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On del Pezzo Fibrations

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Abstract. A Fano–Mori space is a projective morphism with connected fibers and canonical class relatively antiample. These objects are conjecturally the building blocks of uniruled varieties and of projective morphisms between smooth varieties. In the paper are investigated properties of the fundamental divisor of Fano–Mori spaces. It is proved a relative base point freeness result, conjectured by Andreatta and Wiśniewski, and a relative good divisor statement for del Pezzo fibrations.

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Introduction

A projective variety X is called Fano if a multiple of the anticanonical divisor $-K_X$ is ample. To such a variety is naturally associated a Cartier divisor H, the fundamental divisor, and a positive rational number $i(X)$, the index. If $i(X)$ and $\dim(X)$ are "close" enough it is then possible to understand lot of X using properties of $|H|$ such as base point freeness or existence of sections with "good" singularities.

In the same way, to a Fano-Mori space, that is a morphism $f : Y \to T$ with connected fibers and $-K_Y$ f-ample, it is possible to associate a fundamental divisor $L \in \mathrm{Pic}(Y)$ and a positive rational number r. It is quite natural to expect that if r is "close" to $\dim F$, for all fibers F of f, then also in this relative case it is possible to understand better this morphism using properties of the generic section of a fundamental divisor.

While the study of Fano varieties dates back to the beginning of the century, [Fa], the understanding of the generic element of the fundamental divisor is comparatively quite recent even for high index varieties and it has been a breakthrough of the theory since it allowed to reduce the dimension of objects studied, [Sh], [Fu1], [Re1], [Me]. Almost all is known in the relative case is contained in the pioneering works of Kawamata, [Ka2]

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and Andreatta-Wiśniewski, [AW1], which, essentially, deal with relative Projective spaces and Quadrics. In [AW1] the authors study contractions with $\dim F \leq r + 1 - \epsilon(\dim Y - \dim T)$, and prove that under this hypothesis the fundamental divisor is relatively free. The byproducts of this theorem have then been applied to study various and different situations, see [AW2] for an account. We have now a good knowledge of these contractions when Y is smooth.

In the present paper we are interested in the next step, that is contractions with $\dim F \leq r+2-\epsilon(\dim Y - \dim T)$. In this range it is known the classification of possible fibers for:

- Y smooth and $\dim T = 1$, [Fu2],
- isolated 2-dimensional fibers of elementary contractions from smooth 4-folds to 3-folds, [Kac], [AW3],
- isolated 2-dimensional fibers of extremal contractions from smooth n-folds when the fundamental divisor is relatively free, [AW3], we will prove in Theorem 2.6, that this hypothesis is always satisfied,

Furthermore elementary divisorial contractions of smooth 4-folds are mainly understood, [Be], [Fu3].

It is immediate to observe that relative base point freeness is out of consideration in general, since already smooth del Pezzo surfaces of degree 1 fail to have it. In the absolute case Fujita, [Fu1], proved that LT del Pezzo varieties have good divisors, that is the generic element of the fundamental divisor has at worse the same singularities of X. In Section 3 we will prove that the same is true for del Pezzo fibrations and similar results are valid also for more general contractions.

To study the fundamental divisor of an extremal contraction the first task is to understand if there is a section not containing any irreducible component of a fixed fiber. In our case this is by far the most complicate problem and indeed we are able to solve it only for special classes of contractions, see Section 2 for the details. As a by-product we obtain the proof of relative spannedness in some cases. In particular the fundamental divisor is relatively spanned if the contraction is a $(d, 1, 1)$-fibration, with $d \leq 0$, see Definition 1.3. This proves a Conjecture of Andreatta-Wiśniewski and concludes their classification of isolated 2-dimensional fibers of Fano-Mori contractions, [AW3, Section 5]. Observe that $|L|$ may a priori be empty, even for r close to $\dim F$. To overcome this problem we follow [AW1] set up of local contractions.

The second step is to produce a divisor with mild singularities, that allows to start an induction process, this is the content of Proposition 3.3. Finally, in Theorem 3.5, we prove that smooth del Pezzo fibration always have good divisors.

The main tool used all trough the paper is Kawamata's theory of CLC minimal centers of LC singularities. I would like to thank Y. Kawamata for sending me the latest version of his subadjunction formula, [Ka4], this enlarged version helped me to greatly improve and simplify a former manuscript on this topic. This theory is a new dictionary, for Kawamata's Base point free technique, which is particular useful in our situation. Indeed we could roughly say that Kawamata's Bpf reduces the problem of finding sections of a divisor to these of, producing a log variety, and proving a non vanishing on a smaller dimensional variety. In the category of Fano-Mori spaces the latter question is answered using the geometric conditions imposed to the contraction, see the lemmas of Section 2.

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1. – Preliminary results

We use the standard notation from algebraic geometry. In particular it is compatible with that of [KMM] to which we refer constantly and everything is defined over $\mathbb{C}$.

In the following $\equiv$ (respectively $\sim$) will indicate numerical (respectively linear) equivalence of divisors and ϵ will always stand for a sufficiently small positive rational number.

Given a projective morphism $f : X \to Y$ and $A, B \in \mathrm{Div}(X) \otimes \mathbb{Q}$ then A is *f-numerically equivalent* to B ($A \equiv_f B$) if $A \cdot C = B \cdot C$ for any curve contracted by f; and A is *f-linearly* equivalent to B ($A \sim_f B$) if $A - B \sim f^*M$, for some line bundle $M \in \mathrm{Pic}(Y)$, we will suppress the subscript when no confusion is likely to arise.

A *contraction* is a surjective morphism $f : Y \to T$, with connected fibers, between normal varieties. Let f be a contraction and L an f-ample Cartier divisor then f is said to be *supported* by $K_Y + rL$ if there is an $r \in \mathbb{Q}$ such that $K_Y + rL \equiv_f \mathcal{O}_Y$.

A contraction $f : Y \to T$ is called of *fiber type* if $\dim Y > \dim T$ or *birational* otherwise. We will say that a contraction $f : Y \to T$ is *Fano-Mori* (F-M) if Y is LT and $-K_Y$ is f-ample. In the F-M case, which we will treat mainly, by Kawamata-Shokurov base point free theorem, for $m \gg 0$ and $A \in \mathrm{Pic}(T)$ ample, f is the morphism naturally associated to sections in $|m(K_X + rL + f^*A)|$, this is why we say that f is supported by $K_X + rL$.

EXAMPLE 1.1. Let us give some immediate examples of F-M spaces, which justify also their name. If $T = \mathrm{Spec}\,\mathbb{C}$ and f is the constant map, then Y

is just a Fano variety, the opposite of this situation is when f is birational, these morphisms are usually called extremal contractions and were initially investigated by Mori in his celebrated paper, [Mo1].

We will treat various type of contractions to simplify both the treatment and the statement of the results it is convenient to introduce the following definitions.

DEFINITION 1.2. Let $f : Y \to T$ a contraction supported by $K_Y + rL$. Fix a finite set of fibers $F_1, \dots F_k$ of f and an open affine $S \subset T$ such that $f(F_i) \in S$ for $i = 1, \dots, k$. Let $X = f^{-1}S$ then $f : X \to S$ will be called a *local contraction around* $\{F_i\}$. If there is no need to specify fixed fibers then we will simply say that $f : X \to S$ is a local contraction. In particular $S = \mathrm{Spec}(H^0(X, \mathcal{O}_X))$.

DEFINITION 1.3. Let $f : X \to S$ a local contraction around F. Let $r = \inf\{t \in \mathbb{Q} : K_Y + tH \equiv_f 0$ for some ample Cartier divisor $H \in \mathrm{Pic}(X)\}$. Assume that f is supported by $K_Y + rL$. The Cartier divisor L will be called *fundamental divisor* of f. Let G a generic non trivial fiber of f then the *dual-index* of f is

$$d(f) := \dim G - r\,,$$

the *character* of f is

$$\gamma(f) := \begin{cases} 1 & \text{if } \dim X > \dim S \\ 0 & \text{if } \dim X = \dim S \end{cases}$$

and the *difficulty* of f is

$$\Phi(f) = \dim F - r\,.$$

We will say that $(d(f), \gamma(f), \Phi(f))$ is the *type* of f.

REMARK 1.4. If all possible values of a parameter are considered we will simply put a $*$ in its place. In relation to general literature on F-M contractions, we have the following: $(0, 0, 0)$ contractions are smooth blow ups, $(-1, 1, -1)$ fibrations, are scroll, $(0, 1, 0)$ are quadric fibrations, $(1, 1, 1)$ are del Pezzo fibrations, $(-1, 1, *)$ are adjunction scroll and $(0, 1, *)$ are adjunction quadric fibrations. In this notations the main theorem of [AW1] states that the fundamental divisor of a contraction of type $(*, *, 1 - \epsilon\gamma(f))$ is relatively free.

We will use the local set-up developed by Andreatta-Wiśniewski and the notions of horizontal and vertical slicing.

LEMMA 1.5 ([AW1]) (Vertical slicing). *Let $f : X \to S$ be a local contraction supported by $K_X + rL$, with $r \geq -1 + \epsilon\gamma(f)$. Assume that X has LT singularities and let h be a general function on S. Let $X_h = f^*(h)$ then the singularities of X_h are not worse than these of X and any section of L on X_h extends to X.*

Vertical slicing is used to reduce the "bad locus" to a subset of finitely many fibers only.

