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A Relaxation Theorem in the Space of Functions of Bounded Deformation

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Abstract. We obtain an integral representation for the relaxation, in the space of functions of bounded deformation, of the energy

$$\int_{\Omega} f(\mathcal{E}u(x))dx$$

with respect to L^1-convergence. Here $\mathcal{E}u$ represents the absolutely continuous part of the symmetrized distributional derivative Eu and the function f satisfies linear growth and coercivity conditions.

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1. – Introduction

The analysis of variational problems that model interesting elastic and magnetic properties exhibited by certain materials often involves minimization and relaxation of nonconvex energies of the type

$$\int_{\Omega} f(\nabla u)dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded set and the density function f satisfies certain growth and coercivity conditions. In each case the function u in (1.1) represents a specific physical entity and belongs to a space which is appropriate to describe the material phenomenon in question.

In the context of perfect plasticity the function u represents the displacement field of a body occupying the reference configuration Ω with volume energy density f. In this case u belongs to the space BD of functions of bounded deformation composed of integrable vector-valued functions u for which all

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components E_{ij}, $i, j = 1, \dots, N$, of the deformation tensor $Eu := \frac{Du+Du^T}{2}$ are bounded Radon measures. Thus, the study of equilibria leads naturally to questions concerning lower semicontinuity properties and relaxation of

$$\int_\Omega f(\mathcal{E}u(x))dx, \tag{1.2}$$

where $\mathcal{E}u$ represents the absolutely continuous part (with respect to the Lebesgue measure) of the symmetrized distributional derivative Eu.

Our goal in this paper is to show that the relaxation of the bulk energy (1.2), in the space of functions of bounded deformation, admits an integral representation where a surface energy term is naturally produced. In a forthcoming paper we will consider more general linear operators A acting on Du for which $A(Du)$ is a bounded Radon measure (see also [12]).

We remark that lower semicontinuity for (1.2) has been established for convex integrands by Bellettini, Coscia and Dal Maso in [5] and for symmetric quasiconvex integrands (see Definition 3.1) by Ebobisse in [9], where he proves that symmetric quasiconvexity is a necessary and sufficient condition for lower semicontinuity. A relaxation result has been proved by Braides, Defranceschi and Vitali in [6] in the case where the bulk energy density is of the form $\|\mathcal{E}u\|^2$ (i.e. $f(\xi) := |\xi|^2$) or $\|\mathcal{E}^D u\|^2 + (\operatorname{div} u)^2$, up to constants (where for any $N \times N$ matrix A, $A^D := A - \frac{1}{N} tr(A) I$ is the deviator of A), and the total energy also includes a surface energy term. In our work no convexity assumptions are placed on the volume density f.

Precisely, for $u \in \mathrm{BD}(\Omega)$ we consider the energy

$$F(u, \Omega) := \begin{cases} \displaystyle\int_\Omega f(\mathcal{E}u(x))dx & \text{if } u \in W^{1,1}(\Omega, \mathbb{R}^N), \\ +\infty & \text{otherwise}, \end{cases}$$

and the localized functional

$$\mathcal{F}(u,V) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \int_V f(\mathcal{E}u_n(x))dx \mid u_n \in W^{1,1}(V,\mathbb{R}^N),\ u_n \to u \text{ in } L^1(V,\mathbb{R}^N) \right\}$$

defined on the set $\mathcal{O}(\Omega)$ of all open subsets V of Ω.

Assuming that f is continuous and satisfies growth and coercivity conditions of the type

$$\frac{1}{C}|\xi| \le f(\xi) \le C(1 + |\xi|) \quad \text{for all } \xi \in \mathrm{M}^{N\times N}_{\mathrm{sym}},$$

where $C > 0$, we show that $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{O}(\Omega)$ of a Radon measure μ which is absolutely continuous with respect to $\mathcal{L}^N + |E^s u|$, where $Eu = \mathcal{E}u\mathcal{L}^N + E^s u$, and $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure. By the Radon-Nikodym Theorem it follows that

$$\mathcal{F}(u, \Omega) = \int_\Omega \mu_a(x)dx + \int_\Omega \mu_s(x)d|E^s u|(x),$$

and the question now is to identify the densities μ_a and μ_s. The main ingredient of our approach is the blow-up method introduced by Fonseca and Müller [10] which reduces the identification of the energy densities of the relaxed functional to the characterization of $\mathcal{F}(v, Q)$ when Q is a unit cube and v is obtained as the blow-up of the function u around a point x_0. Similar ideas have been used in the context of BV-functions (see, for instance [2], [4], [7], [11]) where, in order to identify the density of the absolutely continuous part μ_a one chooses a point x_0 of approximate differentiability of the function u satisfying

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N+1}} \int_{Q(x_0,\varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|dx = 0. \tag{1.3}$$

Using the L^1-convergence, as $\varepsilon \to 0^+$, of the rescaled functions

$$v_\varepsilon(y) := \frac{u(x_0 + \varepsilon y) - u(x_0)}{\varepsilon}$$

to the homogeneous function $v_0(y) = \nabla u(x_0)y$, which is guaranteed by (1.3), it is possible to reduce the identification of the energy density of the absolutely continuous part μ_a, to a relaxation problem where the target function is homogeneous.

However this reasoning cannot be applied to the BD framework where the equivalent of (1.3), replacing $\nabla u(x_0)$ by $\mathcal{E}u(x_0)$, is, in general, false (see [1]). To overcome this difficulty we will use a Poincaré-type inequality

$$\|u - Pu\|_{L^1(Q,\mathbb{R}^N)} \leq C|Eu|(Q),$$

where P is the orthogonal projection onto the kernel of the operator E (cf. [13], see also Theorem 3.1 in [1]). This, together with a compact embedding result, will ensure the L^1-convergence we need to proceed.

We organize the paper as follows: in Section 2 we recall the main properties of the space BD which will be used in the sequel. In Section 3 we state our main theorem, prove a version of De Giorgi's Slicing Lemma and show some properties of the localized functional $\mathcal{F}(u, V)$. Section 4 is devoted to the characterization of the absolutely continuous density μ_a, whereas μ_s is studied in Section 5.

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2. – The space of functions of bounded deformation

Let Ω be a bounded, open subset of $\mathbb{R}^N$ with Lipschitz boundary Γ. We denote by $\mathcal{B}(\Omega)$, $\mathcal{O}(\Omega)$ and $\mathcal{O}_\infty(\Omega)$ the family of Borel, open and open subsets of Ω with Lipschitz boundary, respectively. We use the standard notation, $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$, for the Lebesgue and Hausdorff measures. We recall that the symmetric product between two vectors u and $v \in \mathbb{R}^N$, $u \odot v$, is the symmetric $N \times N$ matrix defined by $u \odot v := (u \otimes v + v \otimes u)/2$, where $\otimes$ indicates the tensor product. We denote by $B(x, \rho)$ the open ball in $\mathbb{R}^N$ of centre x and radius ρ, by $Q(x, \rho)$ the open cube of centre x and sidelength ρ, while $Q_\nu(x, \rho)$ will be used to indicate the cube with two of its faces perpendicular to the unit vector ν. When $x = 0$ and $\rho = 1$ we shall simply write B and Q. Let $Q' := Q \cap \{x \in \mathbb{R}^N \mid x_N = 0\}$, $S^{N-1} := \partial B$ and $\omega_N := \mathcal{L}^N(B)$.

By C we indicate a constant whose value might change from line to line. As usual, $L^q(\Omega, \mathbb{R}^N)$, $W^{m,q}(\Omega, \mathbb{R}^N)$ denote Lebesgue and Sobolev spaces, respectively, $C_0^\infty(\Omega, \mathbb{R}^N)$ is the space of $\mathbb{R}^N$-valued smooth functions with compact support in Ω, and $C^\infty_{\text{per}}(Q, \mathbb{R}^N)$ are the smooth and Q-periodic functions from Q into $\mathbb{R}^N$.

For $u \in L^1(\Omega, \mathbb{R}^N)$ define Ω_u as the set of the Lebesgue points of u, i.e. the set of points $x \in \Omega$ such that there exists $\tilde{u}(x) \in \mathbb{R}^N$ satisfying

$$\lim_{\rho \to 0^+} \frac{1}{\rho^N} \int_{B(x,\rho)} |u(y) - \tilde{u}(x)| dy = 0.$$

The *Lebesgue discontinuity* set S_u of u is defined as the set of points which are not Lebesgue points, that is $S_u := \Omega \setminus \Omega_u$. By Lebesgue's Differentiation Theorem, S_u is $\mathcal{L}^N$-negligible and the function $\tilde{u} : \Omega \to \mathbb{R}^N$, which coincides with u $\mathcal{L}^N$-almost everywhere in Ω_u, is called the *Lebesgue representative* of u. We say that u has *one sided Lebesgue limits* $u^+(x)$ and $u^-(x)$ at $x \in \Omega$ with respect to a suitable direction $\nu_u(x) \in S^{N-1}$, if

$$\lim_{\rho \to 0^+} \frac{1}{\rho^N} \int_{B_\rho^\pm(x, \nu_u(x))} |u(y) - u^\pm(x)| dy = 0\,,$$

where $B_\rho^\pm(x, \nu_u(x)) := \{y \in B(x, \rho) \mid (y - x) \cdot (\pm \nu_u(x)) > 0\}$. Then the rescaled functions $u_\rho(y) := u(x + \rho y)$ converge in $L^1(B, \mathbb{R}^N)$, as $\rho \to 0^+$, to

$$\overline{u}_{x,\nu_u(x)}(y) := \begin{cases} u^+(x) & \text{if } y \cdot \nu_u(x) > 0, \\ u^-(x) & \text{if } y \cdot \nu_u(x) < 0. \end{cases}$$

When $u^+(x) \neq u^-(x)$, the triplet $(u^+(x), u^-(x), \nu_u(x))$ is uniquely determined up to a change of sign of $\nu_u(x)$ and a permutation of $(u^+(x), u^-(x))$. If $u^+(x) = u^-(x)$ then $x \in \Omega_u$ and the one sided Lebesgue limits coincide with the Lebesgue representative of u. The *jump set* J_u of u is defined as the set of points in Ω where the approximate limits u^+, u^- exist and are not equal. We use $[u](x)$ to denote the jump of u at x, i.e. $[u](x) = u^+(x) - u^-(x)$.

DEFINITION 2.1 ([14]). The space of functions of *bounded deformation*, $BD(\Omega)$, is the set of functions $u \in L^1(\Omega, \mathbb{R}^N)$ whose symmetric part of the distributional derivative Du, $Eu := \frac{Du+Du^T}{2}$, is a matrix-valued bounded Radon measure. We denote by $LD(\Omega)$ the subspace of functions $u \in BD(\Omega)$ for which $Eu \in L^1(\Omega, M^{N\times N}_{sym})$.

We recall that Korn's Inequality holds in certain subspaces of $LD(\Omega)$, e.g. for functions $u \in LD(\Omega)$ such that $u \in L^p(\Omega, \mathbb{R}^N)$, with $1 < p < +\infty$, and $Eu \in L^p(\Omega, M^{N\times N}_{sym})$; precisely, there exists a constant $C = C(\Omega)$ such that

$$\sum_{i,j=1}^{N} \int_\Omega |D_i u_j(x)|^p dx \le C(\Omega) \int_\Omega \left(|u(x)|^p + |Eu(x)|^p\right) dx ,$$

and thus this subspace of $LD(\Omega)$ coincides with $W^{1,p}(\Omega, \mathbb{R}^N)$. A counterexample due to Ornstein [15] shows that Korn's Inequality does not hold for $p = 1$ and that the space $LD(\Omega)$ differs from the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^N)$.

We recall some properties of functions of bounded deformation that will be used in the sequel. For a more detailed study of $BD(\Omega)$ we refer to [1], [5], [13], [17], [18], [19] and [20]. Notice first that the space $BD(\Omega)$ endowed with the norm

$$\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega,\mathbb{R}^N)} + |Eu|(\Omega)$$

is a Banach space.

We cannot expect smooth functions to be dense in $BD(\Omega)$ with respect to this topology but it can be shown, see for instance [19], that such a density result is true in a weaker topology (cf. Theorem 2.6).

Given u, $v \in BD(\Omega)$ we define the distance

$$d(u, v) := \|u - v\|_{L^1(\Omega,\mathbb{R}^N)} + \big|\,|Eu|(\Omega) - |Ev|(\Omega)\big|,$$

and we denote by *intermediate topology* the one determined by this distance: a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $BD(\Omega)$ converges to a function $u \in BD(\Omega)$ with respect to this topology, and we write $u_n \overset{i}{\to} u$, if

$$\begin{cases} u_n \to u \ \text{ in } L^1(\Omega, \mathbb{R}^N), \\ Eu_n \overset{*}{\rightharpoonup} Eu \ \text{ in the sense of measures}, \\ |Eu_n|(\Omega) \to |Eu|(\Omega). \end{cases}$$

If $u_n \to u$ in $L^1(\Omega, \mathbb{R}^N)$ and $|Eu_n|(\Omega) \le C$ then $u \in BD(\Omega)$ and $|Eu|(\Omega) \le \liminf_{n\to\infty} |Eu_n|(\Omega)$. Indeed, Eu_n converges weakly-$*$ in the sense of measures to some measure μ, and by the linearity of the operator E, $\mu = Eu$, so that $u \in BD(\Omega)$. The inequality follows from the lower semicontinuity of the total variation of Radon measures.

