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Blow-up Results for Nonlinear Hyperbolic Inequalities

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Abstract. We study the nonexistence of global weak solutions for equations and systems of the following types (I) $\partial_{tt}u \geq L_m u^p + |u|^q$ and (II) $\partial_{tt}u \geq L_{m_1}u^{p_1} + |v|^{q_1}$ & $\partial_{tt}v \geq L_{m_2}v^{p_2} + |u|^{q_2}$, where the operators L_m and L_{m_i} are homogeneous linear partial differential operators of order $2m$ and $2m_i$. The method relies on a suitable choice of test functions, rescaling techniques and a dimensional analysis.

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1. – Introduction

The study of the existence (or the non-existence) of global solutions to semilinear wave equations has been initiated in the seventies and intensively developed since. The following equation can be considered as the simplest model case

$$\partial_{tt}u - \Delta u = f(u) \tag{1.1}$$

in $\mathbb{R}^N \times \mathbb{R}_+$ where f is usually a continuous real-valued function. There exists a wide class of nonlinearities f for which the Cauchy Problem for (1.1), that is the local solvability in time of (1.1) with given initial data $u(.,0) = u_0$ and $\partial_t u(.,0) = u_1$, is well posed. Defining $F(r) = \int_0^r f(s)ds$, the first observation is that the energy function of a solution u

$$E(u) = \int_{\mathbb{R}^N} \left((\partial_t u)^2 + |\nabla u|^2 - 2F(u)\right)dx \tag{1.2}$$

is independent of time. Therefore a classical approach to prove global existence relies on the study of the energy. In the cases where the nonlinear term $\int_{\mathbb{R}^N} F(u)dx$ can be kept under control by the quadratic term $\int_{\mathbb{R}^N} |\nabla u|^2 dx$, the

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global solvability occurs. In other cases, this is not possible. For example, if f satisfies

$$rf(r) \leq (2+\varepsilon)F(r) \qquad (\forall r \in \mathbb{R}) \tag{1.3}$$

for some $\varepsilon > 0$, Levine [Le] proved that no solution can exist for all time when the energy is negative. Levine's result proof is based on deriving a nonlinear second order inequality that is satisfied by $\|u(t)\|_{L^2}^2$. Another approach for the non-existence of global solutions has been initiated by John and Kato. It is based on an averaging method for positive solutions, usually with compact and propagating support. Much has been devoted to the case of the equation

$$\partial_{tt}u - \Delta u = |u|^q \,. \tag{1.4}$$

Defining $q_{c,N}$ as the positive root of

$$(N-1)\rho^2 - (N+1)\rho - 2 = 0\,, \tag{1.5}$$

John first proved in [Jo1] that when $1 < q \leq q_{c,N}$ there exist smooth initial data, arbitrarily small in $C_c^\infty(\mathbb{R}^N)$ such that no corresponding global solution exists. Actually, when $N = 3$, John's result states that when $1 < q < q_{c,2} = 1 + \sqrt{2}$ all solutions with $C_c^\infty(\mathbb{R}^3)$-initial data blow-up in finite time. Later on the critical value $q = q_{c,2}$ was included in Glassey's proof [Gl2] under the additional assumption that the initial values u_0 and u_1 have both positive average. Glassey's technique was to derive differential inequalities which are satisfied by the function $t \mapsto \int_{\mathbb{R}^N} u(x,t)dx$. Based on these facts Strauss proposed the general conjecture that, when $N \geq 1$, global solutions of (1.4) always exist, provided the initial data are small enough in $C_c^\infty(\mathbb{R}^3)$ and $q > q_{c,N}$. Some other cases were considered by Sideris [Si], and Shaeffer [Sh] who gave extensions of Glassey's results for different dimensions, always with compactly supported initial data, and with positive average. Many works have been devoted to this conjecture which is, up to now, verified if $N \leq 8$ ([LS], [GLS], [Ku]). A slightly less sharp result under much weaker assumptions was obtained by Kato [Ka] with a much easier proof. In particular Kato pointed out the role of the exponent $q = q_0 = (N+1)/(N-1)$ in order to have more general initial data, but still with compact support. The fact that the support of $u(.,t)$ is included in a cone $\{x : |x| \leq t + R\}$ plays a fundamental role in deriving the differential inequalities. Up to now, if $N > 8$, the exponent q_0 is the critical one under which global solutions cannot exist. A comprehensive presentation of these results can be found in [Jo2], [LS] and [St].

In this paper we prove the non-existence of global weak solutions of a very wide class of nonlinear hyperbolic type inequalities of the following type

$$\partial_{tt}u \geq L_m(\varphi_p(u)) + |u|^q \,, \tag{1.6}$$

where φ_p is a locally bounded real-valued function which satisfies

$$|\varphi_p(r)| \le c|r|^p \qquad (\forall r \in \mathbb{R}), \tag{1.7}$$

for some $c > 0$ and $p > 0$, and where $L_m\zeta = \sum_{|\alpha|=m} D^\alpha(a_\alpha(x,t)\zeta)$ is an homogeneous differential operator of order m in which the a_α are merely bounded and measurable functions. By an adequate choice of test functions, rescaling techniques, and a sharp dimensional analysis we prove that there exist no weak solution of (1.6) defined in $\mathbb{R}^N \times \mathbb{R}_+$ with $\int_{\mathbb{R}^N} \partial_t u(x,0)dx \ge 0$ if $q > \max(1,p)$ and

(i) either $2N - m \le 0$, or
(ii) $2N - m > 0$ and $N\frac{q-p}{q+1} \le \frac{m}{2}$.

In opposition with the above mentioned results no assumption on the sign of the average of u_0 (which may not be integrable), or on the support of the solutions is made. In the particular case of the inequality

$$\partial_{tt}u \ge L_2 u + |u|^q, \tag{1.8}$$

our conditions read:

(i) either $N = 1$, or
(ii) $N > 1$ and $1 < q \le \frac{N+1}{N-1}$.

By computing an explicit global solution of (1.8) in the case where $L_2 = \Delta$ and $q > (N+1)/(N-1)$, we prove that our results are sharp in the class of weak solutions. We also give variant of this result when the operator is no longer homogeneous, or have different orders of differentiation with respect to the various variables $(x_1, \dots, x_N)$, or even has unbounded or vanishing coefficients a_α when $|x|^2 + t \to \infty$. An important fact to notice is that the operator L_m is not of any specified type. We note, as far as we know, the first result on the non-existence of global positive solution in $\mathbb{R}^N$ of semilinear equations with linear differential operator with arbitrary type are due to Eidelman and Kondratiev [EK]. In the different context of systems of quasilinear inequalities, a new series of non-existence results for positive solutions have been recently obtained by one of the authors (S. Pohozaev) and E. Mitidieri ([MP1], [MP2]). As in many previous papers their approach is based on sharp energy techniques (multiplication by suitable powers of the solutions) and absorption of those terms by the source zero-order terms via a capacity type estimate.

By using similar arguments, we also prove non-existence results for systems of the type

$$\begin{cases} \partial_{tt}u \ge L_{m_1}\left(\varphi_{p_1}(u)\right) + |v|^{q_1} \\ \partial_{tt}v \ge L_{m_2}\left(\varphi_{p_2}(v)\right) + |u|^{q_2} \end{cases} \tag{1.9}$$

where $q_i > 1$, $L_{m_i}\zeta = \sum_{|\alpha|=m_i} D^\alpha(a_{i,\alpha}(x,t)\zeta)$ and the φ_i satisfy

$$|\varphi_{p_i}(r)| \le c|r|^{p_i} \qquad (\forall r \in \mathbb{R}) \tag{1.10}$$

for some p_i. In that case, the type of results that we obtain is the natural extension of the previous ones on the simple inequality, in the continuation of Kato's work, while, in a different spirit, an extension of John's results has been recently obtained by Del Santo, Georgiev and Mitidieri [DGM], [DM]. We can also handle non-diagonal systems such as

$$\begin{cases} \partial_{tt} u \geq L_{m_1}\big(\varphi_{p_1}(v)\big) + |v|^{q_1} \\ \partial_{tt} v \geq L_{m_2}\big(\varphi_{p_2}(u)\big) + |u|^{q_2} \end{cases} \tag{1.11}$$

or of mixed type hyperbolic-parabolic

$$\begin{cases} \partial_{tt} u \geq L_{m_1}\big(\varphi_{p_1}(u)\big) + |v|^{q_1} \\ \partial_{t} v \geq L_{m_2}\big(\varphi_{p_2}(v)\big) + |u|^{q_2} \end{cases} \tag{1.12}$$

or hyperbolic-elliptic

$$\begin{cases} \partial_{tt} u \geq L_{m_1}\big(\varphi_{p_1}(u)\big) + |v|^{q_1} \\ 0 \geq L_{m_2}\big(\varphi_{p_2}(v)\big) + |u|^{q_2}. \end{cases} \tag{1.13}$$

Again the main point is the fact that the proof heavily relies on a dimensional analysis.

Our paper is organised as follows:

1 – Introduction
2 – Non-existence for equations
3 – Non-existence for hyperbolic systems
4 – Non-existence for systems of mixed type
5 – References

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2. – Non-existence for equations

Let L_m be the homogeneous differential operator of order m defined by

$$L_m \zeta = \sum_{|\alpha|=m} D^\alpha\big(a_\alpha(x,t)\zeta\big), \tag{2.1}$$

where the a_α are bounded and measurable functions defined in $\mathbb{R}^N \times \mathbb{R}_+ = \mathbb{R}_+^{N+1}$ and m is a positive integer, and φ_p a locally bounded real valued function which satisfies for some $p > 0$

$$|\varphi_p(r)| \leq c|r|^p \qquad (\forall r \in \mathbb{R}). \tag{2.2}$$

DEFINITION 1. A weak solution u of the differential inequality

$$\partial_{tt}u \geq L_m(\varphi_p(u)) + |u|^q \tag{2.3}$$

on $\mathbb{R}_+^{N+1}$ with initial data $u(.,0) = u_0(.)$ and $\partial_t u(.,0) = u_1(.)$ belonging to $L^1_{\text{loc}}(\mathbb{R}^N)$, is a locally integrable function such that $u \in L^q_{\text{loc}}(\mathbb{R}_+^{N+1}) \cap L^p_{\text{loc}}(\mathbb{R}_+^{N+1})$ which satisfies

$$\begin{aligned} &-\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx + \int_0^\infty\int_{\mathbb{R}^N} u\partial_{tt}\zeta\, dx\, dt \\ &\leq \int_0^\infty\int_{\mathbb{R}^N} \varphi_p(u)L_m^*\zeta\, dx\, dt + \int_0^\infty\int_{\mathbb{R}^N} |u|^q\zeta\, dx\, dt \end{aligned} \tag{2.4}$$

for any $\zeta \in C_c^\infty(\mathbb{R}_+^{N+1})$, $\zeta \geq 0$, where $L_m^*\zeta = (-1)^m \sum_{|\alpha|=m} a_\alpha(x,t)D^\alpha\zeta$.

