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Painlevé Property of a Degenerate Garnier System of (9/2)-type and of a Certain Fourth Order Non-linear Ordinary Differential Equation

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Abstract. In this paper we prove that a degenerate Garnier system of (9/2)-type has the Painlevé property. The restriction of the system to a complex line gives an example of a fourth order non-linear ordinary differential equation such that all the solutions are meromorphic on the whole complex plane.

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1. – Introduction

The purpose of this paper is to prove that a degenerate Garnier system of $(9/2)$-type has the Painlevé property, which means that, for every solution of it, all the movable singular loci (i.e. singular loci depending on the initial condition) are poles. Furthermore we give an example of a fourth order non-linear ordinary differential equation such that all the solutions are meromorphic on the whole complex plane.

As will be explained below, a Garnier system is derived from the isomonodromic deformation of a linear differential equation of the Fuchsian type. The isomonodromic deformation problem, which was initiated by R. Fuchs [2] and developed by R. Garnier [3] and L. Schlesinger [14], has been formulated and extended by several authors [1], [5], [13], [15]. A formulation by K. Okamoto [13] is described as follows. Consider an equation of the Fuchsian type

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{1.1}$$

with the singularities below:

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(a) $x = 0, 1, \infty, t_\nu$ $(\nu = 1, \dots, N)$ are regular singular points with the characteristic exponents $(0, \kappa_0), (0, \kappa_1), (\rho_\infty, \rho_\infty + \kappa_\infty), (0, \theta_\nu)$, respectively, where none of $\kappa_0, \kappa_1, \kappa_\infty, \theta_\nu$ is an integer;

(b) $x = \lambda_k$ $(k = 1, \dots, N)$ are non-logarithmic regular singular points with the characteristic exponents $(0, 2)$.

In this equation the coefficient $a_2(x)$ contains the accessory parameters

$$\mu_k = \operatorname*{Res}_{x=\lambda_k} a_2(x), \quad K_\nu = -\operatorname*{Res}_{x=t_\nu} a_2(x) \quad (k, \nu = 1, \dots, N),$$

and the non-logarithmic condition (b) means that K_ν $(\nu = 1, \dots, N)$ are certain rational functions of $t = (t_1, \dots, t_N), \lambda = (\lambda_1, \dots, \lambda_N), \mu = (\mu_1, \dots, \mu_N)$. Then the isomonodromic deformation of (1.1) is governed by a completely integrable Hamiltonian system of the form

$$(1.2) \qquad \frac{\partial \lambda_k}{\partial t_\nu} = \frac{\partial K_\nu}{\partial \mu_k}, \quad \frac{\partial \mu_k}{\partial t_\nu} = -\frac{\partial K_\nu}{\partial \lambda_k} \quad (k, \nu = 1, \dots, N);$$

that is to say, there exists a fundamental system of solutions of (1.1) whose monodromy representation is independent of $t = (t_1, \dots, t_N)$, if and only if $(\lambda_1(t), \dots, \lambda_N(t))$ and $(\mu_1(t), \dots, \mu_N(t))$ satisfy (1.2). Furthermore, by a symplectic transformation $q_i = q_i(t, \lambda)$, $p_i = p_i(t, \lambda, \mu)$, $s_\nu = s_\nu(t)$ $(i, \nu = 1, \dots, N)$, system (1.2) is changed into a Hamiltonian system of the form

$$(\mathrm{G}_N) \qquad \frac{\partial q_i}{\partial s_\nu} = \frac{\partial L_\nu}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_\nu} = -\frac{\partial L_\nu}{\partial q_i} \quad (i, \nu = 1, \dots, N),$$

where the Hamiltonian functions L_ν $(\nu = 1, \dots, N)$ are polynomials in $(q_1, \dots, q_N)$, $(p_1, \dots, p_N)$ with coefficients rational in $(s_1, \dots, s_N)$ ([8]). In particular, when $N = 1$, the function $\lambda(t)\,(= \lambda_1(t) = q_1(t))$ satisfies the sixth Painlevé equation (VI) ([2]), which follows from (G_1) or (1.2). We call (G_N) (or (1.2)) an N-dimensional Garnier system. The Painlevé property of (G_N) is verified by using the results of T. Miwa [10] and of B. Malgrange [9] (see [4; p. 229, Corollary 7.3.4]).

It is known that the five Painlevé equations (I) to (V) are given by the isomonodromic deformation of linear differential equations with regular and irregular singular points ([1], [3], [6], [13], [15]). These Painlevé equations are also obtained from the sixth Painlevé equation (VI) by the use of a process of step-by-step degeneration, and the corresponding linear equations are derived from (1.1) with $N = 1$ (or an equivalent one) by confluences of singularities ([3], [13]). For 2-dimensional Garnier system (G_2), H. Kimura [7] carried out the process of degeneration, and consequently obtained seven degenerate Garnier systems. They are completely integrable Hamiltonian systems, and govern the isomonodromic deformation of linear differential equations which are derived from (1.1) with $N = 2$ by confluences of singularities. The most degenerate one is written in the form

$$(\mathrm{dG}_{9/2}) \qquad \frac{\partial q_i}{\partial s_\nu} = \frac{\partial H_\nu}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_\nu} = -\frac{\partial H_\nu}{\partial q_i} \quad (i, \nu = 1, 2),$$

where

$$3H_1 = \left(q_2^2 - q_1 - \frac{s_1}{3}\right) p_1^2 + 2q_2p_1p_2 + p_2^2$$
$$+ 9\left(q_1 + \frac{s_1}{3}\right) q_2 \left(q_2^2 - 2q_1 + \frac{s_1}{3}\right) - 3s_2q_1,$$
$$3H_2 = q_2p_1^2 + 2p_1p_2 + 9\left(q_2^4 - 3q_1q_2^2 + q_1^2 - \frac{s_1}{3}q_1 - \frac{s_2}{3}q_2\right).$$

For an arbitrary $a \in \mathbb{C}$, we put $s_1 = a$, $h_2 = H_2(a, s_2, q_1, q_2, p_1, p_2)$. The restriction of an arbitrary solution of ($dG_{9/2}$) to the complex line $s_1 = a$ satisfies the Hamiltonian system

$$\text{(1.3)} \qquad \frac{dq_i}{ds_2} = \frac{\partial h_2}{\partial p_i}, \quad \frac{dp_i}{ds_2} = -\frac{\partial h_2}{\partial q_i} \quad (i = 1, 2).$$

Eliminating p_1, p_2, q_1 from this system, we arrive at a fourth order non-linear ordinary differential equation of the form

$$(GE_{9/2}) \qquad \eta^{(4)} = 20\eta\eta'' + 10(\eta')^2 - 40\eta^3 - 8a\eta - \frac{8}{3}s \qquad ('= d/ds),$$

where $\eta = q_2$, $s = s_2$. Conversely, for every solution η of ($GE_{9/2}$),

$$\text{(1.4)} \quad (q_1, q_2, p_1, p_2) = (-\eta''/4 + 3\eta^2/2 + a/6,\ \eta,\ 3\eta'/2,\ -3\eta^{(3)}/8 + 3\eta\eta')$$

satisfies (1.3). Recently, by M. Noumi and Y. Yamada, higher order non-linear equations of somewhat different types have been obtained from the isomonodromic deformation of certain systems ([12], see also [11]).