LEMMA 2.2 (Lemma 4.5 [3]). *Let $u \in \mathrm{BD}(\Omega)$ and let $\rho \in C_0^\infty(\mathbb{R}^N)$ be a non-negative function such that* $\operatorname{supp}(\rho) \subset\subset B(0,1)$, *$\rho(-x) = \rho(x)$ for every $x \in \mathbb{R}^N$ and $\int_{\mathbb{R}^N} \rho(x)dx = 1$. For any $n \in \mathbb{N}$ set $\rho_n(x) := n^N \rho(nx)$ and*

$$u_n(x) := (u * \rho_n)(x) = \int_\Omega u(y)\rho_n(x-y)dy, \text{ for } x \in \left\{ y \in \Omega \mid \operatorname{dist}(y, \partial\Omega) > \frac{1}{n} \right\}.$$

Then $u_n \in C^\infty\left(\left\{y \in \Omega \mid \operatorname{dist}(y,\partial\Omega) > \frac{1}{n}\right\}, \mathbb{R}^N\right)$ and

i) *for any non-negative Borel function $h : \Omega \to \mathbb{R}$*

$$\int_{B(x_0,\delta)} h(x)|\mathcal{E}u_n(x)|dx \leq \int_{B\left(x_0,\delta+\frac{1}{n}\right)} (h * \rho_n)(x)d|Eu|(x),$$

whenever $\delta + \frac{1}{n} < \operatorname{dist}(x_0, \partial\Omega)$;

ii) *for any positively homogeneous of degree one, convex function $\theta : \mathrm{M}^{N\times N}_{\mathrm{sym}} \to [0,+\infty[$ and any $\delta \in]0, \operatorname{dist}(x_0,\partial\Omega)[$ such that $|Eu|(\partial B(x_0,\delta)) = 0$,*

$$\lim_{n\to\infty} \int_{B(x_0,\delta)} \theta(\mathcal{E}u_n(x))dx = \int_{B(x_0,\delta)} \theta\left(\frac{d\,Eu}{d|Eu|}\right) d|Eu|;$$

iii) *$\lim_{n\to\infty} u_n(x) = \tilde{u}(x)$ and $\lim_{n\to\infty}(|u_n - u| * \rho_n)(x) = 0$ for every $x \in \Omega \setminus S_u$, whenever $u \in L^\infty(\Omega, \mathbb{R}^N)$.*

PROOF. i) In the same way as it was shown for the distributional derivative of BV functions, it can be proved that $\mathcal{E}u_n(x) = (Eu * \rho_n)(x)$ for every $x \in \Omega$ with $\operatorname{dist}(x,\partial\Omega) > \frac{1}{n}$. Then, by Fubini's Theorem, we get

$$\begin{aligned}
\int_{B(x_0,\delta)} h(z)|\mathcal{E}u_n(z)|dz &\leq \int_{B(x_0,\delta)} \int_\Omega h(z)\rho_n(z-y)d|Eu|(y)dz \\
&\leq \int_{B\left(x_0,\delta+\frac{1}{n}\right)} \int_{\mathbb{R}^N} h(z)\rho_n(z-y)dz d|Eu|(y) \\
&= \int_{B\left(x_0,\delta+\frac{1}{n}\right)} (h * \rho_n)(y)d|Eu|(y).
\end{aligned}$$

ii) Letting $h \equiv 1$ in part i) and since $|Eu|(\partial B(x_0,\delta)) = 0$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \int_{B(x_0,\delta)} |\mathcal{E}u_n(x)|dx &\leq \limsup_{n\to\infty} \int_{B\left(x_0,\delta+\frac{1}{n}\right)} d|Eu|(x) \\
&= |Eu|(B(x_0,\delta)) \\
&\leq \liminf_{n\to\infty} \int_{B(x_0,\delta)} |\mathcal{E}u_n(x)|dx.
\end{aligned}$$

Thus

$$\lim_{n\to\infty}\int_{B(x_0,\delta)} |\mathcal{E}u_n(x)|dx = |Eu|(B(x_0,\delta)),$$

and since $Eu_n \overset{*}{\rightharpoonup} Eu$ in the sense of measures in $B(x_0,\delta)$, the result now follows by applying Reshetnyak's Theorem (see [16]) (note that the hypotheses on θ imply that $0 \le \theta(w) \le C\|w\|$).

The proof of part iii) is carried out exactly as in [3] since it is independent of the operator E. □

The following result summarizes the properties of the trace operator, see [1], [13] and Chapter II of [19].

THEOREM 2.3. *Assume that Ω is a bounded, open subset of $\mathbb{R}^N$ with Lipschitz boundary Γ. There exists a linear, continuous, and surjective trace operator*

$$\mathrm{tr} : \mathrm{BD}(\Omega) \to L^1(\Gamma, \mathbb{R}^N)$$

such that $\mathrm{tr}\, u = u$ *if* $u \in \mathrm{BD}(\Omega) \cap C(\overline{\Omega}, \mathbb{R}^N)$. *This operator is also continuous with respect to the intermediate topology.*

Furthermore, the following Gauss-Green formula holds

$$\int_\Omega (u \odot D\phi)(x)dx + \int_\Omega \phi(x)dEu(x) = \int_\Gamma \phi(x)(\, \mathrm{tr}\, u \odot \nu)(x)d\mathcal{H}^{N-1} \tag{2.1}$$

for every $\phi \in C^1(\overline{\Omega})$.

Adapting the proof of Lemma II.2.2 in [19] it is easy to show the following proposition (see also [1] and [13]).

PROPOSITION 2.4. *If M is a countably $(\mathcal{H}^{N-1}, N-1)$ rectifiable Borel subset of Ω and $u \in \mathrm{BD}(\Omega)$ then*

$$Eu\lfloor M = (u^+ - u^-) \odot \nu_M \mathcal{H}^{N-1}\lfloor M,$$

where ν_M is a unit normal to M and $u^\pm$ are the traces of u on the sides of M determined by ν_M.

In the sequel, given $u \in \mathrm{BD}(\Omega)$ and $V \in \mathcal{O}_\infty(\Omega)$ we use the notation $\mathrm{tr}\, u$ or $u_{|\partial V}$ to indicate the trace of u on the boundary of V.

PROPOSITION 2.5. *If u, $v \in \mathrm{BD}(\Omega)$, and $\omega \subset \Omega$ has Lipschitz boundary, then the function w defined by*

$$w = \begin{cases} u & \text{in } \omega \\ v & \text{in } \Omega \setminus \omega \end{cases}$$

belongs to $\mathrm{BD}(\Omega)$ *and*

$$Ew = Eu\chi_\omega + Ev\chi_{\Omega\setminus\omega} + (v_{|\partial\omega} - u_{|\partial\omega}) \odot \nu\mathcal{H}^{N-1}\lfloor\partial\omega,$$

where ν is the outward unit normal to $\partial\omega$.

For the proof we refer to [19] and [13]. The following result, proved in [19] (see also [13]), asserts that it is possible to approximate a function $u \in \mathrm{BD}(\Omega)$ by a sequence of smooth functions which preserve the trace of u.

THEOREM 2.6. *Let Ω be a bounded, connected, open set with Lipschitz boundary. For every $u \in \mathrm{BD}(\Omega)$ there exists a sequence of smooth functions $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(\Omega, \mathbb{R}^N) \cap W^{1,1}(\Omega, \mathbb{R}^N)$ such that $u_n \xrightarrow{i} u$ and $\operatorname{tr} u_n = \operatorname{tr} u$. If, in addition, $u \in \mathrm{LD}(\Omega)$, then $\|\mathcal{E}u_n - \mathcal{E}u\|_{L^1\left(\Omega, \mathbb{M}^{N\times N}_{\mathrm{sym}}\right)} \to 0$.*

A Poincaré-type inequality holds, precisely

THEOREM 2.7. *Let Ω be a bounded open set with Lipschitz boundary. There exists a constant $C = C(\Omega)$ such that for every $u \in \mathrm{BD}(\Omega)$ with $\operatorname{tr} u = 0$*

$$\|u\|_{L^1(\Omega,\mathbb{R}^N)} \leq C(\Omega)|Eu|(\Omega). \tag{2.2}$$

For the proof see [19], Remark II.2.5, and also [13].

Note that the kernel of the operator E is the class $\mathcal{R}$ of *rigid motions* in $\mathbb{R}^N$, composed of affine maps of the form $Mx + b$ where M is a skew-symmetric $N \times N$ matrix and $b \in \mathbb{R}^N$, and therefore is closed and finite-dimensional. Hence it is possible to define the orthogonal projection $P : \mathrm{BD}(\Omega) \to \mathcal{R}$. This operator belongs to the class considered in the following Poincaré-Friedrichs type inequality for BD functions (see [1], [13] and [19]).

THEOREM 2.8. *Let Ω be a bounded, connected, open set with Lipschitz boundary, and let $R : \mathrm{BD}(\Omega) \to \mathcal{R}$ be a continuous linear map which leaves the elements of $\mathcal{R}$ fixed. Then there exists a constant $C(\Omega, R)$ such that*

$$\int_\Omega |u - R(u)|dx \leq C(\Omega, R)|Eu|(\Omega) \qquad \text{for every } u \in \mathrm{BD}(\Omega). \tag{2.3}$$

REMARK 2.9. Let $C(\Omega)$ denote the smallest constant for which (2.2), (2.3) hold. Then, as usual, a simple homothety argument shows that $C(\lambda\Omega) = \lambda C(\Omega)$.

The following embedding result, proved in [19], Theorem II.2.4 (see also [13]), will be used in Sections 4 and 5 to obtain strong L^1 convergence of a sequence which is uniformly bounded in $\mathrm{BD}(\Omega)$.

THEOREM 2.10. *Let Ω be an open, bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. The space $\mathrm{BD}(\Omega)$ is compactly embedded in $L^q(\Omega, \mathbb{R}^N)$ for every $1 \leq q < \frac{N}{N-1}$. In particular, if $\{u_n\}$ is bounded in $\mathrm{BD}(\Omega)$ then there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$ in $L^1(\Omega, \mathbb{R}^N)$ and $u \in \mathrm{BD}(\Omega)$.*

The next theorem was proved in [1] (see also [13]).

THEOREM 2.11. *For every $u \in \mathrm{BD}(\Omega)$ the jump set J_u is a countably $(\mathcal{H}^{N-1}, N-1)$ rectifiable Borel set.*

This result, together with Proposition 2.4, yields the following decomposition of the Radon measure Eu (see Definition 4.1 and Remark 4.2 in [1]):

$$Eu = \mathcal{E}u\mathcal{L}^N + E^s u = \mathcal{E}u\mathcal{L}^N + ([u] \odot \nu_u)\mathcal{H}^{N-1}\lfloor J_u + E^c u, \tag{2.4}$$

where $\mathcal{E}u$ is the density of the absolutely continuous part of Eu with respect to $\mathcal{L}^N$, $E^s u$ is the singular part, $E^c u$ is the so-called Cantor part and vanishes on Borel sets B with $\mathcal{H}^{N-1}(B) < +\infty$ (see Proposition 4.4 in [1]).

DEFINITION 2.12 (cf. Definition 4.6 in [1]). The space of *special functions of bounded deformation*, denoted by $SBD(\Omega)$, is the set of functions $u \in \mathrm{BD}(\Omega)$ such that the measure $E^c u$ in (2.4) is zero, i.e.,

$$Eu = \mathcal{E}u\mathcal{L}^N + ([u] \odot \nu_u)\mathcal{H}^{N-1}\lfloor J_u.$$

The next lemma will be used in Section 5 to identify the surface term of the relaxed energy $\mathcal{F}(u, \Omega)$.

LEMMA 2.13 (Lemma 2.6 [11]). *Let* $u \in \mathrm{BD}(\Omega)$. *Then for* $\mathcal{H}^{N-1}$*-a.e.* $x_0 \in J_u$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N-1}} \int_{J_u \cap (x_0 + \varepsilon Q_{\nu(x_0)})} |([u] \odot \nu_u)(x)| d\mathcal{H}^{N-1}(x) = |([u] \odot \nu_u)(x_0)|. \tag{2.5}$$

Its proof is based on the definition of rectifiable set and on the version of Lebesgue's Differentiation Theorem proved by Ambrosio and Dal Maso [2] (see [11] for details).

