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PAOLO ALBANO
PIERMARCO CANNARSA

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Structural Properties of Singularities of Semiconcave Functions

PAOLO ALBANO – PIERMARCO CANNARSA

Abstract. A semiconcave function on an open domain of $\mathbb{R}^n$ is a function that can be locally represented as the sum of a concave function plus a smooth one. The local structure of the singular set (non-differentiability points) of such a function is studied in this paper. A new technique is presented to detect singularities that propagate along Lipschitz arcs and, more generally, along sets of higher dimension. This approach is then used to analyze the singular set of the distance function from a closed subset of $\mathbb{R}^n$.

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1. – Introduction

Although the term "singularity" may suggest the idea of a technical topic for specialists, we believe that the object of this paper could be of interest for a general audience. In fact, we shall study the set of points at which a (semi)concave function $u : \Omega \to \mathbb{R}$, defined on an open domain $\Omega \subset \mathbb{R}^n$, fails to be differentiable.

Such a set, hereafter denoted by $\Sigma(u)$, is "small" if estimated by Lebesgue measure: being locally Lipschitz, u is also differentiable almost everywhere in Ω. On the other hand, a set of Lebesgue measure zero can be very "bad" and have no structure whatsoever. For a concave function, however, one might conceivably expect $\Sigma(u)$ to be a more regular set than for a general Lipschitz continuous function.

A possible approach for this analysis is to give upper bounds for the singular set. For instance, one can show that $\Sigma(u)$ can be covered by countably many Lipschitz hypersurfaces of dimension $n - 1$, or, in the language of geometric measure theory, that $\Sigma(u)$ is countably $(n - 1)$-rectifiable. Apart from

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earlier contributions for specific problems like in [16], to our knowledge the first general results in this direction are due to Zajíček [29], [30] and Veselý [25], [26]. Similar rectifiability properties were later extended to semiconcave functions in [3]. Related and improved versions of these results can be found in [1], [2], [5].

Another perspective, opposite to the previous one, is to obtain lower bounds for the singular set. We shall here adopt such a point of view, trying to answer the following natural question: how to find the isolated singularities of a concave function u? Conversely: are there conditions ensuring that a given point $x_0 \in \Sigma(u)$ belongs to a connected component of $\Sigma(u)$ of dimension $\nu \geq 1$? In the latter case, we say that the singularity at x_0 *propagates* along a ν-dimensional set.

This problem was first studied in [10] for semiconcave solutions to Hamilton-Jacobi-Bellman (HJB) equations, and then in [4] for general semiconvex functions. In the first reference, singularities were shown to propagate just along a sequence of points. In the second one, on the contrary, conditions were given to derive estimates for the Hausdorff dimension of the singular set in a neighborhood of a point $x_0 \in \Sigma(u)$. Such conditions were expressed in terms of the superdifferential of u at x_0—a nonempty convex set denoted by $D^+u(x_0)$. The results of [4], however, have at least two drawbacks. The first one is that they still allow the connected component of $\Sigma(u)$ containing x_0 to be a singleton; the second is that they require the assumption $\dim D^+u(x_0) < n$.

In the present paper, we develop a new method to study propagation of singularities addressing each of the issues we have raised above. For this purpose, we introduce a topological condition on $D^+u(x_0)$ that implies no restriction on the dimension of this set. Under such an assumption, we show that $\Sigma(u)$ contains the Lipschitz image of a ν-dimensional convex set, and that such an image has positive ν-dimensional Hausdorff density at x_0, for some integer $\nu \geq 1$ that we compute explicitly.

Besides being more powerful, the construction set forth in this paper seems more straightforward than the one of [4]: the geometric assumption ensuring the propagation of singularities is certainly easier to use and understand in the present form. All this being said, let us also note that a more restrictive semiconcavity property is here required for u than in the above reference.

For the case of $\nu = 1$, an earlier version of our propagation theorem was given by the authors in [1]. Even in this special case, however, the result we here propose improves substantially the one obtained in [1], as explained in Remark 4.6 below.

Since convex analysis occupies such a central position in mathematics, good reasons for studying properties of convex functions should be familiar to most readers.

In addition to the above considerations, we would like to mention the application that motivated our personal interest in this problem, namely the analysis of the singular set of solutions to HJB equations. Except for local results, the theory of these equations is based on suitable notions of weak solutions, as singularities may develop even if the initial data are smooth. On the other hand, the solutions of interest for applications — at least for problems with regular data — are semiconcave. Indeed, the first global existence and uniqueness results for HJB equations were obtained in such a class of functions, see [15], [20] and [22] for further reference. Even in the context of more recent PDE theories, like viscosity solutions or minimax solutions (see [12], [13], [14] and [24]), semiconcavity has a role to play as it represents a maximal regularity property of generalized solutions. Semiconcavity results for viscosity solutions to HJB equations and for the value function of related Optimal Control problems are described in [6], [21] and [8], [9], [10], [18], respectively.

For easier application to HJB equations, we will formulate our propagation results for semiconcave — rather than concave — functions. In order to keep the length of this paper under control, however, we will just present one of the possible applications, discussing the classical eikonal equation. More general types of HJB equations will be studied in a forthcoming paper.

As is well-known, the distance function d_S from a nonempty closed set $S \subset \mathbb{R}^n$ is a semiconcave solution of the eikonal equation $|Du| = 1$. From the point of view of best approximation theory, the singularities of d_S have been investigated by many authors. In [7], the set $\Sigma(d_S)$ is described proving that, in the case of $n = 2$, the non-isolated singularities of the metric projection propagate along Lipschitz arcs. Such a result is extended to Hilbert spaces in [28]. As an application of the propagation result of Section 5, in the present paper we will construct connected components of $\Sigma(d_S)$ of higher dimension.

To conclude this introduction let us explain how the paper is organized. Section 2 contains the general notation used in the sequel. In Section 3 we recall the basic properties of semiconcave functions, of their superdifferentials, and of their singular sets. The next two sections are devoted to propagation of singularities: we treat propagation along arcs in Section 4, and then, in Section 5, along sets of dimension $\nu \geq 1$. The reason for treating the one-dimensional case first, is that the Hausdorff estimate for the density of $\Sigma(u)$ is immediate if $\nu = 1$, and so our method becomes really elementary. Finally, in Section 6, we discuss the above mentioned application to the distance function.

