values of the original functional assumed on sequences converging weakly to the linear function (and equal to it at the boundary). Therefore we have to define the value of the integrand $\tilde{L}$ of the relaxed energy at A as the infimum of the values $\langle L; \nu \rangle$ in the class of gradient Young measures ν centered at A and *associated with* L, i.e. ν have the property

$$L(Du_k) \rightharpoonup \langle L; \nu \rangle, \ k \to \infty, \tag{5.6}$$

where $Du_k \in l_A + W_0^{1,1}(\Omega; \mathbf{R}^m)$ generate ν. Hence the class of Young measures we have to employ is either gradient L-Young measures or those measures which arizes from the gradients of Sobolev functions with the property (5.6). In fact an intermediate class, e.g. the measures generated by the generalized piece-wise affine functions, can be also involved since (5.6) still holds.

Again an ill-possed situation is when infimum of $\langle L; \nu \rangle$ in a wider class of Young measures, which is the class of measures arising from the gradients of functions u_k with finite energy (i.e. $J(u_k) \le c < \infty$, $\forall k \in \mathbf{N}$) and such that $u_k \rightharpoonup l_A$, is lower than $\tilde{L}(A)$. We can not recover the action of such a Young measure by the values of the functional on sequences generating them. However it is not excluded, at least in the general case (H1), that the limit of the values of the relaxed functional $\tilde{J}$ on such a sequence is equal (or is sufficiently close) to the action of the Young measure on $\tilde{L}$. Therefore, if the value $\tilde{J}(l_A)$ is recoverable by the values of J, then $\tilde{J}$ is not lower semicontinuous.

The last point we have to mention is that some difficulties also exist in passing from homogeneous to nonhomogeneous cases. There is a simple way to construct nonhomogeneous gradient Young measures as a limit of piece-wise homogeneous measures. The point is that Young measures $(\nu_x)_{x\in\Omega}$ are just measurable functions

$$\nu : \Omega \to (M, \rho) \ \text{(see (1.3) for the definition of } \rho).$$

Moreover the Lusin property of such functions allow, given $j \in \mathbf{N}$, to decompose Ω into subsets Ω_k^j, $k \in \mathbf{N}$, with diameters less then $1/j$ and such that for some $x_k^j \in \Omega_k^j$

$$\int_{\Omega_k^j} \rho(\nu_x, \nu_{x_k^j})dx \le (1/j)\,\text{meas}\ \Omega_k^j.$$

The form of the metric ρ implies that if $\nu_x^j := \nu_{x_k^j}$, $x \in \Omega_k^j$, then $(\nu_x^j)_{x\in\Omega}$ generates $(\nu_x)_{x\in\Omega}$ as $j \to \infty$. The measures $(\nu_x^j)_{x\in\Omega}$ are piece-wise constant and can be also selected in such a way that

$$\langle L; \nu_{(\cdot)}^j \rangle \rightharpoonup \langle L; \nu_{(\cdot)} \rangle,\ j \to \infty,\ \text{in } L^1,$$

see e.g. [S2] or [S3] for all these properties.

Therefore in case of functions $u_0 \in W^{1,1}(\Omega; \mathbf{R}^m)$, which are piece-wise affine in the standard or generalized senses (see Remark 2), we can generate those measures $(\nu_x)_{x\in\Omega}$ which are composed by gradient homogeneous Young measures $\nu_x, x \in \Omega$, associated with L (i.e. with the property (5.6)) and with the centers of mass at $Du_0(x)$, $x \in \Omega$, by the gradients of perturbations $u_j \in u_0 + W_0^{1,1}(\Omega; \mathbf{R}^m)$ of the function u_0. Moreover in this case we also preserve the property of convergence in energy, i.e.

$$L(Du_j) \rightharpoonup \langle L; \nu_{(\cdot)} \rangle \text{ in } L^1.$$

However the main difficulty is to adjust the measures $(\nu_x^j)_{x\in\Omega}$ to Sobolev functions. One of the possibilities is to find a standard or generalized piecewise affine approximation u_j of the function u_0 with $J(u_j) \to J(u_0)$. We used this approximation in [S3] and [S4]. Another possibility is to modify $(\nu_x^j)_{x\in\Omega}$ to adjust the centers of mass of ν_x^j to $Du_0(x)$, $x \in \Omega$, see [S2]. However in the later case we need extra properties of L to assert that the modified measures still have finite actions on L and that their actions converge to the action of the original measure. Still the issue to switch from the case of linear functions u_0 associated with homogeneous measures to the general case remains nontrivial. However there is a hope to use extra regularity of minimizers of relaxed problems, like in [S4, § 5].

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