3. – Statement of the main result

Let $f : \mathrm{M}^{N\times N}_{\mathrm{sym}} \to \mathbb{R}$ be a continuous function satisfying

$$\frac{1}{C}|\xi| \le f(\xi) \le C(1 + |\xi|) \quad \text{for all } \xi \in \mathrm{M}^{N\times N}_{\mathrm{sym}}, \tag{3.1}$$

and for some $C > 0$. Assume also that there exist $\Lambda, L > 0$, $0 \le \beta < 1$ such that

$$\left| f^\infty(\xi) - \frac{f(t\xi)}{t} \right| \le \Lambda \frac{|\xi|^{1-\beta}}{t^\beta} \tag{3.2}$$
$$\text{for all } \xi \in \mathrm{M}^{N\times N}_{\mathrm{sym}} \text{ and for all } t \text{ such that } t|\xi| > L,$$

where the *recession function*, f^∞, of f is defined as

$$f^\infty(\xi) := \limsup_{t\to\infty} \frac{f(t\xi)}{t}.$$

DEFINITION 3.1. The function $f : M^{N\times N}_{\text{sym}} \to \mathbb{R}$ is said to be *symmetric-quasiconvex* if, for every $\xi \in M^{N\times N}_{\text{sym}}$,

$$f(\xi) \leq \int_Q f(\xi + \mathcal{E}u(x))dx \tag{3.3}$$

whenever $u \in C^\infty_{\text{per}}(Q, \mathbb{R}^N)$.

Note that the symmetric quasiconvexity property (3.3) is independent of the size, orientation, and centre of the cube Q.

Given a function $f : M^{N\times N}_{\text{sym}} \to \mathbb{R}$ we define its symmetric quasiconvex envelope SQf by

$$SQf(\xi) := \inf\left\{\int_Q f(\xi + \mathcal{E}u(x))dx \ : \ u \in C^\infty_{\text{per}}(Q, \mathbb{R}^N)\right\}. \tag{3.4}$$

As it was proven for the usual quasiconvex envelope (see [8] Chapter 5), it is possible to show that SQf is the greatest symmetric quasiconvex function that is less than or equal to f. Moreover, the definition (3.4) does not depend on the domain, i.e.

$$SQf(\xi) = \inf\left\{\frac{1}{\mathcal{L}^N(\Omega)}\int_\Omega f(\xi + \mathcal{E}u(x))dx \ : \ u \in C^\infty_0(\Omega, \mathbb{R}^N)\right\}$$

whenever $\Omega \subset \mathbb{R}^N$ is an open, bounded set, with $\mathcal{L}^N(\partial\Omega) = 0$.

REMARK 3.2. If f is upper semicontinuous and locally bounded from above, it is easy to see, using Theorem 2.6 and Fatou's Lemma, that in the definition of symmetric quasiconvexity $C^\infty_{\text{per}}(Q, \mathbb{R}^N)$ may be replaced by $u \in LD_{\text{per}}(Q)$.

REMARK 3.3. Let $f : M^{N\times N}_{\text{sym}} \to \mathbb{R}$ satisfy the growth and coercivity conditions (3.1). Then SQf also satisfies (3.1) and

$$\frac{1}{C}|\xi| \leq f^\infty(\xi) \leq C|\xi|.$$

Moreover, if f is symmetric quasiconvex then so is f^∞.

The first statement is easily verified. As for the second one, we have for any $u \in C^\infty_{\text{per}}(Q, \mathbb{R}^N)$, and for some sequence $\{t_k\}$ with $t_k \to +\infty$,

$$\begin{aligned}
f^\infty(\xi) &= \limsup_{t\to\infty} \frac{f(t\xi)}{t} = \lim_{k\to\infty} \frac{f(t_k\xi)}{t_k} \\
&\leq \liminf_{k\to\infty} \frac{1}{t_k}\int_Q f(t_k\xi + t_k\mathcal{E}u(x))dx \\
&\leq C \lim_{k\to\infty} \int_Q \frac{1 + t_k|\xi + \mathcal{E}u(x)|}{t_k}dx \\
&\quad - \liminf_{k\to\infty} \int_Q \left[\frac{C(1 + t_k|\xi + \mathcal{E}u(x)|)}{t_k} - \frac{1}{t_k} f(t_k\xi + t_k\mathcal{E}u(x))dx\right].
\end{aligned}$$

By the growth condition (3.1), we may apply Fatou's Lemma to obtain

$$f^\infty(\xi) \le \int_Q \limsup_{k\to\infty} \frac{1}{t_k} f(t_k(\xi + \mathcal{E}u(x)))dx \le \int_Q f^\infty(\xi + \mathcal{E}u(x))dx.$$

The following lemma will be used in Section 4.

LEMMA 3.4. *Let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuous function satisfying $|g(v)| \le C(1 + |v|)$ for some constant $C > 0$. If $u_n \to u_0$ in $L^1(\Omega, \mathbb{R}^d)$, with $u_0 \in L^\infty(\Omega, \mathbb{R}^d)$, and $\sup_n \|v_n\|_\infty < +\infty$, then*

$$\lim_{n\to+\infty} \int_\Omega |g(u_n + v_n) - g(u_0 + v_n)|dx = 0.$$

PROOF. We divide the given integral into two terms

$$\begin{cases} J_1 := \displaystyle\int_{\{x\in\Omega\,:\,|u_n(x)-u_0(x)|>1\}} |g(u_n + v_n) - g(u_0 + v_n)|dx\,, \\ J_2 := \displaystyle\int_{\{x\in\Omega\,:\,|u_n(x)-u_0(x)|\le1\}} |g(u_n + v_n) - g(u_0 + v_n)|dx. \end{cases}$$

The growth condition on g, together with the hypotheses on u_n and v_n, ensure that the sequence $\{|g(u_n + v_n)(\cdot) - g(u_0 + v_n)(\cdot)|\}_{n\in\mathbb{N}}$ is equi-integrable. Thus, since $u_n \to u_0$ strongly in $L^1(\Omega, \mathbb{R}^d)$, the measure of the set $\{x \in \Omega : |u_n(x) - u_0(x)| > 1\}$ can be made arbitrarily small and thus for any $\varepsilon > 0$ there exists an n_0 such that $J_1 < \varepsilon/2$ for every $n \ge n_0$.

On the other hand, using the uniform continuity of g on the closed ball of radius $\|u_0\|_\infty + 1 + \sup_n \|v_n\|_\infty$ centered at 0, it follows that $J_2 < \varepsilon/2$ for every $n \ge n_0$. □

Let $u \in \mathrm{BD}(\Omega)$ and define

$$F(u, \Omega) := \begin{cases} \displaystyle\int_\Omega f(\mathcal{E}u(x))dx & \text{if } u \in W^{1,1}(\Omega, \mathbb{R}^N), \\ +\infty & \text{otherwise.} \end{cases}$$

We consider the functional defined for every $V \in \mathcal{O}(\Omega)$ by

$$\mathcal{F}(u,V) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \int_V f(\mathcal{E}u_n(x))dx \mid u_n \in W^{1,1}(V, \mathbb{R}^N),\, u_n \to u \text{ in } L^1(V,\mathbb{R}^N) \right\}.$$

The main result of this paper is the following integral representation for the relaxed energy $\mathcal{F}(u, \Omega)$ of $F(u, \Omega)$ when $u \in SBD(\Omega)$ (see also Proposition 3.9).

THEOREM 3.5. *If $f : \mathbb{M}^{N\times N}_{\mathrm{sym}} \to \mathbb{R}$ is a continuous function satisfying* (3.1) *and* (3.2)*, then for every $u \in SBD(\Omega)$ we have*

$$\mathcal{F}(u, V) = \int_V SQf(\mathcal{E}u(x))dx + \int_{J_u\cap V} (SQf)^\infty(([u] \odot \nu_u)(x))d\mathcal{H}^{N-1}(x)\,,$$

for every open subset V of Ω.

REMARK 3.6. This integral representation does not provide a full description of the functional since we consider only the case where the Cantor part of the deformation tensor is zero. To extend this representation result to the whole space BD a more complete characterization of the Cantor part of the measure Eu is needed.

In this section we shall establish some properties of the relaxed functional, while the remaining sections will be devoted to the characterization of the volume and surface terms, respectively.

We shall use the following version of De Giorgi's Slicing Lemma (see also [4] and [7]) which allows us to modify a sequence on the boundary without increasing the energy.

PROPOSITION 3.7. *Let $V \subset \mathbb{R}^N$ be an open, bounded set and let $f : \mathrm{M}^{N\times N}_{\mathrm{sym}} \to \mathbb{R}$ satisfy the growth condition (3.1). Let $\{u_n\}$, $\{v_n\}$ be sequences of functions in $BD(V)$ such that $u_n - v_n \to 0$ in $L^1(V, \mathbb{R}^N)$, $\sup_n |Eu_n|(V) < +\infty$ and $|Ev_n| \overset{*}{\rightharpoonup} \mu$, $|Ev_n|(V) \to \mu(V)$. Then there exist a subsequence $\{v_{n_k}\}$ and a sequence $\{w_k\} \subset BD(V)$ such that $w_k = v_{n_k}$ near the boundary of V, $w_k - v_{n_k} \to 0$ in $L^1(V, \mathbb{R}^N)$, and*

$$\limsup_{k\to\infty} \int_V f(\mathcal{E}w_k(x))dx \leq \liminf_{n\to\infty} \int_V f(\mathcal{E}u_n(x))dx.$$

PROOF. Without loss of generality we may assume that

$$\liminf_{n\to\infty} \int_V f(\mathcal{E}u_n(x))dx = \lim_{n\to\infty} \int_V f(\mathcal{E}u_n(x))dx < +\infty.$$

Then, by (3.1), there exists a bounded Radon measure λ such that

$$|Eu_n| + |Ev_n| \overset{*}{\rightharpoonup} \lambda.$$

Choose $r_k \nearrow \infty$ as $k \to \infty$ and $\varepsilon_j^k \searrow 0$ as $j \to \infty$ such that

$$(3.5) \qquad \begin{cases} \lambda\left(\left\{x \in V \mid \operatorname{dist}(x, \partial V) = \frac{1}{r_k}\right\}\right) = 0, \\ \mu\left(\left\{x \in V \mid \operatorname{dist}(x, \partial V) = \frac{1}{r_k+\varepsilon_j^k}\right\}\right) = 0 \\ \lambda\left(\left\{x \in V \mid \operatorname{dist}(x, \partial V) = \frac{1}{r_k\pm\varepsilon_j^k}\right\}\right) = 0. \end{cases}$$

Let $\phi_{k,j}$ be a smooth cut-off function such that $0 \leq \phi_{k,j} \leq 1$, $\phi_{k,j}(x) = 1$ if $x \in V$ and $\operatorname{dist}(x, \partial V) > \frac{1}{r_k-\varepsilon_j^k}$, and $\phi_{k,j}(x) = 0$ if $x \in V$ and $\operatorname{dist}(x, \partial V) < \frac{1}{r_k+\varepsilon_j^k}$, and define

$$w_{n,k,j} := \phi_{k,j}u_n + (1 - \phi_{k,j})v_n.$$

Then $w_{n,k,j} \in BD(V)$, $w_{n,k,j} = v_n$ near the boundary of V, and

$$\|w_{n,k,j} - v_n\|_{L^1(V,\mathbb{R}^N)} \leq \|u_n - v_n\|_{L^1(V,\mathbb{R}^N)}.$$

On the other hand, as $Ew_{n,k,j} = \phi_{k,j} Eu_n + (1-\phi_{k,j})Ev_n + (u_n - v_n) \odot D\phi_{k,j}$, from the growth condition on f it follows that

$$\begin{aligned}
&\int_V f(\mathcal{E}w_{n,k,j}(x))dx \\
&\quad \leq \int_V f(\mathcal{E}u_n(x))dx + \int_{V_{k,j}} f(\mathcal{E}v_n(x))dx + \int_{L_{k,j}} f(\mathcal{E}w_{n,k,j}(x))dx \\
&\quad \leq \int_V f(\mathcal{E}u_n(x))dx + C\mathcal{L}^N\left(V_{k,j}\right) + C|Ev_n|\left(V_{k,j}\right) \\
&\qquad + C\Big(\mathcal{L}^N(L_{k,j}) + (|Eu_n| + |Ev_n|)(L_{k,j}) + \|D\phi_{k,j}\|_\infty \|u_n - v_n\|_{L^1(V,\mathbb{R}^N)}\Big)
\end{aligned}$$

where

$$L_{k,j} := \left\{x \in V \mid \frac{1}{r_k + \varepsilon_j^k} < \operatorname{dist}(x, \partial V) < \frac{1}{r_k - \varepsilon_j^k}\right\},$$

$$V_{k,j} := \left\{x \in V \mid \operatorname{dist}(x, \partial V) < \frac{1}{r_k + \varepsilon_j^k}\right\}.$$

Thus, by the strong convergence of $u_n - v_n$ to zero in $L^1(V, \mathbb{R}^N)$ and (3.5),

$$\begin{aligned}
&\limsup_{k\to\infty} \limsup_{j\to\infty} \limsup_{n\to\infty} \int_V f(\mathcal{E}w_{n,k,j}(x))dx \\
&\quad \leq \lim_{n\to\infty} \int_V f(\mathcal{E}u_n(x))dx + C \limsup_{k\to\infty} \limsup_{j\to\infty} \mathcal{L}^N\left(\tilde{V}_{k,j}\right) \\
&\qquad + C \limsup_{k\to\infty} \limsup_{j\to\infty} \mu\left(V_{k,j}\right) + C \limsup_{k\to\infty} \limsup_{j\to\infty} \lambda(L_{k,j}) \\
&\quad \leq \lim_{n\to\infty} \int_V f(\mathcal{E}u_n(x))dx,
\end{aligned}$$

where

$$\tilde{V}_{k,j} := \left\{x \in V \mid \operatorname{dist}(x, \partial V) < \frac{1}{r_k - \varepsilon_j^k}\right\}.$$

By a standard diagonal argument, we may choose a subsequence $n_k \to \infty$ and a sequence $\varepsilon_{j(k)}^k \to 0$ such that $w_k := w_{n_k,k,\varepsilon_{j(k)}^k}$ satisfies the required properties. □

REMARK 3.8. If the sequences $\{u_n\}$, $\{v_n\}$ are more regular, e.g. if u_n, $v_n \in W^{1,1}(V, \mathbb{R}^N)$ or u_n, $v_n \in C^\infty(V, \mathbb{R}^N)$, then so is the sequence $\{w_k\}$.