LEMMA 1.6 ([AW1]) (Horizontal slicing). *Let $f : X \to S$ be a local contraction around $\{F_i\}$ supported by $K_X + rL$. Let $X_k = \cap_1^k H_i$, with $H_i \in |L|$ generic divisors.*

i) *Let $f_{|X_k} = g \circ f_k$ the Stein factorisation of $f_{|X_k} : X_k \to S$ then $f_k : X_k \to S_k$ is a morphism with connected fibers, around $\{F_i \cap (\cap_1^k H_i)\}$, supported by $K_{X_k} + (r-k)L_{|X_k}$ and S_k is affine. In particular if X_k is normal then f_k is a local contraction.*

Assume that X has LT singularities, $r \geq \epsilon\gamma(f)$ and $k \leq r + 1 - \epsilon\gamma(f)$.

ii) *The singularities of X_k are not worse then that of X outside of $Bsl|L|$, and any section of L on X_k extends to a section of L on X.*

PROOF. We will sketch it since the set up is slightly different from [AW1]. i) is just Stein factorisation and adjunction formula once noticed that $f_{|X_k}(X_k) = \mathrm{Spec}(H^0(X, \mathcal{O}_X \otimes \mathcal{O}_{X_k}))$.

For ii) the first statement is just Bertini theorem, while for the latter consider the exact sequences

$$0 \to \mathcal{O}_{X_i}(-L) \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_{i+1}} \to 0$$
$$0 \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_i}(L) \to \mathcal{O}_{X_{i+1}}(L) \to 0 .$$

Thus to prove the assert it is enough to prove that $H^1(X_i, \mathcal{O}_{X_i}) = 0$, for $i \leq r - \epsilon\gamma(f)$. But this is equivalent, using inductively the first sequence tensored, to $H^j(X, -iL) = 0$, for $i \leq r - \epsilon\gamma(f)$ and $j > 0$, which follows from K-V vanishing. □

Horizontal slicing is used to apply induction arguments, going from the local contraction $f : X \to S$ to the local contraction $f_1 : X_1 \to S_1$.

REMARK 1.7. If $r \geq 1 + \epsilon\gamma(f)$ then $f_{|X_1} = f_1$ and in particular it has connected fibers, [AW1, Lemma 2.6], therefore it is enough to consider the restriction to have a lower dimensional contraction. This is no more true when r is smaller. For instance there are examples due to Mukai, Shepherd-Barron and Wiśniewski of contractions $f : X^4 \to S^3$ with exceptional fiber two copies of $\mathbb{P}^2$ meeting at a single point and with a fundamental divisor relatively free, [Kac] [AW2]. Nonetheless using part (i) of the above lemma we can always associate a lower dimensional contraction f_1 around the fibers we are interested in. That is why we need to consider a finite set of fibers in Definition 1.2, because even if we start with only one fiber after a slicing we could have to study a finite number of disjoint fibers all together. Note that there is a morphism $S_1 \to S$ induced by the morphism of rings $H^0(X, \mathcal{O}_X) \to H^0(X_1, \mathcal{O}_{X_1})$ and when we substitute f by f_1 we are not allowed to use all functions in $\mathcal{O}_{S_1}$ but only those coming from $\mathcal{O}_S$.

The main tool we will use is Kawamata's notion of CLC minimal centers. Let $\mu : Y \to X$ a birational morphism of normal varieties. If D is a $\mathbb{Q}$-divisor

($\mathbb{Q}$-Cartier) then it is well defined the strict transform $\mu_*^{-1}D$ (the pull back μ^*D). For a pair (X, D) of a variety X and a $\mathbb{Q}$-divisor D, a *log resolution* is a proper birational morphism $\mu : Y \to X$ from a smooth Y such that the union of the support of $\mu_*^{-1}D$ and of the exceptional locus is a normal crossing divisor.

DEFINITION 1.8. Let X be a normal variety and $D = \sum_i d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $\mu : Y \to X$ is a log resolution of the pair (X, D), then we can write

$$K_Y + \mu_*^{-1}D = \mu^*(K_X + D) + F$$

with $F = \sum_j \operatorname{disc}(X, E_j, D)E_j$ for the exceptional divisors E_j. We call $e_j := \operatorname{disc}(X, E_j, D) \in \mathbb{Q}$ the discrepancy coefficient for E_j, and regard $-d_i$ as the discrepancy coefficient for D_i.

The pair (X, D) is said to have *log canonical* (LC) (respectively *Kawamata log terminal* (KLT), *log terminal* (LT)) singularities if $d_i \leq 1$ (resp. $d_i < 1$, $d_i = 0$) and $e_j \geq -1$ (resp. $e_j > -1$, $e_j > -1$) for any i, j of a log resolution $\mu : Y \to X$. The *log canonical threshold* of a pair (X, D) is $lct(X, D) := \sup\{t \in \mathbb{Q}\colon (X, tD) \text{ is LC}\}$.

DEFINITION 1.9. A *log-Fano variety* is a KLT pair (X, Δ) such that for some positive integer m, $-m(K_X + \Delta)$ is an ample Cartier divisor. The index of a log-Fano variety $i(X, \Delta) := \sup\{t \in \mathbb{Q} : -(K_X + \Delta) \equiv tH$ for some ample Cartier divisor $H\}$ and the H satisfying $-(K_X + \Delta) \equiv i(X, \Delta)H$ is called fundamental divisor.

PROPOSITION 1.10 ([Am]). *Let (X, Δ) be a log-Fano n-fold of index $i(X)$, H the fundamental divisor in X. If $i(X) > n - 3$ then* $\dim |H| \geq 0$.

Let us now recall the notion and properties of minimal centers of log canonical singularities as introduced in [Ka3]

DEFINITION 1.11 ([Ka3]). Let X be a normal variety and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, D), if there is a birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y with the discrepancy coefficient $e \leq -1$ such that $\mu(E) = W$. For another such $\mu' : Y' \to X$, if the strict transform E' of E exists on Y', then we have the same discrepancy coefficient for E'. The divisor E' is considered to be equivalent to E, and the equivalence class of these prime divisors is called a *place of log canonical singularities* for (X, D). The set of all centers (respectively places) of LC singularities is denoted by $CLC(X, D)$ (resp. $PLC(X, D)$), the locus of all centers of LC singularities is denoted by $LLC(X, D)$.

THEOREM 1.12 ([Ka3], [Ka4]). *Let X be a normal variety and D an effective $\mathbb{Q}$-Cartier divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Assume that X is LT and (X, D) is LC.*

i) *If $W_1, W_2 \in CLC(X, D)$ and W is an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$. In particular, there exist minimal elements in $CLC(X, D)$.*

ii) *If* $W \in CLC(X, D)$ *is a minimal center then* W *is normal*

iii) (*subadjunction formula*) *Let* H *an ample Cartier divisor and* ϵ *a positive rational number. If* W *is a minimal center for* $CLC(X, D)$ *then there exist effective* $\mathbb{Q}$*-divisors* D_W *on* W *such that* $(K_X + D + \epsilon H)_{|W} \equiv K_W + D_W$ *and* (W, D_W) *is KLT.*

In the next section we will frequently use pairs (X, D) which are not LC. To be able to treat this situation let us introduce the following definition and make some useful remarks.

DEFINITION 1.13. The *log canonical threshold related to a scheme* $V \subset X$ of a pair (X, D) is $lct(X, V, D) := \inf\{t \in \mathbb{Q} : V \cap LLC(X, tD) \neq \emptyset\}$. We will say that (X, D) is LC along a scheme V if $lct(X, V, D) \geq 1$.

REMARK 1.14. Let $Z \in CLC(X, lct(X, V, D)D)$ a center and assume that Z intersects V, then $(X, lct(X, V, D)D)$ is LC on the generic point of Z.
If (X, D) is not LC then Theorem 1.12 is in general false. On the other hand the first assertion stays true, also under the weaker hypothesis that (X, D) is LC on the generic point of $W_1 \cap W_2$. In fact the discrepancy is a concept related to a valuation ν, therefore we can always substitute the variety X by an affine neighbourhood of the generic point of the center of ν, see also Claim 2.12.

Before ending this section let us spend a few words on the perturbation of a log variety (X, D) by means of an arbitrarily small $\mathbb{Q}$-divisor.

1.15 (Perturbation of a log variety). Let (X, D) a log variety and assume that (X, D) is LC and $W \in CLC(X, D)$ is a minimal center. Then we will have a log resolution $\mu : Y \to X$ with

$$K_Y = \mu^*(K_X + D) + \sum e_i E_i\,,$$

this time we put also the strict transform of the boundary on the right hand side. Since (X, D) is LC and $W \in CLC(X, D)$ then $e_i \geq -1$ and there is at least one $e_j = -1$ such that $\mu(E_j) = W$. The main problem here is that to apply Kawamata's Bpf we need to have one and only one exceptional divisor with discrepancy -1 and W as a center. To fulfill this requirement it is enough to choose generic very ample M such that $W \subset \mathrm{Supp}(M)$ and no other $Z \in CLC(X, D) \setminus \{W\}$ is contained in $\mathrm{Supp}(M)$, this is always possible since W is minimal in a dimensional sense. We then perturb D to a divisor $D_1 := (1 - \epsilon_1)D + \epsilon_2 M$, with $0 < \epsilon_i \ll 1$ in such a way that

- (X, D_1) is LC
- $CLC(X, D_1) = W$
- $\mu^*\epsilon_2 M = \sum m_i E_i + P$, with P ample, this is possible by Kodaira lemma.