The main results are stated as follows:

THEOREM A. *Every solution of* ($dG_{9/2}$) *is meromorphic on* $\mathbb{C}^2$.

THEOREM B. *Every solution of* ($GE_{9/2}$) *is meromorphic on* $\mathbb{C}$.

Since system ($dG_{9/2}$) is completely integrable, Theorem B immediately follows from Theorem A and (1.4). For ($GE_{9/2}$), local expressions of solutions around a movable pole are given by the following:

THEOREM C. *Around each point* $s = s_0 \in \mathbb{C}$, *equation* ($GE_{9/2}$) *possesses two kinds of families of solutions* $\mathcal{S}(s_0) = \{\varphi(s_0, b, b', b''; s) \mid (b, b', b'') \in \mathbb{C}^3\}$ *and* $\mathcal{S}_*(s_0) = \{\varphi_*(s_0, b, b'; s) \mid (b, b') \in \mathbb{C}^2\}$, *in which* $\varphi(s_0, b, b', b''; s)$ *and* $\varphi_*(s_0, b, b'; s)$ *admit Laurent series expansions in powers of* $\sigma = s - s_0$:

$$\varphi(s_0, b, b', b''; s) = \sigma^{-2} + b + c_2\sigma^2 + b'\sigma^3 + c_4\sigma^4 + c_5\sigma^5 + b''\sigma^6 + \sum_{j\geq 7} c_j\sigma^j,$$

$$c_2 = -3b^2 - a/5, \qquad c_4 = -10b^3 - 4ab/7 + s_0/21, \qquad c_5 = 3bb'/2 + 1/30,$$

$$\varphi_*(s_0, b, b'; s) = 3\sigma^{-2} + c_2^*\sigma^2 + c_4^*\sigma^4 + c_5^*\sigma^5 + b\sigma^6 + b'\sigma^8 + \sum_{j\geq 9} c_j^*\sigma^j,$$

$$c_2^* = -a/35, \qquad c_4^* = -s_0/189, \qquad c_5^* = -1/90.$$

Here the coefficients c_j $(j \geq 7)$ *(or* c_j^* $(j \geq 9)$*) are polynomials in* (s_0, b, b', b'') *(or in* (s_0, b, b')*), which are uniquely determined. Conversely every solution with a pole at* $s = s_0$ *belongs to either* $\mathcal{S}(s_0)$ *or* $\mathcal{S}_*(s_0)$.

Theorem A is proved by the following procedure (Sections 2 to 5). As in the case of (G_N), the result of [10] on the Painlevé property of deformation equations in [5] plays an essential role. But our theorem does not immediately follow from this result. In Section 2, we sum up some known facts which will be used in our proof. In addition to the results of [5], [10] (Theorems 2.2, 2.3), we describe a linear differential equation (denoted by $(\mathrm{L}_{9/2})$) whose isomonodromic deformation yields $(\mathrm{dG}_{9/2})$. In Section 3, we find a Schlesinger system (denoted by (S)) from which $(\mathrm{L}_{9/2})$ follows. Because of the property of highly confluence at an irregular singular point of $(\mathrm{L}_{9/2})$, the coefficient matrix of (S) is of a certain restricted form. Furthermore it has an apparent singularity at $z = 0$. For these reasons we cannot immediately apply the result of [10] to (S). On the other hand the symmetric form of (S) is suitable for deriving the deformation equation and for examining its properties. In Section 4, we give the deformation equation (denoted by (DS)) which governs the isomonodromic deformation of (S). In the process of deriving (DS), we have to check the consistency with the restriction on (S) remarked above. In Section 5, using a Schlesinger transformation, we get a Schlesinger system (denoted by (S*)) which has the same monodromic structure as that of (S) and is free from an apparent singularity. Observing the restriction of the form of (S*) and the relation to $(\mathrm{dG}_{9/2})$ carefully, and applying Theorem 2.3, we show the Painlevé property of $(\mathrm{dG}_{9/2})$. In the final section, Theorem C is proved.

2. – Known results

2.1. – Linear differential equation associated with $(\mathrm{dG}_{9/2})$

The following linear differential equation is obtained from (1.1) with $N = 2$ by confluences of singularities:

$$(\mathrm{L}_{9/2}) \qquad \begin{aligned} &\frac{d^2y}{dx^2} - \left(\sum_{k=1,2} \frac{1}{x - \lambda_k}\right)\frac{dy}{dx} \\ &\quad - \left(9x^5 + 9t_1x^3 + 3t_2x^2 + 3K_2x + 3K_1 - \sum_{k=1,2} \frac{\mu_k}{x - \lambda_k}\right) y = 0. \end{aligned}$$

Here $x = \lambda_1, \lambda_2$ are non-logarithmic regular singular points, μ_1, μ_2 are accessory parameters, and K_1, K_2 are rational functions given by

$$3K_1 = \sum_{k=1,2} \frac{P(\lambda_k)}{\Lambda'(\lambda_k)} \left(\mu_k^2 - \frac{\mu_k}{P(\lambda_k)} - 9\lambda_k^5 - 9t_1\lambda_k^3 - 3t_2\lambda_k^2\right),$$

$$3K_2 = \sum_{k=1,2} \frac{1}{\Lambda'(\lambda_k)}(\mu_k^2 - 9\lambda_k^5 - 9t_1\lambda_k^3 - 3t_2\lambda_k^2),$$

$$\Lambda(\xi) = (\xi - \lambda_1)(\xi - \lambda_2), \quad P(\xi) = \xi - \lambda_1 - \lambda_2,$$

which are non-logarithmic conditions of the singularities $x = \lambda_1, \lambda_2$. Then equation ($\mathrm{L}_{9/2}$) has the formal solutions

$$\exp\left(\pm(6x^{7/2}/7 + t_1x^{3/2} + t_2x^{1/2})\right)x^{-1/4}\sum_{j\geq 0} c_j^{\pm}x^{-j/2}, \qquad c_0^{\pm} = 1$$

near $x = \infty$, and the Riemann scheme of it is described as

$$\left(\begin{array}{c|cccccccc} \lambda_k & \multicolumn{8}{c}{\overbrace{\hphantom{-6/7\;0\;0\;0\;-t_1\;0\;-t_2\;1/2}}^{\infty\ (1/2)}} \\ 0 & -6/7 & 0 & 0 & 0 & -t_1 & 0 & -t_2 & 1/2 \\ 2 & 6/7 & 0 & 0 & 0 & t_1 & 0 & t_2 & 1/2 \end{array}\right) \tag{2.1}$$

(see [7; p. 37, p. 40]).