The localized relaxed functional has the following properties

PROPOSITION 3.9. *Under hypotheses* (3.1) *and* (3.2)

1 – $\mathcal{F}(\cdot, V)$ *is* $L^1(V, \mathbb{R}^N)$ *lower semicontinuous;*
2 – *for every* $u \in \mathrm{BD}(\Omega)$,

$$\frac{1}{C}|Eu|(V) \leq \mathcal{F}(u, V) \leq C(\mathcal{L}^N(V) + |Eu|(V)); \tag{3.6}$$

3 – $\mathcal{F}(u, \cdot)$ *is the restriction to* $\mathcal{O}(\Omega)$ *of a Radon measure.*

We use the same notation for $\mathcal{F}(u, \cdot)$ and its extension to the Borel subsets of Ω.

PROOF. The proof of 1) follows from a standard diagonal argument, whereas 2) is an immediate consequence of the growth conditions on f, of the lower semicontinuity of the total variation of Radon measures and of Theorem 2.6.

To prove 3), we begin by showing that for every $u \in \mathrm{BD}(\Omega)$ and every $U, V, W \in \mathcal{O}(\Omega)$ with $W \subset\subset V \subset\subset U$

$$\mathcal{F}(u, U) \leq \mathcal{F}(u, V) + \mathcal{F}(u, U \setminus \overline{W}). \tag{3.7}$$

Indeed, let $v_n \in W^{1,1}(V, \mathbb{R}^N)$ and $w_n \in W^{1,1}(U \setminus \overline{W}, \mathbb{R}^N)$ be such that $v_n \to u$ in $L^1(V, \mathbb{R}^N)$, $w_n \to u$ in $L^1(U \setminus \overline{W}, \mathbb{R}^N)$,

$$\mathcal{F}(u, V) = \lim_{n\to\infty} \int_V f(\mathcal{E}v_n(x))dx \quad \text{and} \quad \mathcal{F}(u, U \setminus \overline{W}) = \lim_{n\to\infty} \int_{U\setminus\overline{W}} f(\mathcal{E}w_n(x))dx.$$

Let $V_0 \in \mathcal{O}_\infty(\Omega)$ be such that $W \subset\subset V_0 \subset\subset V$ and $|Eu|(\partial V_0) = 0$. Applying Proposition 3.7 and Remark 3.8 to $\{v_n\}$ and u in V_0, we obtain a sequence $\{\overline{v}_n\} \subset W^{1,1}(V_0, \mathbb{R}^N)$ such that $\overline{v}_n = u$ near the boundary of V_0, $\overline{v}_n \to u$ in $L^1(V_0, \mathbb{R}^N)$ and

$$\limsup_{n\to\infty} \int_{V_0} f(\mathcal{E}\overline{v}_n(x))dx \leq \liminf_{n\to\infty} \int_{V_0} f(\mathcal{E}v_n(x))dx.$$

By Proposition 3.7 and Remark 3.8, now applied to $\{w_n\}$ and u in $U \setminus \overline{V}_0$, there exists a sequence $\{\overline{w}_n\} \subset W^{1,1}(U \setminus \overline{V}_0, \mathbb{R}^N)$ such that $\overline{w}_n = u$ in a neighbourhood of ∂V_0, $\overline{w}_n \to u$ in $L^1(U \setminus \overline{V}_0, \mathbb{R}^N)$ and

$$\limsup_{n\to\infty} \int_{U\setminus\overline{V}_0} f(\mathcal{E}\overline{w}_n(x))dx \leq \liminf_{n\to\infty} \int_{U\setminus\overline{V}_0} f(\mathcal{E}w_n(x))dx.$$

Define

$$z_n := \begin{cases} \overline{v}_n & \text{in } V_0, \\ \overline{w}_n & \text{in } U \setminus V_0. \end{cases}$$

Then $z_n \in W^{1,1}(U,\mathbb{R}^N)$, $z_n \to u$ in $L^1(U,\mathbb{R}^N)$, and

$$\begin{aligned}\mathcal{F}(u,U) &\le \liminf_{n\to\infty}\int_U f(\mathcal{E}z_n(x))dx\\ &\le \limsup_{n\to\infty}\int_{V_0} f(\mathcal{E}\overline{v}_n(x))dx + \limsup_{n\to\infty}\int_{U\setminus\overline{V}_0} f(\mathcal{E}\overline{w}_n(x))dx\\ &\le \liminf_{n\to\infty}\int_{V_0} f(\mathcal{E}v_n(x))dx + \liminf_{n\to\infty}\int_{U\setminus\overline{V}_0} f(\mathcal{E}w_n(x))dx\\ &\le \liminf_{n\to\infty}\int_{V} f(\mathcal{E}v_n(x))dx + \liminf_{n\to\infty}\int_{U\setminus\overline{W}} f(\mathcal{E}w_n(x))dx\\ &= \mathcal{F}(u,V)+\mathcal{F}(u,U\setminus\overline{W}).\end{aligned}$$

Now let $u_n \in W^{1,1}(\Omega,\mathbb{R}^N)$ be such that $u_n \to u$ in $L^1(\Omega,\mathbb{R}^N)$ and

$$\mathcal{F}(u,\Omega) = \lim_{n\to\infty}\int_\Omega f(\mathcal{E}u_n(x))dx. \tag{3.8}$$

Since $\{f(\mathcal{E}u_n)(\cdot)\}_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, it follows that there exists a bounded Radon measure μ on $\overline{\Omega}$ such that for a subsequence (not relabelled) $f(\mathcal{E}u_n)\chi_\Omega\mathcal{L}^N \overset{*}{\rightharpoonup} \mu$ in the sense of measures, and so

$$\mathcal{F}(u,\Omega) = \mu(\overline{\Omega}). \tag{3.9}$$

By the definition of $\mathcal{F}(u,\cdot)$, for every $U \in \mathcal{O}(\Omega)$

$$\mathcal{F}(u,U) \le \liminf_{n\to\infty}\int_U f(\mathcal{E}u_n(x))dx \le \mu(\overline{U}). \tag{3.10}$$

Let now $V \in \mathcal{O}(\Omega)$ and $\varepsilon > 0$ be fixed, and consider $W \in \mathcal{O}(\Omega)$ with $W \subset\subset V$ and $\mu(V\setminus W) < \varepsilon$. Then

$$\mu(V) \le \mu(W)+\varepsilon = \mu(\overline{\Omega}) - \mu(\overline{\Omega}\setminus W)+\varepsilon.$$

By (3.9), applying (3.10) with $U := \Omega\setminus\overline{W}$, and then (3.7) for $U := \Omega$, we get

$$\mu(V) \le \mathcal{F}(u,\Omega) - \mu(\overline{\Omega}\setminus W)+\varepsilon \le \mathcal{F}(u,\Omega) - \mathcal{F}(u,\Omega\setminus\overline{W})+\varepsilon \le \mathcal{F}(u,V)+\varepsilon.$$

Letting $\varepsilon \to 0^+$ we obtain $\mu(V) \le \mathcal{F}(u,V) \le \mu(\overline{V})$ for every $V \in \mathcal{O}(\Omega)$. It remains to show that $\mathcal{F}(u,U) \le \mu(U)$ for every $U \in \mathcal{O}(\Omega)$. In order to do this, we fix again an $\varepsilon > 0$ and choose $V, W \in \mathcal{O}(\Omega)$ such that $W \subset\subset V \subset\subset U$ and $\mathcal{L}^N(U\setminus\overline{W}) + |Eu|(U\setminus\overline{W}) < \varepsilon$. By (3.6), (3.7), and (3.10), we get

$$\begin{aligned}\mathcal{F}(u,U) &\le \mathcal{F}(u,U\setminus\overline{W}) + \mathcal{F}(u,V)\\ &\le C(\mathcal{L}^N(U\setminus\overline{W}) + |Eu|(U\setminus\overline{W})) + \mathcal{F}(u,V) \le \varepsilon + \mu(\overline{V}) \le \varepsilon + \mu(U).\end{aligned}$$

To conclude it suffices to let $\varepsilon \to 0^+$. □

For any $u \in \mathrm{BD}(\Omega)$ and $V \in \mathcal{O}_\infty(\Omega)$ define

$$\mathbf{m}(u, V) := \inf\left\{\mathcal{F}(v, V) \mid v_{|\partial V} = u_{|\partial V}\,,\ \ v \in \mathrm{BD}(\Omega)\right\}.$$

In order to identify the surface term of the relaxed energy we will follow the general method for relaxation introduced by Bouchitté, Fonseca and Mascarenhas [7]. The idea is to compare the asymptotic behaviours of $\mathbf{m}(u, Q(x_0, \varepsilon))$ and $\mathcal{F}(u, Q(x_0, \varepsilon))$ as $\varepsilon \to 0^+$, and to show, via a blow-up argument, that relaxation reduces to solving a Dirichlet problem (see Lemma 3.12). The following three lemmas are entirely similar to Lemmas 3.1, 3.3 and 3.5 in [7]; for completeness of the presentation we include their proofs here.

LEMMA 3.10. *There exists a positive constant C such that for any u_1, $u_2 \in$ $\mathrm{BD}(\Omega)$ and any $V \in \mathcal{O}_\infty(\Omega)$ we have*

$$|\mathbf{m}(u_1, V) - \mathbf{m}(u_2, V)| \leq C \int_{\partial V} |\ \mathrm{tr}(u_1 - u_2)(x)| d\mathcal{H}^{N-1}(x).$$

PROOF. Given $\delta > 0$ consider the set $V_\delta := \{x \in V \mid \mathrm{dist}\,(x, \partial V) > \delta\}$ and let $v \in \mathrm{BD}(\Omega)$ be such that $v = u_2$ on ∂V. Define $v_\delta \in \mathrm{BD}(\Omega)$ by $v_\delta := v$ in V_δ and $v_\delta := u_1$ in $\Omega \setminus V_\delta$. Then, by the definition of $\mathbf{m}$, and by virtue of the additivity and locality of $\mathcal{F}$,

$$\mathbf{m}(u_1, V) \leq \mathcal{F}(v_\delta, V) = \mathcal{F}(v, V_\delta) + \mathcal{F}(v_\delta, V \setminus V_\delta). \tag{3.11}$$

By (3.6),

$$\begin{aligned} \mathcal{F}(v_\delta, V \setminus V_\delta) &\leq C\left(\mathcal{L}^N(V \setminus V_\delta) + |Ev_\delta|(V \setminus V_\delta)\right) \\ &= C\left(\mathcal{L}^N(V \setminus V_\delta) + |Eu_1|(V \setminus \overline{V_\delta}) + |E^s v_\delta|(\partial V_\delta)\right). \end{aligned} \tag{3.12}$$

The first two terms in (3.12) tend to zero as $\delta \to 0$, and as for the third one we have

$$\begin{aligned} |E^s v_\delta|(\partial V_\delta) &= \int_{\partial V_\delta} |((v_{|\partial V_\delta} - u_{1|\partial V_\delta}) \odot \nu_\delta)(x)| d\mathcal{H}^{N-1}(x) \\ &\to \int_{\partial V} |((\ \mathrm{tr}\, u_2 -\ \mathrm{tr}\, u_1) \odot \nu)(x)| d\mathcal{H}^{N-1}(x) \ \text{ as } \delta \to 0. \end{aligned} \tag{3.13}$$