2. – Notation

Let n be a positive integer. We denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the Euclidean scalar product and norm in $\mathbb{R}^n$. For any $R > 0$ and $x_0 \in \mathbb{R}^n$ we set

$$B_R(x_0) = \{x \in \mathbb{R}^n \, : \, |x - x_0| < R\}$$

and we abbreviate $B_R = B_R(0)$. We denote by $\overline{B}_R(x_0)$ the closure of $B_R(x_0)$. Let A be a subset of $\mathbb{R}^n$. We use the notation $\operatorname{diam}(A)$ for the diameter of A. We write

$$A \ni x \to x_0$$

to mean that $x \in A$ and $x \to x_0$. Moreover, for a family $\{A_i\}_{i\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$, we use the symbol

$$A_i \ni x_i \to x$$

to denote a sequence $x_i \in A_i$ converging to x.
Let N be another positive integer. We denote by $\mathrm{Lip}(A;\mathbb{R}^N)$ the space of all Lipschitz continuous maps defined on A, that is the space of all functions $f : A \to \mathbb{R}^N$ satisfying

$$|f(x) - f(y)| \leq L|x - y| \qquad \forall x, y \in A$$

for some constant $L \geq 0$. We refer to any constant L verifying the above inequality as a Lipschitz constant for f. The infimum of such constants is the Lipschitz seminorm of f, denoted by $\mathrm{Lip}(f)$.
Given an integer $\nu \in \{1, \dots, n\}$, we recall that A is said to be ν-rectifiable if $A \subset f(\mathbb{R}^\nu)$ for some mapping $f \in \mathrm{Lip}(\mathbb{R}^\nu;\mathbb{R}^n)$. We also agree with the convention of saying that A is 0-rectifiable if A is a singleton. More generally, we call a set A countably ν-rectifiable, $\nu \in \{0, \dots, n\}$, if $A = \cup_i A_i$ for some family $\{A_i\}_{i\in\mathbb{N}}$ of ν-rectifiable sets.
For any real number $\nu \in [0, n]$, the ν–dimensional Hausdorff measure of A is defined as

$$\mathcal{H}^\nu(A) := \frac{\alpha_\nu}{2^\nu} \sup_{\delta>0} \inf \left\{ \sum_{j=0}^{\infty} (\mathrm{diam}(A_j))^\nu \;:\; A \subset \bigcup_{j=0}^{\infty} A_j\,,\ \mathrm{diam}(A_j) < \delta \right\},$$

where

$$\alpha_\nu = \frac{\pi^{\frac{\nu}{2}}}{\Gamma\left(\dfrac{\nu}{2} + 1\right)}$$

and $\Gamma(t) = \int_0^{+\infty} e^{-s} s^{t-1} ds$ is the Euler function. Moreover, the Hausdorff dimension of A is defined as $\mathcal{H}$–dim $A = \inf\{\nu > 0 : \mathcal{H}^\nu(A) = 0\}$. If A is convex, then $\mathcal{H}$–dim A coincides with the classical dimension of A, that is the dimension of the smaller affine hyperplane containing A.
If $f \in \mathrm{Lip}(\mathbb{R}^n;\mathbb{R}^N)$, then it is easy to show that

$$\mathcal{H}^\nu(f(A)) \leq (\mathrm{Lip}(f))^\nu \mathcal{H}^\nu(A) \qquad \forall A \subset \mathbb{R}^n\,. \tag{2.1}$$

Given a nonempty closed set $S \subset \mathbb{R}^n$, we denote by d_S the Euclidean distance function from S, i.e.

$$d_S(x) := \inf_{y\in S} |x - y|\,.$$

If S is convex, then $N_S(x)$ denotes the normal cone to S at x, that is

$$N_S(x) = \{q \in \mathbb{R}^n \;:\; \langle q, y - x\rangle \leq 0, \forall\, y \in S\} \qquad \forall x \in S\,.$$

3. – Basic properties of semiconcave functions

For any $x_0, x_1 \in \mathbb{R}^n$, we denote by $[x_0, x_1]$ the line segment

$$[x_0, x_1] = \{ t x_1 + (1-t) x_0 \,:\, t \in [0,1] \}.$$

The open segment $]x_0, x_1[$ is defined similarly.

Let $A \subset \mathbb{R}^n$. A function $u : A \to \mathbb{R}$ is *concave on* A if

$$t u(x_1) + (1-t) u(x_0) - u\,(t x_1 + (1-t) x_0) \leq 0 \qquad \forall t \in [0,1]$$

for any pair $x_0, x_1 \in A$ such that $[x_0, x_1] \subset A$. Clearly, the above notion reduces to the standard definition of concave function if A is convex. We now define a more general class of functions.

Definition 3.1. A function $u : A \to \mathbb{R}$ is called *semiconcave* if, for some constant $C \in \mathbb{R}$,

$$t u(x_1) + (1-t) u(x_0) - u(t x_1 + (1-t) x_0) \leq C t (1-t) |x_1 - x_0|^2 \qquad \forall t \in [0,1]$$

for any $x_0, x_1 \in A$ such that $[x_0, x_1] \subset A$. We refer to such a constant C as a *semiconcavity constant* for u on A.

It is easy to check that

$$\text{(3.1)} \quad u \text{ is semiconcave on } A \quad \Longleftrightarrow \quad x \mapsto u(x) - C|x|^2 \text{ is concave on } A\,.$$

For an open set of $\Omega \subset \mathbb{R}^n$ we can further extend the above class of functions introducing locally semiconcave functions.

Definition 3.2. A function $u : \Omega \to \mathbb{R}$ is called *locally semiconcave* in Ω if u is semiconcave on every compact subset of Ω. We denote by $SC(\Omega)$ the class of all locally semiconcave functions defined in Ω.

We now proceed to review some differentiability properties of locally semiconcave functions in an open set. To begin, let us recall that any $u \in SC(\Omega)$ is locally Lipschitz continuous (see e.g. [17]). Hence, by Rademacher's Theorem, u is differentiable a.e. in Ω and the gradient of u is locally bounded. Then, the set

$$\text{(3.2)} \qquad D^* u(x) = \left\{ p \in \mathbb{R}^n \,:\, \Omega \ni x_i \to x\,,\ \ Du(x_i) \to p \right\}$$

is nonempty for any $x \in \Omega$. The elements of $D^*u(x)$ will be called the *reachable gradients* of u at x. The *superdifferential* of u at x is given by the convex hull of $D^*u(x)$, that is

$$\text{(3.3)} \qquad D^+ u(x) = \operatorname{co} D^* u(x)\,.$$

It is shown in [10] that $D^*u(x)$ is contained in the (topological) boundary of $D^+u(x)$, i.e.

$$D^*u(x) \subset \partial D^+u(x) \qquad \forall x \in \Omega\,. \tag{3.4}$$

From (3.3) it follows that $D^+u(x)$ is a nonempty compact convex set and that

$$\max\{|p| : p \in D^+u(x)\} \le L \tag{3.5}$$

where L is any Lipschitz constant for u in a neighborhood of x. Like for concave functions, we have that u is continuously differentiable at x if and only if $D^+u(x)$ is a singleton.

Now, let C be a semiconcavity constant for u on an open set $\Omega' \subset\subset \Omega$. Then, using (3.1), one can show that a vector $p \in \mathbb{R}^n$ belongs to the superdifferential of u at a point $x \in \Omega'$ if and only if

$$u(x') - u(x) - \langle p, x' - x\rangle \le C|x' - x|^2 \qquad \forall x' \in \Omega'\,. \tag{3.6}$$

The above inequality has several consequences. An immediate corollary is the fact that, for any pair $x, x' \in \Omega'$,

$$\langle p' - p, x' - x\rangle \le 2C|x' - x|^2 \qquad \forall p \in D^+u(x) \quad \forall p' \in D^+u(x')\,. \tag{3.7}$$

Inequality (3.6) is also useful to check the validity of many calculus rules for the superdifferential, such as Fermat's rule, that is $0 \in D^+u(x)$ at any local maximum or minimum point x for u, and the sum rule

$$D^+u(x) + D^+v(x) \subset D^+(u+v)(x) \qquad \forall x \in \Omega\,.$$

Notice that the above inclusion reduces to an equality if at least one of the functions u, v is continuously differentiable at x. Another easy consequence of (3.6) is the upper semicontinuity of D^+u as a set-valued map, that is

$$\left.\begin{array}{r}\Omega \ni x_i \to x\\ D^+u(x_i) \ni p_i \to p\end{array}\right\} \Longrightarrow p \in D^+u(x) \tag{3.8}$$