After this perturbation the log resolution looks like the following

$$K_Y + \sum_{j=0} E_j + \Delta - A = \mu^*(K_X + D_1) - P\,,$$

where the E_j's are integral irreducible divisors and $\mu(E_j) = W$, A is a μ-exceptional integral divisor and $\lfloor \Delta \rfloor = 0$. It is now enough to use the ampleness of P to choose just one of the E_j. Indeed for small enough $\delta_j > 0$ $P' := P - \sum_{j=1} \delta_j E_j$ is still ample therefore we produce the desired resolution

$$K_Y + E_0 + \Delta' - A = \mu^*(K_X + D') - P', \tag{1.1}$$

here and all trough the paper after a perturbation we will always gather together all the fractional part with negative log discrepancy in P and Δ, respectively the ample part of it and the remaining. If instead of an ample M we choose a nef and big divisor, we can repeat the above argument with Kodaira lemma, but this time we cannot choose the center $\mu(E_0)$ like before, and in particular we cannot assume that at the end we are on a minimal center for (X, D). An instructive example is the following.

EXAMPLE 1.16 (see also [AW3]). Let Y a smooth degree 1 del Pezzo n-fold and H the fundamental divisor. Consider $X := \mathrm{Spec}_Y(\oplus_{k\geq 0}\mathcal{O}(kH)) \xrightarrow{\pi} Y$, the total space of the dual bundle $\mathcal{O}(H)^\vee$ with the zero section $Y_0 \subset X$. Let $f : X \to Z$ the contraction of Y_0 to a point, that is $Z = \mathrm{Spec}(\oplus_{k\geq 0} H^0(X, kH))$. This is a F-M space and f is a birational contraction around Y_0 supported by $K_X + (n-2)L$, where $L = \pi^* H$. Let $H_i \in |L|$ be generic sections and consider the divisor $D = Y_0 + \sum_1^n H_i$, then $D \equiv_f (n-1)L$ and by standard properties of del Pezzo varieties (X, D) is LC and $CLC(X, D) = \{Y_0, H_i, Y_0 \cap_i H_i, x\}$ where $x = Bsl|L|$. In particular $\{x\}$ is the minimal center for (X, D). Let $M = \mathcal{O}_X$, then M is f-nef and f-big and if we perturb D by means of M we get a log resolution as in (1.1) but $\mu(E_0) = Z \neq x$. This can be easily seen by a direct calculation or using the results of Section 2 to derive the contradiction that $|L|$ should be spanned if $Z = x$.

Nevertheless all results, like normality and subadjunction stay true for $\mu(E_0)$ since the existence of a resolution as in (1.1) implies that it is minimal for (X, D').

We can also extend the above arguments to log varieties (X, D) which are only LC at the generic point of a subvariety W, even if in this case nothing can be said about singularities or subadjunction.

2. – Existence of sections on local contractions

The main result of this section is that a local contraction of type $(d, 1, 1)$ has sections in the fundamental divisor non vanishing on any irreducible component of the fixed fiber F when $d \leq 1$. This is crucial since it is the first step of the inductive procedure we are aiming to apply.

Let us start with some technical lemmas. The following is just a restatement of [AW1, Th 2.1], in the dictionary of CLC minimal centers.

LEMMA 2.1. *Let $f : X \to S$ a local contraction supported by $K_X + rL$ around F, and D a $\mathbb{Q}$-divisor, $D \equiv_f \gamma L$. Assume X is LT and that $Z \in CLC(X, D)$, with $Z \subseteq F$ and (X, D) is LC on the generic point of Z. If $\gamma < r + 1 - \dim Z$, then there exists a section of $|L|$ not vanishing identically on Z.*

PROOF. Let us perturb D to $D_1 := (1 - \epsilon_1)D + \epsilon_2 M$, for some $\epsilon_i \ll 1$ and $M \in |mL|$ f-very ample. Then we may assume that (X, D_1) is LC at the generic point of Z and $D_1 \equiv \gamma_1 L$ with $\gamma_1 < r + 1 - \dim Z$. Furthermore by Kodaira lemma we have a log resolution $\mu : Y \to X$, of (X, D_1) with

$$K_Y + E - A + B + \Delta = \mu^*(K_X + D_1) - P\,,$$

where E is an irreducible integral divisor, A, and B are integral divisors, Δ and P are $\mathbb{Q}$-divisors such that: $\mu(E) = W \subseteq Z$, A is μ-exceptional, $\lfloor \Delta \rfloor = 0$, $W \not\subseteq \mu(B)$ and P is $(f \circ \mu)$-ample. We stress that E, A, B, P and Δ have not common irreducible components. Let

$$N(t) := -K_Y - E - \Delta - B + A + \mu^*(tL)\,,$$

then $N(t) \equiv_{f\circ\mu} \mu^*(t+r-\gamma_1)L + P$ and $N(t)$ is $(f \circ \mu)$-ample whenever $t+r-\gamma_1 \geq 0$. Thus K-V vanishing yields

$$H^i(Y, \mu^*(tL) - E + A - B) = 0 \quad H^i(E, (\mu^*(tL) + A - B)_{|E}) = 0 \tag{2.1}$$

for $i > 0$ and $t + r - \gamma_1 \geq 0$. Consequently

$$H^0(Y, \mu^*L + A - B) \to H^0(E, \mu^*L + A - B) \to 0\,.$$

A is effective and μ-exceptional and $W \not\subset \mu(B)$, thus any section in $H^0(Y, \mu^*L + A - B)$, not vanishing on E, pushes forward to give a section of L not vanishing on Z. To conclude the proof it is, therefore, enough to prove that $h^0(E, \mu^*L + A - B) > 0$. Let $p(t) = \chi(E, \mu^*tL + A - B)$, then $\deg p(t) \leq \dim Z$. Furthermore $W \not\subset \mu(B)$ thus $p(t) = h^0(E, \mu^*tL + A - B) > 0$ for $t \gg 0$. If W is a point then $p(1) = p(t) > 0$. If $\dim W > 0$ then by equation (2.1), $p(0) \geq 0$ and $p(t) = 0$ for $0 > t \geq -\dim Z + 1$. Therefore we have enough zeros to ensure that $h^0(E, \mu^*L + A - B) = p(1) > 0$. □

LEMMA 2.2. *Let $f : X \to S$ a local contraction supported by $K_X + rL$ around F. Fix a subvariety $Z \subset F$, and a $\mathbb{Q}$-divisor D, with $D \equiv_f \gamma L$. Assume that X is LT, (X, D) is LC along Z, and $W \in CLC(X, D)$ is a minimal center contained in Z. Assume that one of the following conditions is satisfied:*

i) $r - \gamma > \max\{0, \dim W - 3\}$,
ii) $\dim W \leq 1$ *and* $r - \gamma > -1$.

Then there exists a section of $|L|$ not vanishing identically on W.

PROOF. Since D is LC along W we can assume that there exists a log resolution $\mu : Y \to X$ of (X, D) with

$$K_Y - A + E + \Delta + B = \mu^*(K_X + D) - P\,,$$

where E is an irreducible integral divisor, A, and B are integral divisors, Δ and P are $\mathbb{Q}$-divisors such that: $\mu(E) = W$, A is μ-exceptional, $\lfloor \Delta \rfloor = 0$, $Z \cap \mu(B) = \emptyset$ and P is $(f \circ \mu)$-ample. Let

$$N(t) := \mu^* tL + A - \Delta - E - B - K_Y \equiv_{f \circ \mu} \mu^*(t + r - \gamma)L + P\,, \tag{2.2}$$

then $N(t)$ is $(f \circ \mu)$-ample whenever $t + r - \gamma \geq 0$. In particular if any of the conditions of the lemma are satisfied by K-V vanishing we have the following surjection

$$H^0(Y, \mu^*L + A - B) \to H^0(E, (\mu^*L + A)_{|E})\,.$$

Since A does not contain E and is effective then

$$H^0(W, L_{|W}) \hookrightarrow H^0(E, (\mu^*L + A)_{|E})\,.$$

In particular any section of $H^0(W, L_{|W})$ gives rise to a section in $H^0(X, L)$ not vanishing identically on W. Therefore to conclude the proof it is enough to produce a section in $H^0(W, L_{|W})$. By subadjunction formula of Theorem 1.12 there exists a $\mathbb{Q}$-divisor D_W such that

$$K_W + D_W \equiv (K_X + D + \delta L)_{|W} \equiv -(r - \gamma - \delta)L_{|W}\,, \tag{2.3}$$

for any $0 < \delta \ll 1$.

In case (i) since $r - \gamma > 0$ then by equation (2.3), for sufficiently small δ, (W, D_W) is a log Fano variety of index $i(W, D_W) = r - \gamma - \delta > \dim W - 3$. Therefore we can apply Proposition 1.10.

If $\dim W = 1$ then W is smooth. Since $r - \gamma - \delta > -1$ by relation (2.3) $0 < L \cdot W \geq 2g(W) - 2$ thus $h^0(W, L_{|W}) > 0$ by R-R formula. □

In case of birational contractions we "gain a vanishing more" from perturbing with $f^*\mathcal{O}_S$.