Therefore from (3.11), (3.12) and (3.13) it follows that

$$\mathbf{m}(u_1, V) \leq \mathcal{F}(v, V) + C \int_{\partial V} |(\ \mathrm{tr}(u_2 - u_1) \odot \nu)(x)| d\mathcal{H}^{N-1}(x).$$

Taking the infimum over v and using the fact that $|(\ \mathrm{tr}(u_2 - u_1) \odot \nu| \leq C|\ \mathrm{tr}(u_2 - u_1)|$, we obtain

$$\mathbf{m}(u_1, V) - \mathbf{m}(u_2, V) \leq C \int_{\partial V} |\ \mathrm{tr}(u_1 - u_2)(x)| d\mathcal{H}^{N-1}(x).$$

Interchanging the roles of u_1 and u_2 the proof is concluded. □

As in [7], fix $u \in \mathrm{BD}(\Omega)$, $\nu \in S^{N-1}$, and define $\mu := \mathcal{L}^N + |E^s u|$. Let

$$O^* := \{Q_\nu(x, \varepsilon) \mid x \in \Omega,\ \nu \in S^{N-1},\ \varepsilon > 0\},$$

and for $\delta > 0$, $V \in \mathcal{O}(\Omega)$, set

$$\mathbf{m}^\delta(u, V) := \inf\left\{\sum_{i=1}^{\infty} \mathbf{m}(u, Q_i) : Q_i \in O^*,\ Q_i \cap Q_j = \emptyset,\ Q_i \subset V,\right.$$
$$\left. \operatorname{diam}(Q_i) < \delta,\ \mu(V \setminus \cup_{i=1}^{\infty} Q_i) = 0\right\}.$$

Since $\delta \to \mathbf{m}^\delta(u, V)$ is a decreasing function, we define

$$\mathbf{m}^*(u, V) := \sup\left\{\mathbf{m}^\delta(u, V) \ :\ \delta > 0\right\} = \lim_{\delta \to 0^+} \mathbf{m}^\delta(u, V).$$

LEMMA 3.11 (Lemma 3.3 [7]). *Let* $u \in \mathrm{BD}(\Omega)$. *Under conditions* 1), 2), 3) *of Proposition* 3.9, *for any* $V \in \mathcal{O}(\Omega)$

$$\mathcal{F}(u, V) = \mathbf{m}^*(u, V).$$

PROOF. Since $\mathcal{F}(u, \cdot)$ is a Radon measure and $\mathbf{m}(u, V) \leq \mathcal{F}(u, V)$, the inequality $\mathbf{m}^*(u, V) \leq \mathcal{F}(u, V)$ is clear. To show the reverse inequality, fix $\delta > 0$ and let (Q_i^δ) be an admissible family for $\mathbf{m}^\delta(u, V)$ such that

$$\sum_{i=1}^{\infty} \mathbf{m}(u, Q_i^\delta) < \mathbf{m}^\delta(u, V) + \delta \quad \text{and} \quad \mu(N^\delta) = 0, \tag{3.14}$$

where $N^\delta := V \setminus \cup_{i=1}^{\infty} Q_i$. Let now $v_i^\delta \in \mathrm{BD}(\Omega)$ be such that $v^\delta_{i\,|\partial Q_i^\delta} = u_{|\partial Q_i^\delta}$ and

$$\mathcal{F}(v_i^\delta, Q_i^\delta) \leq \mathbf{m}(u, Q_i^\delta) + \delta \mathcal{L}^N(Q_i^\delta). \tag{3.15}$$

Define

$$v^\delta := \sum_{i=1}^{\infty} v_i^\delta \chi_{Q_i^\delta} + u \chi_{N_0^\delta},$$

where $N_0^\delta := \Omega \setminus \cup_{i=1}^{\infty} Q_i^\delta$. Clearly $v^\delta \in L^1(\Omega, \mathbb{R}^N)$. On the other hand, given $\phi \in C_0^\infty(\Omega)$, using the fact that $v^\delta - u = 0$ in N_0^δ, and integrating by parts (see (2.1)) over the cubes Q_i^δ we get

$$\begin{aligned}\langle E(v^\delta - u), \phi\rangle &= \left\langle E\left(\sum_{i=1}^\infty \chi_{Q_i^\delta}(v^\delta - u)\right), \phi\right\rangle \\ &= \sum_{i=1}^\infty \left\langle E\left(\chi_{Q_i^\delta}(v^\delta - u)\right), \phi\right\rangle \\ &= \sum_{i=1}^\infty \int_{\partial\Omega} \phi \operatorname{tr}(\chi_{Q_i^\delta}(v^\delta - u)) \odot \nu d\mathcal{H}^{N-1} - \sum_{i=1}^\infty \int_\Omega \chi_{Q_i^\delta}(v^\delta - u) \odot D\phi dx \\ &= -\sum_{i=1}^\infty \int_{Q_i^\delta} v^\delta \odot D\phi dx + \sum_{i=1}^\infty \int_{Q_i^\delta} u \odot D\phi dx \\ &= -\sum_{i=1}^\infty \int_{\partial Q_i^\delta} \phi(x) \ \operatorname{tr} v_i^\delta \odot \nu d\mathcal{H}^{N-1} + \sum_{i=1}^\infty \int_{Q_i^\delta} \phi dEv_i^\delta \\ &\quad + \sum_{i=1}^\infty \int_{\partial Q_i^\delta} \phi(x) \ \operatorname{tr} v_i^\delta \odot \nu d\mathcal{H}^{N-1} - \sum_{i=1}^\infty \int_{Q_i^\delta} \phi dEu \\ &= \sum_{i=1}^\infty \left\langle Ev_i^\delta \lfloor Q_i^\delta, \phi\right\rangle - \left\langle Eu \lfloor (\Omega \setminus N_0^\delta), \phi\right\rangle.\end{aligned}$$

Hence

$$Ev^\delta = \sum_{i=1}^\infty Ev_i^\delta \lfloor Q_i^\delta + Eu \lfloor N_0^\delta \tag{3.16}$$

in the sense of distributions, and due to the lower bound in (3.6) and (3.14), (3.15), the right-hand side is a bounded Radon measure on Ω, and we conclude that $v^\delta \in \mathrm{BD}(\Omega)$.

By (3.14) and (3.16) $|Ev^\delta|(N^\delta) = |Eu|(N^\delta) \le \mu(N^\delta) = 0$. Thus, by (3.6),

$$\mathcal{F}(v^\delta, N^\delta) \le C(\mathcal{L}^N(N^\delta) + |Ev^\delta|(N^\delta)) = 0.$$

Since $\mathcal{F}(v^\delta, \cdot)$ is the restriction of a measure, by (3.14) and (3.15) it follows that

$$\begin{aligned}\mathcal{F}(v^\delta, V) &= \sum_{i=1}^\infty \mathcal{F}(v_i^\delta, Q_i^\delta) + \mathcal{F}(v^\delta, N^\delta) \\ &\le \sum_{i=1}^\infty \mathbf{m}(u, Q_i^\delta) + \delta\mathcal{L}^N(V) \le \mathbf{m}^\delta(u, V) + \delta + \delta\mathcal{L}^N(V).\end{aligned} \tag{3.17}$$

The result is now a consequence of the lower semicontinuity of $\mathcal{F}(\cdot, V)$ provided we can show that $\{v^\delta\}$ converges to u strongly in $L^1(V, \mathbb{R}^N)$. Indeed, if this is the case, then

$$\mathcal{F}(u, V) \le \liminf_{\delta\to 0} \mathcal{F}(v^\delta, V) \le \liminf_{\delta\to 0} \mathbf{m}^\delta(u, V) = \mathbf{m}^*(u, V).$$

Since $v_i^\delta = u$ on ∂Q_i^δ, by Poincaré's Inequality (2.2) and Remark 2.9, there exists a constant $C > 0$ such that

$$\|v_i^\delta - u\|_{L^1(Q_i^\delta, \mathbb{R}^N)} \leq C\delta |E(v_i^\delta - u)|(Q_i^\delta).$$

Hence

$$\begin{aligned}\|v^\delta - u\|_{L^1(V, \mathbb{R}^N)} &= \sum_{i=1}^{\infty} \|v_i^\delta - u\|_{L^1(Q_i^\delta, \mathbb{R}^N)} \leq C\delta |E(v^\delta - u)|(\cup_{i=1}^\infty Q_i^\delta) \\ &\leq C\delta(|Ev^\delta|(V) + |Eu|(V)).\end{aligned}$$

Since the sequence $\{|Ev^\delta|(V)\}_\delta$ is bounded by the coercivity condition (3.6) and (3.17), it follows that $\{v^\delta\}$ converges to u strongly in $L^1(V, \mathbb{R}^N)$. □

LEMMA 3.12 (Lemma 3.5 [7]). *If $\mathcal{F}$ satisfies conditions* 1)-3) *of Proposition* 3.9 *then*

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}(u, Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} = \lim_{\varepsilon \to 0} \frac{\mathbf{m}(u, Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} \quad \mu \text{ a.e. } x_0 \in \Omega \text{ and for all } \nu \in S^{N-1}.$$

The proof is the same as in [7] since it does not depend on whether we are considering the distributional derivative D or its symmetrized part E.

REMARK 3.13. The conclusion of this lemma still holds replacing $Q_\nu(x_0, \varepsilon)$ by $U(x_0, \varepsilon) := x_0 + \varepsilon U$, where U is any bounded, open, convex subset of $\mathbb{R}^N$ containing the origin.

The following result shows that in the definition of the Dirichlet functional $\mathbf{m}$ one may consider more regular functions.

LEMMA 3.14. *If the function f satisfies* (3.1), *then for every $u \in \mathrm{BD}(\Omega)$ and every $V \in \mathcal{O}_\infty(\Omega)$*

$$\mathbf{m}(u, V) = \mathbf{m_0}(u, V) := \inf\left\{F(v, V) \ : \ v = u \text{ on } \partial V, \ v \in W^{1,1}(\Omega, \mathbb{R}^N)\right\}.$$

PROOF. Clearly $\mathbf{m_0}(u, V) \geq \mathbf{m}(u, V)$. To show the reverse inequality, fix $\varepsilon > 0$ and let $v \in \mathrm{BD}(\Omega)$ be such that $v = u$ on ∂V and $\mathbf{m}(u, V) \geq \mathcal{F}(v, V) - \varepsilon$. Consider a sequence $u_n \in W^{1,1}(\Omega, \mathbb{R}^N)$ such that $u_n \to v$ in $L^1(\Omega, \mathbb{R}^N)$,

$$\mathcal{F}(v, V) = \lim_{n \to \infty} \int_V f(\mathcal{E}u_n(x))dx\,,$$

and, using Theorem 2.6, let $v_n \in W^{1,1}(\Omega, \mathbb{R}^N)$ satisfy $v_n \to v$ in $L^1(\Omega, \mathbb{R}^N)$, $v_n = v$ on ∂V – thus $v_n = u$ on ∂V – and $|Ev_n|(V) \to |Ev|(V)$. It suffices now to apply Proposition 3.7 to $\{u_n\}$ and $\{v_n\}$, and, in light of Remark 3.8, there exists a sequence $\{w_k\}$ of functions in $W^{1,1}(\Omega, \mathbb{R}^N)$ such that $w_k = u$ on ∂V and

$$\mathbf{m_0}(u, V) \leq \liminf_{k \to \infty} \int_V f(\mathcal{E}w_k(x))dx \leq \liminf_{n \to \infty} \int_V f(\mathcal{E}u_n(x))dx \leq \mathbf{m}(u, V) + \varepsilon,$$

and we conclude by letting $\varepsilon \searrow 0^+$. □

4. – The volume term

PROPOSITION 4.1. *If* $u \in \mathrm{BD}(\Omega)$ *then*

$$\mathcal{F}(u, \Omega) \geq \int_\Omega SQf(\mathcal{E}u(x))dx,$$

where $Eu = \mathcal{E}u\mathcal{L}^N + E^s u$.

This result follows immediately from Theorem 3.1 in [9], where f is assumed to be symmetric quasiconvex. For completeness, we provide below a slightly modified argument.