THEOREM 1. *Assume that $q > \max(1,p)$. Then there exists no solution u of inequality* (2.3) *defined on $\mathbb{R}^N \times \mathbb{R}_+$ such that $\int_{\mathbb{R}^N} u_1\, dx \geq 0$, if one of the following assumptions is fulfilled:*

(i) *either $2N - m \leq 0$, or*
(ii) *$2N - m > 0$ and $N \leq \frac{m}{2}\frac{q+1}{q-p}$.*

PROOF. Let u be such a weak solution and ζ be a smooth nonnegative test function. From (2.4) we get

$$\begin{aligned} &\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_0^\infty\int_{\mathbb{R}^N} |u|^q\zeta dx\, dt \\ &\qquad \leq \int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx + \int_0^\infty\int_{\mathbb{R}^N} u\partial_{tt}\zeta\, dx\, dt \\ &\qquad\quad - \int_0^\infty\int_{\mathbb{R}^N} \varphi_p(u)L_m^*\zeta\, dx\, dt\,. \end{aligned} \tag{2.5}$$

If ζ is chosen such that

$$\int_0^\infty\int_{\mathbb{R}^N} \left(|\partial_{tt}\zeta|^{q'}\zeta^{1-q'} + |L_m^*\zeta|^{q/(q-p)}\zeta^{p/(q-p)}\right)dx\, dt < \infty\,, \tag{2.6}$$

where $q' = q/(q-1)$, then

$$\begin{aligned} \int_0^\infty\int_{\mathbb{R}^N} u\partial_{tt}\zeta dx\, dt &\leq \left(\int_0^\infty\int_{\mathbb{R}^N} |u|^q\zeta dx\, dt\right)^{1/q} \left(\int_0^\infty\int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q'}\zeta^{1-q'} dx\, dt\right)^{1/q'}, \\ &\leq \frac{1}{4}\int_0^\infty\int_{\mathbb{R}^N} |u|^q\zeta dx\, dt + C_1\int_0^\infty\int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q'}\zeta^{1-q'} dx\, dt, \end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
(2.8)\quad & -\int_0^\infty\int_{\mathbb{R}^N} \varphi_p(u)L_m^*\zeta\,dx\,dt \\
& \leq c\left(\int_0^\infty\int_{\mathbb{R}^N} |u|^q\zeta\,dx,dt\right)^{p/q}\left(\int_0^\infty\int_{\mathbb{R}^N} |L_m^*\zeta|^{q/(q-p)}\zeta^{-p/(1-p)}dx\,dt\right)^{(q-p)/q}, \\
& \leq \frac{1}{4}\int_0^\infty\int_{\mathbb{R}^N} |u|^q\zeta dx\,dt + C_2\int_0^\infty\int_{\mathbb{R}^N} |L_m^*\zeta|^{q/(q-p)}\zeta^{-p/(q-p)}dx\,dt\,.
\end{aligned}
$$

We assume now that ζ is also chosen such that

$$
\int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx = 0\,. \tag{2.9}
$$

Then (2.5)-(2.9) imply

$$
\begin{aligned}
(2.10)\quad & \int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \frac{1}{2}\int_0^\infty\int_{\mathbb{R}^N} |u|^q\zeta\,dx\,dt \\
& \qquad \leq C_1\int_0^\infty\int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q'}\zeta^{1-q'}dx\,dt \\
& \qquad\quad + C_2\int_0^\infty\int_{\mathbb{R}^N} |L_m^*\zeta|^{q/(q-p)}\zeta^{-p(q-p)}dx\,dt\,.
\end{aligned}
$$

Now we take $\zeta(x,t) = \varphi(\frac{t^\kappa+|x|^\mu}{R^2})$ where $\varphi \in C_c^\infty(\mathbb{R}_+)$ satisfies $0 \leq \varphi \leq 1$ and

$$
\varphi(r) = \begin{cases} 0 & \text{if } r \geq 2\,, \\ 1 & \text{if } 0 \leq r \leq 1\,, \end{cases} \tag{2.11}
$$

R is a positive parameter, while $\kappa > 1$ and $\mu > 0$ will be determined later on.

Since $\partial_t\zeta(x,t) = \kappa t^{\kappa-1}R^{-2}\varphi'(\frac{t^\kappa+|x|^\mu}{R^2})$ estimate (2.9) holds. In order to estimate the right-hand side of (2.10) we consider the change of variables

$$
\begin{cases} R^{-2}t^\kappa = \tau^\kappa\,, \\ R^{-2}|x|^\mu = |y|^\mu\,, \end{cases} \iff \begin{cases} t = R^{2/\kappa}\tau\,, \\ x = R^{2/\mu}y\,. \end{cases} \tag{2.12}
$$

Denoting $\Omega = \{(y,\tau) \in \mathbb{R}^N\times\mathbb{R}_+ : \tau^\kappa+|y|^\mu \leq 2\}$ and setting $\rho(y,\tau) = \tau^\kappa+|y|^\mu$, there holds

$$
\begin{aligned}
(2.13)\quad & \int_0^\infty\int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q'}\zeta^{1-q'}dx\,dt \\
& = R^{-4q'/\kappa+2/\kappa+2N/\mu}\iint_\Omega\left(|\partial_{\tau\tau}\varphi\circ\rho|^{q'}(\varphi\circ\rho)^{1-q'}\right)dy\,d\tau,
\end{aligned}
$$

and (by a straghtforward computation)

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} |L_m^*\zeta|^{q/(q-p)}\zeta^{-p/(q-p)}dx\,dt \\
&= R^{-2mq/(\mu(q-p))+2/\kappa+2N/\mu}\iint_\Omega\Big(|L_m^*(\varphi\circ\rho)|^{q/(q-p)}(\varphi\circ\rho)^{-p/(q-p)}\Big)dy\,d\tau.
\end{aligned}
\tag{2.14}
$$

Therefore there exist two positive constants C_3 and C_4 such that

$$
\begin{aligned}
&2\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_0^\infty\int_{\mathbb{R}^N}|u|^q\zeta\,dx\,dt \\
&\le C_3R^{2N/\mu+2/\kappa-4q'/\kappa} + C_4R^{2N/\mu+2/\kappa-2mq/(\mu(q-p))},
\end{aligned}
\tag{2.15}
$$

for any $R>0$. We choose κ such that $\frac{2N}{\mu}+\frac{2}{\kappa}-\frac{4q'}{\kappa} = \frac{2N}{\mu}+\frac{2}{\kappa}-\frac{2mq}{\mu(q-p)}$, or equivalently,

$$
4/\kappa = 2m(q-1)/(\mu(q-p))\,. \tag{2.16}
$$

Such a choice gives a common value α of the exponents of R in (2.15), namely

$$
\alpha = \frac{1}{\mu}\left(2N - \frac{m(q+1)}{(q-p)}\right). \tag{2.17}
$$

The sign of (2.17) does not depend on $\mu>0$ while the condition $\kappa>1$ is then equivalent to $m(q-1)/(q-p) < 2\mu$. This is insured by taking μ large enough.

If $\alpha<0$, the right-hand side of (2.15) goes to 0 when R goes to infinity, while the left-hand side converges to $\int_{\mathbb{R}^N} u_1dx + \int_0^\infty\int_{\mathbb{R}^N}|u|^q dx\,dt$. Clearly this implies that u cannot exist.

If $\alpha=0$, then $\int_0^\infty\int_{\mathbb{R}^N}|u|^q dx\,dt<\infty$. We return to inequality (2.5), which actually reads

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^N}|u|^q\zeta\,dx\,dt + \int_{\mathbb{R}^N} u_1dx \\
&\le \iint_{R^2\le t^\kappa+|x|^2\le 2R^2} u\partial_{tt}\zeta\,dx\,dt - \iint_{R^2\le t^\kappa+|x|^2\le 2R^2}\varphi_p(u)L_m^*\zeta\,dx\,dt \\
&\le \left(\iint_{R^2\le t^\kappa+|x|^2\le 2R^2}|u|^q dx\,dt\right)^{1/q}\left(\iint_{1\le\tau^\kappa+|y|^2\le 2}|\partial_{tt}\varphi|^{q'}\varphi^{1-q'}dy\,d\tau\right)^{1/q'} \\
&\quad + \left(\iint_{R^2\le t^\kappa+|x|^2\le 2R^2}|u|^q\zeta\,dx\,dt\right)^{p/q} \\
&\qquad \times\left(\iint_{1\le\tau^\kappa+|y|^2\le 2}|L_m^*\varphi|^{q/(q-p)}\varphi^{-p/(q-p)}dy\,d\tau\right)^{(q-p)/q}
\end{aligned}
\tag{2.18}
$$

by using the fact that the exponent α of R in (2.13), (2.14) vanishes. But $\int_0^\infty \int_{\mathbb{R}^N} |u|^q dx\, dt < \infty$ implies that $\lim_{R\to\infty} \iint_{R^2 \le t^\kappa + |x|^2 \le 2R^2} |u|^q dx\, dt = 0$. This infers that $\int_0^\infty \int_{\mathbb{R}^N} |u|^q dx\, dt = 0$.

Now the assumption $\alpha \le 0$ means

$$N \le \frac{m}{2} \frac{q+1}{q-p} \iff (2N-m)q \le 2Np + m\,. \tag{2.19}$$

It must be noticed that if $2N - m \le 0$, then (2.19) is obviously fulfilled. □

REMARK 1. The integrability assumption on u_1 can be relaxed and the sign condition replaced by the following weaker one:

$$\limsup_{R\to\infty} \int_{\mathbb{R}^N} u_1(x)\varphi(|x|^2/R^2)dx \in [0,\infty]\,.$$

REMARK 2. The positive sign assumption on u_1 can hardly be avoided since for any $\delta > 0$ there exists a unique positive function z satisfying

$$\frac{d^2 z}{dt^2} = |z|^q \quad \text{on} \quad [0,\infty)\,, \quad \text{and} \quad z(0) = \delta\,. \tag{2.20}$$

For such a function there always holds $z'(t) < 0$ $(\forall t \ge 0)$ and particular $z'(0) = \gamma(\delta) < 0$. Therefore the Cauchy problem associated with equation (2.20) and initial data $z(0) = \delta$, and $z'(0) = \gamma(\delta)$ has a global solution defined on $\mathbb{R}_+$.

The sign of the derivative of a solution at initial time is important since it is easily shown that, for this inequality, there exists some q satisfying $1 < q < (N+1)/(N-1)$ and a positive global solution u of

$$\partial_{tt} u \ge \Delta u + u^q \tag{2.21}$$

in $\mathbb{R}_+^{N+1}$ under the form

$$u(x,t) = A(t+t_0)^\alpha \left((t+t_0)^2 + |x|^2\right)^\beta,$$

for $t_0 > 0$ and some $A > 0$, α and β such that $\alpha + 2\beta = -2/(q-1)$, $-2/(q-1) < \beta < 0$, provided there holds

$$2\beta(\beta-1)\xi^2 + \beta\xi\eta(2\alpha - 2\beta + 3 - N) + 2\alpha(\alpha-1)\eta^2 > 0 \quad (\forall(\eta,\xi) : \xi^2 + \eta^2 > 0)\,.$$

However the derivative of this solution is negative for $t = 0$.

The next result is an immediate consequence of Theorem 1.

COROLLARY 1. *Assume that* $q > 1$ *and that the* $a_{i,j}$ *are measurable uniformly bounded functions in* $\mathbb{R}_+^{N+1}$. *Then, if* $1 < q \leq (N+1)/(N-1)$, *there exists no weak solution u of*

$$\partial_{tt} u \geq \sum_{i,j=1}^{N} \partial_{x_i, x_j}\big(a_{i,j}(x,t)u\big) + |u|^q \quad in \quad \mathbb{R}_+^{N+1}, \tag{2.22}$$

such that $\int_{\mathbb{R}^N} \partial_t u(x,0)dx \geq 0$.