LEMMA 2.3. *Let $f : X \to S$ a local contraction supported by $K_X + rL$ around F, and D a $\mathbb{Q}$-divisor, $D \equiv_f \gamma L$. Assume X is LT and f birational.*

Let $\dim(LLC(X, D) \cap F) = w$ *and assume that there exists $W \in CLC(X, D)$ with $W \subset F$ satisfying one of the following conditions:*

i) *(X, D) is LC on the generic point of W and $\gamma \leq r + 1 - w$;*
ii) *(X, D) is LC along W, $w = 2$ and $r - \gamma = 0$;*
iii) *(X, D) is LC along W, $w \leq 1$ and $r - \gamma = -1$.*

Then there exists a section of $|L|$ not vanishing identically on $LLC(X, D) \cap F$.

PROOF. The proof of i) and iii) is exactly as in Lemmas 2.1, 2.2 once noticed that since f is birational we can perturb D with the f-nef and f-big divisor $\mathcal{O}_X$. The difference here is that we cannot choose the minimal center after the perturbation, see example 1.16, but just one center contained in $LLC(X, D) \cap F$. For ii), following the proof of Lemma 2.2, we have only to prove that $H^0(W, L_{|W}) > 0$. Let $L_W := L_{|W}$ then by K-V vanishing applied to (2.2) we have

$$H^i(E, \mathcal{O}_E(\mu^*(tL) + A)) = R^i\mu_*\mathcal{O}_E(\mu^*(tL) + A) = 0\,,$$

for $i > 0$ and $t \geq 0$. Furthermore by projection formula $\mu_*\mathcal{O}_E(\mu^*(tL) + A)) = \mathcal{O}_W(tL_W)$, see for instance the proof of the theorem in the Appendix. Thus by Leray spectral sequence we have $\chi(W, \mathcal{O}_W) = 1$ and $\chi(W, L_W) = h^0(W, L_W)$. Let $\nu : V \to W$ a log resolution for (W, D_W), then by subadjunction formula (2.3)

$$\nu^*L_W \cdot (\nu^*L_W - K_V) = \nu^*L_W \cdot (\nu^*L_W + \nu^*(r - \gamma - \delta)L_W + \text{ effective}) > 0\,.$$

W has rational singularities hence $\chi(V, \nu^*L_W) = \chi(W, L_W)$. Thus by R-R formula

$$h^0(W, L_{|W}) = \chi(V, \nu^*L_W) > 1\,. \qquad \square$$

Just to warm up let us start to prove that there is a section non vanishing on the whole fiber, even for a broader class of contractions.

PROPOSITION 2.4. *Let $f : X \to S$ be a local contraction of type $(*, *, \Phi)$, supported by $K_X + rL$ around a fiber F. Assume that X is LT and one of the following holds:*

i) $r > 0$ *and* $\Phi < 3$, *i.e.* $\dim F < r + 3$;

ii) $\Phi \leq 2 - \epsilon\gamma(f)$, *i.e.* $\dim F \leq r + 2 - \epsilon(\dim X - \dim S)$.

Then there is a section of $|L|$ not vanishing identically along F.

PROOF. The claim is immediate when f is finite, thus we can assume that $r \geq -1+\epsilon\gamma(f)$ in ii). Let $\{g_i\}$ be general functions on S vanishing at $f(F)$ and $D = \sum l_i f^*(g_i)$, with $l_i \ll 1$. We can assume that (X, D) is LC along F with minimal center W and $LLC(X, D) \subseteq F$. Then W satisfies the assumptions of Lemma 2.2 or those of Lemma 2.3, thus there is a section of $|L|$ not vanishing along F. More in detail we conclude by: Lemma 2.2 i) for $r > 0$, Lemma 2.3 ii) for $r = 0$ and $\dim F = 2$, Lemma 2.2 ii) for $0 > r > -1$, Lemma 2.3 iii) for $r = -1$ and $\dim F = 1$. $\square$

REMARK 2.5. The above is a sort of ideal proof for this kind of results. Unfortunately, arguing as in Proposition 2.4 we cannot predict in which irreducible component of F is contained W.

We can now state the main result of this section.

THEOREM 2.6. *Let $f : X \to S$ be a local contraction supported by $K_X + rL$ around a fiber F. Assume that X is LT, and either f is of type $(d, 1, 1)$, with $d \leq 0$ or F is reducible and f is of type $(1, 1, 1)$. Then $|L|$ is relatively spanned by global sections around F. That is $Bsl|L| :=$ Supp(Coker($f^*f_*L \to L$)) does not meet F.*

REMARK 2.7. The above theorem was proved by Kachi, [Kac, Theorem 4.1], in case X smooth, $\dim X = 4$, dim $S = 3$ and f elementary with isolated 2 dimensional fibers. The general statement for type $(d, 1, 1)$-fibrations was conjectured by Andreatta-Wiśniewski, [AW2, Conj 1.13].

PROOF. Let $V = Bsl|L| \cap F$. Let $\{g_i\}$ be general functions on S vanishing at $f(F)$ and $D_0 = \sum l_i f^*(g_i)$, with $l_i \ll 1$. Define $c_0 = lct(X, V, D_0)$ and

$$\mathcal{Z} = \{W \in CLC(X, c_0 D_0) : W \cap V \neq \emptyset\}.$$

Let $Z_0 \in \mathcal{Z}$ a center such that $\dim Z_0 \leq \dim W$, for any $W \in \mathcal{Z}$. Our plan is to produce a section non vanishing identically along V so to derive a contradiction. We have to distinguish between various cases.

CASE 2.8 ($Z_0 \subset V$). Then $(X, c_0 D_0)$ is LC along Z_0 and by Lemma 2.2 we have the desired section.

Therefore $Z_0 \not\subset V$. Let $H_1 \in |L|$ a generic divisor and

$$c_1 = \sup\{t \in \mathbb{Q} : (X, c_0 D_0 + t H_1) \text{ is LC along } Z_0 \cap V\}.$$

Since H_1 is a Cartier divisor containing V then $c_1 \leq 1$.

CASE 2.9 ($c_1 < 1$). Then

$$CLC(X \setminus Bsl|L|, c_0 D_0 + c_1 H_1) = CLC(X \setminus Bsl|L|, c_0 D_0).$$

Moreover $Z_0 \not\subset V$ and it has the smallest dimension between all centers in $\mathcal{Z}$ thus $c_1 > 0$. So that any $W \in CLC(X, c_0 D_0 + c_1 H_1) \setminus CLC(X, c_0 D_0)$ is contained in $Bsl|L|$. By construction $(X, c_0 D_0 + c_1 H_1)$ is LC along $Z_0 \cap V$, therefore by Theorem1.12 (i), keep in mind the Remark after Definition1.13, there exists a minimal center W satisfying the following conditions:

- $W \in CLC(X, c_0 D_0 + c_1 H_1)$,
- $W \subseteq (Z_0 \cap V)$,
- $\dim W < \dim Z_0 \leq r + 1$.

Hence Lemma 2.2 produces again the section needed.

Let $D_1 = c_0 D_0 + H_1$ and

$$\mathcal{Z}_1 = \{W \in CLC(X, D_1) : W \subseteq (Z_0 \cap H_1) \text{ and } W \cap (Z_0 \cap V) \neq \emptyset\}.$$

Let $Z_1 \in \mathcal{Z}_1$ a center such that $\dim Z_1 \leq \dim W$, for any $W \in \mathcal{Z}_1$. Observe that any irreducible component of $Z_0 \cap H_1$ belongs to $\mathcal{Z}_1$, therefore $\dim Z_1 < \dim Z_0 \leq r + 1$.

CASE 2.10 ($Z_1 \subset V$ and $r > 0$). Then (X, D_1) is LC along Z_1 and we conclude by Lemma 2.2, this is as in Case 2.8.

We can iterate the above procedure substituting the log variety $(X, c_0 D_0)$ and the center Z_0 with the log variety (X, D_1) and the center Z_1. Repeating the same arguments either we derive a contradiction as in (2.9) (observe that $\dim Z_1 \leq r+1-1$ thus the inequalities of Lemma 2.2 are fulfilled for $r>1$), or we produce a log variety (X, D_2) and a new center Z_2. More generally we can iterate finitely many times to produce log varieties (X, D_k) and minimal centers Z_k such that:

- $Z_k \in CLC(X, D_k)$ and $Z_k \subseteq Z_{k-1} \cap H_k$;
- $Z_k \cap (Z_{k-1} \cap V) \neq \emptyset$;
- (X, D_k) is LC along $Z_{k-1} \cap V$;
- $\dim Z_k \leq r+1-k$;
- $D_k = c_0 D_0 + \sum_1^k H_j$, with $H_j \in |L|$ generic.

By this procedure we can assume that (X, D_{r+1}) exists, Z_r is a curve, which is an irreducible component of a fiber of $f_{|X_r}$, and Z_{r+1} is a set of points in V. Let $\mu : Y \to X$ a log resolution of (X, D_{r+1}) and $G_j := \mu_*^{-1} H_j$, $X_k = X \cap (\cap_1^k H_j)$ and $Y_k = Y \cap (\cap_1^k G_j)$. Furthermore we can assume that $\mu_k := \mu_{|Y_k} : Y_k \to X_k$ is a log resolution for all k's, where all relevant divisors have simple normal crossings. It is important to stress two points here.