PROOF. Let $u_n \in W^{1,1}(\Omega, \mathbb{R}^N)$ be such that $u_n \to u$ in $L^1(\Omega, \mathbb{R}^N)$ and

$$\mathcal{F}(u, \Omega) = \lim_{n\to\infty} \int_\Omega f(\mathcal{E}u_n(x))dx < +\infty.$$

By passing to a subsequence (not relabelled), there exists a finite non-negative Radon measure λ such that

$$f(\mathcal{E}u_n)\mathcal{L}^N \overset{*}{\rightharpoonup} \lambda \text{ in the sense of measures.}$$

Thus, it suffices to show that for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq SQf(\mathcal{E}u(x_0)).$$

Consider a point $x_0 \in \Omega$ such that

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0} \frac{\lambda(Q(x_0,\varepsilon))}{\varepsilon^N} \text{ exists and is finite,} \tag{4.1}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} |\mathcal{E}u(x) - \mathcal{E}u(x_0)|dx = 0, \tag{4.2}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} |E^s u|(Q(x_0,\varepsilon)) = 0, \tag{4.3}$$

where the sequence of $\varepsilon \searrow 0^+$ is chosen such that $\lambda(\partial Q(x_0,\varepsilon)) = 0$. It is well-known that properties (4.1)-(4.3) hold for $\mathcal{L}^N$-almost every $x_0 \in \Omega$.

Then

$$\begin{aligned} \frac{d\lambda}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} f(\mathcal{E}u_n(x))dx \\ &= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_Q f(\mathcal{E}u_{n,\varepsilon}(y))dy \end{aligned} \tag{4.4}$$

where

$$u_{n,\varepsilon}(y) := \frac{u_n(x_0 + \varepsilon y) - u(x_0)}{\varepsilon}.$$

By (4.1), (4.4), and the coercivity hypothesis on f, we obtain

$$\sup_{n,\varepsilon} |Eu_{n,\varepsilon}|(Q) < +\infty \tag{4.5}$$

and so, applying Theorem 2.8,

$$\sup_{n,\varepsilon} \|u_{n,\varepsilon} - Pu_{n,\varepsilon}\|_{L^1(Q,\mathbb{R}^N)} \leq C,$$

where P is the projection of $BD(Q)$ onto the kernel of the operator E. Theorem 2.10, together with (4.5) and the fact that $EPu_{n,\varepsilon} = 0$, implies the existence of a function $v \in L^1(Q, \mathbb{R}^N)$ such that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \|u_{n,\varepsilon} - Pu_{n,\varepsilon} - v\|_{L^1(Q,\mathbb{R}^N)} = 0, \tag{4.6}$$

and this, in turn, entails

$$Eu_{n,\varepsilon} \overset{*}{\rightharpoonup} Ev \text{ in the sense of distributions.} \tag{4.7}$$

On the other hand, given $\phi \in C_0(Q)$, from the strong convergence in L^1 of u_n to u and the convergence of Eu_n to Eu in the sense of measures, we deduce that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_Q \phi(y)\mathcal{E}u_{n,\varepsilon}(y)dy &= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_Q \phi(y)\mathcal{E}u_n(x_0+\varepsilon y)dy \\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} \phi\left(\frac{x-x_0}{\varepsilon}\right)\mathcal{E}u_n(x)dx \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} \phi\left(\frac{x-x_0}{\varepsilon}\right)dEu(x) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} \phi\left(\frac{x-x_0}{\varepsilon}\right)\mathcal{E}u(x)dx \\
&\quad + \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} \phi\left(\frac{x-x_0}{\varepsilon}\right)dE^s u(x) \\
&= \lim_{\varepsilon\to 0} \int_Q \phi(y)\left[\mathcal{E}u(x_0+\varepsilon y) - \mathcal{E}u(x_0)\right]dy \\
&\quad + \int_Q \phi(y)\mathcal{E}u(x_0)dy \\
&\quad + \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} \phi\left(\frac{x-x_0}{\varepsilon}\right)dE^s u(x) \\
&= \int_Q \phi(y)\mathcal{E}u(x_0)dy
\end{aligned}$$

by (4.2) and (4.3). We have thus showed that $Eu_{n,\varepsilon} \overset{*}{\rightharpoonup} \mathcal{E}u(x_0)\mathcal{L}^N$ in the sense of measures, which, together with (4.7), yields

$$Ev = \mathcal{E}u(x_0)\mathcal{L}^N. \tag{4.8}$$

Extracting a diagonal subsequence $\{\overline{u}_m\}$ of $\{u_{n,\varepsilon} - Pu_{n,\varepsilon}\}$, according to (4.6), (4.7) and (4.8), we then have, by (4.4)

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq \lim_{m\to\infty} \int_Q f(\mathcal{E}\overline{u}_m(x))dx, \qquad \overline{u}_m \to v \text{ in } L^1(Q, \mathbb{R}^N).$$

By virtue of Proposition 3.7, without loss of generality, we may assume that $\overline{u}_m = v$ on ∂Q. Hence by the definition of SQf, Remark 3.2 and (4.8), we have

$$\begin{aligned}\frac{d\lambda}{d\mathcal{L}^N}(x_0) &\geq \lim_{m\to\infty} \int_Q f(\mathcal{E}\overline{u}_m(x))dx \\ &= \liminf_{m\to\infty} \int_Q f(\mathcal{E}u(x_0) + \mathcal{E}(\overline{u}_m - v)(x))dx \\ &\geq SQf(\mathcal{E}u(x_0)).\end{aligned}$$

□

PROPOSITION 4.2. *Let $u \in \mathrm{BD}(\Omega)$. Then for $\mathcal{L}^N$-almost every $x_0 \in \Omega$*

$$\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \leq SQf(\mathcal{E}u(x_0)).$$

PROOF. Choose a point $x_0 \in \Omega$ such that (4.2) and (4.3) are satisfied, and

$$\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0} \frac{\mathcal{F}(u, Q(x_0,\varepsilon))}{\varepsilon^N} \quad \text{exists and is finite,}$$

where the sequence of $\varepsilon \searrow 0^+$ is chosen such that $|Eu|(\partial Q(x_0,\varepsilon)) = 0$.

Fix $\delta > 0$ and let $\phi \in C^\infty_{\text{per}}(Q, \mathbb{R}^N)$ be such that

$$\int_Q f(\mathcal{E}u(x_0) + \mathcal{E}\phi(x))dx \leq SQf(\mathcal{E}u(x_0)) + \delta. \tag{4.9}$$

Extend ϕ to $\mathbb{R}^N$ by periodicity and define

$$\phi_n(x) := \frac{1}{n}\phi(nx).$$

Then $\mathcal{E}\phi_n(x) = \mathcal{E}\phi(nx)$, and we define

$$\overline{u}_n^\varepsilon(x) := (\rho_n * u)(x) + \varepsilon\phi_n\left(\frac{x - x_0}{\varepsilon}\right).$$

Then, as $n\to\infty$, $\overline{u}_n^\varepsilon\to u$ in $L^1(Q(x_0,\varepsilon),\mathbb{R}^N)$, so that $\{\overline{u}_n^\varepsilon\}\subset W^{1,1}(Q(x_0,\varepsilon),\mathbb{R}^N)$ is admissible for $\mathcal{F}(u,Q(x_0,\varepsilon))$. Hence we obtain,

$$\begin{aligned}
\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon\to 0}\frac{\mathcal{F}(u,Q(x_0,\varepsilon))}{\varepsilon^N}\\
&\leq \liminf_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(\mathcal{E}(u*\rho_n)(x)+\mathcal{E}\phi_n\left(\frac{x-x_0}{\varepsilon}\right)\right)dx\\
&\leq \limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} f\left(\mathcal{E}u(x_0)+\mathcal{E}\phi_n\left(\frac{x-x_0}{\varepsilon}\right)\right)dx\\
&\quad+\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}\left[f\left(\mathcal{E}(u*\rho_n)(x)+\mathcal{E}\phi_n\left(\frac{x-x_0}{\varepsilon}\right)\right)\right.\\
&\quad\left.-f\left(\mathcal{E}u(x_0)+\mathcal{E}\phi_n\left(\frac{x-x_0}{\varepsilon}\right)\right)\right]dx\\
&=: I_1+I_2,
\end{aligned}$$

where, by changing variables and using the periodicity of ϕ_n and (4.9),

$$I_1=\lim_{n\to\infty}\int_Q f(\mathcal{E}u(x_0)+\mathcal{E}\phi_n(y))dy=\int_Q f(\mathcal{E}u(x_0)+\mathcal{E}\phi(x))dx\leq SQf(\mathcal{E}u(x_0))+\delta.$$

It remains, therefore, to show that $I_2=0$ and, finally, to let $\delta\to 0^+$.

By Lemma 2.2 and setting $u_0(x)\cdot:=\mathcal{E}u(x_0)x$,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}|\mathcal{E}(u*\rho_n)(x)-\mathcal{E}u(x_0)|dx\\
&=\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}|\mathcal{E}((u-u_0)*\rho_n)(x)|dx\\
&\leq\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q\left(x_0,\varepsilon+\frac{1}{n}\right)}d|E(u-u_0)|(x)\\
&\leq\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q\left(x_0,\varepsilon+\frac{1}{n}\right)}|\mathcal{E}u(x)-\mathcal{E}u(x_0)|dx\\
&\quad+\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q\left(x_0,\varepsilon+\frac{1}{n}\right)}d|E^su|(x)\\
&=\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}|\mathcal{E}u(x)-\mathcal{E}u(x_0)|dx\\
&\quad+\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^N}|E^su|(\overline{Q}(x_0,\varepsilon))=0
\end{aligned}$$

where we have used (4.2), the fact that $\varepsilon\searrow 0^+$ were chosen such that $|Eu|(\partial Q(x_0,\varepsilon))=0$, and (4.3). Then, since $\sup_n\|\mathcal{E}\phi_n\|_{L^\infty(Q)}=||\mathcal{E}\phi||_{L^\infty(Q)}<$

$+\infty$, using Lemma 3.4 we conclude that

$$I_2 = \limsup_{\varepsilon\to 0}\limsup_{n\to\infty} \int_Q [f(\mathcal{E}(u*\rho_n)(x_0+\varepsilon y)+\mathcal{E}\phi_n(y)) \\ -f(\mathcal{E}u(x_0)+\mathcal{E}\phi_n(y))]\,dy = 0.$$

□

5. - The surface term

PROPOSITION 5.1. *Let $u \in SBD(\Omega)$. Then, under hypotheses* (3.1), (3.2),

$$\mathcal{F}(u, V\cap J_u) = \int_{V\cap J_u} g(x, u^+(x), u^-(x), \nu_u(x))d\mathcal{H}^{N-1}(x) \quad \textit{for all } V \in \mathcal{O}(\Omega),$$

where $g(x,\lambda,\theta,\nu) := \limsup_{\varepsilon\to 0} \dfrac{\mathbf{m}(u_{\lambda,\theta,\nu}(\cdot - x), Q_\nu(x,\varepsilon))}{\varepsilon^{N-1}}$

and $u_{\lambda,\theta,\nu}(y) = \begin{cases} \lambda & \textit{if } y\cdot\nu > 0, \\ \theta & \textit{if } y\cdot\nu < 0. \end{cases}$

PROOF. Since u^+, u^- are Borel functions and J_u is rectifiable, writing $\nu := \nu_u(x_0)$, for $\mathcal{H}^{N-1}$ almost every $x_0 \in J_u$ we have (see [1]) $|[u]\odot\nu_u|(x_0) > 0$, and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\int_{Q_\nu^+(x_0,\varepsilon)} |u(x)-u^+(x_0)|\,dx = 0, \tag{5.1}$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\int_{Q_\nu^-(x_0,\varepsilon)} |u(x)-u^-(x_0)|\,dx = 0, \tag{5.2}$$

$$\frac{d\mathcal{F}(u,\cdot)}{d|[u]\odot\nu_u|\mathcal{H}^{N-1}\lfloor J_u}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mathcal{F}(u, Q_\nu(x_0,\varepsilon))}{|[u]\odot\nu_u|\mathcal{H}^{N-1}\lfloor J_u(Q_\nu(x_0,\varepsilon))} \tag{5.3}$$

exists and is finite.