THEOREM 2.1 ([7; pp. 69–73]). *The isomonodromic deformation of* ($\mathrm{L}_{9/2}$) *is governed by* ($\mathrm{dG}_{9/2}$) *with*

$$(q_1, q_2) = (\lambda_1\lambda_2 - t_1/3, \lambda_1 + \lambda_2), \quad (p_1, p_2) = \left(\frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1}, \frac{\lambda_1\mu_1 - \lambda_2\mu_2}{\lambda_1 - \lambda_2}\right),$$
$$(s_1, s_2) = (t_1, -t_2).$$

2.2. – Isomonodromic deformation of a Schlesinger system

Consider a Schlesinger system of the form

$$\frac{d\mathbf{y}}{dz} = A(z)\mathbf{y}, \qquad A(z) = -\sum_{\nu=0}^{6} A_{-\nu}z^{\nu}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{2.2}$$

which has an irregular singular point at $z = \infty$. Here $A_{-\nu}$ $(0 \leq \nu \leq 6)$ are 2 by 2 matrices. Assume that (2.2) possesses the formal fundamental matrix solution

$$Y^{\infty}(z) \exp T^{\infty}(z) \tag{2.3}$$

with

$$Y^{\infty}(z) = \sum_{j\ge 0} Y_j^{\infty} z^{-j}, \quad Y_0^{\infty} = I,$$

$$T^{\infty}(z) = \sum_{j=1}^{7} T_{-j}^{\infty} z^j/(-j) + T_0^{\infty} \log(z^{-1}),$$

$$T_{-j}^{\infty} = \mathrm{diag}[\tau_{-j}^{(1)}, \tau_{-j}^{(2)}] \quad (1 \le j \le 7), \qquad T_0^{\infty} = \mathrm{diag}[\alpha_0, -\alpha_0].$$

By d we denote the exterior differentiation with respect to $\tau = (\tau_{-j}^{(i)})$ ($i = 1, 2$; $1 \le j \le 7$). Consider a matrix of 1-forms with respect to τ expressed as

$$\Omega(z, \tau) = \sum_{\substack{1\le j\le 7\\ i=1,2}} Z_{-j}^{i}(z) d\tau_{-j}^{(i)} = \sum_{k=0}^{7} \Phi_{-k}(\tau) z^k,$$

where $\Phi_{-k}(\tau)$ $(0 \le k \le 7)$ are defined by

$$\sum_{k=-\infty}^{7} \Phi_{-k}(\tau) z^k = Y^{\infty}(z) dT^{\infty}(z) Y^{\infty}(z)^{-1}.$$

We take the entries of τ as the deformation parameters and those of $A_{-\nu}$ $(0 \le \nu \le 6)$ as the unknowns. Then we have the following ([5; Theorem 1 or 3.3]):

THEOREM 2.2. *The isomonodromic deformation of* (2.2) *is governed by*

$$dA(z) = \frac{\partial}{\partial z}\Omega(z, \tau) + [\Omega(z, \tau), A(z)], \tag{2.4}$$

which is completely integrable.

By [10], this system has the Painlevé property:

THEOREM 2.3. *Every solution of* (2.4) *is meromorphic on the universal covering space of* $\{\tau = (\tau_{-j}^{(i)}) \in \mathbb{C}^{14} \mid \tau_{-7}^{(1)} - \tau_{-7}^{(2)} \neq 0\}$.

3. – Schlesinger system which yields ($L_{9/2}$)

We wish to choose a Schlesinger system from which equation ($L_{9/2}$) follows. Consider a system of the form

$$\frac{d\xi}{dz} = B(z)\xi, \qquad B(z) = -\sum_{\nu=-1}^{6} B_{-\nu} z^{\nu}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{S}$$

In this expression the coefficients are given by

$$(3.1)\quad \begin{aligned} &B_{-6}=6J, && B_{-5}=uL, && B_{-4}=PK-QJ, && B_{-3}=vL, \\ &B_{-2}=RK-SJ, && B_{-1}=wL, && B_0=-r(J+K), && B_1=(I-L)/2, \end{aligned}$$

where u, v, w, r, P, Q, R, S are complex parameters and

$$(3.2)\quad I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad J=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad K=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad L=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

The background of the choice of $B_{-\nu}$ $(-1 \le \nu \le 6)$ is explained as follows. For a 2-dimensional system of the form

$$(3.3)\quad \frac{d\mathbf{y}}{dx}=U(x)\mathbf{y},\qquad U(x)=\begin{pmatrix}U_{11}(x)&U_{12}(x)\\U_{21}(x)&U_{22}(x)\end{pmatrix},\quad \mathbf{y}=\begin{pmatrix}y_1\\y_2\end{pmatrix},$$

we have the following:

LEMMA 3.1. *For an arbitrary solution* $\mathbf{y}$ *of* (3.3), *the first entry* $y=y_1$ *satisfies the linear differential equation*

$$(3.4)\quad \begin{aligned} &\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0,\\ &P_1(x)=-U_{11}-U_{22}-\frac{d}{dx}\log U_{12},\\ &P_2(x)=U_{11}U_{22}-U_{12}U_{21}+U_{11}\frac{d}{dx}\log U_{12}-\frac{dU_{11}}{dx}. \end{aligned}$$

By this lemma, if we take $U(x)$ such that

$$\begin{aligned} &U_{11}(x)=-U_{22}(x)=a_2x^2+a_1x+a_0,\\ &U_{12}(x)=x^2+a_1^{12}x+a_0^{12}=(x-\tilde{\lambda}_1)(x-\tilde{\lambda}_2),\\ &U_{21}(x)=9x^3+a_2^{21}x^2+a_1^{21}x+a_0^{21}, \end{aligned}$$

then the coefficients of the corresponding linear equation are

$$P_1(x)=-\sum_{k=1,2}\frac{1}{x-\tilde{\lambda}_k},\qquad P_2(x)=-9x^5-\sum_{\nu=0}^{4}\beta_\nu x^\nu+\sum_{k=1,2}\frac{\tilde{\mu}_k}{x-\tilde{\lambda}_k}$$

(compare with those of ($L_{9/2}$)). On the other hand, for the same $U(x)$, the change of variables

$$x=z^2,\quad \mathbf{y}=\begin{pmatrix}1&0\\0&z\end{pmatrix}H^{-1}\boldsymbol{\xi},\quad H=\begin{pmatrix}3&-1\\3&1\end{pmatrix}$$

takes (3.3) into a system with the same form as of (S), in which u, v, w, r, P, Q, R, S are written as linear combinations of a_m, a_m^{ij}. Therefore, for our purpose, we start from system (S) satisfying (3.1).