Remark. The notation for superdifferentials and reachable gradients is by no means standard in the literature. Here, we have adopted a terminology that is used in the viscosity solution context we are more familiar with, but analogous concepts — though denoted differently — are widely used in the literature, see e.g. [11]. To help the reader who is accustomed to the language of nonsmooth analysis, let us add a few comments. First, we observe that the usual definition of the superdifferential $D^+u(x)$ is usually different from ours, and can be applied to any function u. For semiconcave functions, however, it can be proved that $D^+u(x)$ is given by formula (3.3), and coincides in turn with the *proximal superdifferential* $\partial^P u(x)$. Moreover, we note that reachable gradients could be defined for locally Lipschitz functions as well. In this case, the convex hull of $D^*u(x)$ is a possible characterization of Clarke's *generalized gradient* $\partial u(x)$ ([11, Theorem 8.1 p. 93]), which is therefore equal to $D^+u(x)$ whenever u is semiconcave. Finally, for any $u \in SC(\Omega)$, $D^*u(x)$ coincides with the *limiting subdifferential* $\partial_L u(x)$ of nonsmooth analysis.

We conclude this brief review of the generalized differentials of a semiconcave function with a characterization of reachable gradients that will be used in the sequel.

PROPOSITION 3.4. *Let $u \in SC(\Omega)$. Then, for any $x \in \Omega$,*

$$D^*u(x)=\left\{\lim_{i\to\infty} p_i \ : \ p_i \in D^+u(x_i)\,,\ \Omega \ni x_i \to x\,,\ \operatorname{diam}(D^+u(x_i)) \to 0\right\}. \tag{3.9}$$

PROOF. Let us denote by $D^\sharp u(x)$ the right hand side of (3.9). Since D^+u reduces to the gradient at any differentiability point of u, we have that $D^*u(x) \subset D^\sharp u(x)$. Now, to show the reverse inclusion, let us fix a point $p = \lim_{i\to\infty} p_i$, with $p_i \in D^+u(x_i)$ and $x_i \in \Omega$ as in (3.9). Recalling the definition of D^*u, for any $i \in \mathbb{N}$ we can find a vector $p_i^* \in D^*u(x_i)$ and a point $x_i^* \in \Omega$, at which u is differentiable, such that

$$|x_i - x_i^*| + |Du(x_i^*) - p_i^*| \le \frac{1}{i}\,.$$

Then, $x_i^* \to x$ and

$$\begin{aligned}|Du(x_i^*) - p| &\le |Du(x_i^*) - p_i^*| + |p_i^* - p_i| + |p_i - p|\\ &\le \frac{1}{i} + \operatorname{diam}(D^+u(x_i)) + |p_i - p| \to 0\,,\end{aligned}$$

as $i \to \infty$. Hence, $p \in D^*u(x)$. □

The last part of this preliminary section will be focused on the *singular set* of a semiconcave function u. More precisely, given $u \in SC(\Omega)$, we denote by $\Sigma(u)$ the set of all points $x \in \Omega$ at which u fails to be differentiable. In light of the above remarks, such a set could be also described as the set of points $x \in \Omega$ at which $D^+u(x)$ is not a singleton or, equivalently, as

$$\Sigma(u) = \{x \in \Omega : \dim D^+u(x) \ge 1\}\,.$$

The points of $\Sigma(u)$ are the *singular points*, or singularities, of u. For a more accurate analysis of the singular set let us define, for any integer $k \in \{0, \dots, n\}$,

$$\Sigma^k(u) = \{x \in \Omega : \dim D^+u(x) = k\}\,.$$

Then, the family $\{\Sigma^k(u)\}_{k=0}^n$ is a partition of Ω, and

$$\Sigma(u) = \bigcup_{k=1}^{n} \Sigma^k(u)\,.$$

The next definition introduces the magnitude of a point, a natural concept that will be used in the sequel.

DEFINITION 3.5. The *magnitude* of a point $x \in \Omega$ (with respect to u) is the integer

$$k(x) = \dim D^+u(x)\,.$$

If x is a singular point, then $1 \leq k(x) \leq n$.

As we noted above, the set of all differentiability points of u, $\Sigma^0(u)$, has full measure in Ω. Therefore, $\Sigma(u)$ has measure 0. Even a superficial analysis of the behaviour of a concave function suggests the idea that the "size" of $\Sigma^k(u)$ should decrease as k increases. A rigorous result for such a conjecture can be given in terms of rectifiable sets.

THEOREM 3.6. *Let $u \in SC(\Omega)$. Then, for any $k \in \{0,\dots,n\}$, $\Sigma^k(u)$ is countably $(n-k)$-rectifiable. Consequently, $\Sigma(u)$ is countably $(n-1)$-rectifiable and $\Sigma^n(u)$ is at most countable.*

The above theorem is essentially due to Zajíček [30]. Extensions to more general classes of functions have been obtained by several authors, as reported in Section 1.

4. – Propagation of singularities along Lipschitz arcs

Let u be a semiconcave function in an open domain $\Omega \subset \mathbb{R}^n$. The rectifiability properties of $\Sigma(u)$, recalled in the last section, could be regarded as "upper bounds" for $\Sigma(u)$. From now on, we shall study the singular set of u trying to obtain "lower bounds" for such a set. More precisely, given a point $x_0 \in \Sigma(u)$, we are interested in conditions ensuring the existence of other singular points approaching x_0. The following example explains the nature of such conditions.

EXAMPLE 4.1. The functions

$$u_1(x_1,x_2) = -\sqrt{x_1^2+x_2^2}\,, \qquad u_2(x_1,x_2) = -|x_1| - |x_2|$$

are concave in $\mathbb{R}^2$, and $(0,0)$ is a singular point for both of them. Moreover, $(0,0)$ is the only singularity for u_1 while

$$\Sigma(u_2) = \{(x_1,x_2)\,:\, x_1x_2 = 0\}\,.$$

So, $(0,0)$ is the intersection point of two singular lines of u_2. Notice that $(0,0)$ has magnitude 2 with respect to both functions as

$$D^+u_1(0,0) = \{(p_1,p_2) \,:\, p_1^2 + p_2^2 \le 1\}$$
$$D^+u_2(0,0) = \{(p_1,p_2) \,:\, |p_1| \le 1\,,\ |p_2| \le 1\}\,.$$

The different structure of $\Sigma(u_1)$ and $\Sigma(u_2)$ in a neighborhood of x_0 is captured by the reachable gradients. In fact,

$$D^*u_1(0,0) = \{(p_1,p_2) \,:\, p_1^2 + p_2^2 = 1\} = \partial D^+u_1(0,0)$$
$$D^*u_2(0,0) = \{(p_1,p_2) \,:\, |p_1| = 1\,,\ |p_2| = 1\} \neq \partial D^+u_2(0,0)\,.$$

In other words, (3.4) is an equality for u_1, and a proper inclusion for u_2.