Choose one such point $x_0 \in J_u$. Let $\alpha := |[u]\odot\nu_u|\mathcal{H}^{N-1}\lfloor J_u$ and set $\mu := \mathcal{L}^N + \alpha$. By Lemma 3.12 the limit in (5.3) is equal to

$$\lim_{\varepsilon\to 0^+} \frac{\mathbf{m}(u, Q_\nu(x_0,\varepsilon))}{\alpha(Q_\nu(x_0,\varepsilon))} \quad \text{for } \alpha \text{ a.e. } x_0 \in \Omega.$$

Recall that

$$\mathbf{m}(u,V) = \inf\left\{\mathcal{F}(v,V) \mid v_{|\partial V} = u_{|\partial V},\ v \in BD(\Omega)\right\}.$$

By Lemma 2.13 the point x_0 can be chosen to satisfy also

$$(5.4)\quad \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon^{N-1}}|Eu|(Q_\nu(x_0,\varepsilon)) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon^{N-1}}\alpha(Q_\nu(x_0,\varepsilon)) = |([u]\odot \nu_u)(x_0)|.$$

Let $u_\varepsilon : Q_\nu \to \mathbb{R}^N$ be defined by $u_\varepsilon(y) := u(x_0+\varepsilon y)$. Then, by (5.1) and (5.2) $u_\varepsilon \to \overline{u}_{x_0,\nu}$ strongly in $L^1(Q_\nu, \mathbb{R}^N)$, where

$$\overline{u}_{x_0,\nu}(y) := \begin{cases} u^+(x_0) & \text{if } y\cdot\nu > 0, \\ u^-(x_0) & \text{if } y\cdot\nu < 0. \end{cases}$$

Also, from Lemma 2.13 it follows that

$$|Eu_\varepsilon|(Q_\nu) = \frac{1}{\varepsilon^{N-1}}|Eu|(Q_\nu(x_0,\varepsilon)) \longrightarrow |([u]\odot\nu)(x_0)| = |E\overline{u}_{x_0,\nu}|(Q_\nu)\,,$$

so that $u_\varepsilon \to \overline{u}_{x_0,\nu}$ in the intermediate topology of $BD(Q_\nu)$, and since the trace operator is continuous with respect to this topology (cf. Theorem 2.3), we deduce that

$$(5.5)\quad \begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \int_{\partial Q_\nu(x_0,\varepsilon)} |\ \mathrm{tr}(u - \overline{u}_{x_0,\nu}(\cdot - x_0))|\, d\mathcal{H}^{N-1} \\ &= \lim_{\varepsilon\to 0} \int_{\partial Q_\nu} |\ \mathrm{tr}(u_\varepsilon - \overline{u}_{x_0,\nu})|\, d\mathcal{H}^{N-1} = 0. \end{aligned}$$

We have, by (5.3), Lemma 3.10 and (5.4), (5.5),

$$(5.6)\quad \begin{aligned} &\frac{d\mathcal{F}(u,\cdot)}{d\alpha}(x_0) = \lim_{\varepsilon\to 0+} \frac{\mathcal{F}(u, Q_\nu(x_0,\varepsilon))}{\alpha(Q_\nu(x_0,\varepsilon))} \\ &= \lim_{\varepsilon\to 0+} \frac{\mathbf{m}(u, Q_\nu(x_0,\varepsilon))}{\alpha(Q_\nu(x_0,\varepsilon))} \\ &= \lim_{\varepsilon\to 0+} \frac{\mathbf{m}(u,Q_\nu(x_0,\varepsilon)) - \mathbf{m}(\overline{u}_{x_0,\nu}(\cdot - x_0), Q_\nu(x_0,\varepsilon)) + \mathbf{m}(\overline{u}_{x_0,\nu}(\cdot - x_0), Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \\ &\quad\times \frac{\varepsilon^{N-1}}{\alpha(Q_\nu(x_0,\varepsilon))} \\ &= \lim_{\varepsilon\to 0+} \frac{\mathbf{m}(\overline{u}_{x_0,\nu}(\cdot - x_0), Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} (|([u]\odot\nu)(x_0)|)^{-1}. \end{aligned}$$

Hence

$$\mathcal{F}(u, V\cap J_u) = \int_{V\cap J_u} \frac{d\mathcal{F}(u,\cdot)}{d\alpha}(x)\, d\alpha(x) = \int_{V\cap J_u} g(x, u^+(x), u^-(x), \nu(x))\, d\mathcal{H}^{N-1}. \ \square$$

Remark 5.2 In view of Remark 3.13 and (5.6), setting

$$Q_{k,\varepsilon}(x_0):=\left\{x\in\mathbb{R}^N \mid |(x-x_0)\cdot\nu_i|<\frac{\varepsilon k}{2},\ \ |(x-x_0)\cdot\nu|<\frac{\varepsilon}{2}, i=1,\dots,N-1\right\},$$

where $\{\nu_1,\dots,\nu_{N-1},\nu\}$ is an orthonormal basis of $\mathbb{R}^N$, the surface energy density g is given by

$$g(x_0,\lambda,\theta,\nu)=\lim_{\varepsilon\to 0^+}\frac{\mathbf{m}(u_{\lambda,\theta,\nu}(\cdot-x_0),Q_{k,\varepsilon}(x_0,\varepsilon))}{\varepsilon^{N-1}k^{N-1}},$$

for all $k\in\mathbb{N}$.

We will show now that the surface energy density may be more explicitly characterized.

Proposition 5.3. *If* (3.1), (3.2) *hold and if* $u\in SBD(\Omega)$, *then for* $\mathcal{H}^{N-1}$*-almost every* $x_0\in J_u$

$$g(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0))=(SQf)^\infty(([u]\odot\nu_u)(x_0)).$$

Proof. By Lemma 3.14, for $\mathcal{H}^{N-1}$-almost every $x_0\in J_u$ the function g is given by

$$g(x_0,u^+(x_0),u^-(x_0),\nu)=\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^{N-1}}\inf\left\{\int_{Q_\nu(x_0,\varepsilon)}f(\mathcal{E}w(x))dx\right.$$
$$\left. w\in W^{1,1}(\Omega,\mathbb{R}^N),\ w=\overline{u}_{x_0,\nu}(\cdot-x_0)\text{ on }\partial Q_\nu(x_0,\varepsilon)\right\}$$

where $\nu:=\nu_u(x_0)$, and $\overline{u}_{x_0,\nu}(y):=\begin{cases}u^+(x_0) & \text{if } y\cdot\nu>0,\\ u^-(x_0) & \text{if } y\cdot\nu<0.\end{cases}$ Hence, for any $k\in\mathbb{N}$ there exists a function $w_{k,\varepsilon}$, depending also on ε, such that $w_{k,\varepsilon}=\overline{u}_{x_0,\nu}(\cdot-x_0)$ on $\partial Q_\nu(x_0,\varepsilon)$, and setting $\overline{w}_{k,\varepsilon}(y):=w_{k,\varepsilon}(x_0+\varepsilon y)$,

$$\begin{aligned} g(x_0,u^+(x_0),u^-(x_0),\nu)+\frac{1}{k} &\geq \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^{N-1}}\int_{Q_\nu(x_0,\varepsilon)}f(\mathcal{E}w_{k,\varepsilon}(x))dx\\ &=\limsup_{\varepsilon\to 0}\int_{Q_\nu}\varepsilon f\left(\frac{1}{\varepsilon}\mathcal{E}\overline{w}_{k,\varepsilon}(y)\right)dy \qquad (5.7)\\ &\geq \limsup_{\varepsilon\to 0}\int_{Q_\nu}\varepsilon SQf\left(\frac{1}{\varepsilon}\mathcal{E}\overline{w}_{k,\varepsilon}(y)\right)dy,\end{aligned}$$

and

$$\begin{aligned}&\int_{Q_\nu}\varepsilon SQf\left(\frac{1}{\varepsilon}\mathcal{E}\overline{w}_{k,\varepsilon}(y)\right)dy\\ &=\int_{Q_\nu}\left[\varepsilon SQf\left(\frac{1}{\varepsilon}\mathcal{E}\overline{w}_{k,\varepsilon}(y)\right)-(SQf)^\infty(\mathcal{E}\overline{w}_{k,\varepsilon}(y))\right]dy+\int_{Q_\nu}(SQf)^\infty(\mathcal{E}\overline{w}_{k,\varepsilon}(y))dy\\ &=: I_1+I_2.\end{aligned}$$

Using hypothesis (3.2), the growth condition of $(SQf)^\infty$, and Hölder's inequality, it follows that

$$\begin{aligned}
|I_1| &\le \int_{Q_\nu\cap\{|\mathcal{E}\overline{w}_{k,\varepsilon}|\le L\varepsilon\}} \left|\varepsilon SQf\left(\frac{1}{\varepsilon}\mathcal{E}\overline{w}_{k,\varepsilon}(y)\right) - (SQf)^\infty(\mathcal{E}\overline{w}_{k,\varepsilon}(y))\right| dy \\
&\quad + \int_{Q_\nu\cap\{|\mathcal{E}\overline{w}_{k,\varepsilon}|> L\varepsilon\}} \left|\varepsilon SQf\left(\frac{1}{\varepsilon}\mathcal{E}\overline{w}_{k,\varepsilon}(y)\right) - (SQf)^\infty(\mathcal{E}\overline{w}_{k,\varepsilon}(y))\right| dy \\
&\le \int_{Q_\nu\cap\{|\mathcal{E}\overline{w}_{k,\varepsilon}|\le L\varepsilon\}} \left[\varepsilon C\left(1+\frac{1}{\varepsilon}|\mathcal{E}\overline{w}_{k,\varepsilon}(y)|\right) + C|\mathcal{E}\overline{w}_{k,\varepsilon}(y)|\right] dy \\
&\quad + \int_{Q_\nu} \Lambda\varepsilon^\beta|\mathcal{E}\overline{w}_{k,\varepsilon}(y)|^{1-\beta}dy \\
&\le C\varepsilon + CL\varepsilon + \Lambda\varepsilon^\beta\left(\int_{Q_\nu}|\mathcal{E}\overline{w}_{k,\varepsilon}(y)|dy\right)^{1-\beta}.
\end{aligned}$$

By the coercivity of SQf (cf. Remark 3.3) and (5.7) it follows that

$$+\infty > \sup_{\varepsilon,k}\varepsilon\int_{Q_\nu} SQf\left(\frac{1}{\varepsilon}\mathcal{E}\overline{w}_{k,\varepsilon}(y)\right)dy \ge \sup_{\varepsilon,k}\frac{1}{C}\int_{Q_\nu}|\mathcal{E}\overline{w}_{k,\varepsilon}(y)|dy,$$

and thus

$$\int_{Q_\nu}\varepsilon SQf\left(\frac{1}{\varepsilon}\mathcal{E}\overline{w}_{k,\varepsilon}(y)\right)dy = o(1) + \int_{Q_\nu}(SQf)^\infty(\mathcal{E}\overline{w}_{k,\varepsilon}(y))dy. \tag{5.8}$$

Set

$$z_{k,\varepsilon}(y) := \overline{w}_{k,\varepsilon}(y) - \left[([u]\otimes\nu_u)(x_0)y + \frac{u^-(x_0)+u^+(x_0)}{2}\right].$$

Then $z_{k,\varepsilon}\in W^{1,1}_{\text{per}}(Q_\nu,\mathbb{R}^N)$, so using the symmetric quasiconvexity of $(SQf)^\infty$ (cf. Remark 3.3) we obtain

$$\begin{aligned}
(SQf)^\infty(([u]\odot\nu_u)(x_0)) &\le \int_{Q_\nu}(SQf)^\infty(([u]\odot\nu_u)(x_0)+\mathcal{E}z_{k,\varepsilon}(y))dy \\
&= \int_{Q_\nu}(SQf)^\infty(\mathcal{E}\overline{w}_{k,\varepsilon}(y))dy.
\end{aligned}\tag{5.9}$$

From (5.7), (5.8) and (5.9) we conclude that

$$\begin{aligned}
g(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)) + \frac{1}{k} &\ge \limsup_{\varepsilon\to 0}\left(o(1)+\int_{Q_\nu}(SQf)^\infty(\mathcal{E}\overline{w}_{k,\varepsilon}(y))dy\right) \\
&\ge (SQf)^\infty(([u]\odot\nu_u)(x_0)).
\end{aligned}$$

It suffices to let $k\to\infty$.