In view of Riemann scheme (2.1), in what follows except Proposition 3.2 and its proof, we impose on (S) the condition below containing the deformation parameters t_1, t_2:

(MSC) *System* (S) *possesses the formal fundamental matrix solution*

$$\Xi(z) = Y(z)\exp T(z) \tag{3.5}$$

with

$$\begin{aligned} Y(z) &= \sum_{j\geq 0} Y_j z^{-j}, \qquad Y_0 = I, \\ T(z) &= T_{-7}z^7/(-7) + T_{-3}z^3/(-3) + T_{-1}z/(-1) + T_0\log(z^{-1}), \\ T_{-7} &= 6J, \quad T_{-3} = 3t_1 J, \quad T_{-1} = t_2 J, \quad T_0 = (1/2)I. \end{aligned}$$

The following proposition guarantees the feasibility of imposing such a condition.

PROPOSITION 3.2. *System* (S) *fulfills* (MSC), *if and only if, between the parameters, there exist the relations below:*

$$Q = u^2/12, \tag{3.6}$$

$$S = -3t_1 + uv/6 + (Q^2 - P^2)/12, \tag{3.7}$$

$$r = -t_2 + uw/6 + v^2/12 + (QS - PR)/6. \tag{3.8}$$

Then the relation between (S) and ($L_{9/2}$) is described as follows:

PROPOSITION 3.3. *For an arbitrary solution* $\xi = \xi(z)$ *of* (S), *the first entry of*

$$\mathbf{y}(x) = \begin{pmatrix} 1 & 0 \\ 0 & x^{1/2} \end{pmatrix} H^{-1}\xi(x^{1/2}), \qquad H = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix} \tag{3.9}$$

satisfies equation ($L_{9/2}$) *with* λ_k, μ_k ($k = 1, 2$) *defined by*

$$\lambda_1 + \lambda_2 = (P + Q)/6, \qquad \lambda_1\lambda_2 = -(R + S)/6, \tag{3.10}$$

$$\mu_k = -(u\lambda_k^2 + v\lambda_k + w)/2. \tag{3.11}$$

PROOF OF PROPOSITION 3.2. Note the facts below:

LEMMA 3.4. *Between the matrices given by* (3.2), *the following relations hold:*

$$\begin{aligned} J^2 = I, \qquad K^2 = -I, \qquad L^2 = I, \\ JK = -KJ = L, \qquad KL = -LK = J, \qquad LJ = -JL = -K. \end{aligned}$$

LEMMA 3.5 ([5; Proposition 2.2 and the proof]). *The formal power series matrix* $Y(z) = \sum_{j\geq 0} Y_j z^{-j}$ *in* (3.5) *is decomposed into a product of the form*

$$Y(z) = F(z)D(z), \qquad F(z) = \sum_{j\geq 0} F_j z^{-j}, \qquad D(z) = \sum_{j\geq 0} D_j z^{-j},$$

where

$$F_0 = D_0 = I, \qquad F_j = f_j L + g_j K, \qquad D_j = \mathrm{diag}[d_j^1, d_j^2] \quad (j \ge 1).$$

Moreover the condition (MSC) *is written in the form*

$$F'(z) + F(z)\left(D'(z)D(z)^{-1} + T'(z)\right) = B(z)F(z). \tag{3.12}$$

Suppose the condition (MSC). Comparing the coefficients of z^{6-m} $(1 \le m \le 7)$ in (3.12), we see that $F_j = f_j L + g_j K$ $(1 \le j \le 7)$ satisfy

$$B_{-6+m} = 6[F_m, J] - \sum_{j=1}^{m-1} B_{-6+m-j}F_j + \sum_{j=0}^{m-1} F_j T_{-7+m-j}, \tag{3.13}$$

where $T_{-6} = T_{-5} = T_{-4} = T_{-2} = 0$. Using Lemma 3.4, from (3.13) with $m = 1$, we have

$$uL = -12f_1K - 12g_1L,$$

and hence $F_1 = g_1K$, $g_1 = -u/12$. From (3.13) with $m = 2$, we obtain

$$PK - QJ = -12f_2K - 12g_2L + ug_1J.$$

This implies $Q = -ug_1$, $F_2 = f_2L$, $f_2 = -P/12$. In this way, comparing the coefficients of J, K, L in (3.13) with $1 \le m \le 7$, we derive the relations below:

$$\begin{aligned}
&F_1 = g_1K, && g_1 = -u/12;\\
&Q = -ug_1, &&\\
&F_2 = f_2L, && f_2 = -P/12;\\
&F_3 = g_3K, && g_3 = (Qg_1 - v)/12;\\
&S = -3t_1 - ug_3 + Pf_2 - vg_1, &&\\
&F_4 = f_4L, && f_4 = (Qf_2 - R)/12;\\
&F_5 = g_5K, && g_5 = (Qg_3 + Sg_1 - 3t_1g_1 - w)/12;\\
&r = -t_2 - ug_5 + Pf_4 - vg_3 + Rf_2 - wg_1, &&\\
&F_6 = f_6L, && f_6 = (Qf_4 + Sf_2 - 3t_1f_2 + r)/12;\\
&F_7 = g_7K, && g_7 = (1/2 + Qg_5 + Sg_3 + rg_1 - 3t_1g_3 - t_2g_1)/12,
\end{aligned}$$

from which (3.6), (3.7) and (3.8) immediately follow.