The above example suggests that a sufficient condition to exclude the existence of an isolated singular point $x_0 \in \Sigma(u)$ should be that $D^*u(x_0)$ fails to cover the whole boundary of $D^+u(x_0)$. As we shall see, such a condition implies a much stronger property, namely that x_0 is the initial point of a Lipschitz singular arc.

We recall that an *arc* $\mathbf{x}$ in $\mathbb{R}^n$ is a continuous map $\mathbf{x} : [0,\rho] \to \mathbb{R}^n\,,\ \rho > 0$. We shall say that $\mathbf{x}$ is *singular* for u if the support of $\mathbf{x}$ is contained in Ω and $\mathbf{x}(s) \in \Sigma(u)$ for every $s \in [0,\rho]$. The following result describes the "arc structure" of the singular set $\Sigma(u)$.

THEOREM 4.2. *Let $x_0 \in \Omega$ be a singular point of a function $u \in SC(\Omega)$. Suppose that*

$$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset\,. \tag{4.1}$$

Then there exist a Lipschitz singular arc $\mathbf{x} : [0,\rho] \to \mathbb{R}^n$ for u, with $\mathbf{x}(0) = x_0$, and a positive number δ such that

$$\lim_{s\to 0^+} \frac{\mathbf{x}(s) - x_0}{s} \neq 0 \tag{4.2}$$

$$\operatorname{diam}\big(D^+u(\mathbf{x}(s))\big) \ge \delta \qquad \forall s \in [0,\rho]\,. \tag{4.3}$$

Moreover, $\mathbf{x}(s) \neq x_0$ for any $s \in (0,\rho]$.

Observe that Theorem 4.2 gives no information on the magnitude of $\mathbf{x}(s)$ as a singular point, besides the trivial estimate $1 \le k(\mathbf{x}(s)) \le n$. However, Theorem 3.6 implies that the support of $\mathbf{x}$ contains at most countably many singular points of magnitude n. In other words, there exists no singular arc of constant magnitude equal to n.

REMARK. We note that condition (4.1) is equivalent to the existence of two vectors, $p_0 \in \mathbb{R}^n$ and $q \in \mathbb{R}^n \setminus \{0\}$, such that

$$p_0 \in D^+u(x_0) \setminus D^*u(x_0) \tag{4.4}$$

$$\langle q, p - p_0\rangle \ge 0 \qquad \forall\, p \in D^+u(x_0)\,. \tag{4.5}$$

Indeed, (4.4) and (4.5) imply that $p_0 \in \partial D^+u(x_0)\setminus D^*u(x_0)$. Conversely, if (4.1) holds true, then (4.4) is trivially satisfied by any vector $p_0 \in \partial D^+u(x_0)\setminus D^*u(x_0)$, and (4.5) follows taking $-q$ in the normal cone to the convex set $D^+u(x_0)$ at p_0. The importance of a condition of type (4.4) was initially pointed out in [4].

REMARK. It is easy to see that the support of the singular arc $\mathbf{x}$, given by Theorem 4.2, is a connected set of Hausdorff dimension 1. Indeed, from the Lipschitz continuity of $\mathbf{x}$ it follows that the support of $\mathbf{x}$ is 1-rectifiable, while property (4.2) implies that the 1-dimensional Hausdorff measure of $\mathbf{x}([0,T])$ is positive.

The idea of the proof of Theorem 4.2 relies on the geometric intuition that the distance between the graph of u and a suitable plane through $(x_0, u(x_0))$, transverse the graph of u, should be maximized along the singular arc we expect to find. We will be able to construct such a transverse plane using a vector of the form $p_0 - q$, where p_0 and q are chosen as in Remark 4.3. Indeed, condition (4.5) implies that $p_0 - q \notin D^+u(x_0)$, and so the graph of

$$x \mapsto u(x_0) + \langle p_0 - q, x - x_0\rangle \qquad x \in \mathbb{R}^n \tag{4.6}$$

is transverse to the graph of u. We single out this step of the proof in the next lemma, as such a technique applies to any point x_0 of the domain of a concave function, regardless of the regularity of u at x_0. For $x_0 \in \Sigma(u)$, we will then show that the arc we construct in this way is singular for u.

LEMMA 4.5. *Let C be a semiconcavity constant for u in $\overline{B}_R(x_0) \subset \Omega$. Fix $p_0 \in \partial D^+u(x_0)$ and let $q \in \mathbb{R}^n \setminus \{0\}$ be such that*

$$\langle q, p - p_0\rangle \geq 0 \qquad \forall p \in D^+u(x_0)\,. \tag{4.7}$$

Define

$$\sigma = \min\left\{\frac{R}{4|q|}, \frac{1}{4C}\right\}. \tag{4.8}$$

Then there exists a Lipschitz arc $\mathbf{x} : [0,\sigma] \to B_R(x_0)$, with $\mathbf{x}(0) = x_0$, such that

$$0 < |\mathbf{x}(s) - x_0| < 4|q|s \qquad \forall s \in (0,\sigma] \tag{4.9}$$

$$\lim_{s\downarrow 0} \frac{\mathbf{x}(s) - x_0}{s} = q \tag{4.10}$$

$$\mathbf{p}(s) := p_0 + \frac{\mathbf{x}(s) - x_0}{s} - q \in D^+u(\mathbf{x}(s)) \qquad \forall s \in (0,\sigma]\,. \tag{4.11}$$

Moreover,

$$\mathrm{Lip}(\mathbf{x}) \leq 4L + 2|q|$$

where L is a Lipschitz constant for u in $B_R(x_0)$.

PROOF. Let us define, for any $s > 0$,

$$\phi_s(x) = u(x) - u(x_0) - \langle p_0 - q, x - x_0 \rangle - \frac{1}{2s}|x - x_0|^2 \qquad x \in \overline{B}_R(x_0).$$

Notice that ϕ_s is the difference between the affine function in (4.6) and the function $x \mapsto u(x) - \frac{1}{2s}|x - x_0|^2$, which is a concave perturbation of u.

Being strictly concave, ϕ_s has a unique maximum point in $\overline{B}_R(x_0)$, that we term x_s. For technical reasons that will be clear in the sequel, we restrict our attention to the interval $0 \le s \le \sigma$, where σ is the number given by (4.8). Let us define $\mathbf{x}$ by

$$\mathbf{x}(s) = \begin{cases} x_0 & \text{if } s = 0 \\ x_s & \text{if } s \in (0, \sigma]. \end{cases}$$

We now proceed to show that $\mathbf{x}$ possesses all the required properties.

First, we claim that $\mathbf{x}$ satisfies estimate (4.9). Indeed, by the characterization of D^+u given in (3.6), we have that

$$\phi_s(x) \le \langle q, x - x_0 \rangle + \left(C - \frac{1}{2s}\right)|x - x_0|^2 \tag{4.13}$$

for any $x \in \overline{B}_R(x_0)$. Moreover, $p_0 - q \notin D^+u(x_0)$ in view of condition (4.7). Since this fact implies that there are points in $\overline{B}_R(x_0)$ at which ϕ_s is positive, we conclude that $\phi_s(\mathbf{x}(s)) > 0$. The last estimate and (4.13) yield

$$0 < |\mathbf{x}(s) - x_0| < \frac{2s|q|}{1 - 2Cs} \qquad \forall s \in (0, \sigma] \tag{4.14}$$

and so (4.9) follows from (4.8).