REMARK 3. The results concerning the nonlinear wave equations (1.4) ([Jo1], [G], [Sc], [Si]) with compactly supported initial data give a larger upper limit for the non-existence of global solutions. However their assumptions are more restrictive since they also need $\int_{\mathbb{R}^N} u_0 dx \geq 0$ and $\int_{\mathbb{R}^N} u_1 dx \geq 0$. Kato's result [Ka] is closer to our since $\int_{\mathbb{R}^N} u_1 dx \geq 0$. If we analyse Kato's proof, the key ingredient is the fact that for any $t \geq 0$ the solutions have a compact support in a ball which propagates at constant speed. The type of the differential operator is not fundamental in his proof.

The next result points out the sharpness of our results when we only deal with weak solutions without any compact support assumption.

THEOREM 2. *Let* $N > 1$ *and* $q > (N+1)/(N-1)$. *Then there exists a positive weak solution u of*

$$\partial_{tt} u \geq \Delta u + |u|^q \quad in \quad \mathbb{R}_+^{N+1} \tag{2.23}$$

defined on $\mathbb{R}_+^{N+1}$ *and such that* $\int_{\mathbb{R}^N} \partial_1 u(x,0)dx \geq 0$.

PROOF. Let $v(x,t) = s^\lambda - s_0^\lambda$ with $s = 2^{-1}(|x|^2 - t^2) + s_0$, $\lambda = -1/(q-1)$ and $s_0 > 0$. A straightforward computation yields

$$\partial_{tt} v - \Delta v = -\lambda(2\lambda + N - 1)s^{\lambda-1} + 2\lambda(\lambda-1)s_0 s^{\lambda-2} \tag{2.24}$$

in $C_+ = \{(x,t) \in \mathbb{R}_+^{N+1} : |x|^2 > t^2\}$. If $q > (N+1)/(N-1)$, then $(N-1)q - (N+1) > 0$. Clearly the function $\tilde{u} = Av$ satisfies

$$\begin{aligned} \partial_{tt}\tilde{u} - \Delta\tilde{u} - \tilde{u}^q &= As^{\lambda-1}\frac{q(N-1)-(N+1)}{(q-1)^2} - A^q(s^\lambda - s_0^\lambda)^q + \frac{2q}{(q-1)^2}As_0 s^{\lambda-2}, \\ &\geq As^{\lambda-1}\left(\frac{q(N-1)-(N+1)}{(q-1)^2} - A^{q-1}\right) + \frac{2q}{(q-1)^2}As_0 s^{\lambda-2}. \end{aligned} \tag{2.25}$$

If we take $0 < A < A_{N,q} = \left(\frac{(N-1)q-(N+1)}{(q-1)^2}\right)^{1/(q-1)}$, there holds

$$\partial_{tt}\tilde{u} - \Delta\tilde{u} \geq \tilde{u}^q \quad \text{in} \quad C_+. \tag{2.26}$$

We define u in $\mathbb{R}_+^{N+1}$ by setting

$$u(x,t) = \begin{cases} \tilde{u}(x,t) & \text{if } (x,t) \in C_+, \\ 0 & \text{if } (x,t) \in \mathbb{R}_+^{N+1} \setminus C_+, \end{cases} \tag{2.27}$$

then $u \in L^q_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ since it is bounded. Denoting $C_- = \mathbb{R}^{N+1}_+ \setminus \overline{C}_+$, then

$$
\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} \big(u(\partial_{tt}\zeta - \Delta\zeta) - u^q\zeta\big)dx\,dt \\
(2.28)\quad &= \int_{C_+} \big(\tilde{u}(\partial_{tt}\zeta - \Delta\zeta) - \tilde{u}^q\zeta\big)dx\,dt + \int_{C_-} \big(u(\partial_{tt}\zeta - \Delta\zeta) - u^q\zeta\big)dx\,dt\,, \\
&= \int_{C_+} \big(\tilde{u}(\partial_{tt}\zeta - \Delta\zeta) - \tilde{u}^q\zeta\big)dx\,dt\,,
\end{aligned}
$$

for any $\zeta \in C^\infty_c(\mathbb{R}^{N+1}_+)$ such that $\zeta \geq 0$. Because u is radial with respect to x, we can assume that the same holds for the test function ζ. Therefore an integration by parts gives

$$
\begin{aligned}
&\int_{C_+} \zeta(\partial_{tt}\tilde{u} - \Delta\tilde{u})dx\,dt = \int_{\mathbb{R}^N}\int_0^{|x|} \zeta\,\partial_{tt}\tilde{u}\,dt\,dx - \int_{C_+} \zeta\,\Delta\tilde{u}\,dx\,dt\,, \\
(2.29)\quad &= \int_{\mathbb{R}^N}\left(\big[\zeta\,\partial_t\tilde{u}\big]_{t=0}^{|x|} - \int_0^{|x|} \partial_t\tilde{u}\partial_t\zeta dt\right)dx - \int_{C_+} \zeta\,\Delta\tilde{u}\,dx\,dt\,, \\
&= \int_{\mathbb{R}^N}\left(\big[\zeta\,\partial_t\tilde{u} - \tilde{u}\partial_t\zeta\big]_{t=0}^{|x|} + \int_0^{|x|} \tilde{u}\partial_{tt}\zeta dt\right)dx - \int_{C_+} \zeta\,\Delta\tilde{u}\,dx\,dt\,.
\end{aligned}
$$

Up to the multiplicative factor $N\omega_N = |S^{N-1}|$, which will be always forgotten, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \big[\zeta\,\partial_t\tilde{u} - \tilde{u}\partial_t\zeta\big]_{t=0}^{|x|} = \int_0^\infty \big[\zeta\,\partial_t\tilde{u} - \tilde{u}\partial_t\zeta\big]_{t=0}^{r} r^{N-1}dr\,, \\
(2.30)\quad &= \int_0^\infty (\zeta\,\partial_t\tilde{u} - \tilde{u}\partial_t\zeta)(r,r)r^{N-1}dr - \int_{\mathbb{R}^N} (\zeta\,\partial_t\tilde{u} - \tilde{u}\partial_t\zeta)(x,0)dx\,,
\end{aligned}
$$

(here we denote $\zeta(x,t) = \zeta(r,t)$ with $r > 0$) and

$$
\begin{aligned}
&\int_{C_+} \zeta\,\Delta\tilde{u}\,dx\,dt = \int_0^\infty \int_{|x|\geq t} \zeta\,\Delta\tilde{u}\,dx\,dt\,, \\
&= \int_0^\infty \int_t^\infty \zeta\,\big(\partial_r(r^{N-1}\partial_r\tilde{u})\big)dr\,dt\,, \\
(2.31)\quad &= \int_0^\infty \big[\zeta r^{N-1}\partial_r\tilde{u}\big]_t^\infty dt - \int_0^\infty\int_t^\infty \partial_r\zeta\,\partial_r\tilde{u} r^{N-1}dr\,dt\,, \\
&= \int_0^\infty \big[\zeta\,\partial_r\tilde{u} - \tilde{u}\partial_r\zeta\big]_t^\infty t^{N-1}dt + \int_0^\infty\int_t^\infty \tilde{u}\big(\partial_r(r^{N-1}\partial_r\zeta)\big)dr\,dt\,, \\
&= -\int_0^\infty \big(\zeta\,\partial_r\tilde{u} - \tilde{u}\partial_r\zeta\big)(t,t)t^{N-1}dt + \int_0^\infty\int_t^\infty \tilde{u}\big(\partial_r(r^{N-1}\partial_r\zeta)\big)dr\,dt\,.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\int_{C_+} \zeta(\partial_{tt}\tilde{u}-\Delta\tilde{u})dx\,dt \\
&= \int_{C_+} \tilde{u}(\partial_{tt}\zeta u-\Delta\zeta)dx\,dt-\int_{\mathbb{R}^N}(\zeta\partial_t\tilde{u}-\tilde{u}\partial_t\zeta)(x,0)dx \\
&\quad +\int_0^{\infty}(\zeta\partial_t\tilde{u}-\tilde{u}\partial_t\zeta)(r,r)r^{N-1}dr+\int_0^{\infty}(\zeta\partial_r\tilde{u}-\tilde{u}\partial_r\zeta)(t,t)t^{N-1}dt, \\
&= \int_{C_+} \tilde{u}(\partial_{tt}\zeta-\Delta\zeta)dx\,dt-\int_{\mathbb{R}^N}(\zeta\partial_t\tilde{u}-\tilde{u}\partial_t\zeta)(x,0)dx \\
&\quad +\int_0^{\infty}((\partial_t\tilde{u}+\partial_r\tilde{u})\zeta)(\tau,\tau)\tau^{N-1}d\tau-\int_0^{\infty}((\partial_t\zeta+\partial_r\zeta)\tilde{u})(\tau,\tau)\tau^{N-1}d\tau.
\end{aligned} \tag{2.32}
$$

From the definition of $\tilde{u}$, we have

$$\tilde{u}(\tau,\tau)=0 \quad \text{and} \quad (\partial_t\tilde{u}+\partial_r\tilde{u})(\tau,\tau)=0\,. \tag{2.33}$$

Since $\int_{C_+}\zeta(\partial_{tt}\tilde{u}-\Delta\tilde{u}-\tilde{u}^q)dx\,dt\geq 0$, we obtain

$$\int_{C_+}\tilde{u}(\partial_{tt}\zeta-\Delta\zeta)dx\,dt+\int_{\mathbb{R}^N}(\tilde{u}\partial_t\zeta-\zeta\partial_t\tilde{u})(x,0)dx\geq\int_{C_+}\zeta\tilde{u}^q dx\,dt\,. \tag{2.34}$$

It follows from (2.28) and (2.34) that

$$\int_{\mathbb{R}_+^{N+1}}u(\partial_{tt}\zeta-\Delta\zeta)dx\,dt+\int_{\mathbb{R}^N}(u\partial_t\zeta-\zeta\partial_t u)(x,0)dx\geq\int_{\mathbb{R}_+^{N+1}}\zeta u^q dx\,dt. \tag{2.35}$$

Moreover $\partial_t u(x,0)\equiv 0$. Therefore the function u is a global weak positive solution of (2.23) and it satifies $\int_{\mathbb{R}^N}\partial_t u(x,0)dx\geq 0$. □

By using the techniques developed in Theorem 1, we can handle the case of a non-homogeneous partial differential operator $\mathcal{L}$ determined by

$$\mathcal{L}\zeta=\sum_{k=\ell}^{m}L_k\zeta \tag{2.36}$$

with $\ell\in\mathbb{N}^*$ and

$$L_k\zeta=\sum_{|\alpha|=k}D^{\alpha}(a_{\alpha,k}(x,t)\zeta)\,. \tag{2.37}$$

We define a weak solution u of the differential inequality

$$\partial_{tt}u\geq\mathcal{L}(\varphi_p(u))+|u|^q\,, \tag{2.38}$$

on $\mathbb{R}_+^{N+1}$ with initial data $u(.\,,0)=u_0$ and $\partial_t u(.\,,0)=u_1(.)$ belonging to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, as a locally integrable function such that $u\in L^q_{\mathrm{loc}}(\mathbb{R}_+^{N+1})\cap L^p_{\mathrm{loc}}(\mathbb{R}_+^{N+1})$ which satisfies

$$
\begin{aligned}
&-\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx+\int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx+\int_0^\infty\int_{\mathbb{R}^N} u\partial_{tt}\zeta dx\,dt\\
&\geq\int_0^\infty\int_{\mathbb{R}^N}\varphi_p(u)\mathcal{L}^*\zeta dx\,dt+\int_0^\infty\int_{\mathbb{R}^N}|u|^q\zeta dx\,dt
\end{aligned}
\tag{2.39}
$$

for any $\zeta\in C_c^\infty(\mathbb{R}_+^{N+1})$, $\zeta\geq 0$ where

$$
\mathcal{L}^*\zeta=\sum_{\ell=1}^m(-1)^k\sum_{|\alpha|=k}a_{\alpha,k}(x,t)D^\alpha\zeta=\sum_{k=\ell}^m(-1)^kL_\ell^*\zeta\,.
$$

The main fact is that the conclusion of Theorem 1 still holds for the coefficients corresponding to the lower order term L_ℓ in $\mathcal{L}$.