Conversely suppose that (3.6), (3.7) and (3.8) hold. Choose F_m $(1 \le m \le 7)$ as above. Then it is easy to see that $F_*(z) = \sum_{j=0}^{7} F_j z^{-j}$ satisfies

$$F_*(z)T'(z) = \left(B(z) + (\delta_3 z^3 + \delta_1 z + \delta_{-1}z^{-1})I\right) F_*(z) + \sum_{j\ge 2} E_j z^{-j},$$

where δ_i $(i = -1, 1, 3)$ are linear combinations of f_j, g_j $(1 \le j \le 6)$, and E_j $(j \ge 2)$ are 2 by 2 matrices. Observing that $\mathrm{tr}(F_*(z)T'(z)F_*(z)^{-1}) = \mathrm{tr}\,T'(z) = -T_0z^{-1}$, we have $\delta_{-1} = \delta_1 = \delta_3 = 0$, which implies that (3.13) is valid for $1 \le m \le 7$. Furthermore, comparing the coefficients of z^{6-m} $(m \ge 8)$ in (3.12), we can recursively determine F_m $(m \ge 8)$ and D_j $(j \ge 1)$ in such a way that (3.12) holds ([5; Proposition 2.2]). Then the condition (MSC) is fulfilled. Thus the proof is completed. □

In the proof of Proposition 3.3, we use the following:

LEMMA 3.6. *If a linear equation of the form*

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0,$$

where $P_k(x)$ $(k = 1, 2)$ *are rational functions, possesses the formal solutions*

$$(3.14)\quad \varphi_\pm(x) = \exp\big(\pm(6x^{7/2}/7 + t_1x^{3/2} + t_2x^{1/2})\big)x^{-1/4}\sum_{j\geq 0} d_j^\pm x^{-j/2}, \quad d_0^\pm = 1,$$

then near $x = \infty$,

$$P_1(x) = O(x^{-1}), \qquad P_2(x) = -9x^5 - 9t_1x^3 - 3t_2x^2 + O(x).$$

PROOF. Substitute (3.14) into

$$P_1(x) = -\frac{d}{dx}\log W(\varphi_-, \varphi_+), \quad P_2(x) = -\frac{\varphi_-''(x)}{\varphi_-(x)} + \frac{\varphi_-'(x)}{\varphi_-(x)} \cdot \frac{d}{dx}\log W(\varphi_-, \varphi_+),$$

where $W(\varphi_-, \varphi_+) = \varphi_-(x)\varphi_+'(x) - \varphi_-'(x)\varphi_+(x)$. Then we obtain the lemma. □

PROOF OF PROPOSITION 3.3. By a straightforward computation, we can verify that $\mathbf{y} = \mathbf{y}(x)$ given by (3.9) satisfies a system of the form (3.3) with

$$U(x) = \sum_{\nu=0}^{3} U_{-\nu}x^\nu,$$

$$U_{-3} = \begin{pmatrix} 0 & 0 \\ 9 & 0 \end{pmatrix}, \qquad U_{-2} = \begin{pmatrix} -u/2 & 1 \\ 3(P-Q)/2 & u/2 \end{pmatrix},$$

$$U_{-1} = \begin{pmatrix} -v/2 & -(P+Q)/6 \\ 3(R-S)/2 & v/2 \end{pmatrix}, \qquad U_0 = \begin{pmatrix} -w/2 & -(R+S)/6 \\ -3r & w/2 \end{pmatrix}.$$

By Lemma 3.1, the first entry of $\mathbf{y}(x)$ satisfies the linear equation

$$(3.15)\quad \frac{d^2y}{dx^2} - \left(\sum_{k=1,2}\frac{1}{x-\lambda_k}\right)\frac{dy}{dx} - \left(9x^5 + \sum_{j=0}^{4}\gamma_j x^j - \sum_{k=1,2}\frac{\mu_k}{x-\lambda_k}\right)y = 0,$$

where λ_k, μ_k are given by (3.10), (3.11), and γ_j $(0 \leq j \leq 4)$ are polynomials in u, v, w, r, P, Q, R, S. By (MSC) equation (3.15) possesses the same Riemann scheme as (2.1) of $(\mathrm{L}_{9/2})$. Hence, by Lemma 3.6, $\gamma_4 = 0$, $\gamma_3 = 9t_1$, $\gamma_2 = 3t_2$. The non-logarithmic property of the singularities λ_k $(k = 1, 2)$ implies that $\gamma_1/3$ and $\gamma_0/3$ are determined to be rational functions of λ_k, μ_k, t_ν, which are equal to K_2 and K_1 of $(\mathrm{L}_{9/2})$, respectively. Hence (3.15) coincides with $(\mathrm{L}_{9/2})$. Thus the proof is completed. □

4. – Isomonodromic deformation of (S)

The following proposition gives a Pfaffian system which governs the isomonodromic deformation of system (S).

PROPOSITION 4.1. *There exists a fundamental matrix solution of* (S) *whose monodromy representation is independent of* (t_1, t_2), *if and only if* (u, V, v, W, w) *with* $V = P + Q$, $W = R + S$ *satisfies a completely integrable Pfaffian system of the form*

$$\text{(DS)}\qquad \begin{aligned} du &= 2(S - W)dt_1 + 2(Q - V)dt_2, \\ dV &= -(2w + uW/3)dt_1 - (2v + uV/3)dt_2, \\ dv &= \big(2r + (QW - SV)/3\big)dt_1 + 2(S - W)dt_2, \\ dW &= \big(-2 + (wV - vW)/3\big)dt_1 - (2w + uW/3)dt_2, \\ dw &= ((u - rV)/3)dt_1 + 2r dt_2, \end{aligned}$$

where

$$\begin{aligned} Q &= u^2/12, \\ S &= -3t_1 + uv/6 + V(2Q - V)/12, \\ r &= -t_2 + uw/6 + v^2/12 + (QW + SV - VW)/6, \end{aligned}$$

and d denotes the exterior differentiation with respect to (t_1, t_2).

To prove this proposition, we start from the fact below. This is verified by the same argument as in the proof of [5; Theorem 1 or 3.3], though our system (S) has an apparent singularity at $z = 0$.

PROPOSITION 4.2. *Consider a matrix of* 1*-forms with respect to* $t = (t_1, t_2)$ *written in the form*

$$\Omega(z, t) = \sum_{k=0}^{3} \Phi_{-k}(t) z^k$$

with $\Phi_{-k}(t)$ $(0 \le k \le 3)$ *defined by*

$$\text{(4.1)}\qquad \sum_{k=-\infty}^{3} \Phi_{-k}(t) z^k = Y(z)(-z^3 dt_1 - z dt_2) J Y(z)^{-1}.$$

Then the isomonodromic deformation of (S) *is governed by*

$$\text{(4.2)}\qquad dB(z) = \frac{\partial}{\partial z}\Omega(z, t) + [\Omega(z, t), B(z)].$$

We compute $\Phi_{-k}(t)$ $(0 \le k \le 3)$. Substituting the formal series

$$Y(z) = \sum_{j\ge 0} Y_j z^{-j}, \qquad Y(z)^{-1} = \sum_{j \ge 0} Y_j^- z^{-j}, \qquad Y_0 = Y_0^- = I$$

into $B(z) = Y(z)T'(z)Y(z)^{-1} + Y'(z)Y(z)^{-1}$ $('= d/dz)$, and comparing the coefficients, we have, for $3 \le m \le 5$,

$$B_{-m} = 6\sum_{j=0}^{6-m} Y_{6-m-j}JY_j^-. \tag{4.3}$$

On the other hand, from (4.1), we have, for $0 \le k \le 3$,

$$\Phi_{-k}(t) = -\left(\sum_{j=0}^{3-k} Y_j JY_{3-k-j}^-\right)dt_1 - \left(\sum_{j=0}^{1-k} Y_j JY_{1-k-j}^-\right)dt_2. \tag{4.4}$$