Second, we proceed to check (4.11). For this purpose we note that, on account of estimate (4.14), the choice of σ forces $\mathbf{x}(s) \in B_R(x_0)$ for any $s \in [0, \sigma]$. Hence, $\mathbf{x}(s)$ is also a local maximum point of ϕ_s. So, by the calculus rules for D^+u we recalled in Section 3,

$$0 \in D^+\phi_s(\mathbf{x}(s)) = D^+u(\mathbf{x}(s)) - p_0 + q - \frac{\mathbf{x}(s) - x_0}{s}$$

for any $s \in (0, \sigma]$. Clearly, the last inclusion can be recast in the desired form (4.11).

Third, to prove (4.10), we show that $\lim_{s\to 0} \mathbf{p}(s) = p_0$, where $\mathbf{p}$ is defined in (4.11). Let $\overline{p} = \lim_{k\to\infty} \mathbf{p}(s_k)$ for some sequence $s_k \downarrow 0$. Then, taking the scalar product of both sides of the identity

$$\mathbf{p}(s_k) - p_0 + q = \frac{\mathbf{x}(s_k) - x_0}{s_k}.$$

with $\mathbf{p}(s_k) - p_0$ and recalling property (3.7), we obtain

$$
(4.15)\qquad
\begin{aligned}
|\mathbf{p}(s_k) - p_0|^2 + \langle q, \mathbf{p}(s_k) - p_0\rangle &= \frac{1}{s_k}\langle \mathbf{p}(s_k) - p_0, \mathbf{x}(s_k) - x_0\rangle \\
&\le \frac{2C}{s_k}|\mathbf{x}(s_k) - x_0|^2 .
\end{aligned}
$$

Now, observe that the right-hand side above tends to 0 as $k \to \infty$, in view of (4.14). Moreover, $\overline{p} \in D^+u(x_0)$ since D^+u is upper semicontinuous; so, $\langle q, \overline{p} - p_0\rangle \ge 0$ by assumption (4.7). Therefore, (4.15) yields $|\overline{p} - p_0|^2 \le 0$ in the limit as $k \to \infty$. This proves that $\overline{p} = p_0$ as required.

Finally, let us derive the Lipschitz estimate (4.12). Let $r, s \in [0, \sigma]$. Using (4.11) to evaluate $\mathbf{x}(s)$ and $\mathbf{x}(r)$ one can easily compute that

$$
(4.16)\qquad \mathbf{x}(s) - \mathbf{x}(r) = s[\mathbf{p}(s) - \mathbf{p}(r)] + (s - r)[\mathbf{p}(r) - p_0 + q] .
$$

Now, taking the scalar product of both sides of above equality with $\mathbf{x}(s) - \mathbf{x}(r)$, and recalling (3.7), we obtain

$$
|\mathbf{x}(s) - \mathbf{x}(r)|^2 \le 2Cs|\mathbf{x}(s) - \mathbf{x}(r)|^2 + |s - r|\,|\mathbf{x}(s) - \mathbf{x}(r)|\,|\mathbf{p}(r) - p_0 + q| .
$$

Hence, for any $r, s \in [0, \sigma]$,

$$
(1 - 2Cs)|\mathbf{x}(s) - \mathbf{x}(r)| \le |\mathbf{p}(r) - p_0 + q|\,|s - r| \le (2L + |q|)\,|s - r| ,
$$

because L provides a bound for D^+u in $B_R(x_0)$. This completes the proof. □

Proof of Theorem 4.2. To begin, let us fix a radius $R > 0$ such that $\overline{B}_R(x_0) \subset \Omega$. We recall that, as noted in Remark 4.3, the geometric assumption that $\partial D^+u(x_0) \setminus D^*u(x_0)$ be nonempty is equivalent to the existence of vectors $p_0 \in \mathbb{R}^n$ and $q \in \mathbb{R}^n \setminus \{0\}$ satisfying

$$
(4.17)\qquad p_0 \in D^+u(x_0) \setminus D^*u(x_0) \quad \& \quad \langle q, p - p_0\rangle \ge 0 \quad \forall\, p \in D^+u(x_0) .
$$

Applying Lemma 4.5 to such a pair, we can construct two arcs, $\mathbf{x}$ and $\mathbf{p}$, that enjoy properties (4.10) and (4.11). Moreover, the same lemma ensures that $\mathbf{x}$ is Lipschitz continuous and that $\mathbf{x}(s) \ne x_0$ for any $s \in (0, \sigma]$.

Therefore, it remains to show that the restriction of $\mathbf{x}$ to a suitable subinterval $[0, \rho]$ is singular for u, and that the diameter of $D^+u(\mathbf{x}(s))$ is bounded away from 0 for all $s \in [0, \rho]$. In fact, it suffices to check the latter point. Let us argue by contradiction: suppose that a sequence $s_k \downarrow 0$ exists such that $\operatorname{diam}(D^+u(\mathbf{x}(s_k))) \to 0$ as $k \to \infty$. Then, by Proposition 3.4,

$$
D^+u(\mathbf{x}(s_k)) \ni \mathbf{p}(s_k) \to p_0 \quad \text{as} \quad k \to \infty \qquad \Longrightarrow \qquad p_0 \in D^*u(x_0) ,
$$

contrary to (4.17). □

REMARK 4.6. An earlier version of the above technique of proof was used in [1, Theorem 4.1] in order to construct a Lipschitz singular arc $\mathbf{x}$ for u. The new idea introduced in the present paper, is the different choice of the function ϕ_s to maximize. A direct consequence of such a choice is the existence of the right derivative of $\mathbf{x}$ at 0, that we were unable to obtain in [1]. Such a property will play a crucial role for future applications to Hamilton-Jacobi-Bellman equations. Apart from (4.2), another entirely new result is estimate (4.3) for the diameter of D^+u along the singular arc.

5. – Lipschitz singular sets of higher dimension

In the previous section we proved that the singularities of a function $u \in SC(\Omega)$ propagate along a Lipschitz image of an interval $[0, \rho]$, from any point $x_0 \in \Omega$ at which $D^*u(x_0)$ fails to cover the whole boundary of $D^+u(x_0)$. Such a result describes a sort of basic structure for the propagation of singularities of a concave function. However, concave functions may well present singular sets of dimension greater than 1. The next example is a case in point.

EXAMPLE 5.1. It is easy to check that the singular set of the concave function

$$u(x) = -|x_3| \qquad x = (x_1, x_2, x_3) \in \mathbb{R}^3$$

is given by the coordinate plane

$$\Sigma(u) = \{x \in \mathbb{R}^3 : x_3 = 0\}.$$

Moreover,

$$D^*u(0) = \{(0,0,1), (0,0,-1)\} \qquad D^+u(0) = \{(0,0,p_3) : |p_3| \leq 1\}.$$

Therefore, one can apply Theorem 4.2 with $x_0 = 0$ and $p_0 = 0$, but this procedure only gives a Lipschitz singular arc starting at 0, whereas one would expect a 2-dimensional singular set. Actually, a more careful application of Lemma 4.5 suggests that a singular arc for u should correspond to any vector $q \neq 0$ satisfying (4.7). Moreover, such a correspondence should be 1-to-1 in light of (4.10). Since (4.7) is satisfied by all vectors $q \in \mathbb{R}^3$ such that $q_1^2 + q_2^2 = 1$, $q_3 = 0$, one can imagine to construct the whole singular plane $x_3 = 0$ in this way.