- The resolution has only simple normal crossing thus, when discrepancies on X_k are defined, for any exceptional divisor E_j such that $E_j \cap Y_k \neq \emptyset$, by adjunction formula, $\operatorname{disc}(X_k, c_0 D_{0|X_k}, E_{j|Y_k}) = \operatorname{disc}(X, D_k, E_j)$.
- $Z_k \subset Z_{k-1} \cap H_k$ is irreducible, (X, D_k) is LC on the generic point of Z_k and $Z_k \not\subset V$ if $k \leq r$.

Let us briefly sketch the conclusion of the proof, before going into technical details. We will produce a birational morphism $\varphi : X_r \to S_\varphi$, with Z_r as a fiber. This can be done, at least in a complex neighbourhood of Z_r, and tells us that $Z_r \simeq \mathbb{P}^1$. Then the general principle, see for instance Lemma 2.3, is that in this way we gain a vanishing more perturbing with $\varphi^* \mathcal{O}_{S_\varphi}$. To gain this vanishing we have to overcome the problem that the perturbation is only defined in a complex neighbourhood of Z_r. To do this we will carefully perturb our starting $\mathbb{Q}$-divisor D_r in such a way that we can compare it with the one perturbed directly on X_r and then extend the section found in Y_r to a section defined on the whole of Y. To accomplish the latter we need to work all trough the proof with the fixed resolution μ, therefore when we have a center of LC singularities, we have also to exhibit a place of LC singularities on the variety upstairs, see Claim 2.12.

Let us now deep into technicalities.

CLAIM 2.11. There exists a complex neighbourhood $U \supset Z_r$ in X_r and a birational contraction $\varphi : U \to S_\varphi$ such that U is LT and Z_r is a fiber of φ. In particular $Z_r \simeq \mathbb{P}^1$.

PROOF (of the Claim). Let us prove by induction that X_k is LT in a neighbourhood of Z_k, this is true by hypothesis for $X_0 := X$ and Z_0. Then X_k is a Cartier divisor of X_{k-1} and, by Bertini theorem, it can be not LT only along $Bsl|L| \cap X_{k-1}$. By our construction (X, D_k) is LC along $Z_{k-1} \cap V$ and $Z_{k-1} \subset \operatorname{Supp}(D_0)$ therefore by adjunction formula we have only to prove that

X_k is non singular in codimension 1 in a neighbourhood of Z_{k-1}. Assume that this is not true, then there is a codimension 2 subvariety $W \subset X_{k-1}$ along which X_k is singular. In particular $W \subset (Bsl|L| \cap X_{k-1})$ and $Z_k \not\subset W$. Moreover since terminal singularities are smooth in codimension 2 then $W \in CLC(X, D_k)$. Indeed if X_{k-1} is smooth along W then $\mathrm{mult}_W H_{|X_{k-1}} \geq 2$ while if X_{k-1} is singular along W then there exists a valuation with center W of non positive discrepancy and $\mathrm{mult}_W H_{|X_{k-1}} \geq 1$. So in any case we have a place of LC singularities over W. But then $W \cap Z_k \in CLC(X, D_k)$ and this is impossible by our assumption on the dimension of Z_k.

If $d(f) \leq 0$ then $f_{|X_r}$ is already birational. If $d(f) = 1$ and F is reducible then choose a very ample divisor $M \in \mathrm{Pic}(X_r)$ such that $\mathrm{Supp}(M)$ does not contain the points of intersection of Z_r with the other components of the fiber $F \cap X_r$. Then we can shrink S to a complex neighbourhood of $f(F)$ in such a way that $M = M_Z + M_F$, where M_Z and M_F are effective Cartier divisors such that $M_Z \cdot C = 0$ for any curve $C \subset (F \cap X_r) \setminus Z_r$, and $M_F \cap Z_r = \emptyset$. Using the sections of M_F we define a birational contraction $\varphi : U \subset X_r \to S_\varphi$, from a complex neighbourhood U of Z_r, supported by K_{X_r} with Z_r as a fiber. Shrinking S_φ, to a complex neighbourhood of $\varphi(Z_r)$ we can assume that U is LT and therefore by K-V vanishing $Z_r \simeq \mathbb{P}^1$, all necessary vanishing are still true in this analytic situation, for details see [Ka1] and [Na]. $\square$

This tells us that μ is an isomorphism on the generic point of X_k containing Z_k. Furthermore by the above Claim we are allowed to talk about discrepancies in a neighbourhood of Z_r. We will take advantage of this in Claim 2.12 and in (2.13), to shift our attention on the manifold Y_r.

CLAIM 2.12. There exists a divisor $\widetilde{E} \subset Y_r$ such that $\widetilde{E} \in PLC(X_r, c_0\, D_{0|X_r})$ and $\mu_r(\widetilde{E}) = Z_r$. In particular $lct(X_r, Z_r, c_0 D_{0|X_r}) = 1$.

PROOF (of the Claim). The idea of the proof of this claim is the following. We want to present a place of log canonical singularities for (X, D_r) in Y with center Z_r and which intersects Y_r. Since $Y_r = \cap G_i$, $Z_r \subset \mu(Y_r)$ and the $G_i \in PLC(X, D_r)$ we will use Shokurov Connectedness and induction to prove that Y_r cannot be disjoined from the places of Log Canonical singularities over Z_r. We are interested in properties of valuations with centers Z_r. Therefore we can assume that the whole of X shares the same properties of the generic point of Z_r, that is we can assume that X is LT, L is spanned and (X, D_k) is LC for $k \leq r+1$. We can now argue by induction since X_k is LT by Bertini theorem. By the induction hypothesis there exists a divisor $E_k \subset Y_k$ such that $E_k \in PLC(X_k, c_0 D_{0|X_k})$ and $\mu(E_k) = Z_k$. Let $M_k \in |mL_{|X_k}|$, a generic divisor whose support contains Z_k, $e = \mathrm{mult}_{E_k} c_0 D_0$ and $e_1 = \mathrm{mult}_{E_k} M_k$. Let $a_1 = e/e_1$ then for any $\epsilon \ll 1$ $E_k \in PLC(X_k, ((1-\epsilon)c_0 D_0 + H_{k+1})_{|X_k} + a_1 \epsilon M_k)$. Furthermore the only element of $PLC(X_k \setminus Z_k, ((1-\epsilon)c_0 D_0 + H_{k+1})_{|X_k} + a_1 \epsilon M_1)$ which exists on Y_k is $G_{k+1|Y_k}$. By Connectedness theorem, [Sh2] see also [Ka3], there exists a divisor $\widetilde{E}_{k\epsilon} \subset Y_k$ such that $\widetilde{E}_{k\epsilon} \in PLC(X_k, ((1-\epsilon)c_0 D_0 + H_{k+1})_{|X_k} + a_1 \epsilon M_1)$, $\mu(\widetilde{E}_{k\epsilon}) = Z_k$ and $\widetilde{E}_{k\epsilon} \cap G_{k+1_{|Y_k}} \neq \emptyset$. Since there are only finitely many exceptional divisors on Y then there is a fixed $\widetilde{E}_k = \widetilde{E}_{k\epsilon}$ for a sequence $\epsilon \to 0$. To proceed with the induction let $E_{k+1} = \widetilde{E}_{k|Y_{k+1}}$, and to conclude let $\widetilde{E} = E_r$. $\square$

2.13 (Perturbation on X). Let $D' = (1-\epsilon_0)c_0 D_0 + \sum_1^r H_j$, with $\epsilon_0 \ll 1$, such that

$$K_Y + \sum_1^r G_j = \mu^*(K_X + D') + \sum_i e'_i E_i\,,$$

with all $e'_i \notin \mathbb{Z}$.

Let $\overline{D} = D' + \epsilon_1 M_1 + \epsilon_2 M_2$, with $\epsilon_i \ll 1$ and $M_i \in |mL|$, such that

$$K_Y + \sum_1^r G_j = \mu^*(K_X + \overline{D}) - \epsilon_2 \mu^* M_2 + \sum_i e_i E_i - P\,, \tag{2.4}$$

with all $e_i \notin \mathbb{Z}$, $\lceil e_i \rceil = \lceil e'_i \rceil$ for any i, $Z_{r+1} \cap LLC(X, \overline{D}) = \emptyset$, and P is $(f \circ \mu)$-ample. In the perturbation we have distinguished two divisors M_1 and M_2 to stress that M_1 is used as always, cfr. (1.15), to introduce the ample divisor P, while M_2 will be used in the next step (2.14).

Let $\overline{D}_k := (\overline{D} - \sum_1^k H_j)_{|X_k}$ and define $A - B := \lceil \sum e_i E_i - \epsilon_2 \mu^* M_2 \rceil$. Then A is μ-exceptional and $Z_{r+1} \cap \mu(B) = \emptyset$. Using the ampleness of P and adjunction formula we also have, for $k \le r-1$ and $\mathbb{Q}$-divisors Δ_k, P_k

$$K_{Y_k} + (B - A + G_{k+1})_{|Y_k} + \Delta_k = \mu_k^*(K_{X_k} + \overline{D}_k) - P_k\,.$$

Let

$$N_k := -K_{Y_k} - (B - A + G_{k+1})_{|Y_k} - \Delta_k + \mu_k^* L \equiv_f \delta_k \mu^* L + P_k\,, \tag{2.5}$$

with $\delta_k > 0$, for $k \le r-1$. Here and in the following, $\equiv_f$ means that we are considering numerical equivalence with respect to the initial contraction f, keep in mind the Remark after Lemma 1.6.