THEOREM 3. *Assume that* $q>\max(1,p)$. *Then there exists no solution u of inequality* (2.38) *defined on* $\mathbb{R}^N\times\mathbb{R}_+$ *and such that* $\int_{\mathbb{R}^N}u_1\,dx\geq 0$, *if one of the following assumptions is fulfilled:*

(i) *either* $2N-\ell\leq 0$, *or*
(ii) $2N-\ell>0$ *and* $N\leq\frac{\ell}{2}\frac{q+1}{q-p}$.

PROOF. We first have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx+\int_0^\infty\int_{\mathbb{R}^N}|u|^q\zeta\,dx\,dt\\
&\quad\leq\int_{\mathbb{R}^N}u_0(x)\partial_t\zeta(x,0)dx+\int_0^\infty\int_{\mathbb{R}^N}u\partial_{tt}\zeta dx\,dt-\int_0^\infty\int_{\mathbb{R}^N}\varphi_p(u)\mathcal{L}^*\zeta\,dx\,dt
\end{aligned}
$$

for any nonnegative test function. Choosing ζ such that

$$
\int_0^\infty\int_{\mathbb{R}^N}\left(|\partial_{tt}\zeta|^{q'}\zeta^{1-q'}+|L_k^*\zeta|^{q/(q-p)}\zeta^{p/(q-p)}\right)dx\,dt<\infty
\tag{2.40}
$$

for any $k=\ell,\dots,m$, yields

$$
\begin{aligned}
&-\int_0^\infty\int_{\mathbb{R}^N}\varphi_p(u)L_k^*\zeta\,dx\,dt\\
&\leq c\left(\int_0^\infty\int_{\mathbb{R}^N}|u|^q\zeta\,dx\,dt\right)^{p/q}\left(\int_0^\infty\int_{\mathbb{R}^N}|L_k^*\zeta|^{q/(q-p)}\zeta^{-p/(q-p)}dx\,dt\right)^{(q-p)/q},\\
&\leq\frac{1}{4m}\int_0\int_{\mathbb{R}^N}|u|^q\zeta\,dx\,dt+C_2\int_0^\infty\int_{\mathbb{R}^N}|L_k^*\zeta|^{q/(q-p)}\zeta^{-p/(q-p)}dx\,dt.
\end{aligned}
\tag{2.41}
$$

Therefore

$$
\begin{aligned}
&\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^N} |u|^q \zeta \, dx\, dt \\
&\quad \le C_1 \int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q'} \zeta^{1-q'} dx\, dt \\
&\quad + C_2 \int_0^\infty \int_{\mathbb{R}^N} \sum_{k=\ell}^{m} |L_k^* \zeta|^{q/(q-p)} \zeta^{-p/(q-p)} dx\, dt\,.
\end{aligned}
\tag{2.42}
$$

Taking ζ as in the proof of Theorem 1, with an exponent κ to be determined later on, yields (2.13) and

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} |L_k^* \zeta|^{q/(q-p)} \zeta^{-p/(q-p)} dx\, dt \\
&= R^{-2kq/(\mu(q-p))+2/\kappa+2N/\mu} \iint_\Omega \left(|L_k^*(\varphi\circ\rho)|^{q/(q-p)} (\varphi\circ\rho)^{-p/(q-p)}\right) \circ \rho \, dy\, d\tau.
\end{aligned}
\tag{2.43}
$$

We choose κ such that $2N/\mu + 2/\kappa - 4q'/\kappa = 2N/\mu + 2/\kappa - 2\ell q/(\mu(q-p))$, that is

$$
4/\kappa = 2\ell(q-1)/(\mu(q-p))\,, \tag{2.44}
$$

and we get

$$
\int_0^\infty \int_{\mathbb{R}^N} |L_k^* \zeta|^{q/(q-p)} \zeta^{-p/(q-p)} dx\, dt = D_k R^{\alpha_k}\,, \tag{2.45}
$$

with

$$
\alpha_k = \frac{2}{\mu}\left(N - \ell\frac{q+1}{2(q-p)} - (k-\ell)\frac{q}{q-p}\right). \tag{2.46}
$$

Therefore, the conditions $2N-\ell \le 0$, or $2N-\ell > 0$ and $N \le \frac{\ell}{2}\frac{q+1}{q-p}$ imply that $\alpha_\ell \le 0$ and $\alpha_k < 0$ for $k > \ell$. We end the proof as in Theorem 1. □

Our technique can also be applied to anisotropic operators, that means operators with different orders of partial differentiation in the variables. For such a task, we write $x = (x_1, \dots, x_N)$ the variable in $\mathbb{R}^N$, and introduce operators with order of differentiation β_j in the variable x_j of the following form

$$
\mathfrak{L}\zeta = \sum_{j=1}^{N} \frac{\partial^{\beta_j}}{\partial x_j^{\beta_j}} \left(a_{\beta_j}(x,t)\zeta\right) \tag{2.47}
$$

We consider the following inequality

$$
\partial_{tt} u \ge \mathfrak{L}\left(\varphi_p(u)\right) + |u|^q\,, \tag{2.48}
$$

on $\mathbb{R}_+^{N+1}$ with initial data $u(.,0) = u_0(.)$ and $\partial_t u(.,0) = u_1(.)$, two locally integrable functions in $\mathbb{R}^N$. This means again that $u \in L^q_{\mathrm{loc}}(\mathbb{R}_+^{N+1}) \cap L^p_{\mathrm{loc}}(\mathbb{R}_+^{N+1})$ and (2.4) holds with L_m^* replaced by $\mathfrak{L}^*$, the formal adjoint of $\mathfrak{L}$.

THEOREM 4. *Assume that* $q > \max(1, p)$. *Then there exists no solution* u *of inequality* (2.48) *defined on* $\mathbb{R}_+^{N+1}$ *and such that* $\int_{\mathbb{R}^N} u_1\,dx \geq 0$, *if* $\sum_{j=1}^N \beta_j^{-1} \leq \frac{q+1}{2(q-p)}$.

PROOF. As in the previous theorems, we start from the estimate

$$\begin{aligned}&\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} |u|^q \zeta\,dx\,dt\\ &\leq \int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} u\partial_{tt}\zeta\,dx\,dt\\ &\quad - \sum_{j+1}^N (-1)^{\beta_j} \int_0^\infty \int_{\mathbb{R}^N} \varphi_p(u) a_{\beta_j} \frac{\partial^{\beta_j}\zeta}{\partial x_j^{\beta_j}} dx\,dt\,.\end{aligned} \tag{2.49}$$

Choosing ζ such that

$$\int_0^\infty \int_{\mathbb{R}^N} \left(|\partial_{tt}\zeta|^{q'} \zeta^{1-q'} + |a_{\beta_j} \partial^{\beta_j}\zeta / \partial x_j^{\beta_j}|^{q/(q-p)} \zeta^{p/(q-p)} \right) dx\,dt < \infty \tag{2.50}$$

for any $j = 1, \dots, N$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^N} |u|^q \zeta\,dx\,dt\\ &\leq C_1 \int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q'} \zeta^{1-q'} dx\,dt\\ &\quad + C_2 \sum_{j=1}^N \int_0^\infty \int_{\mathbb{R}^N} |a_{\beta_j} \partial^{\beta_j}\zeta / \partial x_j^{\beta_j}|^{q/(q-p)} \zeta^{p/(q-p)} dx\,dt < \infty\,.\end{aligned} \tag{2.51}$$

We now take $\zeta(x,t) = \varphi((t^\kappa + \sum_{j=1}^N |x_j|^{2\kappa_j})/R^2)$, with φ as before, $\kappa > 1$ and the $\kappa_j > 0$ to be determined later on. We consider the change of variables

$$\left\{\begin{aligned} R^{-2}t^\kappa &= \tau^\kappa\,,\\ R^{-2}|x_j|^{2\kappa_j} &= |y_j|^{2\kappa_j}\,,\end{aligned}\right. \iff \left\{\begin{aligned} t &= \tau R^{2/\kappa}\,,\\ x_j &= y_j R^{1/\kappa_j}\,.\end{aligned}\right. \tag{2.52}$$

Denoting $\Omega = \{(y,\tau) \in \mathbb{R}^N \times \mathbb{R}_+ : \tau^\kappa + \sum_{j=1}^N |y_j|^{2\kappa_j} \leq 2\}$ and $\rho(y,\tau) = \tau^\kappa + \sum_{j=1}^N |y_j|^{2\kappa_j}$, then

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q'} \zeta^{1-q'} dx\,dt\\ &= R^{-4q'/\kappa + 2/\kappa + \sum_{j=1}^N \kappa_j^{-1}} \iint_\Omega \left(|\partial_{\tau\tau}\varphi \circ \rho|^{q'} (\varphi \circ \rho)^{1-q'} \right) dy\,d\tau\,,\end{aligned} \tag{2.53}$$

and

$$\text{(2.54)}\qquad \begin{aligned}&\int_0^\infty \int_{\mathbb{R}^N} \left|a_{\beta_i} \frac{\partial^{\beta_i}\zeta}{\partial x_i^{\beta_i}}\right|^{\frac{q}{q-p}} \zeta^{\frac{p}{q-p}} dx\, dt \\ &= R^{-\frac{\beta_i q}{\kappa_i(q-p)}+\frac{2}{\kappa}+\sum_{j=1}^N \kappa_j^{-1}} \iint_\Omega \left(\left|a_{\beta_i} \frac{\partial^{\beta_i}(\varphi\circ\rho)}{\partial y_i^{\beta_i}}\right|^{\frac{q}{q-p}} (\varphi\circ\rho)^{-\frac{p}{q-p}} \right) dy\, d\tau\,.\end{aligned}$$

Consequently, there exist positive constants C_i ($i = 0, \ldots, N$) such that

$$\text{(2.55)}\qquad \begin{aligned}&2\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} |u|^q \zeta\, dx\, dt \\ &\le C_0 R^{\sum_{j=1}^N \kappa_j^{-1}+2/\kappa-4q'/\kappa} + \sum_{i=1}^N C_j R^{-\beta_i q/(\kappa_i(q-p))+2/\kappa+\sum_{j=1}^N \kappa_j^{-1}}\,.\end{aligned}$$

We now choose the $\kappa_j > 0$ and κ in order all the exponents of R be the same in (2.54), that is

$$\text{(2.56)}\qquad \begin{gathered}\sum_{i=1}^N \kappa_i^{-1} + 2/\kappa - 4q'/\kappa = -\beta_j q/\kappa_j(q-p) + 2/\kappa + \sum_{i=1}^N \kappa_i^{-1}\,, \\ \Longleftrightarrow \\ 4\kappa_j/\kappa = \beta_j(q-1)/(q-p)\,.\end{gathered}$$

In that case, the common value ε of the exponents is

$$\text{(2.57)}\qquad \varepsilon = \frac{2}{\kappa(q-1)}\left(2(q-p)\sum_{j=1}^N \frac{1}{\beta_j} - q - 1\right)\,,$$

and the constraint on κ is $\beta_j(q-1)/(q-p) < 4\kappa_j$, which can always be fulfilled by taking the κ_j large enough. Since by assumption $\gamma \le 0$, we conclude as in the previous theorems by letting R go to infinity. □

The techniques developed above can also be used to handle the case of operators the coefficients of which have a strong dependence upon the variables x and t at infinity. Instead of assuming that the coefficients a_α in the operator L_m defined in (2.1) are merely bounded, we assume that

$$\text{(2.58)}\qquad |a_\alpha(x,t)| \le C|x|^\delta t^\gamma \qquad (\forall\,(x,t) \in \mathbb{R}_+^{N+1} | |x|^2 + t \ge 1)\,,$$

for some positive C and δ and γ belonging to $\mathbb{R}$.