The next result generalizes Theorem 4.2, showing propagation of singularities along a ν-dimensional set. The integer $\nu \geq 1$ is given by the number of linearly independent directions of the normal cone to the superdifferential of u at the initial singular point, as conjectured in the above example.

THEOREM 5.2. *Let $x_0 \in \Omega$ be a singular point of a function $u \in SC(\Omega)$. Suppose that*

$$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset$$

and, having fixed a point $p_0 \in \partial D^+u(x_0) \setminus D^*u(x_0)$, define

$$\nu := \dim\ N_{D^+u(x_0)}(p_0)\,.$$

Then a number $\rho > 0$ and a Lipschitz map $\mathbf{f} : N_{D^+u(x_0)}(p_0) \cap B_\rho \to \Sigma(u)$ exist such that

$$\mathbf{f}(q) = x_0 - q + |q|\mathbf{h}(q) \text{ with } \mathbf{h}(q) \to 0 \text{ as } N_{D^+u(x_0)} \cap B_\rho \ni q \to 0 \tag{5.1}$$

$$\liminf_{r\to 0^+} r^{-\nu}\mathcal{H}^\nu\left(\mathbf{f}(N_{D^+u(x_0)}(p_0) \cap B_\rho) \cap B_r(x_0)\right) > 0\,. \tag{5.2}$$

Moreover,

$$\operatorname{diam}\left(D^+u(\mathbf{f}(q))\right) \geq \delta \qquad \forall q \in N_{D^+u(x_0)}(p_0) \cap B_\rho \tag{5.3}$$

for some $\delta > 0$.

REMARK 5.3. As one can easily realize, Theorem 5.2 is an extension of Theorem 4.2. The property

$$\mathbf{f}(q) \neq x_0 \quad \forall q \in N_{D^+u(x_0)}(p_0) \cap B_\rho \setminus \{0\},$$

though absent from the statement of Theorem 5.2, is valid in the present case as well. In fact, it can also be derived from (5.1), possibly restricting the domain of $\mathbf{f}$.

As in the previous section, we first prove a preliminary result.

LEMMA 5.4. *Let C be a semiconcavity constant for u in $\overline{B}_R(x_0) \subset \Omega$. Fix $p_0 \in \partial D^+u(x_0)$ and define*

$$\sigma = \min\left\{\frac{1}{4C}, \frac{R}{4}\right\}. \tag{5.4}$$

Then a Lipschitz map $\mathbf{f} : N_{D^+u(x_0)} \cap B_\sigma \to \mathbb{R}^n$ exists such that

$$\mathbf{f}(q) = x_0 - q + |q|\mathbf{h}(q) \text{ with } \mathbf{h}(q) \to 0 \text{ as } N_{D^+u(x_0)} \cap B_\sigma \ni q \to 0 \tag{5.5}$$

$$p_0 + \mathbf{h}(q) \in D^+u(\mathbf{f}(q)) \qquad \forall\, q \in N_{D^+u(x_0)}(p_0) \cap B_\sigma\,. \tag{5.6}$$

Moreover,

$$\mathrm{Lip}(\mathbf{f}) \leq 4\,L + 2 \tag{5.7}$$

where L is a Lipschitz constant for u in $B_R(x_0)$.

PROOF. Let us set, for brevity,

$$N := N_{D^+u(x_0)}(p_0).$$

We proceed as in the proof of Lemma 4.5 and consider, for any $q \in N \setminus \{0\}$, the function

$$\phi_q(x) = u(x) - u(x_0) - \left\langle p_0 + \frac{q}{|q|}, x - x_0 \right\rangle - \frac{1}{2|q|}|x - x_0|^2 \qquad x \in \overline{B}_R(x_0)$$

and the point x_q given by the relation

$$\phi_q(x_q) = \max_{x \in \overline{B}_R(x_0)} \phi_q(x).$$

Then, we define σ as in (5.4) and $\mathbf{f}$ by

$$\mathbf{f}(q) = \begin{cases} x_0 & \text{if } q = 0 \\ x_q & \text{if } q \in N \cap B_\sigma \setminus \{0\}. \end{cases}$$

Arguing as in the proof of Lemma 4.5, we obtain

$$0 < |\mathbf{f}(q) - x_0| < \frac{2|q|}{1 - 2C|q|} \qquad \forall q \in N \cap B_\sigma \setminus \{0\} \tag{5.8}$$

and

$$p_0 + \frac{\mathbf{f}(q) - x_0 + q}{|q|} \in D^+u(\mathbf{f}(q)) \qquad \forall q \in N_{D^+u(x_0)}(p_0) \cap B_\sigma. \tag{5.9}$$

Hence, denoting by $\mathbf{h}(q)$ the quotient in (5.9), assertion (5.6) follows.

To prove (5.5) we must show that

$$\mathbf{h}(q) \to 0 \qquad \text{as} \qquad N \cap B_\sigma \ni q \to 0. \tag{5.10}$$

For this purpose, let $\{q_i\}$ be an arbitrary sequence in $N \cap B_\sigma \setminus \{0\}$ such that $q_i \to 0$. Since $\mathbf{h}$ is bounded, we can extract a subsequence (still termed $\{q_i\}$) such that $\lim_{i\to\infty} \mathbf{h}(q_i)$ exists and

$$\lim_{i\to\infty} \frac{q_i}{|q_i|} = \overline{q}$$

for some $\overline{q} \in N$ satisfying $|\overline{q}| = 1$. We claim that $\lim_{i\to\infty} \mathbf{h}(q_i) = 0$, which in turn implies (5.10). Indeed, let us set

$$\overline{p} := p_0 + \lim_{i\to\infty} \mathbf{h}(q_i)$$

and observe that $\overline{p} \in D^+u(x_0)$ as D^+u is upper semicontinuous and $\mathbf{f}$ is continuous at 0. Taking the scalar product of both sides of the identity

$$\mathbf{h}(q_i) - \frac{q_i}{|q_i|} = \frac{\mathbf{f}(q_i) - x_0}{|q_i|}$$

with $\mathbf{h}(q_i)$ and applying inequality (3.7), we deduce that

$$\begin{aligned}|\mathbf{h}(q_i)|^2 - \left\langle \frac{q_i}{|q_i|}, \mathbf{h}(q_i)\right\rangle &= \frac{1}{|q_i|}\langle \mathbf{f}(q_i) - x_0, \mathbf{h}(q_i)\rangle \\ &\leq \frac{2C}{|q_i|}|\mathbf{f}(q_i) - x_0|^2 \leq \frac{2C}{|q_i|}\left(\frac{2|q_i|}{1 - 2C|q_i|}\right)^2\end{aligned}$$

where the last estimate follows from (5.8). In the limit as $i \to \infty$, the above inequality yields

$$|\overline{p} - p_0|^2 - \langle \overline{q}, \overline{p} - p_0\rangle \leq 0\,.$$

Hence, recalling that $\overline{q} \in N$, we conclude that $\overline{p} = p_0$. Our claim is thus proved.

The reasoning that shows the Lipschitz estimate (5.7) is the same as in the proof of Lemma 4.5, and is therefore omitted. □

We are now ready to prove the main result of this section.