A is μ-exceptional and $\mu^{-1}(Z_{r+1}) \cap B = \emptyset$ thus to conclude the proof of the theorem it is enough to produce a section of $H^0(Y, \mu^* L - B + A)$ not vanishing identically along $\mu^{-1}(Z_{r+1})$. By K-V vanishing applied to (2.5) we have for any $0 \le k \le r-1$ a surjection

$$H^0(Y_k, (\mu^* L - B + A)_{|Y_k}) \to H^0(Y_{k+1}, (\mu^* L - B + A)_{|Y_{k+1}})\,.$$

Therefore to prove the theorem it is enough to prove that there exists a section in $H^0(Y_r, (\mu^* L - B + A)_{|Y_r})$ non vanishing identically along $\mu_r^{-1}(Z_{r+1})$.

What remains to be done is to prove the nonvanishing on the manifold Y_r. This is the argument of the next and final step.

2.14 (Non vanishing on Y_r). To simplify notations let us assume that the complex neighbourhood U, of Claim 2.11, where φ is defined is the whole

X_r. We can do it since we will only compare divisors restricted to $\widetilde{E} \subset Y_r$, see (2.12). Let

$$\widetilde{D} := (1-\epsilon_0)c_0 D_{0|X_r} + \epsilon_3 M_1$$

and

$$D_\varphi := (1-\epsilon_0)c_0 D_{0|X_r} + \epsilon_3 \varphi^*\mathcal{O}_{S_\varphi}\,.$$

Then we can assume that:

- $lct(X_r, Z_r, \tilde{D}) = lct(X_r, Z_r, D_\varphi) = 1$,
- μ_r is a log resolution for both divisors,
- since $M_1 - \varphi^*\mathcal{O}_{S_\varphi}$ is ample then the result of the two perturbations is the same, that is to say

$$K_{Y_r} = \mu_r^*(K_{X_r} + \tilde{D}) + \sum \tilde{e}_i E_{i|Y_r} - \tilde{P}$$
$$K_{Y_r} = \mu_r^*(K_{X_r} + D_\varphi) + \sum \tilde{e}_i E_{i|Y_r} - P_\varphi\,,$$

- $\widetilde{E} \in PLC(X_r, \tilde{D}) = PLC(X_r, D_\varphi)$,

Furthermore $\widetilde{D}$ and D_φ are obtained from $(1-\epsilon_0)c_0 D_{0|X_r}$ just adding an arbitrarily small effective $\mathbb{Q}$-divisor therefore

$$\lceil e \rceil - 1 = \lceil e' \rceil - 1 \le \lceil \tilde{e}_i \rceil \le \lceil e' \rceil = \lceil e \rceil\,. \tag{2.6}$$

If we define $\tilde{A} - \tilde{B} - \widetilde{E} := \lceil \sum \tilde{e}_i E_{i|Y_r} \rceil$ then, by (2.6),

(2.7) $A_{|Y_r} - \tilde{A}$ and $\tilde{B} - B_{|Y_r}$ are effective and $\widetilde{E}$ is not contained in $A_{|Y_r}$.

Using the perturbation divisor M_2 we can moreover assume that

$$\mu^{-1} Z_{r+1} \not\subset \mathrm{Supp}(A_{|Y_r} - \tilde{A})\,.$$

It is enough to choose a generic divisor M_2 such that $\mathrm{mult}_{E_i} M_2 = \mathrm{mult}_{E_i} \varphi^*\mathcal{O}_{S_\varphi}$ for those i such that $\mu(E_i) \subset Z_{r+1}$ (this can always be achieved choosing a generic M_2 containing a prescribed ideal supported on Z_{r+1}), define $\epsilon_2 = \epsilon_3 - \epsilon_1$ and then rescale everything such that $m(\epsilon_1 + \epsilon_2) \ll 1$.

For the log resolution $\mu_r : Y_r \to X_r$ we have

$$K_{Y_r} + \widetilde{E} + \tilde{B} - \tilde{A} + \tilde{\Delta} = \mu_r^*(K_{X_r} + \tilde{D}) - \tilde{P}$$
$$K_{Y_r} + \widetilde{E} + \tilde{B} - \tilde{A} + \tilde{\Delta} = \mu_r^*(K_{X_r} + D_\varphi) - P_\varphi$$

where $\mu_r(\widetilde{E}) = Z_r$ and $Z_{r+1} \cap \mu_r(\tilde{B}) = \emptyset$.

We can therefore define a divisor

$$\tilde{N} := -K_{Y_r} - \widetilde{E} - \tilde{B} + \tilde{A} - \tilde{\Delta} + \mu_r^* L \equiv_f (1-\epsilon_3)\mu_r^* L + \tilde{P} \tag{2.8}$$
$$\equiv_\varphi \mu_r^* L + P_\varphi \tag{2.9}$$

Using the vanishing gained with φ in (2.9), arguing as in Lemma 2.3 (i) we prove that $H^0(\widetilde{E}, (\mu_r^* L + \tilde{A} - \tilde{B})_{|\widetilde{E}})$ is not empty. Furthermore, see for instance the appendix,

$$0 \neq H^0(\widetilde{E}, (\mu_r^* L + \tilde{A} - \tilde{B})_{|\widetilde{E}}) \simeq H^0(Z_r, L_{|Z_r} \otimes \mathcal{I}_{\mu_{r*}\tilde{B}}) .$$

Since $Z_r \simeq \mathbb{P}^1$ and $Z_{r+1} \cap \mu(\tilde{B}) = \emptyset$ then there exists a section

$$\sigma_1 \in H^0(\widetilde{E}, (\mu_r^* L + \tilde{A} - \tilde{B})_{|\widetilde{E}}) ,$$

which is not vanishing identically along $\mu_r^{-1}(Z_{r+1})$. By K-V applied to (2.8) we have the following surjection

$$H^0(Y_r, \mu_r^* L - \tilde{B} + \tilde{A}) \to H^0(\widetilde{E}, (\mu_r^* L - \tilde{B} + \tilde{A})_{|\widetilde{E}}) ,$$

and by relations (2.7) we have an injection

$$H^0(Y_r, \mu_r^* L - \tilde{B} + \tilde{A}) \hookrightarrow H^0(Y_r, (\mu^* L - B + A)_{|Y_r}) .$$

$\mu^{-1} Z_{r+1}$ is not contained in $\mathrm{Supp}(A_{|Y_r} - \tilde{A})$, thus the above section extends to the required section in

$$H^0(Y_r, (\mu^* L - B + A)_{|Y_r}) ,$$

not vanishing identically along $\mu_r^{-1}(Z_{r+1})$. □

An immediate corollary is the classification of isolated 2-dimensional fibers of Fano-Mori contractions.

COROLLARY 2.15. *Let $f : X \to S$ a local contraction around F from a smooth n-fold X. Assume that f is supported by $K_X + L$ and F is an isolated 2-dimensional fiber. Then the only possibilities for $(F, L_{|F})$ are those listed in [AW2, Proposition 4.3.2].*

REMARK 2.16. Moreover all the arguments of [AW3, Section 5] regarding fiber type contractions are true without the assumption on relative base point freeness of the fundamental divisor. This seems a good starting point for an higher dimensional generalisation of A-W classification, but if one tries to use an inductive method, by means of horizontal slicing, to study higher dimensional isolated fibers, soon realizes that after the first slice the jumping fibers are no more isolated, see also [AW3, Section 4], I would like to thank Marco Andreatta for signalling me this point. A simple example is the following. Let $\mathbb{C}^n$, with coordinates z_i's and $\mathbb{P}^{n-1}$ with coordinates t_i's. Let us consider the variety given by $X := (\sum t_i^2 z_i = 0) \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$. Then X is smooth and admits a morphism $f : X \to \mathbb{C}^n$ such that all fibers on $\mathbb{C}^n \setminus \{0\}$ are $\mathbb{Q}^{n-2}$ and the special fiber over $(0, \dots, 0)$ is $\mathbb{P}^{n-1}$, for any $n \geq 3$. The contraction f is supported by $K_X + (n-2)\mathcal{O}(1)$ and after any slice, for $n \geq 4$ there is a 1-dimensional component of fibers $\mathbb{P}^{n-2}$. It is my feeling that, due to this behaviour, the study of higher dimensional isolated fibers is inseparable from that of more general $(d, *, 1)$ fibrations, with $d \leq 0$.

As a first step toward a better comprehension of these morphisms we have the following.

THEOREM 2.17. *Let $f : X \to S$ be a local contraction supported by $K_X + rL$ around a fiber F. Assume that X is LT and $\overline{F}$ is any irreducible component of F_{red}.*

i) *If f is of type $(d, 1, 1)$, with $d \leq 0$, then $\Delta(\overline{F}, L_{|\overline{F}}) = 0$, in particular $\overline{F}$ is normal.*

ii) *If F is reducible and f is of type $(1, 1, 1)$, let $\nu : \tilde{F} \to \overline{F}$ the normalisation of $\overline{F}$, then $\Delta(\widetilde{F}, \nu^* L) = 0$.*

REMARK 2.18. The first part of the above theorem has been proved in [AW3, Th 1.10] under the assumption of relative base point freeness.