By (4.3) and (4.4),

$$\begin{aligned} &\Phi_{-3}(t) = -Jdt_1, && \Phi_{-2}(t) = -(B_{-5}/6)dt_1, \\ &\Phi_{-1}(t) = -(B_{-4}/6)dt_1 - Jdt_2, && \Phi_0(t) = -(B_{-3}/6)dt_1 - (B_{-5}/6)dt_2. \end{aligned} \tag{4.5}$$

Proof of Proposition 4.1. Using (4.5) and

$$\frac{\partial}{\partial z}\Omega(z,t) = 3\Phi_{-3}(t)z^2 + 2\Phi_{-2}(t)z + \Phi_{-1}(t),$$

we can verify that (4.2) is equivalent to the following:

(4.6.1) $du = -2Rdt_1 - 2Pdt_2,$

(4.6.2) $dP = -(2w + Su/3)dt_1 - (2v + Qu/3)dt_2,$

(4.6.3) $dQ = -(Ru/3)dt_1 - (Pu/3)dt_2,$

(4.6.4) $dv = (2r + (QR - PS)/3)dt_1 - 2Rdt_2,$

(4.6.5) $dR = (1 - ur/3 + Qw/3 - Sv/3)dt_1 - (2w + Su/3)dt_2,$

(4.6.6) $dS = (-3 + ur/3 + Pw/3 - Rv/3)dt_1 - (Ru/3)dt_2,$

(4.6.7) $dw = (u/3 - Pr/3 - Qr/3)dt_1 + 2rdt_2,$

(4.6.8) $dr = (Q/6 - P/6 + vr/3)dt_1 - (1 - ur/3)dt_2.$

First we regard the system of these equations as a Pfaffian system with the unknown (u, P, Q, v, R, S, w, r), and denote it by (Pf). Then the completely integrability of (Pf) is verified by a straightforward computation. From (4.6.1) and (4.6.3), we have $d(Q - u^2/12) = dQ - (u/6)du = 0$, which implies that (Pf) has the integral expressed as (3.6). Let (Pf_1) be the Pfaffian system generated by (4.6.m) ($1 \le m \le 8$, $m \ne 3$) with Q given by (3.6). Then (Pf_1) is also completely integrable, and has the integral expressed as (3.7). In fact, by (3.6), (4.6.1), (4.6.2), (4.6.4), (4.6.6),

$$\begin{aligned} &d\left(S - \left(-3t_1 + uv/6 + (Q^2 - P^2)/12\right)\right) \\ =&dS - \left(-3dt_1 + (udv + vdu)/6 + (QdQ - PdP)/6\right) \\ =&dS - (-3 + ur/3 + Pw/3 - Rv/3)dt_1 + (Ru/3)dt_2 = 0. \end{aligned}$$

Repeating such a procedure, we arrive at the completely integrable Pfaffian system (Pf_*) with the unknown (u, P, v, R, w) which is generated by (4.6.m) ($m = 1, 2, 4, 5, 7$) and contains (Q, S, r) given by (3.6), (3.7), (3.8). Every solution of (Pf_*) satisfies system (Pf). Then, by Propositions 3.2 and 4.2, the isomonodromic deformation of (S) is governed by (Pf_*) (with (3.6), (3.7), (3.8)). It is easy to see that the transformation $(V, W) = (P + Q, R + S)$ takes (Pf_*) into system (DS). Thus the proof is completed. □

5. – Proof of Theorem A

We give a Schlesinger transformation, by which the apparent singularity $z = 0$ of (S) is removed. (For the procedure of finding the transformation see [6].)

PROPOSITION 5.1. *By the Schlesinger transformation*

$$\zeta = \Psi(z)\xi, \qquad \Psi(z) = \begin{pmatrix} 1 & 1 \\ -u/12 & -u/12 + z \end{pmatrix}, \tag{5.1}$$

system (S) *is changed into*

$$\frac{d\zeta}{dz} = C(z)\zeta, \qquad C(z) = -\sum_{\nu=0}^{6} C_{-\nu} z^{\nu}, \tag{S*}$$

where

$$C_{-6} = 6J, \quad C_{-5} = \begin{pmatrix} 0 & -12 \\ -P - u^2/12 & 0 \end{pmatrix}, \qquad C_{-4} = \begin{pmatrix} -V & 0 \\ v + uV/6 & V \end{pmatrix},$$

$$C_{-3} = \begin{pmatrix} v + uV/6 & 2V \\ -(R + uv/6 + u^2V/72) & -(v + uV/6) \end{pmatrix}, C_{-2} = \begin{pmatrix} -W & 0 \\ w + uW/6 & W \end{pmatrix},$$

$$C_{-1} = \begin{pmatrix} w + uW/6 & 2W \\ r - uw/6 - u^2W/72 & -(w + uW/6) \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix}.$$

Furthermore this system admits the formal fundamental matrix solution

$$\Xi_*(z) = Y_*(z) \exp T_*(z) \tag{5.2}$$

with

$$Y_*(z) = \sum_{j \geq 0} Y_{*j} z^{-j}, \qquad Y_{*0} = I,$$

$$T_*(z) = -\big((6/7)z^7 + t_1 z^3 + t_2 z - (1/2)\log(z^{-1})\big)J.$$

Let (u, V, v, W, w) be an arbitrary solution of (DS). Then, by Propositions 4.1 and 5.1, system (S*) also has the isomonodromic property. By Theorem 2.2 with $\alpha_0 = 1/2$ of (2.3), deformation equation (2.4) admits a special solution $(A_{-\nu}(\tau); 0 \le \nu \le 6)$ such that, for each ν, the restriction of $A_{-\nu}(\tau)$ to the subspace

$$\tau_{-7}^{(1)} = -\tau_{-7}^{(2)} = 6, \qquad \tau_{-3}^{(1)} = -\tau_{-3}^{(2)} = 3t_1, \qquad \tau_{-1}^{(1)} = -\tau_{-1}^{(2)} = t_2,$$
$$\tau_{-j}^{(i)} = 0 \quad (i = 1, 2;\ j = 2, 4, 5, 6)$$

coincides with $C_{-\nu} = C_{-\nu}(t_1, t_2)$. Hence by Theorem 2.3, the entries V, W, $v + uV/6$, $w + uW/6$ are meromorphic functions of (t_1, t_2). From this fact combined with Proposition 3.3 and Theorem 2.1, it follows that (q_1, q_2, p_1, p_2) with