PROOF OF THEOREM 5.2. Keeping the notation $N = N_{D^+u(x_0)}(p_0)$ as in the proof of Lemma 5.4, let us denote by $L(N)$ the linear subspace generated by N, and by $\pi : \mathbb{R}^n \to L(N)$ the ortoghonal projection of $\mathbb{R}^n$ onto $L(N)$.

Having fixed $R > 0$ so that $\overline{B}_R(x_0) \subset \Omega$, let $\mathbf{f} : N \cap B_\sigma \to \mathbb{R}^n$ be the map given by Lemma 5.4. Arguing by contradiction — as in the proof of Theorem 4.2 — the reader can easily show that a suitable restriction of $\mathbf{f}$ to $N \cap B_\rho$, $0 < \rho \leq \sigma$, satisfies (5.3). In particular, $\mathbf{f}(q) \in \Sigma(u)$ for any $q \in N \cap B_\rho$. Moreover, (5.1) is an immediate consequence of (5.5).

Therefore, the only point of the conclusion that needs to be demonstrated, is estimate (5.2) for the ν–dimensional Hausdorff density of the singular set $\mathbf{f}(N \cap B_\rho)$, or, since $\mathcal{H}^\nu$ is traslation invariant,

$$\liminf_{r\to 0^+} r^{-\nu}\mathcal{H}^\nu\left([\mathbf{f}(N \cap B_\rho) - x_0] \cap B_r\right) > 0\,.$$

We note that the above inequality can be deduced from the lower bound

$$\liminf_{r\to 0^+} r^{-\nu}\mathcal{H}^\nu\left(\pi([\mathbf{f}(N \cap B_\rho) - x_0] \cap B_r)\right) > 0 \tag{5.11}$$

since

$$\mathcal{H}^\nu\left([\mathbf{f}(N \cap B_\rho) - x_0] \cap B_r\right) \geq \mathcal{H}^\nu\left(\pi([\mathbf{f}(N \cap B_\rho) - x_0] \cap B_r)\right)\,.$$

Now, using the shorter notation $F(q) = \pi(x_0 - \mathbf{f}(q))$, we observe that (5.11) can in turn be derived from the lower bound

$$\liminf_{r\to 0^+} r^{-\nu}\mathcal{H}^\nu(F(N\cap B_r)) > 0\,. \tag{5.12}$$

Indeed, for any sufficiently small r, say $0 < r \le M\rho$ where $M := \mathrm{Lip}(\mathbf{f})$, we have that

$$F(N\cap B_{r/M}) = \pi\left[x_0 - \mathbf{f}\left(N\cap B_{r/M}\right)\right] \subset \pi\left[\left(x_0 - \mathbf{f}(N\cap B_\rho)\right)\cap B_r\right]\,.$$

The rest of our reasoning will therefore be devoted to the proof of (5.12). To begin, we note that, in view of (5.1), the map $F : N\cap B_\rho \to L(N)$ introduced above can be also represented as

$$F(q) = q + H(q)$$

where $H(q)/|q| \to 0$ as $N\cap B_\rho \ni q \to 0$. Consequently, the function

$$\lambda(r) := \sup\{|H(q)| \, : \, q\in N\cap B_r\}\,.$$

satisfies

$$\lim_{r\downarrow 0}\frac{\lambda(r)}{r} = 0\,. \tag{5.13}$$

Let us define

$$N_r := \left\{q\in N\cap B_r \, : \, d_{\Gamma_r}(q) \ge \lambda(r)\right\} \qquad 0<r\le\rho\,. \tag{5.14}$$

where Γ_r denotes the relative boundary of $N\cap B_r$ and d_{Γ_r} the distance from Γ_r. Using the limit (5.13), it is easy to check that $N_r \neq \emptyset$ provided r is sufficiently small, say $0<r<\rho_0\le\rho$. We claim that

$$N_r \subset F(N\cap B_r) \qquad \forall\, r\in(0,\rho_0)\,. \tag{5.15}$$

Indeed, having fixed $y\in N_r$, let us rewrite the equation $F(q) = y$ as

$$q = y - H(q) =: H_y(q)\,.$$

Now, observe that $\overline{B}_{\lambda(r)}(y) \subset N\cap B_r$ and that the continuous map $H_y(q)$ satisfies

$$H_y\left(\overline{B}_{\lambda(r)}(y)\right) \subset y - H(\overline{N\cap B_r}) \subset y + \overline{B}_{\lambda(r)} = \overline{B}_{\lambda(r)}(y)\,.$$

Therefore, applying Brouwer's Fixed Point Theorem, we conclude that $q = H_y(q)$ has a solution $q\in \overline{B}_{\lambda(r)}(y)$. So, our claim (5.15) follows.

Our next step is to obtain the lower bound

$$\liminf_{r\to 0+} \frac{\mathcal{H}^\nu(N_r)}{r^\nu} > 0 \tag{5.16}$$

for the density of the set N_r introduced in (5.14). To verify the above estimate let $\overline{q}$ be a point in the relative interior of N, with $|\overline{q}| = 1$. Then, using the notation $\widehat{B}_\alpha := B_\alpha \cap L(N)$ to denote ν-dimensional balls, we have that

$$\overline{q} + \widehat{B}_{2\alpha} \subset N$$

for some $\alpha \in (0, 1/2]$. On account of (5.13), there exists $r_0 \in (0, \rho_0)$ such that

$$\lambda(r) \leq \frac{r\alpha}{2} \qquad \forall r \in (0, r_0]. \tag{5.17}$$

Hence,

$$\frac{r}{2}\left(\overline{q} + \widehat{B}_{2\alpha}\right) \subset N \cap B_r \qquad \forall r \in (0, r_0). \tag{5.18}$$

Now, combining (5.17), (5.18) and the definition of N_r, we discover

$$\frac{r}{2}\left(\overline{q} + \widehat{B}_{\alpha}\right) \subset N_r \quad \forall r \in (0, r_0).$$

Estimate (5.16) is an immediate consequence of the last inclusion.

Finally, to complete the proof, it suffices to observe that (5.12) follows from (5.15) and (5.16). □

Remark 5.5. Though clear from (5.2), we explicitly note that Theorem 5.2 ensures that, in a neighbourood of x_0, $\Sigma(u)$ covers a rectifiable set of Hausdorff dimension ν. Moreover, from formula (5.1) it follows that the set $\mathbf{f}(N_{D^+u(x_0)}(p_0) \cap B_\rho) \subset \Sigma(u)$ possesses tangent space at x_0 whenever the normal cone $N_{D^+u(x_0)}(p_0)$ is actually a vector space. This happens, for instance, when x_0 is a singular point of magnitude $k(x_0) < n$ and one can find a point p_0 in the relative interior of $D^+u(x_0)$, but not in $D^*u(x_0)$. Then, it is easy to check that $N_{D^+u(x_0)}(p_0)$ is a vector space of dimension $\nu = n - k(x_0)$.

6. – Application to the distance function

In this section we examine the singular points of the distance function d_S associated to a nonempty closed subset S of $\mathbb{R}^n$.

We denote by $\text{proj}_S(x)$ the set of closest points in S to x, i.e.

$$\text{proj}_S(x) = \{y \in S \ : \ d_S(x) = |x - y|\} \qquad x \in \mathbb{R}^n.$$

The next proposition collects the properties of d_S we need for the analysis.