THEOREM 5. *Assume that $q > \max(1,p)$, $\delta < m$ and $\gamma > -2(q-p)/(q-1)$. Then there exists no solution u of inequality* (2.3) *defined on $\mathbb{R}_+^{N+1}$ and such that $\int_{\mathbb{R}^N} u_1 dx \ge 0$, if the inequality $N \le \frac{(m\delta)(q+1)}{2(q-p)+\gamma(q-1)}$ holds.*

PROOF. We follow Theorem 1 and, by using the same test function, we first obtain (2.10). Performing the change of variables (2.12), the relation (2.13) remains valid while (2.14) is replaced by

$$
\begin{aligned}
(2.59)\quad & \int_0^\infty \int_{\mathbb{R}^N} |L_m^*\zeta|^{q/(q-p)}\zeta^{-p/(q-p)}dx\,dt \\
&\le \iint_{|x|^2+t\le 1} |L_m^*\zeta|^{q/(q-p)}\zeta^{-p/(q-p)}dx\,dt \\
&+ CR^{-\frac{2mq}{\mu(q-p)}+\frac{2}{\kappa}+\frac{2N}{\mu}+(\frac{2\delta}{\mu}+\frac{2\gamma}{\kappa})\frac{q}{q-p}} \iint_\Omega \left(\sum_{|\alpha|=m} |D^\alpha(\varphi\circ\rho)|^{\frac{q}{q-p}}(\varphi\circ\rho)^{-\frac{p}{q-p}} \right) dy\,d\tau.
\end{aligned}
$$

Consequently

$$
(2.60)\quad
\begin{aligned}
& 2\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} |u|^q\zeta\,dx\,dt \\
&\le \iint_{|x|^2+t\le 1} |L_m^*\zeta|^{q/(q-p)}\zeta^{-p/(q-p)}dx\,dt + C_3R^{2N/\mu+2/\kappa-4q'/\kappa} \\
&+ C_5R^{-2mq/(\mu(q-p))+2/\kappa+2N/\mu+(2\delta/\mu+2\gamma/\kappa)q/(q-p)}.
\end{aligned}
$$

Since $\iint_{|x|^2+t\le 1} |L_m^*\zeta|^{q/(q-p)}\zeta^{-p/(q-p)}dx\,dt = 0$ for $R \ge R_0 > 1$, depending on κ and $\mu > 0$, the problem is reduced to equalise the two exponents of R in (2.60). This means

$$
(2.61)\quad
\begin{aligned}
-2mq/(\mu(q-p)) + 2/\kappa + 2N/\mu + (2\delta/\mu + 2\gamma/\kappa)q/(q-p) \\
= 2N/\mu + 2/\kappa - 4q'/\kappa\,.
\end{aligned}
$$

We choose

$$
(2.62)\quad \theta\left(\gamma + 2\frac{q-p}{q-1}\right) = m - \delta\,,
$$

with $\theta = \mu/\kappa$, and this gives the common value ω for the exponents of R:

$$
(2.63)\quad \omega = \frac{1}{\mu}\left(N - \frac{(m-\delta)(q+1)}{2(q-p)+\gamma(q-1)}\right).
$$

The end of the proof is as in Theorem 1. □

APPLICATION 1. We take $m = 2$, $p = 1$, $\delta = 1$ and $\gamma = 0$. Then the conditions on γ and δ are satisfied, while the condition on N reads $1 \le N \le (q+1)/(2(q-1))$ or, equivalently, $1 < q \le (2N+1)/(2N-1)$.

REMARK 4. By using the same method, we can also derive non-existence results for stationary solutions of the preceding inequalities (that is solutions independent of the t variable and only depending on $x \in \mathbb{R}^N$, and in such a case, we assume that $a_\alpha = a_\alpha(x)$). Since the proofs are straightforward imitations of the above ones, we just state the results without any proof.

A – Assume that $q > \max(1, p)$. Then there exists no stationary solution of (2.3) defined in $\mathbb{R}^N$ if $N \le mq/(q-p)$.

B – Assume that $q > \max(1, p)$. Then there exists no stationary solution of (2.38) defined in $\mathbb{R}^N$ if $N \le \ell q/(q-p)$.

C – Assume that $q > \max(1, p)$. Then there exists no stationary solution of (2.48) defined in $\mathbb{R}^N$ if $\sum_{i=1}^N \beta_i^{-1} \le q/(q-p)$.

D – Assume that $q > \max(1, p)$, $\delta < m$ and (2.58) holds with $\gamma = 0$. Then there exists no stationary solution of (2.3) defined in $\mathbb{R}^N$ if $N \le q(m-\delta)/(q-p)$.

3. – Non-existence for hyperbolic systems

Let L_{m_i} be differential operators of order m_i $(i = 1, 2)$ defined by

$$L_{m_i}\zeta = \sum_{|\alpha|=m_i} D^\alpha(a_{i,\alpha}(x,t)\zeta)\,, \tag{3.1}$$

where the $a_{i,\alpha}$ are bounded and measurable functions defined in $\mathbb{R}^N \times \mathbb{R}_+ = \mathbb{R}_+^{N+1}$, and the φ_{p_i} are real-valued continuous functions which satisfy

$$|\varphi_{p_i}(r)| \le c|r|^{p_i} \qquad (\forall r \in \mathbb{R})\,, \tag{3.2}$$

for some $p_i > 0$ and $c > 0$.

Definition 2. The couple $(u, v) \in L^{p_1}_{\mathrm{loc}}(\mathbb{R}_+^{N+1}) \cap L^{q_1}_{\mathrm{loc}}(\mathbb{R}_+^{N+1}) \times L^{p_2}_{\mathrm{loc}}(\mathbb{R}_+^{N+1}) \cap L^{q_2}_{\mathrm{loc}}(\mathbb{R}_+^{N+1})$ is a weak solution of

$$\begin{cases} \partial_{tt}u \ge L_{m_1}(\varphi_{p_1}(u)) + |v|^{q_1}\,, \\ \partial_{tt}v \ge L_{m_2}(\varphi_{p_2}(v)) + |u|^{q_2}\,, \end{cases} \tag{3.3}$$

in $\mathbb{R}_+^{N+1}$ with initial data $(u(.,0), v(.,0)) = (u_0(.), v_0(.))$ and $(\partial_t u(.,0)\partial_t v(.,0)) = (u_1(.), v_1(.))$, all data belonging to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, if for any $\zeta \in C_c^\infty(\mathbb{R}_+^{N+1})$, $\zeta \ge 0$, there holds

$$\begin{aligned} &-\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx + \int_0^\infty\int_{\mathbb{R}^N} u\partial_{tt}\zeta\,dx\,dt \\ &\ge \int_0^\infty\int_{\mathbb{R}^N} \varphi_{p_1}(u)L^*_{m_1}\zeta\,dx\,dt + \int_0^\infty\int_{\mathbb{R}^N} |v|^{q_1}\zeta\,dx\,dt\,, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &-\int_{\mathbb{R}^N} v_1(x)\zeta(x,0)dx + \int_{\mathbb{R}^N} v_0(x)\partial_t\zeta(x,0)dx + \int_0^\infty\int_{\mathbb{R}^N} v\partial_{tt}\zeta\,dx\,dt \\ &\ge \int_0^\infty\int_{\mathbb{R}^N} \varphi_{p_2}(v)L^*_{m_2}\zeta\,dx\,dt + \int_0^\infty\int_{\mathbb{R}^N} |u|^{q_2}\zeta\,dx\,dt\,, \end{aligned} \tag{3.5}$$

where $L^*_{m_i} = (-1)^{m_i}\sum_{|\alpha|=m_i} a_{i,\alpha}(x,t)D^\alpha$.

THEOREM 6. *Assume that* $q_1 > \max(1, p_2)$ *and* $q_2 > \max(1, p_1)$. *Then there exists no solution* (u, v) *of the differential inequalities* (3.3) *such that* $\int_{\mathbb{R}^N} u_1 dx \geq 0$ *and* $\int_{\mathbb{R}^N} v_1 dx \geq 0$, *if one of the following assumptions is fulfilled:*

I – *If* $q_1 \geq q_2$ *and* $m_2 q_1/(q_1 - p_2) \geq m_1 q_2/(q_2 - p_1)$, *then* $N \leq m_1 q_2 (q_1 + 1)/(2q_1(q_2 - p_1))$.

II – *If* $q_1 \geq q_2$ *and* $m_2 q_1/(q_1 - p_2) < m_1 q_2/(q_2 - p_1)$, *then* $N \leq m_2 (q_1 + 1)/(2(q_1 - p_2))$.

III – *If* $q_2 > q_1$ *and* $m_1 q_2/(q_2 - p_1) \leq m_2 q_1/(q_1 - p_2)$, *then* $N \leq m_1 (q_2 + 1)/(2(q_2 - p_1))$.

IV – *If* $q_2 > q_1$ *and* $m_1 q_2/(q_2 - p_1) > m_2 q_1/(q_1 - p_2)$, *then* $N \leq m_2 q_1 (q_2 + 1)/(2q_2(q_1 - p_2))$.