PROOF. Let $\delta = L^{\dim \overline{F}} \cdot \overline{F}$.

For i) we have to prove that $h^0(\overline{F}, L_{|\overline{F}}) \geq \delta + r + 1$. To do this we will prove that there are at least $\delta + r + 1$ independent sections of $H^0(X, L)$ not vanishing identically on F_0. By Theorem 2.6 and Lemma 1.6 we reduce to the case of a contraction $f : X \to S$ supported by K_X with one dimensional fiber F and irreducible component $\overline{F} \simeq \mathbb{P}^1$, with $L \cdot \overline{F} = \delta$. Then by assumption f is birational. Let $H \in |L|$ a generic section then $H \cap F$ is a reduced scheme of length δ. Furthermore by Lemma 1.6 all sections of $|L_{|H}|$ extends to sections of $|L|$, therefore this is enough to conclude.

In case ii) we cannot use the same arguments as in i) since to produce the birational contraction φ, as in the proof of Theorem 2.6, we have to switch to an analytic neighbourhood which may well not contain the whole fiber F. To overcome this problem we will directly work on sections of $\widetilde{F}$ in the following way. Let $H_j \in |L|$ generic sections. Then by Bertini theorem and Theorem 2.6, $X_k := X \cap (\cap_1^k H_j)$ is LT and there is a contraction $f_k : X_k \to S_k$ supported by $K_{X_k} + (r - k)L_{|X_k}$. Let $F_k := \overline{F} \cap X_k$, as observed during the proof of Theorem 2.6, there is also a birational contraction $\varphi : U_r \to S_\varphi$, from a complex neighbourhood U_r of F_r, which has F_r as an irreducible fiber, therefore $F_r \simeq \mathbb{P}^1$. Let $g : \widehat{F} \to \overline{F}$ be a resolution of $\overline{F}$. Since $L_{|\overline{F}}$ is spanned then we have also the embedded resolutions $g_k : \widehat{F}_k \to F_k$ and an isomorphism $g_r : \widehat{F}_r \to F_r \simeq \mathbb{P}^1$, in particular

$$H^1(\widehat{F}_r, \mathcal{O}_{\widehat{F}_r}) = 0\,. \tag{2.10}$$

Let $\hat{L}_k = g_k^* L$ then $\hat{L}_k$ is nef and big therefore

$$H^1(\widehat{F}_k, -\hat{L}_k) = 0\,, \tag{2.11}$$

by K-V vanishing. For any $k \leq r - 1$ we have the following exact sequence

$$H^1(\widehat{F}_k, -\hat{L}_k) \to H^1(\widehat{F}_k, \mathcal{O}_{\widehat{F}_k}) \to H^1(\mathcal{O}_{\widehat{F}_{k+1}})\,,$$

and by vanishings (2.11) and (2.10) we inductively prove that $H^1(\widehat{F}_k, \mathcal{O}_{\widehat{F}_k}) = 0$, for any $k \leq r$. This means that $\Delta(\widehat{F}, \hat{L}) = 0$ and therefore implies our conclusion. □

EXAMPLE 2.19. Let us give some examples of these contractions. Let $W := \mathbb{C}^n \times \mathbb{Q}^{n-2} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$ and define $X := (\sum t_i z_i = 0) \cap W$. Then X is smooth and admits a contraction, supported by $K_X + (n-3)\mathcal{O}(1)$, onto $\mathbb{C}^n$ such that all fibers on $\mathbb{C}^n \setminus \{0\}$ are $\mathbb{Q}^{n-3}$ and the special fiber over $(0, \dots, 0)$ is $\mathbb{Q}^{n-2}$. Let $V := (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, with $t_1, \dots, t_6$ the coordinates of the $\mathbb{P}^1$'s. Define $X = (t_1 z_1 + t_2 z_2 = 0) \subset V \times \mathbb{C}^2 \subset \mathbb{P}^5 \times \mathbb{C}^2$ then X is smooth and the contraction onto $\mathbb{C}^2$ is supported by $K_X + 2\mathcal{O}(1)$ and has generic fiber $\mathbb{Q}^2$ and one special fiber $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

For reducible or non reduced fibers let $X = (f_0(t_i) + \sum_1^k s_i f_i = 0)$, where f_i are degree 3 homogeneous polynomials of $\mathbb{P}^l$ and s_i are affine parameters. Then $X \subset \mathbb{P}^l_{\mathbb{C}^k}$ is a cubic hypersurface and if $k \geq l$ one can choose f_i in such a way that the special fiber is either reducible or non reduced even if X is smooth. In the non equidimensional case one can consider a quadric bundle over $\mathbb{C}^k$ with discriminant passing trough the origin, then an hyperplane section as above, $(\sum t_i z_i)$ gives the desired contraction.

Summing up all we have done in this section we can state the following result about the fundamental divisor of (d, γ, Φ)-fibrations.

COROLLARY 2.20. *Let $f : X \to S$ a contraction supported by $K_Y + rL$ around a fiber F. Let $H \in |L|$ a generic section and assume that f satisfies one of the following:*

- $(*, 1, 1)$.
- $(*, *, 2 - \epsilon\gamma(f))$ *with F irreducible.*
- $(*, *, 3 - \epsilon\gamma(f))$ *with F irreducible and $r > 0$.*

Then H does not vanish identically on any irreducible component of F.

REMARK 2.21. The assumption that L is a Cartier divisor is crucial for this kind of results and cannot be relaxed to Weil divisor. In fact there are flipping contractions of terminal 3-folds for which all divisors in $|-K_X|$ contain the whole flipping curve, [Mo2, Section 9].

3. – Good divisors on $(1, 1, 1)$-contractions

DEFINITION 3.1. Let $f : X \to S$ a local contraction of type (d, γ, Φ), supported by $K_X + rL$. Then we will say that f has *good divisors* if, after maybe shrinking S, the generic element $H \in |L|$ has at worst the same singularities as X and $f_{|H} : H \to S_H$ is of type $(*, *, \Phi)$.

REMARK 3.2. Note that both the character and the dual index of the contraction can change after an horizontal slice, think to non equidimensional contractions with one dimensional generic fiber.

In this section we are interested in answering the good divisor problem for contractions of type $(1, 1, 1)$, i.e. del Pezzo fibrations.

Proposition 3.3. *Let $f : X \to S$ be a local contraction supported by $K_X + rL$ around a fiber F. Assume that X is LT and either f is of type $(1, 1, 1)$ or F is irreducible and f is of type $(*, *, 2 - \epsilon\gamma(f))$, with $r \geq 0$. Then f has good divisors.*

Proof. Let $H \in |L|$ a generic section and assume that H is not LT. By Bertini theorem $LLC(X, H) \subset Bsl|L|$, thus by Theorem 2.6 we can assume that F is irreducible and, maybe shrinking S, that all fibers are irreducible since X is normal and f has connected fibers. Then by vertical slicing we can assume that $LLC(X, H) \subset F$. Let $D = H + \delta f^*(g)$, for some $\delta \ll 1$ and g function on S vanishing at $f(F)$. Then again $LLC(X, D) \subset F$ and $\gamma = l.c.t.(X, D) < 1$. Let $W \in CLC(X, \gamma D)$ a minimal center. Then by Corollary 2.20 $\dim W < \dim F$ thus by Lemma 2.2 we derive a contradiction. More in detail if $r \geq 1$ by Lemma 2.2 i), if $1 > r \geq 0$ by Lemma 2.2 ii). □

Corollary 3.4. *Let $f : X \to S$ be a local contraction supported by $K_X + rL$ around a fiber F. Assume that X is either canonical or terminal , $r \geq 1$ and f is either a $(1, 1, 1)$ contraction or F is irreducible and f is of type $(*, *, 2 - \epsilon\gamma(f))$. Then f has good divisors.*

Proof. This is just a direct consequence of Proposition 3.3 and the definition of canonical and terminal singularities. In other words one could say that the generic section of $|L|$ has only terminal singularities along $Bsl|L|$. □

We are now ready to prove the main result of this section.

Theorem 3.5. *Let $f : X \to S$ a local contraction of type $(1, 1, 1)$, supported by $K_X + rL$ around F. Assume that X is smooth. Then f has good divisors.*

Proof. Let $H \in |L|$ a generic section, by vertical slicing we can assume that $\mathrm{Sing}(H) \subset F$. Furthermore by horizontal slicing, using inductively Proposition 3.3 and Theorem 2.6 we know that $\dim Bsl|L| \cap F \leq 0$. Let $x \in Bsl|L| \cap F$, $k = \mathrm{cod}_X F$ and $\rho = \max\{1, r\}$. Let $l_i \ll 1$, g_i's generic functions vanishing at $s = f(F)$ with $\mathrm{mult}_x \sum l_i f^*(g_i) = k$. Let $D_0 = \sum l_i f^*(g_i)$ and

$$D_m = \sum l_i f^*(g_i) + \sum_1^{\rho m} (1/m) H_j \equiv_f \rho L\,,$$

for $m > 0$ and $H_j \in |L|$ generic elements.

Claim 3.6. (X, D_m) is LC in a punctured neighbourhood of x, for $m \gg 0$.