To show the reverse inequality, for $x \in Q_{k,\varepsilon}(x_0)$, set

$$w_\varepsilon(x) := ([u] \otimes \nu)(x_0) \left(\frac{x - x_0}{\varepsilon} \right) + \frac{u^+(x_0) + u^-(x_0)}{2},$$

where the parallelipiped $Q_{k,\varepsilon}$ was defined in Remark 5.2. Then w_ε has the same trace as $\overline{u}_{x_0,\nu}(\cdot - x_0)$ when $(x - x_0) \cdot \nu = \pm\frac{\varepsilon}{2}$ but not on the remaining lateral faces of the boundary of $Q_{k,\varepsilon}(x_0)$. In order to obtain an admissible sequence for $\mathbf{m}(\overline{u}_{x_0,\nu}(\cdot - x_0), Q_{k,\varepsilon}(x_0))$, we consider a smooth function $\varphi_{k,\varepsilon} \in C^\infty(Q_{k,\varepsilon}(x_0))$, depending only on $x \cdot \nu_i$, $i = 1, \dots, N-1$, such that $0 \le \varphi_{k,\varepsilon} \le 1$, $\varphi_{k,\varepsilon}(x) = 1$ if $x \in Q_{k-1,\varepsilon}(x_0)$, $\varphi_{k,\varepsilon}(x) = 0$ if $|(x - x_0) \cdot \nu_i| = \frac{\varepsilon k}{2}$ and $\|D\varphi_{k,\varepsilon}\|_\infty = O(\frac{1}{\varepsilon})$. Thus $\varphi_{k,\varepsilon} w_\varepsilon + (1 - \varphi_{k,\varepsilon})\overline{u}_{x_0,\nu}(\cdot - x_0)$ has the same trace as $\overline{u}_{x_0,\nu}(\cdot - x_0)$ on $\partial Q_{k,\varepsilon}(x_0)$, and, in view of Remark 5.2,

$$\begin{aligned} & g(x_0, u^+(x_0), u^-(x_0), \nu) \\ (5.10) \qquad & = \lim_{k\to\infty} \lim_{\varepsilon\to 0} \frac{\mathbf{m}(\overline{u}_{x_0,\nu}(\cdot - x_0), Q_{k,\varepsilon}(x_0))}{(k\varepsilon)^{N-1}} \\ & \le \limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \frac{\mathcal{F}(\varphi_{k,\varepsilon} w_\varepsilon + (1 - \varphi_{k,\varepsilon})\overline{u}_{x_0,\nu}(\cdot - x_0), Q_{k,\varepsilon}(x_0))}{(k\varepsilon)^{N-1}}. \end{aligned}$$

By definition of symmetric quasiconvex envelope, fix $\delta > 0$ and let $\phi \in C^\infty_{\text{per}}(Q_{k,1}(0), \mathbb{R}^N)$ be such that

$$\begin{aligned} (5.11) \qquad & \int_{Q_{k,1}(0)} f\left(\frac{1}{\varepsilon}([u] \odot \nu_u)(x_0) + \mathcal{E}\phi(x) \right) dx \\ & \le k^{N-1}(SQf)\left(\frac{1}{\varepsilon}([u] \odot \nu_u)(x_0) \right) + k^{N-1}\delta. \end{aligned}$$

Extend ϕ to $\mathbb{R}^N$ by periodicity and define $\phi_n(x) := \frac{1}{n}\phi(nx)$.

By Theorem 2.6, let $u_n \in C^\infty(Q_{k,\varepsilon}(x_0), \mathbb{R}^N) \cap W^{1,1}(Q_{k,\varepsilon}(x_0), \mathbb{R}^N)$ be such that $u_n \xrightarrow{i} \overline{u}_{x_0,\nu}(\cdot - x_0)$, $\operatorname{tr} u_n = \operatorname{tr} \overline{u}_{x_0,\nu}(\cdot - x_0)$ on $\partial Q_{k,\varepsilon}(x_0)$, and define

$$u_{k,\varepsilon,n}(x) := (\varphi_{k,\varepsilon} w_\varepsilon + (1 - \varphi_{k,\varepsilon})u_n)(x) + \varepsilon\phi_n\left(\frac{x - x_0}{\varepsilon} \right).$$

Then $u_{k,\varepsilon,n} \to \varphi_{k,\varepsilon} w_\varepsilon + (1 - \varphi_{k,\varepsilon})\overline{u}_{x_0,\nu}(\cdot - x_0)$ strongly in $L^1(Q_{k,\varepsilon}(x_0), \mathbb{R}^N)$ as $n \to \infty$, and $u_{k,\varepsilon,n} \in W^{1,1}(Q_{k,\varepsilon}(x_0), \mathbb{R}^N)$. Hence $\{u_{k,\varepsilon,n}\}$ is admissible for $\mathcal{F}(\varphi_{k,\varepsilon} w_\varepsilon + (1 - \varphi_{k,\varepsilon})\overline{u}_{x_0,\nu}(\cdot - x_0), Q_{k,\varepsilon}(x_0))$ and therefore, as

$$\mathcal{E}u_{k,\varepsilon,n}(x) = \varphi_{k,\varepsilon}(x)(\mathcal{E}w_\varepsilon - \mathcal{E}u_n)(x) + \mathcal{E}u_n(x) + \big((w_\varepsilon - u_n) \odot D\varphi_{k,\varepsilon}\big)(x) + \mathcal{E}\phi_n\left(\frac{x - x_0}{\varepsilon} \right),$$

it follows from (5.10) that

$$
\begin{aligned}
&g(x_0, u^+(x_0), u^-(x_0), \nu)\\
&\leq \limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{(k\varepsilon)^{N-1}}\liminf_{n\to\infty}\int_{Q_{k,\varepsilon}(x_0)} f(\mathcal{E}u_{k,\varepsilon,n}(x))dx\\
&\leq \limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{(k\varepsilon)^{N-1}}\limsup_{n\to\infty}\int_{Q_{k-1,\varepsilon}(x_0)} f\left(\mathcal{E}w_\varepsilon(x)+\mathcal{E}\phi_n\left(\frac{x-x_0}{\varepsilon}\right)\right)dx\\
&\quad+\limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{(k\varepsilon)^{N-1}}\\
&\quad\times\limsup_{n\to\infty}\int_{L_{k,\varepsilon}} C\left(1+|\mathcal{E}w_\varepsilon|(x)+|\mathcal{E}u_n|(x)+\frac{1}{\varepsilon}|w_\varepsilon-u_n|(x)+|\mathcal{E}\phi_n|\left(\frac{x-x_0}{\varepsilon}\right)\right)dx\\
&=: I_1+I_2,
\end{aligned}
$$

where $L_{k,\varepsilon} := Q_{k,\varepsilon}(x_0)\setminus Q_{k-1,\varepsilon}(x_0)$. Now,

$$
\begin{aligned}
I_1 &\leq \limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{(k\varepsilon)^{N-1}}\lim_{n\to\infty}\int_{Q_{k,\varepsilon}(x_0)} f\left(\frac{1}{\varepsilon}([u]\odot\nu_u)(x_0)+\mathcal{E}\phi_n\left(\frac{x-x_0}{\varepsilon}\right)\right)dx\\
&= \limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\frac{\varepsilon}{k^{N-1}}\lim_{n\to\infty}\int_{Q_{k,1}(0)} f\left(\frac{1}{\varepsilon}([u]\odot\nu_u)(x_0)+\mathcal{E}\phi_n(y)\right)dy\\
&\leq \limsup_{\varepsilon\to 0}\left(\varepsilon(SQf)\left(\frac{1}{\varepsilon}([u]\odot\nu_u)(x_0)\right)+\varepsilon\delta\right) = (SQf)^\infty(([u]\odot\nu_u)(x_0)),
\end{aligned}
$$

where we used the periodicity of ϕ and (5.11).

On the other hand, as $\mathcal{L}^N(L_{k,\varepsilon}) = O(\varepsilon^N k^{N-2})$,

$$
\frac{1}{(k\varepsilon)^{N-1}}\int_{L_{k,\varepsilon}} C\left(1+\frac{1}{\varepsilon}|([u]\odot\nu_u)(x_0)|\right)dx = O\left(\frac{\varepsilon}{k}\right)+O\left(\frac{1}{k}\right).
$$

Also

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\frac{1}{(k\varepsilon)^{N-1}}\lim_{n\to\infty}\int_{L_{k,\varepsilon}}|\mathcal{E}\phi_n|\left(\frac{x-x_0}{\varepsilon}\right)dx\\
&= \limsup_{\varepsilon\to 0}\frac{\varepsilon^N}{(k\varepsilon)^{N-1}}\lim_{n\to\infty}\int_{Q_{k,1}(0)}|\mathcal{E}\phi|\left(\frac{y}{n}\right)dy = \limsup_{\varepsilon\to 0}\frac{\varepsilon}{k^{N-1}}|\mathcal{E}\phi|(Q_{k,1}(0)) = 0.
\end{aligned}
$$

From the convergence of u_n to $\overline{u}_{x_0,\nu}(\cdot - x_0)$ in the intermediate topology, together with the fact that

$$
([u]\odot\nu_u)(x_0)\mathcal{H}^{N-1}\lfloor J_{\overline{u}_{x_0,\nu}}(\partial L_{k,\varepsilon}) = 0,
$$

we deduce that

$$
\lim_{n\to\infty}\int_{L_{k,\varepsilon}}|\mathcal{E}u_n|(x)dx = \int_{L_{k,\varepsilon}\cap J_{\overline{u}_{x_0,\nu}}}|([u]\odot\nu_u)(x_0)|d\mathcal{H}^{N-1}(x);
$$

also

$$\lim_{n\to\infty}\int_{L_{k,\varepsilon}} |w_\varepsilon - u_n|(x)dx = \int_{L_{k,\varepsilon}} |w_\varepsilon - \overline{u}_{x_0,\nu}(\cdot - x_0)|(x)dx,$$

with

$$\limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{(k\varepsilon)^{N-1}}|([u]\odot\nu_u)(x_0)|\mathcal{H}^{N-1}(L_{k,\varepsilon}\cap J_{\overline{u}_{x_0,\nu}})$$
$$= \limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{(k\varepsilon)^{N-1}}O(\varepsilon^{N-1}k^{N-2}) = 0,$$

and

$$\limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{(k\varepsilon)^{N-1}}\int_{L_{k,\varepsilon}}\frac{1}{\varepsilon}|w_\varepsilon - \overline{u}_{x_0,\nu}(\cdot - x_0)|(x)dx = 0,$$

since $|(x-x_0)\cdot\nu| < \frac{\varepsilon}{2}$ in $L_{k,\varepsilon}$, which implies that $|w_\varepsilon - \overline{u}_{x_0,\nu}(\cdot - x_0)| \leq C$, and where we used once again the fact that $\mathcal{L}^N(L_{k,\varepsilon}) = O(\varepsilon^N k^{N-2})$. Thus we conclude that $I_2 = 0$. □

REFERENCES

[1] L. Ambrosio – A. Coscia – G. Dal Maso, *Fine properties of functions with bounded deformation*, Arch. Rat. Mech. Anal. **139** (1997), 201-238.

[2] L. Ambrosio – G. Dal Maso, *On the relaxation in $BV(\Omega;\mathbf{R}^m)$ of quasi-convex integrals*, J. Funct. Anal. **109** (1992), 76-97.

[3] L. Ambrosio – S. Mortola – V. M. Tortorelli, *Functionals with linear growth defined on vector valued BV functions*, J. Math. Pures et Appl. **70** (1991), 269-323.

[4] A. C. Barroso – G. Bouchitté – G. Buttazzo – I. Fonseca, *Relaxation of bulk and interfacial energies*, Arch. Rat. Mech. Anal. **135** (1996), 107-173.

[5] G. Bellettini – A. Coscia – G. Dal Maso, *Special functions of bounded deformation*, Math. Z. **228** (1998), 337-351.

[6] A. Braides – A. Defranceschi – E. Vitali, *A relaxation approach to Hencky's plasticity*, Appl. Math. Optimization **35** (1997), 45-68.

[7] G. Bouchitté – I. Fonseca – L. Mascarenhas, *A global method for relaxation*, Arch. Rat. Mech. Anal. **145** (1998), 51-98.

[8] B. Dacorogna, "Direct Methods in the Calculus of Variations", Springer, 1989.

[9] F. B. Ebobisse, *On lower semicontinuity of integral functionals in $LD(\Omega)$*, Ricerche Mat. (to appear).

[10] I. Fonseca – S. Müller, *Quasiconvex integrands and lower semicontinuity in L^1*, SIAM J. Math. Anal. **23** (1992), 1081-1098.

[11] I. Fonseca – S. Müller, *Relaxation of quasiconvex functionals in $BV(\Omega;\mathbf{R}^p)$*, Arch. Rat. Mech. Anal. **123** (1993), 1-49.

[12] I. Fonseca – S. Müller, *$\mathcal{A}$-quasiconvexity, lower semicontinuity and Young measures*, SIAM J. Math. Anal. **30** (1999), 1355-1390.

[13] R. V. Kohn, "New Estimates for Deformations in Terms of Their Strains", Ph.D. Thesis, Princeton University, 1979.

[14] H. Matthies – G. Strang – E. Christiansen, "The Saddle Point of a Differential Program", Energy Methods in Finite Element Analysis, Wiley, New York, 1979.

[15] D. A. Ornstein, *Non-inequality for differential operators in the L^1-norm*, Arch. Rat. Mech. Anal. **11** (1962), 40-49.

[16] Y. G. Reshetnyak, *Weak convergence of completely additive vector functions on a set*, Siberian Math. J. **9** (1968), 1039-1045 (translation of Sibirsk. Mat. Z. **9** (1968), 1386-1394).

[17] P. M. Suquet, *Existence et régularité des solutions des équations de la plasticité parfaite*, C. R. Acad. Sci. Paris Sér. A **286** (1978), 1201-1204.

[18] P. M. Suquet, *Un espace fonctionel pour les équations de la plasticité*, Ann. Fac. Sci. Toulouse Math (6) **1** (1979), 77-87.

[19] R. Temam, "Problèmes Mathématiques en Plasticité", Gauthier-Villars, 1983.

[20] R. Temam – G. Strang, *Functions of bounded deformation*, Arch. Rat. Mech. Anal. **75** (1980), 7-21.

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