PROOF. As in Section 2, we consider a nonnegative test function $\zeta \in C_c^\infty(\mathbb{R}_+^{N+1})$ such that

$$\begin{aligned}(3.6)\qquad &\int_0^\infty \int_{\mathbb{R}^N} \left(|\partial_{tt}\zeta|^{q_1'} \zeta^{1-q_1'} + |L_{m_1}^* \zeta|^{q_1'/(q_1'-p_2)} \zeta^{-p_2/(q_1'-p_2)} \right) dx\, dt \\ &+ \int_0^\infty \int_{\mathbb{R}^N} \left(|\partial_{tt}\zeta|^{q_2'} \zeta^{1-q_2'} + |L_{m_2}^* \zeta|^{q_2'/(q_2'-p_1)} \zeta^{-p_1/(q_2'-p_1)} \right) dx\, dt < \infty,\end{aligned}$$

where $q_i' = q_i/(q_i - 1)$. We first have

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^N} u \partial_{tt}\zeta \, dx\, dt \\ (3.7)\qquad &\leq \left(\int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2} \zeta \, dx\, dt \right)^{1/q_2} \left(\int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_2'} \zeta^{1-q_2'} dx\, dt \right)^{1/q_2'}, \\ &\leq \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2} \zeta \, dx\, dt + C_{1,1} \int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_2'} \zeta^{1-q_2'} dx\, dt,\end{aligned}$$

and

$$\begin{aligned}&- \int_0^\infty \int_{\mathbb{R}^N} \varphi_{p_1}(u) L_{m_1}^* \zeta \, dx\, dt \\ (3.8)\qquad &\leq c \left(\int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2} \zeta \, dx\, dt \right)^{\frac{p_1}{q_2}} \left(\int_0^\infty \int_{\mathbb{R}^N} |L_{m_1}^* \zeta|^{\frac{q_2}{q_2 - p_1}} \zeta^{-\frac{p_1}{q_2 - p_1}} dx\, dt \right)^{\frac{q_2 - p_1}{q_2}}, \\ &\leq \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2} \zeta \, dx\, dt + C_{1,2} \int_0^\infty \int_{\mathbb{R}^N} |L_{m_1}^* \zeta|^{\frac{q_2}{q_2 - p_1}} \zeta^{-\frac{p_1}{q_2 - p_1}} dx\, dt.\end{aligned}$$

Similarly

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^N} v \partial_{tt}\zeta \, dx\, dt \\ (3.9)\qquad &\leq \left(\int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1} \zeta \, dx\, dt \right)^{1/q_1} \left(\int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_1'} \zeta^{1-q_1'} dx\, dt \right)^{1/q_1'}, \\ &\leq \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1} \zeta \, dx\, dt + C_{2,1} \int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_1'} \zeta^{1-q_1'} dx\, dt,\end{aligned}$$

and

$$
\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}^N} \varphi_{p_2}(v) L^*_{m_2}\zeta \, dx\, dt \\
(3.10)\quad &\le c\Big(\int_0^\infty\int_{\mathbb{R}^N} |v|^{q_1}\zeta \, dx\, dt\Big)^{\frac{p_2}{q_1}} \Big(\int_0^\infty\int_{\mathbb{R}^N} |L^*_{m_2}\zeta|^{\frac{q_1}{q_1-p_2}} \zeta^{-\frac{p_2}{q_1-p_2}} dx\, dt\Big)^{\frac{q_1-p_2}{q_1}}, \\
&\le \frac{1}{4}\int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1}\zeta \, dx\, dt + C_{2,2}\int_0^\infty \int_{\mathbb{R}^N} |L^*_{m_2}\zeta|^{\frac{q_1}{q_1-p_2}} \zeta^{-\frac{p_2}{q_1-p_2}} dx\, dt.
\end{aligned}
$$

We choose ζ such that

$$
(3.11)\qquad \int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} v_0(x)\partial_t\zeta(x,0)dx = 0 .
$$

By summing the two next inequalities (derived from (3.7)-(3.10)),

$$
\begin{aligned}
&\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1}\zeta \, dx\, dt \\
(3.12)\quad &\le \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2}\zeta \, dx\, dt + C_{1,1}\int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_2'}\zeta^{1-q_2'} dx\, dt \\
&\quad + C_{1,2}\int_0^\infty \int_{\mathbb{R}^N} |L^*_{m_1}\zeta|^{q_2/(q_2-p_1)}\zeta^{-p_1/(q_2-p_1)} dx\, dt ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} v_1(x)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2}\zeta \, dx\, dt \\
(3.13)\quad &\le \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1}\zeta \, dx\, dt + C_{2,1}\int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_1'}\zeta^{1-q_1'} dx\, dt \\
&\quad + C_{2,2}\int_0^\infty \int_{\mathbb{R}^N} |L^*_{m_2}\zeta|^{q_1/(q_1-p_2)}\zeta^{-p_2/(q_1-p_2)} dx\, dt ,
\end{aligned}
$$

we deduce that the following estimate holds

$$
\begin{aligned}
&2\int_{\mathbb{R}^N} (u_1+v_1)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} (|v|^{q_1}+|u|^{q_2})\zeta \, dx\, dt \\
(3.14)\quad &\le C_1\int_0^\infty \int_{\mathbb{R}^N} \big(|\partial_{tt}\zeta|^{q_2'}\zeta^{1-q_2'} + |\partial_{tt}\zeta|^{q_1'}\zeta^{1-q_1'}\big) dx\, dt \\
&\quad + C_2\int_0^\infty\int_{\mathbb{R}^N} \Big(|L^*_{m_1}\zeta|^{\frac{q_2}{q_2-p_1}}\zeta^{-\frac{p_1}{q_2-p_1}} + |L^*_{m_2}\zeta|^{\frac{q_1}{q_1-p_2}}\zeta^{-\frac{p_2}{q_1-p_2}}\Big) dx\, dt ,
\end{aligned}
$$

We take again $\zeta(x,t) = \varphi(\frac{t^\kappa + |x|^\mu}{R^2})$, where $\varphi \in C_c^\infty(\mathbb{R}_+)$ satisfies $0 \le \varphi \le 1$ and

$$
(3.15)\qquad \varphi(r) = \begin{cases} 0 & \text{if } r \ge 2, \\ 1 & \text{if } 0 \le r \le 1, \end{cases}
$$

R is a positive parameter and $\kappa > 1$ and $\mu > 0$ will be determined later on. Clearly (3.9) holds.

In order to estimate the right-hand side of (3.14) we perform the same change of variables as in Theorem 1, that is

$$\left\{\begin{array}{l} R^{-2}t^{\kappa} = \tau^{\kappa}\,, \\ R^{-2}|x|^{\mu} = |y|^{\sigma}\,, \end{array}\right. \iff \left\{\begin{array}{l} t = R^{2/\kappa}\tau\,, \\ x = R^{2/\mu}y\,. \end{array}\right. \tag{3.16}$$

This yields

$$\begin{aligned} &2\int_{\mathbb{R}^N}(u_1+v_1)\zeta(x,0)dx + \int_0^{\infty}\int_{\mathbb{R}^N}\left(|v|^{q_1}+|u|^{q_2}\right)\zeta\,dx\,dt \\ &\leq C_3\left(R^{-4q_1'/\kappa+2/\kappa+2N/\mu} + R^{-4q_2'/\kappa+2/\kappa+2N/\mu}\right) \\ &\quad + C_4\left(R^{-2m_1q_2/(\mu(q_2-p_1))+2/\kappa+2N/\mu} + R^{-2m_2q_1/(\mu(q_1-p_2))+2/\kappa+2N/\mu}\right) \end{aligned} \tag{3.17}$$

Setting $\theta = \mu/\kappa > 0$ the exponents of R will be nonpositive if the following inequalities are all satisfied for some $\theta > 0$.

$$\left\{\begin{array}{ll} \text{(i)} & N \leq \theta(q_1+1)/(q_1-1)\,, \\ \text{(ii)} & N \leq \theta(q_2+1)/(q_2-1)\,, \\ \text{(iii)} & N \leq m_2q_1/(q_1-p_2)-\theta\,, \\ \text{(iv)} & N \leq m_1q_2/(q_2-p_1)-\theta\,, \end{array}\right. \tag{3.18}$$

Therefore, if we define $\widetilde{N}$ by

$$\widetilde{N} = \max\left\{r \geq 0 : \exists\theta > 0 \text{ s.t. (3.18) holds}\right\}, \tag{3.19}$$

then $N = E(\widetilde{N})$ is the largest integer such that (3.18) holds. For example, if $q_1 \geq q_2$ and $m_2q_1/(q_1-p_2) \geq m_1q_2/(q_2-p_1)$, θ is defined by the relation

$$\theta(q_1-1)/(q_1+1) = m_1q_2/(q_2-p_1)-\theta \iff \theta = 2^{-1}m_1q_2(q_1+1)/(q_1(q_2-p_1)), \tag{3.20}$$

and the conditions (3.18) read $N \geq m_1q_2(q_1+1)/(2q_1(q_2-p_1))$, which is I. While if $q_1 \geq q_2$ and $m_2q_1/(q_1-p_2) < m_1q_2/(q_2-p_1)$, θ is defined by

$$\theta(q_1+1)/(q_1-1) = m_2q_1/(q_1-p_2)-\theta \iff \theta = 2^{-1}m_2(q_1-1)/(q_1-p_2)\,,$$

and the conditions (3.18) read $N \leq m_2(q_1+1)/(2(q_1-p_2))$, which is II. We proceed similarly if $q_1 < q_2$. We conclude as in the previous theorems: by letting R go to infinity, and since $\int_{\mathbb{R}^N}(u_1+v_1)\zeta(x,0)dx \geq 0$, we infer that $\int_0^{\infty}\int_{\mathbb{R}^N}(|v|^{q_1}+|u|^{q_2})\zeta\,dx\,dt = 0$. □

APPLICATION 2. In the case where $m_i = 2$ and $p_i = 1$ for $i = 1, 2$, the relation $q_1 \geq q_2$ implies $m_2 q_1/(q_1 - p_2) = 2q_1/(q_1 - 1) \leq m_1 q_2/(q_2 - p_1) = 2q_2/(q_2 - 1)$. The blow-up occurs when $N \leq (q_1 + 1)/(q_1 - 1) \iff q_1 \leq (N + 1)/(N - 1)$, which the same condition as in the single equation case.

REMARK 5. This result has to be compared with the recent ones of Del Santo, Georgiev and Mitidieri [DGM], [DM] dealing with the case $m_i = 2$, $N = 3$ and $p_i = 1$ and $L_{m_i} = \Delta$. In this case it is proved that if

$$p > 1\,,\ q > 1\,,\ \max\left\{\frac{p + 2 + q^{-1}}{pq - 1}\,,\ \frac{q + 2 + p^{-1}}{pq - 1}\right\} = A(p, q) \geq 1\,, \tag{3.21}$$

any solution of the above mentionned inequalities, with compactly-supported small enough initial data, blows-up in finite time. On the contrary, if $A(p, q) < 1$, the corresponding system of equations admits global solutions provided the initial data are small enough. This type of results, with stronger assumptions but obviously sharper statements, are the natural extension to systems of the ones of John, Glassey or Shaeffer ([Jo1], [Gl2], [Sc]). Our results, dealing with weak solutions with general initial data, are much more in the continuation of Kato's works ([Ka]).