$$\begin{aligned} &q_1 = -W/6 - s_1/3, \quad q_2 = V/6, \\ &p_1 = (uV/6 + v)/2, \quad p_2 = -V(uV/6 + v)/12 - (uW/6 + w)/2 \end{aligned} \tag{5.3}$$

is a solution of $(\mathrm{dG}_{9/2})$ meromorphic for $(s_1, s_2) = (t_1, -t_2) \in \mathbb{C}^2$. Furthermore the quadruplets with entries (5.3) range over all the solutions of $(\mathrm{dG}_{9/2})$. In fact, for an arbitrary $(q_1^0, q_2^0, p_1^0, p_2^0) \in \mathbb{C}^4$, if we choose a solution $(u^*, V^*, v^*, W^*, w^*)$ of (DS) such that it takes the value $\big(0, 6q_2^0, 2p_1^0, -6q_1^0 - 2t_1^0, -2(p_2^0 + p_1^0 q_2^0)\big)$ at $(t_1, t_2) = (t_1^0, t_2^0)$, then the solution $(q_1^*, q_2^*, p_1^*, p_2^*)$ of $(\mathrm{dG}_{9/2})$ derived by the argument above takes the initial value $(q_1^0, q_2^0, p_1^0, p_2^0)$ at $(s_1, s_2) = (t_1^0, -t_2^0)$. Therefore every solution of $(\mathrm{dG}_{9/2})$ is meromorphic on $\mathbb{C}^2$. This completes the proof of Theorem A.

6. – Proof of Theorem C

6.1. – Formal series expansions

We write

$$\mathcal{D}[\eta] = \eta^{(4)} - 20\eta\eta'' - 10(\eta')^2 + 40\eta^3 + 8a\eta + (8/3)(\sigma + s_0), \qquad \sigma = s - s_0.$$

Observe the following fact:

LEMMA 6.1. *Let $\chi(\sigma) = \sum_{j \ge -2} r_j \sigma^j$ and $\omega_n(\sigma) = \sum_{j \ge n} \rho_j \sigma^j$ be formal Laurent series, where $r_{-2} = 1$ and $n \in \{-1, 0\} \cup \mathbb{N}$. Then we have*

$$\mathcal{D}[\chi(\sigma) + \omega_n(\sigma)] - \mathcal{D}[\chi(\sigma)] = n(n-3)(n-6)(n+3)\rho_n \sigma^{n-4} + \sum_{i \ge n-3} \Gamma_i \sigma^i.$$

PROOF. Observing that

$$\begin{aligned} \mathcal{D}[\chi(\sigma) + \omega_n(\sigma)] - \mathcal{D}[\chi(\sigma)] = {} & \omega_n^{(4)}(\sigma) - 20\big(\omega_n(\sigma)\chi''(\sigma) + \omega_n''(\sigma)\chi(\sigma)\big) \\ & -20\omega_n'(\sigma)\chi'(\sigma) + 120\omega_n(\sigma)\chi(\sigma)^2 + \sum_{i \ge n-3} \tilde{\Gamma}_i \sigma^i, \end{aligned}$$

we have the lemma. □

Similarly we obtain the following:

LEMMA 6.2. *Let* $\chi^*(\sigma) = \sum_{j\ge -2} r_j^* \sigma^j$ *and* $\omega_n^*(\sigma) = \sum_{j\ge n} \rho_j^* \sigma^j$ *be formal Laurent series, where* $r_{-2}^* = 3$ *and* $n \in \{-1, 0\} \cup \mathbb{N}$. *Then we have*

$$\mathcal{D}[\chi^*(\sigma)+\omega_n^*(\sigma)]-\mathcal{D}[\chi^*(\sigma)] = (n-6)(n-8)(n+3)(n+5)\rho_n^*\sigma^{n-4}+\sum_{i\ge n-3}\Gamma_i^*\sigma^i.$$

Suppose that $\eta = \sum_{j\ge -m} c_j\sigma^j$, $c_{-m} \ne 0$, $(m \in \mathbb{N})$ is a formal solution of $(\mathrm{GE}_{9/2})$. Substituting this into $\mathcal{D}[\eta] = 0$ and comparing the terms of the lowest degree, we have $m = 2$, and $c_{-2} = 1$ or 3. Consider the case where $c_{-2} = 1$. By a straightforward computation we can verify that, for arbitrary constants b, b', b'', the function

$$\chi_6(\sigma) = \sigma^{-2} + b + c_2\sigma^2 + b'\sigma^3 + c_4\sigma^4 + c_5\sigma^5 + b''\sigma^6$$

with c_2, c_4, c_5 of the theorem satisfies $\mathcal{D}[\chi_6(\sigma)] = \sum_{i\ge 3}\Gamma_i\sigma^i$, where Γ_i $(i \ge 3)$ are constants depending on s_0, b, b', b''. By Lemma 6.1, $\mathcal{D}[\chi_6(\sigma) + c_7\sigma^7] = (280c_7+\Gamma_3)\sigma^3+\sum_{i\ge 4}\Gamma_i'\sigma^i$. Take $c_7 = -\Gamma_3/280$. Then $\mathcal{D}[\chi_7(\sigma)] = \sum_{i\ge 4}\Gamma_i'\sigma^i$, where $\chi_7(\sigma) = \chi_6(\sigma) + c_7\sigma^7$. Similarly, for $j \ge 7$, we can successively determine c_j $(j \ge 7)$ in such a way that $\chi_j(\sigma) = \chi_6(\sigma) + \sum_{\nu=7}^{j} c_\nu\sigma^\nu$ satisfies $\mathcal{D}[\chi_j(\sigma)] = \sum_{i\ge j-3}\Gamma_i''\sigma^i$. In this way we construct a formal solution of the form $\varphi(s_0, b, b', b''; s) = \chi_6(\sigma) + \sum_{j\ge 7} c_j\sigma^j$. Suppose that $\tilde{\varphi}(s_0, b, b', b''; s) = \sum_{j\ge -2}\tilde{c}_j\sigma^j$ is another formal solution satisfying $\tilde{c}_{-2} = 1$, $\tilde{c}_0 = c_0 = b$, $\tilde{c}_3 = c_3 = b'$, $\tilde{c}_6 = c_6 = b''$, and that $\varphi(s_0, b, b', b''; s) - \tilde{\varphi}(s_0, b, b', b''; s) = \sum_{j\ge j_0}(c_j - \tilde{c}_j)\sigma^j$, $c_{j_0} - \tilde{c}_{j_0} \ne 0$, $j_0 \ne -2, 0, 3, 6$. Then, by Lemma 6.1, we have $j_0(j_0-3)(j_0-6)(j_0+3)(c_{j_0} - \tilde{c}_{j_0}) = 0$, which is a contradiction. Therefore $(\mathrm{GE}_{9/2})$ possesses a unique formal solution of the form $\varphi(s_0, b, b', b''; s)$ $(c_{-2} = 1)$. Using Lemma 6.2, we can similarly verify the existence and uniqueness of a formal solution of the form $\varphi_*(s_0, b, b'; s)$ $(c_{-2}^* = 3)$ as well.