PROPOSITION 6.1. *Let S be a nonempty closed subset of $\mathbb{R}^n$. Then the following holds:*

(i) $d_S \in SC(\mathbb{R}^n \setminus S)$.
(ii) *d_S is differentiable at $x \notin S$ if and only if* $\mathrm{proj}_S(x)$ *is a singleton.*
(iii) *For any $x \notin S$ and any $y \in$* $\mathrm{proj}_S(x)$, *d_S is differentiable along the segment* $]x, y[$.
(iv) *For any $x \notin S$*

$$D^*d_S(x) = \left\{ \frac{x-y}{|x-y|} \ : \ y \in \mathrm{proj}_S(x) \right\} . \tag{6.1}$$

The above assertions are either known in the literature, or can be easily deduced from known properties. In particular:

(i) is proved in [9, Proposition 3.2] (see also [6, Chap. 2, Example 4.5]);
(ii) is a well known property, see e.g. [19, p. 62]
(iii) is very easy to check;
(iv) holds at any differentiability point of d_S (see e.g. [11, Chap. 1, Theorem 6.1]), and so $D^*d_S(x)$ is contained in the set in the right-hand side of (6.1); the reverse inclusion follows from (iii).

The following theorem characterizes the isolated singularities of d_S.

THEOREM 6.2. *Let S be a nonempty closed subset of $\mathbb{R}^n$ and $x \notin S$ be a singular point of d_S. Then the following properties are equivalent:*

(a) *x is an isolated point of $\Sigma(d_S)$.*
(b) $\partial D^+ d_S(x) = D^* d_S(x)$.
(c) $\mathrm{proj}_S(x) = \partial B_r(x)$ *where* $r := d_S(x)$.

Property (c) above was observed by Motzkin [23] in the case of $n = 2$; it was later extended to Hilbert spaces in [28]. Here, we give an independent proof of this result in Euclidean spaces, using our analysis of singularities for semiconcave functions.

PROOF. $\boxed{\text{(a)} \Rightarrow \text{(b)}}$ This implication is an immediate corollary of the propagation result of Section 4. Indeed, should $\partial D^+ d_S(x) \setminus D^* d_S(x)$ be nonempty, then Theorem 4.2 would ensure the existence of a non-costant singular arc with initial point x. In particular, x could not be isolated.

$\boxed{\text{(b)} \Rightarrow \text{(c)}}$ Assume (b). First, we claim that x must be a singular point of magnitude $k(x) = n$, i.e. $\dim D^+ d_S(x) = n$. For suppose the strict inequality $k(x) < n$ is verified. Then, the whole superdifferential would be made of boundary points and so, owing to (b), $D^+ d_S(x) = D^* d_S(x)$. Therefore, $D^+ d_S(x) \subset \partial B_1$ as all reachable gradients of d_S are unit vectors. But the last inclusion contradicts the fact that $D^+ d_S(x)$ is a convex set of dimension at least 1. Our claim is thus proved.

Now, use the fact that $D^+d_S(x)$ is an n-dimensional convex set with

$$\partial D^+d_S(x) = D^*d_S(x) \subset \partial B_1$$

to conclude that $D^+d_S(x) = \overline{B}_1$ and $D^*d_S(x) = \partial B_1$.

Finally, invoke formula (6.1) to discover

$$\mathrm{proj}_S(x) = x - d_S(x)D^*d_S(x) = \partial B_r(x)\,, \tag{6.2}$$

as stated in (c).

$\boxed{\text{(c)} \Rightarrow \text{(a)}}$ From Proposition 6.1 (iii) we know that d_S is differentiable along each segment $]x, y[$ with $y \in \mathrm{proj}_S(x) = \partial B_r(x)$. So, $d_S \in C^1(B_r(x) \setminus \{x\})$ and the proof is complete. □

We note that, in particular, the distance from a simply connected set $S \subset \mathbb{R}^2$ has no isolated singularities in $\mathbb{R}^2 \setminus S$.

The propagation of the non-isolated singularities of d_S along Lipschitz arcs has been established in both Euclidean and Hilbert spaces, see [7] and [27], [28] respectively. To construct higher dimensional singular sets for the metric projection onto S, we will now apply the theory of Section 5.

We recall that an exposed face E of a convex set $D \subset \mathbb{R}^n$ is the intersection $E = D \cap H$, where H is any support hyperplane to D. Equivalently, a vector p belongs to an exposed face of D iff $\langle p, q\rangle = \max_{p' \in D}\langle p', q\rangle$ for some unit vector $q \in \mathbb{R}^n$.

In the following, we will use the shorter notation

$$P_S(x) = \mathrm{co}[\mathrm{proj}_S(x)].$$

In view of the first identity in (6.2),

$$P_S(x) = x - d_S(x)D^+d_S(x)\,. \tag{6.3}$$

THEOREM 6.3. *Let S be a nonempty closed subset of $\mathbb{R}^n$ and $x \notin S$ be a non-isolated singular point of d_S. Then $P_S(x)$ has an exposed face of dimension at least* 1. *Moreover, if y is in the relative interior of an exposed face E of $P_S(x)$ satisfying* $\dim E \geq 1$, *then* $\dim N_{P_S(x)}(y)$ *is a lower bound for the Hausdorff dimension of the connected component of x in $\Sigma(d_S)$.*

PROOF. Since x is a non-isolated singular point of d_S, Theorem 6.2 ensures that

$$\partial D^+d_S(x) \setminus D^*d_S(x) \neq \emptyset\,.$$

Now, simple arguments of convex analysis show that any vector $p \in \partial D^+d_S(x) \setminus D^*d_S(x)$ belongs to some exposed face V of $D^+d_S(x)$, with $\dim V \geq 1$. Then, recalling (6.2) and (6.3) we conclude that $x - d_S(x)V$ is an exposed face of $P_S(x)$.

By similar arguments we have that, if y is in the relative interior of an exposed face E of $P_S(x)$ with $\dim E \geq 1$, then

$$p := \frac{x-y}{d_S(x)} \in \partial D^+ d_S(x) \setminus D^* d_S(x)$$

and $N_{P_S(x)}(y) = -N_{D^+d_S(x)}(p)$. Then, to complete the proof, it suffices to apply Theorem 5.2 to the distance function. □

The next result immediately follows from the previous one since, owing to (6.3), the dimension of $P_S(x)$ coincides with $k(x)$, the magnitude of x.

COROLLARY 6.4. *Let S be a nonempty closed subset of $\mathbb{R}^n$ and $x \notin S$ be a singular point of d_S. Then $n - k(x)$ is a lower bound for Hausdorff dimension of the connected component of x in $\Sigma(d_S)$.*

A typical situation that is covered by Corollary 6.4 is when $\mathrm{proj}_S(x) = \{y_0, \ldots, y_k\}$ for some $k \in \{1, \ldots, n-1\}$ provided that the vectors $y_1 - y_0, \ldots, y_k - y_0$ are linearly independent. In this case the connected component of x in $\Sigma(d_S)$ has dimension $\geq n - k$.

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Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica
00133 Roma, Italy
albano@axp.mat.uniroma2.it
cannarsa@axp.mat.uniroma2.it