Proof (of the Claim). Let $Z \subset Bsl|L|$ be any positive dimensional subvariety. Since $\dim Z \cap F = 0$ then (X, D_m) is LC along Z by vertical slicing and Lemma 2.2 (ii). We have therefore only to care about subvarieties of F. To do this we will work with a fixed log resolution $\mu : Y \to X$ obtained in the following way. Let m_s the maximal ideal of $s \in S$ and $\mathcal{I} \subset \mathcal{O}_X$, the inverse image ideal. Let $X' \to X$ the blow up of $\mathcal{I}$, and $\overline{X} \to X'$ a resolution of singularities. Finally let $Y \to \overline{X}$ a log resolution of the base locus of $|L|$ and of

(X, D_0). Observe that μ is a log resolution of (X, D_m) for any m, furthermore, by Hironaka theorem we can assume that μ is a sequence of blow ups with smooth centers in smooth varieties. Finally since μ factors trough the blow up of $\mathcal{I}$ and the g_i's are generic function vanishing at s then (X, D_m) is LC at the generic point of F.

Let now $E \subset Y$ a μ-exceptional divisor, and $Z = \mu(E)$. Assume that $Z \subset F$ is a positive dimensional subvariety, and $x \in Z$. Let $h = \mathrm{cod}_F Z$. Since H_j are generic then $\mathrm{mult}_Z D \leq k + h/m$. Let $Y \xrightarrow{\alpha} \widetilde{Y} \xrightarrow{\nu} X$ any factorisation of μ with $\widetilde{Y}$ smooth and $\widetilde{Z} \subset \widetilde{Y}$ any subvariety with $\alpha(E) = \widetilde{Z}$ and $\nu(\widetilde{Z}) = Z$. Let $\widetilde{D}_m = \nu_*^{-1} D_m$ and, if ν is an isomorphism on the generic point of F, $\widetilde{F} = \nu_*^{-1} F$. If $\widetilde{Z} \not\subset \widetilde{F}$ or $\widetilde{F}$ does not exist, then $\mathrm{mult}_{\widetilde{Z}} \widetilde{D}_m \leq h/m + \epsilon$. If $\widetilde{Z} \subset \widetilde{F}$ then $\mathrm{cod}_{\widetilde{Y}} \widetilde{Z} \geq k+1$ and $\mathrm{mult}_{\widetilde{Z}} \widetilde{D}_m \leq k + h/m$. Therefore for any valuation E, exceptional for μ, we have $\mathrm{disc}(X, E, D_m) \geq -\sum_1^e h/m - \epsilon_E$. We are working with a fixed resolution μ, therefore , for some fixed integer N and rational $\epsilon_1 \ll 1$, independent on E and m, we have $\epsilon_E < \epsilon_1$ and $e < N$. In particular $E \notin CLC(X, D_m)$, for $m \gg 0$. □

If all H_i are singular at x then

$$\begin{aligned} &\mathrm{mult}_x D_m \geq k + 2r \geq \mathrm{cod}_X x && \text{if } r > 0 \\ &\mathrm{mult}_x D_m \geq k + 2\mathrm{cod}_X x && \text{if } r = 0 \end{aligned}$$

Thus $x \in CLC(X, D_m)$ and we derive a contradiction by Lemma 2.2 (ii). □

REMARK 3.7. The same result is true for contraction of type $(*, *, 2 - \epsilon\gamma(f))$, with $r \geq 0$, around an irreducible fiber F. The main difficulty is to prove that $\dim Bsl|L| \cap F \leq 0$, this time we cannot use Theorem 2.6. This can be done by a long and quite involved study of all the possible reducible fibers that can appear after an horizontal slice. Since at the moment there are not interesting applications for this variant of Theorem 3.5, we prefer to leave it in the author's keyboard.

REMARK 3.8. To complete the understanding of the fundamental divisor of $(1, 1, 1)$ fibrations, it would be important to understand the irreducible non reduced fibers. The main point here is to understand if there are non reduced fibers whose reduced structure is a degree 1 del Pezzo, the stronger statement that the reduced structure has 0 Δ-genus should hold. If the answer were no then the fundamental divisor would be spanned whenever the generic fiber has degree ≥ 2, reproducing the classical result of del Pezzo surfaces. I tried to prove this, at least in the smooth case using deformation of rational curves, but the only result I had is that if such a fiber exists then the reduced component has to be not normal.

Appendix

Let (X, D) a LC pair and W a minimal center, then by Kawamata's subadjunction formula in Theorem 1.12 (W, D_W) is KLT, therefore in particular W has rational singularities. In this appendix we give a direct proof of this fact. The idea of such a proof originated reading [Ko, Section 11] and we think it is interesting in its own even if it is weaker than Kawamata's assertion.

Let us start with a lemma which is probably well known.

LEMMA A.1. *Let $f : Y \to X$ a projective surjective morphism of normal schemes. Assume that Y is CM and A is a locally free sheaf on Y, then f_*A is locally free in codimension 1.*

PROOF. Let $k = \dim X$, $n = \dim Y$ and $x \in X$ be a point with $\dim x = k-1$. Our hypothesis are stable under localisation in X, therefore we can assume that X is a smooth curve and $x \in X$ a closed point. Let $F = f^{-1}(x)$ then $H^0_x(X, f_*A) = H^0_F(Y, A)$ and we have therefore to prove that $H^0_F(Y, A) = 0$, [Ko, Lemma 11.4]. By formal function theorem for any coherent sheaf $\mathcal{F}$ on Y $R^{n-k+1} f_*\mathcal{F} = 0$ thus $H^0_F(Y, A) \overset{d}{\sim} R^{n-k+1} f_*(K_Y - A) = 0$ [Ko, Prop 11.6]. □

THEOREM A.2. *Let (X, D) be LC and W a minimal center in $CLC(X, D)$. Then W has rational singularities.*

PROOF. After perturbing D we can assume that there exists a log resolution $\mu : Y \to X$ of (X, D) with the following properties:

- $K_Y + E + \Delta - A = \mu^*(K_X + D)$, where $\lfloor \Delta \rfloor = 0$, $\mu(E) = W$, A is μ-exceptional.
- the projection $\mu_{|E} : E \to W$ is factorised as $E \subset Y \overset{\alpha}{\to} V \subset \tilde{Y} \overset{\beta}{\to} W \subset X$ with V smooth and $\beta_{|V}$ birational.

Furthermore the assertion is local therefore we can also assume that X is affine. Let us now follow [Ka3, Th 1.6], by K-V vanishing

$$H^i(Y, \mathcal{O}_Y(A - E)) = R^i \alpha_* \mathcal{O}_Y(A - E) = 0 \quad \text{for } i > 0 .$$

Replace D by $(1 - \epsilon)D$ to get

$$H^i(Y, \mathcal{O}_Y(A)) = R^i \alpha_* \mathcal{O}_Y(A) = 0 \quad \text{for } i > 0$$

The structure sequence of E and the above vanishing yields to a surjection

$$H^0(Y, \mathcal{O}_Y(A)) \to H^0(E, \mathcal{O}_E(A))$$

and vanishings

$$H^i(E, \mathcal{O}_E(A)) = R^i \alpha_* \mathcal{O}_E(A) = 0 \quad \text{for } i > 0 .$$

Since A is exceptional $H^0(X, \mathcal{O}_X) \simeq H^0(Y, A)$, therefore $\mathcal{O}_W \sim \mu_* \mathcal{O}_E(A)$.

CLAIM A.3. $\alpha_*\mathcal{O}_E(A)$ is a line bundle.

PROOF (of the Claim). Let $G := \alpha_*\mathcal{O}_E(A)$. G is locally free in codimension1 by Lemma A.1. β is birational and $\beta_*G \sim \mathcal{O}_W$ thus G is a rank one coherent sheaf. Since V is smooth the claim is equivalent to prove that G is S_2.

Let B the support of the exceptional locus of β, then G is locally free outside of B. By [Gr, Prop 3.7] it is enough to prove that $H^1_B(V, G) = 0$. By Leray spectral sequence $H^1(V, G) = 0$, let us consider the long exact sequence associated to local cohomology

$$0 \to H^0(V, G) \to H^0(V \setminus B, G) \to H^1_B(V, G) \to 0 .$$

Since W is normal and $\beta_*G \sim \mathcal{O}_W$ then

$$H^0(V, G) \simeq H^0(W, \mathcal{O}_W) \simeq H^0(W \setminus \mu(B), \mathcal{O}_W) \simeq H^0(V \setminus B, G) .$$

We can therefore conclude that $H^1_B(V, G) = 0$. □

We want now to apply Elkik vanishing theorem [El], see also [Ko], to this end let us write the canonical class in the following way

$$K_V \sim \mathcal{O}_V \otimes (K_V \otimes G^{-1}) \otimes G .$$

By the claim G is a β-exceptional divisor and by the above vanishing and Grauert theorem we have

$$R^i\beta_*(\mathcal{O}_V \otimes G) = R^i\beta_*G = H^i(V, \mathcal{O}_V(G)) = H^i(E, \mathcal{O}_E(A)) = 0 \text{ for } i > 0$$
$$R^i\beta_*(K_V \otimes G^{-1} \otimes G) = R^i\beta_*K_V = 0 \text{ for } i > 0 .$$

Therefore the following vanishing are also true, [El],

$$R^i\beta_*K_V \otimes G^{-1} = R^i\beta_*\mathcal{O}_V = 0 .$$

We already now that W is normal therefore by the latter we conclude that W has rational singularities. □

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