In the next theorems we give blow-up results for non-diagonal hyperbolic systems of the following types

$$\begin{cases} \partial_{tt} u \geq L_{m_1}(\varphi_{p_1}(v)) + |u|^{q_1} \\ \partial_{tt} v \geq L_{m_2}(\varphi_{p_2}(u)) + |v|^{q_2} \end{cases} \tag{3.22}$$

and

$$\begin{cases} \partial_{tt} u \geq L_{m_1}(\varphi_{p_1}(v)) + |v|^{q_1} \\ \partial_{tt} v \geq L_{m_2}(\varphi_{p_2}(u)) + |u|^{q_2} \end{cases} \tag{3.23}$$

A couple $(u, v) \in L^{q_1}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+) \cap L^{p_2}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+) \times L^{q_2}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+) \cap L^{p_1}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ as a weak solution in $\mathbb{R}^{N+1}_+$ of (3.22) with initial data $(u(.\,, 0), v(.\,, 0)) = (u_0(.), v_0(.))$ and $(\partial_t u(.\,, 0), \partial_t v(.\,, 0)) = (u_1(.), v_1(.))$, if for any $\zeta \in C^\infty_c(\mathbb{R}^{N+1}_+)$, $\zeta \geq 0$, there holds

$$\begin{aligned} &- \int_{\mathbb{R}^N} u_1(x)\zeta(x, 0)dx + \int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x, 0)dx + \int_0^\infty \int_{\mathbb{R}^N} u\partial_{tt}\zeta\, dx\, dt \\ &\geq \int_0^\infty \int_{\mathbb{R}^N} \varphi_{p_1}(v)L^*_{m_1}\zeta\, dx\, dt + \int_0^\infty \int_{\mathbb{R}^N} |u|^{q_1}\zeta\, dx\, dt\,, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} &- \int_{\mathbb{R}^N} v_1(x)\zeta(x, 0)dx + \int_{\mathbb{R}^N} v_0(x)\partial_t\zeta(x, 0)dx + \int_0^\infty \int_{\mathbb{R}^N} v\partial_{tt}\zeta\, dx\, dt \\ &\geq \int_0^\infty \int_{\mathbb{R}^N} \varphi_{p_2}(u)L^*_{m_2}\zeta\, dx\, dt + \int_0^\infty \int_{\mathbb{R}^N} |v|^{q_2}\zeta\, dx\, dt\,, \end{aligned} \tag{3.25}$$

Similarly $(u, v) \in L^{q_2}_{\text{loc}}(\mathbb{R}^{N+1}_+) \cap L^{p_2}_{\text{loc}}(\mathbb{R}^{N+1}_+) \times L^{q_1}_{\text{loc}}(\mathbb{R}^{N+1}_+) \cap L^{p_1}_{\text{loc}}(\mathbb{R}^{N+1}_+)$ is a weak solution in $\mathbb{R}^{N+1}_+$ of (3.23) with initial data $(u(.\,,0), v(.\,,0)) = (u_0(.), v_0(.))$ and $(\partial_t u(.\,,0).\partial_t v(.\,,0)) = (u_1(.), v_1(.))$, if for any $\zeta \in C^\infty_c(\mathbb{R}^{N+1}_+)$, $\zeta \geq 0$, there holds

$$\begin{aligned}
&-\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx + \int_0^\infty\int_{\mathbb{R}^N} u\partial_{tt}\zeta\, dx\, dt \\
&\geq \int_0^\infty\int_{\mathbb{R}^N} \varphi_{p_1}(v)L^*_{m_1}\zeta\, dx\, dt + \int_0^\infty\int_{\mathbb{R}^N} |v|^{q_1}\zeta\, dx\, dt\,,
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}^N} v_1(x)\zeta(x,0)dx + \int_{\mathbb{R}^N} v_0(x)\partial_t\zeta(x,0)dx + \int_0^\infty\int_{\mathbb{R}^N} v\partial_{tt}\zeta\, dx\, dt \\
&\geq \int_0^\infty\int_{\mathbb{R}^N} \varphi_{p_2}(u)L^*_{m_2}\zeta\, dx\, dt + \int_0^\infty\int_{\mathbb{R}^N} |u|^{q_2}\zeta\, dx\, dt\,,
\end{aligned} \tag{3.27}$$

Since the techniques involved are the same as the one of Theorem 6, we just state with a very reduced proof our results concerning these non-diagonal systems.

THEOREM 7. *Assume that* $q_1 > \max(1, p_2)$ *and* $q_2 > \max(1, p_1)$, *then there exists no solution* (u, v) *of the differential inequalities* (3.22) *such that* $\int_{\mathbb{R}^N} u_1 dx \geq 0$ *and* $\int_{\mathbb{R}^N} v_1 dx \geq 0$ *if one of the following assumptions is fulfilled:*

I – *If* $q_1 \geq q_2$ *and* $m_2q_1/(q_1 - p_2) \geq m_1q_2/(q_2 - p_1)$, *then* $N \leq m_1q_2(q_1 + 1)/(2q_1(q_2 - p_1))$.

II – *If* $q_1 \geq q_2$ *and* $m_2q_1/(q_1 - p_2) < m_1q_2/(q_2 - p_1)$, *then* $N \leq m_2(q_1 + 1)/(2(q_1 - p_2))$.

III – *If* $q_2 > q_1$ *and* $m_1q_2/(q_2 - p_1) \leq m_2q_1/(q_1 - p_2)$, *then* $N \leq m_1(q_2 + 1)/(2(q_2 - p_1))$.

IV – *If* $q_2 > q_1$ *and* $m_1q_2/(q_2 - p_1) > m_2q_1/(q_1 - p_2)$, *then* $N \leq m_2q_1(q_2 + 1)/(2q_2(q_1 - p_2))$.

PROOF. The only difference with the proof of Theorem 6 is that (3.7) and (3.9) are respectively replaced by

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^N} u\partial_{tt}\zeta\, dx\, dt \\
&\leq \left(\int_0^\infty\int_{\mathbb{R}^N} |u|^{q_1}\zeta\, dx\, dt\right)^{1/q_1}\left(\int_0^\infty\int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_1'}\zeta^{1-q_1'}dx\, dt\right)^{1/q_1'}, \\
&\leq \frac{1}{4}\int_0^\infty\int_{\mathbb{R}^N} |u|^{q_1}\zeta\, dx\, dt + C_{1,1}\int_0^\infty\int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_1'}\zeta^{1-q_1'}dx\, dt\,,
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} v\partial_{tt}\zeta \, dx\, dt \\
&\le \left(\int_0^\infty \int_{\mathbb{R}^N} |v|^{q_2}\zeta \, dx\, dt\right)^{1/q_2} \left(\int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_2'}\zeta^{1-q_2'} dx\, dt\right)^{1/q_2'}, \\
&\le \frac{1}{4}\int_0^\infty \int_{\mathbb{R}^N} |v|^{q_2}\zeta \, dx\, dt + C_{2,1}\int_0^\infty \int_{\mathbb{R}^N} |\partial_{tt}\zeta|^{q_2'}\zeta^{1-q_2'} dx\, dt,
\end{aligned} \tag{3.29}$$

with a test function a test function satisfying (3.6) while (3.8) and (3.10) holds. By summation the inequality (3.14) is still valid, and the conclusion follows by the same discussion. □

THEOREM 8. *Assume that $q_1 > \max(1, p_1)$ and $q_2 > \max(1, p_2)$. Then there exists no solution (u, v) of the differential inequalities (3.23) such that $\int_{\mathbb{R}^N} u_1 dx \ge 0$ and $\int_{\mathbb{R}^N} v_1 dx \ge 0$ if one of the following assumptions is fulfilled:*

- I – *If $q_1 \ge q_2$ and $m_2q_2/(q_2 - p_2) \ge m_1q_1/(q_1 - p_1)$, then $N \le m_1(q_1 + 1)/(2(q_1 - p_1))$.*
- II – *If $q_1 \ge q_2$ and $m_2q_2/(q_2 - p_2) < m_1q_1/(q_1 - p_1)$, then $N \le m_2q_2(q_2 + 1)/(2q_1(q_2 - p_2))$.*
- III – *If $q_2 > q_1$ and $m_2q_2/(q_2 - p_2) \le m_1q_1/(q_1 - p_1)$, then $N \le m_2(q_2 + 1)/(2(q_2 - p_2))$.*
- IV – *If $q_2 > q_1$ and $m_2q_2/(q_2 - p_2) > m_1q_1/(q_1 - p_1)$, then $N \le m_1q_1(q_2 + 1)/(2q_2(q_1 - p_1))$.*

PROOF. We consider a nonnegative test function $\zeta \in C_c^\infty(\mathbb{R}_+^{N+1})$ such that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} \left(|\partial_{tt}\zeta|^{q_1'}\zeta^{1-q_1'} + |L_{m_1}^*\zeta|^{q_1'/(q_1'-p_1)}\zeta^{-p_1/(q_1'-p_1)}\right) dx\, dt \\
&+ \int_0^\infty \int_{\mathbb{R}^N} \left(|\partial_{tt}\zeta|^{q_2'}\zeta^{1-q_2'} + |L_{m_2}^*\zeta|^{q_2'/(q_2'-p_2)}\zeta^{-p_2/(q_2'-p_2)}\right) dx\, dt < \infty.
\end{aligned} \tag{3.30}$$

Estimates (3.7) and (3.9) hold, while (3.8) and (3.10) are replaced by

$$\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}^N} \varphi_{p_2}(u) L_{m_2}^*\zeta \, dx\, dt \\
&\le c\left(\int_0^\infty\int_{\mathbb{R}^N} |u|^{q_2}\zeta \, dx\, dt\right)^{\frac{p_2}{q_2}} \left(\int_0^\infty\int_{\mathbb{R}^N} |L_{m_2}^*\zeta|^{\frac{q_2}{q_2-p_2}}\zeta^{-\frac{p_2}{q_2-p_2}} dx\, dt\right)^{\frac{q_2-p_2}{q_2}}, \\
&\le \frac{1}{4}\int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2}\zeta \, dx\, dt + C_{1,2}\int_0^\infty \int_{\mathbb{R}^N} |L_{m_2}^*\zeta|^{\frac{q_2}{q_2-p_2}}\zeta^{-\frac{p_2}{q_2-p_2}} dx\, dt.
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}^N} \varphi_{p_1}(v) L^*_{m_1}\zeta \, dx\, dt \\
(3.32)\quad &\leq c\left(\int_0^\infty\int_{\mathbb{R}^N} |v|^{q_1}\zeta\, dx\, dt\right)^{\frac{p_1}{q_1}} \left(\int_0^\infty\int_{\mathbb{R}^N} |L^*_{m_1}\zeta|^{\frac{q_1}{q_1-p_1}}\zeta^{-\frac{p_1}{q_1-p_1}} dx\, dt\right)^{\frac{q_1-p_1}{q_1}}, \\
&\leq \frac{1}{4}\cdot\int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1}\zeta\, dx\, dt + C_{2,2}\int_0^\infty \int_{\mathbb{R}^N} |L^*_{m_1}\zeta|^{\frac{q_1}{q_1-p_1}}\zeta^{-\frac{p_1}{q_1-p_1}} dx\, dt.
\end{aligned}$$

Consequently, (3.14) is replaced by

$$\begin{aligned}
&2\int_{\mathbb{R}^N} (u_1+v_1)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} (|v|^{q_1} + |u|^{q_2})\zeta\, dx\, dt \\
(3.33)\quad &\leq C_1 \int_0^\infty \int_{\mathbb{R}^N} \left(\left|\partial_{tt}\zeta\right|^{q_2'}\zeta^{1-q_2'} + \left|\partial_{tt}\zeta\right|^{q_1'}\zeta^{1-q_1'}\right) dx\, dt \\
&\quad + C_2 \int_0^\infty \int_{\mathbb{R}^N} \left(\left|L^*_{m_1}\zeta\right|^{\frac{q_1}{q_1-p_1}}\zeta^{-\frac{p_1}{q_1-p_1}} + \left|L^*_{m_2}\zeta\right|^{\frac{q_2}{q_2-p_2}}\zeta^{-\frac{p_2}{q_2-p_2}}\right) dx\, dt,
\end{aligned}$$

and (3.17) becomes

$$\begin{aligned}
&2\int_{\mathbb{R}^N} (u_1+v_1)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} \left(|v|^{q_1} + |u|^{q_2}\right)\zeta\, dx\, dt \\
(3.34)\quad &\leq C_3\left(R^{-4q_1'/\kappa+2/\kappa+2N/\mu} + R^{-4q_2'/\kappa+2/\kappa+2N/\mu}\right) \\
&\quad + C_4\left(R^{-2m_1q_1/(\mu(q_1-p_1))+2/\kappa+2N/\mu} + R^{-2m_2q_2/(\mu(q_2-p_2))+2/\kappa+2N/\mu}\right).
\end{aligned}$$