6.2. – Proof of convergence

For a matrix $A = (a_{pq})$ $(1 \le p \le \lambda; 1 \le q \le \mu)$, we define the norm of A by $\|A\| = \max\{\sum_{q=1}^{\mu}|a_{pq}| \mid 1 \le p \le \lambda\}$; in particular, for a row vector $\mathbf{v} = (v_1, \dots, v_\mu)$, $\|\mathbf{v}\| = |v_1| + \cdots + |v_\mu|$. We shall show the convergence of the formal solution $\varphi(s_0, b, b', b''; s)$. Consider a column vector given by

$$\psi = {}^t(\psi_1, \psi_2, \psi_3, \psi_4),$$
$$\psi_1 = \eta - \sigma^{-2}, \quad \psi_2 = \sigma\psi_1', \quad \psi_3 = \sigma\psi_2', \quad \psi_4 = \sigma\psi_3', \quad \sigma = s - s_0.$$

Then $(\mathrm{GE}_{9/2})$ is written in the form

$$\sigma\psi' = \mathbf{a_0}(\sigma) + M\psi + \sum_{1\le\|\mathbf{j}\|\le 3}\mathbf{a_j}(\sigma)\psi^{\mathbf{j}}, \tag{6.1}$$

$$\mathbf{j} = (j_1, j_2, j_3, j_4) \in (\mathbb{N}\cup\{0\})^4, \quad \psi^{\mathbf{j}} = \psi_1^{j_1}\psi_2^{j_2}\psi_3^{j_3}\psi_4^{j_4}.$$

Here $\mathbf{a_j}(\sigma)$ $(0 \le \|\mathbf{j}\| \le 3)$ are 4-dimensional column vectors with entries polynomial in σ such that $\mathbf{a_j}(0) = \mathbf{o}$, and M is a matrix of the form

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -54 & 9 & 6 \end{pmatrix}.$$

Note that (6.1) has the formal solution

$$\begin{aligned}\psi &= {}^t\big(\varphi_0(\sigma),\ \sigma\varphi_0'(\sigma),\ \sigma^2\varphi_0''(\sigma) + \sigma\varphi_0'(\sigma),\ \sigma^3\varphi_0^{(3)}(\sigma) + 3\sigma^2\varphi_0''(\sigma) + \sigma\varphi_0'(\sigma)\big) \\ &\qquad - {}^t(\sigma^{-2}, -2\sigma^{-2}, 4\sigma^{-2}, -8\sigma^{-2}) \\ &= \sum_{k\ge 0} \mathbf{c}_k\sigma^k, \qquad \mathbf{c}_k = {}^t(c_{k1}, c_{k2}, c_{k3}, c_{k4}) \in \mathbb{C}^4,\end{aligned}$$

where $\varphi_0(\sigma) = \varphi(s_0, b, b', b''; s)$. Now we choose an integer k_0 so large that, for $k \ge k_0$, $\|(kI - M)^{-1}\| \le 1$. Then, the formal series $\theta(\sigma) = \sum_{k\ge k_0} \mathbf{c}_k\sigma^k$ satisfies

$$\sigma\theta' - M\theta = \sum_{\|\mathbf{j}\|\le 3} \mathbf{b_j}(\sigma)\theta^{\mathbf{j}}, \tag{6.2}$$

where

$$\mathbf{b_j}(\sigma) = \sum_{i=0}^{d_0} \mathbf{b}_{\mathbf{j}i}\sigma^i, \quad \mathbf{b}_{\mathbf{j}i} = {}^t(b_{\mathbf{j}i1}, b_{\mathbf{j}i2}, b_{\mathbf{j}i3}, b_{\mathbf{j}i4}) \in \mathbb{C}^4, \quad d_0 \in \mathbb{N},$$
$$\mathbf{b_j}(0) = \mathbf{o} \quad \text{for } \|\mathbf{j}\| = 1, \qquad \mathbf{b}_{\mathbf{o}i} = \mathbf{o} \quad \text{for } 0 \le i \le k_0 - 1.$$

By (6.2), for each $k \ge k_0$,

$$\mathbf{c}_k = (kI - M)^{-1}\mathbf{v}_k(\mathbf{b}_{\mathbf{j}i}, \mathbf{c}_\kappa;\ \|\mathbf{j}\| \le 3, 0 \le i \le d_0, k_0 \le \kappa \le k-1),$$

where $\mathbf{v}_k$ is a 4-dimensional column vector function whose entries are polynomials in $b_{\mathbf{j}i\iota}, c_{\kappa\iota}$ $(\|\mathbf{j}\| \le 3, 0 \le i \le d_0, k_0 \le \kappa \le k-1, 1 \le \iota \le 4)$ with positive coefficients. Put

$$\gamma_k = \|\mathbf{v}_k(\beta_0\mathbf{l}, \gamma_\kappa\mathbf{l};\ \|\mathbf{j}\| \le 3, 0 \le i \le d_0, k_0 \le \kappa \le k-1)\| \quad (k \ge k_0)$$

with $\beta_0 = \max\{\|\mathbf{b}_{\mathbf{j}i}\| \mid \|\mathbf{j}\| \le 3, 0 \le i \le d_0\}$, $\mathbf{l} = {}^t(1,1,1,1)$. Then, we see that $\gamma_k \ge \|\mathbf{c}_k\|$ $(k \ge k_0)$, and that the formal series $\Theta(\sigma) = \sum_{k\ge k_0} \gamma_k\sigma^k$ satisfies

$$\Theta = \beta_0\left[\sum_{i=k_0}^{d_0}\sigma^i + \left(\sum_{i=1}^{d_0}\sigma^i\right)\Theta + \left(\sum_{i=0}^{d_0}\sigma^i\right)(10\Theta^2 + 20\Theta^3)\right]. \tag{6.3}$$

By the implicit function theorem, near $\sigma = 0$, equation (6.3) possesses a unique holomorphic solution whose series expansion coincides with $\Theta(\sigma)$. This implies that the series $\varphi(s_0, b, b', b''; s)$ converges around $s = s_0$. The convergence of $\varphi_*(s_0, b, b'; s)$ is also verified in the same way.

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