In order to have the exponents of R nonpositive, we have to find a positive $\theta = \mu/\kappa$ such that

$$(3.35)\qquad \begin{cases} \text{(i)} & N \leq \theta(q_1+1)/(q_1-1), \\ \text{(ii)} & N \leq \theta(q_2+1)/(q_2-1), \\ \text{(iii)} & N \leq m_2q_2/(q_2-p_2) - \theta, \\ \text{(iv)} & N \leq m_1q_1/(q_1-p_1) - \theta, \end{cases}$$

Defining

$$(3.36)\qquad \widehat{N} = \max\left\{r \geq 0 : \exists\theta > 0 \text{ s.t. } (3.35) \text{ holds}\right\},$$

then $N = E(\widehat{N})$ is the largest integer such that (3.35) holds. The conclusion follows. □

4. – Non-existence for systems of mixed type

In this Section we consider systems of mixed type parabolic-hyperbolic

$$\tag{4.1} \begin{cases} \partial_{tt}u \geq L_{m_1}\big(\varphi_{p_1}(u)\big) + |v|^{q_1} \\ \partial_t v \geq L_{m_2}\big(\varphi_{p_2}(v)\big) + |u|^{q_2} \end{cases}$$

and elliptic-hyperbolic

$$\tag{4.2} \begin{cases} \partial_{tt}u \geq L_{m_1}\big(\varphi_{p_1}(u)\big) + |v|^{q_1} \\ 0 \geq L_{m_2}\big(\varphi_{p_2}(v)\big) + |u|^{q_2} \end{cases}$$

We define the operators L_{m_1}, L_{m_2}, $L^*_{m_1}$ and $L^*_{m_2}$ as in Section III.

Definition 3. We say that $(u, v) \in L^{p_1}_{\text{loc}}(\mathbb{R}^{N+1}_+) \cap L^{q_1}_{\text{loc}}(\mathbb{R}^{N+1}_+) \times L^{p_2}_{\text{loc}}(\mathbb{R}^{N+1}_+) \cap L^{q_2}_{\text{loc}}(\mathbb{R}^{N+1}_+)$ is a weak solution of (4.1) (resp. (4.2)) in $\mathbb{R}^{N+1}_+$ with initial data $(u(.,0), v(.,0)) = (u_0(.), v_0(.))$ and $\partial_t u(.,0) = u_1(.)$, all belonging to $L^1_{\text{loc}}(\mathbb{R}^N)$ (resp. $u(.,0) = u_0(.)$ and $\partial_t u(.,0) = u_1(.)$)), if the two inequalities

$$\tag{4.3} \begin{aligned} &-\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} u\partial_{tt}\zeta\, dx\, dt \\ &\quad \geq \int_0^\infty \int_{\mathbb{R}^N} \varphi_{p_1}(u) L^*_{m_1}\zeta\, dx\, dt + \int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1}\zeta\, dx\, dt\,, \end{aligned}$$

and

$$\tag{4.4} \begin{aligned} &-\int_{\mathbb{R}^N} v_0(x)\partial_t\zeta(x,0)dx - \int_0^\infty \int_{\mathbb{R}^N} v\partial_t\zeta\, dx\, dt \\ &\quad \geq \int_0^\infty \int_{\mathbb{R}^N} \varphi_{p_2}(v) L^*_{m_2}\zeta\, dx\, dt + \int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2}\zeta\, dx\, dt\,, \end{aligned}$$

hold for any $\zeta \in C^\infty_c(\mathbb{R}^{N+1}_+)$, $\zeta \geq 0$ (resp. inequality (4.3) and

$$\tag{4.5} 0 \geq \int_0^\infty \int_{\mathbb{R}^N} \varphi_{p_2}(v) L^*_{m_2}\zeta\, dx\, dt + \int_0^\infty \int_{\mathbb{R}^N} |u|^{q_2}\zeta\, dx\, dt\Big).$$

Theorem 9. *Assume that* $q_1 > \max(1, p_2)$ *and* $q_2 > \max(1, p_1)$. *Then there exists no solution* (u, v) *of the differential inequalities* (4.1) *such that* $\int_{\mathbb{R}^N} u_1 dx \geq 0$ *and* $\int_{\mathbb{R}^N} v_0 dx \geq 0$ *if one of the following assumptions is fulfilled:*

I – *If* $q_1 \leq 2q_2/(q_2+1)$ *and* $m_2q_1/(q_1-p_2) \geq m_1q_2/(q_2-p_1)$, *then* $N \leq m_1(q_2+1)/(2(q_2-p_1))$.

II – *If* $q_1 \leq 2q_2/(q_2+1)$ *and* $m_2q_1/(q_1-p_2) < m_1q_2/(q_2-p_1)$, *then* $N \leq m_2q_1(q_2+1)/(2q_2(q_1-p_2))$.

III – *If* $q_1 \geq 2q_2/(q_2+1)$ *and* $m_1q_2/(q_2-p_1) \leq m_2q_1/(q_1-p_2)$, *then* $N \leq m_1q_2/(q_1(q_2-p_1))$.

IV – *If* $q_1 \geq 2q_2/(q_2+1)$ *and* $m_1q_2/(q_2-p_1) > m_2q_1/(q_1-p_2)$, *then* $N \leq m_2/(q_1-p_2)$.

PROOF. We consider a nonnegative test function $\zeta \in C_c^\infty(\mathbb{R}_+^{N+1})$ such that

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} \left(|\partial_{tt}\zeta|^{q_1'}\zeta^{1-q_1'} + |L_{m_1}^*\zeta|^{q_1'/(q_1'-p_2)}\zeta^{-p_2/(q_1'-p_2)} \right) dx\,dt \\
&+ \int_0^\infty \int_{\mathbb{R}^N} \left(|\partial_{tt}\zeta|^{q_2'}\zeta^{1-q_2'} + |L_{m_2}^*\zeta|^{q_2'/(q_2'-p_1)}\zeta^{-p_1/(q_2'-p_1)} \right) dx\,dt < \infty,
\end{aligned} \tag{4.6}
$$

Inequality (3.7)-(3.8)-(3.10) holds while (3.10) is replaced by

$$
\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}^N} v\partial_t\zeta \, dx\,dt \\
&\leq \left(\int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1}\zeta \, dx\,dt \right)^{1/q_1} \left(\int_0^\infty \int_{\mathbb{R}^N} |\partial_t\zeta|^{q_1'}\zeta^{1-q_1'} dx\,dt \right)^{1/q_1'}, \\
&\leq \frac{1}{4}\int_0^\infty \int_{\mathbb{R}^N} |v|^{q_1}\zeta \, dx\,dt + C_{2,1}\int_0^\infty \int_{\mathbb{R}^N} |\partial_t\zeta|^{q_1'}\zeta^{1-q_1'} dx\,dt.
\end{aligned} \tag{4.7}
$$

We choose ζ such that

$$
\int_{\mathbb{R}^N} u_0(x)\partial_t\zeta(x,0)dx = 0, \tag{4.8}
$$

and obtain

$$
\begin{aligned}
&2\int_{\mathbb{R}^N} (u_1+v_0)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} (|v|^{q_1}+|u|^{q_2})\zeta \, dx\,dt \\
&\leq C_1 \int_0^\infty \int_{\mathbb{R}^N} \left(|\partial_{tt}\zeta|^{q_2'}\zeta^{1-q_2'} + |\partial_{tt}\zeta|^{q_1'}\zeta^{1-q_1'} \right) dx\,dt \\
&\quad + C_2 \int_0^\infty \int_{\mathbb{R}^N} \left(|L_{m_1}^*\zeta|^{\frac{q_2}{q_2-p_1}}\zeta^{-\frac{p_1}{q_2-p_1}} + |L_{m_2}^*\zeta|^{\frac{q_1}{q_1-p_2}}\zeta^{-\frac{p_2}{q_1-p_2}} \right) dx\,dt.
\end{aligned} \tag{4.9}
$$

Taking ζ, φ, κ and μ as above yields

$$
\begin{aligned}
&2\int_{\mathbb{R}^N} (u_1+v_0)\zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} \left(|v|^{q_1}+|u|^{q_2}\right)\zeta \, dx\,dt \\
&\leq C_3\left(R^{-2q_1'/\kappa+2/\kappa+2N/\mu} + R^{-4q_2'/\kappa+2/\kappa+2N/\mu} \right) \\
&\quad + C_4\left(R^{-2m_1q_2/(\mu(q_2-p_1))+2/\kappa+2N/\mu} + R^{-2m_2q_1/(\mu(q_1-p_2))+2/\kappa+2N/\mu} \right).
\end{aligned} \tag{4.10}
$$

The inequality (3.18) is transformed into (with $\theta = \mu/\kappa > 0$)

$$
\begin{cases}
\text{(i)} & N \leq \theta/(q_1-1), \\
\text{(ii)} & N \leq \theta(q_2+1)/(q_2-1), \\
\text{(iii)} & N \leq m_2q_1/(q_1-p_2) - \theta, \\
\text{(iv)} & N \leq m_1q_2/(q_2-p_1) - \theta,
\end{cases} \tag{4.11}
$$

Then N is the largest integer such that (4.11) holds. Since the discussion occurs according $1/(q_1-1) \geq (q_2+1)/(q_2-1) \iff q_1 \leq 2q_2/(q_2+1)$, the conclusion follows. □

THEOREM 10. *Assume that* $q_1 > \max(1, p_2)$ *and* $q_2 > \max(1, p_1)$. *Then there exists no solution* (u, v) *of the differential inequalities* (4.2) *such that* $\int_{\mathbb{R}^N} u_1 dx \geq 0$ *if one of the following assumptions is fulfilled:*

I – *If* $m_1 q_2/(q_2 - p_1) \leq m_2 q_1/(q_1 - p_2)$, *then* $N \leq m_1(q_2 + 1)/(2(q_2 - p_1))$.

II – *If* $m_1 q_2/(q_2 - p_1) > m_2 q_1/(q_1 - p_2)$, *then* $N \leq m_2 q_1(q_2 + 1)/(2q_2(q_1 - p_2))$.

PROOF. Using the same techniques as in Theorem 8, the inequality (4.10) is replaced by

$$\begin{aligned} &2\int_{\mathbb{R}^N} u_1 \zeta(x,0)dx + \int_0^\infty \int_{\mathbb{R}^N} \left(|v|^{q_1} + |u|^{q_2}\right)\zeta \, dx\, dt \\ &\leq C_3 R^{-4q_2'/\kappa + 2/\kappa + 2N/\mu} \\ &\quad + C_4\left(R^{-2m_1 q_2/(\mu(q_2-p_1))+2/\kappa+2N/\mu} + R^{-2m_2 q_1/(\mu(q_1-p_2))+2/\kappa+2N/\mu}\right). \end{aligned} \tag{4.12}$$

Then (4.11) reduces to

$$\begin{cases} \text{(i)} & N \leq \theta(q_2 + 1)/(q_2 - 1)\,, \\ \text{(iii)} & N \leq m_2 q_1/(q_1 - p_2) - \theta\,, \\ \text{(iii)} & N \leq m_1 q_2/(q_2 - p_1) - \theta\,. \end{cases} \tag{4.13}$$

The end of the proof is the same as in Theorem 8, case III or IV. □

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