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The Effect of Variational Framework on the Spectral Asymptotics for Nonlinear Elliptic Two-Parameter Problems

TETSUTARO SHIBATA

Abstract. We study the structure of the set $\{(\lambda, \mu)\} \subset \mathbb{R}_+^2$ such that the nonlinear elliptic two-parameter problem

$$-\Delta u + \lambda g(u) = \mu f(u),\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

has a solution $u \in W_0^{1,2}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an appropriate smooth bounded domain. For this purpose, viewing $\lambda > 0$ as a given parameter, we define two kinds of variational eigenvalues $\mu = \mu_1(\lambda, \alpha)$ and $\mu = \mu_2(\lambda, \beta)$ by the Lagrange multipliers of the variational problems subject to the constraints depending on positive parameters α and β. Then for $\mu_1(\lambda, \alpha)$ and $\mu_2(\lambda, \beta)$, we establish two kinds of asymptotic formulas as $\lambda \to \infty$, which are explicitly represented by means of λ, α, β, and are effected by the asymptotic behavior of f and g as $u \to \infty$ and $u \to 0$, respectively. We emphasize that there are noticeable differences between the asymptotic behavior of $\mu_1(\lambda, \alpha)$ and $\mu_2(\lambda, \beta)$ as $\lambda \to \infty$.

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1. – Introduction

We consider the nonlinear elliptic two-parameter problem

$$(1.1) \qquad \begin{aligned} -\Delta u + \lambda g(u) &= \mu f(u),\ u > 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\lambda, \mu > 0$ are parameters, and $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with an appropriately smooth boundary $\partial\Omega$. We assume:

(A.1) $f, g \in C^1(\mathbb{R})$ are odd in u, and $f(u), g(u) > 0$ for $u > 0$. Furthermore, there exist constants $1 < q \leq p < (N+2)/(N-2)$ and $K_0, J_0, K_1, J_1 > 0$ such

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that

$$\frac{g(u)}{u} \to K_0, \quad \frac{f(u)}{u^p} \to K_1 \quad \text{as } u \to \infty, \tag{1.2}$$

$$\frac{g(u)}{u} \to J_0, \quad \frac{f(u)}{u^q} \to J_1 \quad \text{as } u \downarrow 0. \tag{1.3}$$

The typical example of f, g is

$$f(u) = |u|^{p-1}u + |u|^{q-1}u, \ g(u) = u. \quad (1 < q \le p < (N+2)/(N-2)) \tag{1.4}$$

The purpose of this paper is to investigate and understand the structure of the set $\{(\lambda, \mu)\} \subset \mathbb{R}_+^2$ such that (1.1) has a solution $u \in W_0^{1,2}(\Omega)$ by variational methods, where $W_0^{1,2}(\Omega)$ is the usual real Sobolev space. To this end, viewing $\lambda > 0$ as a given parameter, we apply the following two variational problems subject to the constraints depending on positive parameters α, β and λ:

(M.1)
$$\text{Maximize } \int_\Omega \left(\int_0^{u(x)} f(s)ds \right) dx \text{ under the constraint}$$
$$u \in N_{\lambda,\alpha} := \left\{ u \in W_0^{1,2}(\Omega) : \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega \left(\int_0^{u(x)} g(s)ds \right) dx = \alpha \right\},$$

(M.2)
$$\text{Minimize } \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega \left(\int_0^{u(x)} g(s)ds \right) dx \text{ under the constraint}$$
$$u \in M_\beta := \left\{ u \in W_0^{1,2}(\Omega) : \int_\Omega \left(\int_0^{u(x)} f(s)ds \right) dx = \beta \right\}.$$

Then we obtain two solutions trio $(\lambda, \mu_1(\lambda, \alpha), u_{1,\lambda,\alpha}), (\lambda, \mu_2(\lambda, \beta), u_{2,\lambda,\beta}) \in \mathbb{R}_+^2 \times W_0^{1,2}(\Omega)$ corresponding to the problems (M.1) and (M.2), respectively, by the Lagrange multiplier theorem. A natural problem in this context is to clarify the difference between $\mu_1(\lambda, \alpha)$ and $\mu_2(\lambda, \beta)$. To do this, we shall establish two asymptotic formulas for $\mu_1(\lambda, \alpha)$ and $\mu_2(\lambda, \beta)$ as $\lambda \to \infty$, respectively, which are explicitly represented by means of λ and α, β. Under the suitable conditions on (λ, α) (resp. (λ, β)), one of them for $\mu_1(\lambda, \alpha)$ (resp. $\mu_2(\lambda, \beta)$) depends only on the asymptotic behavior of f and g as $u \to \infty$, and another depends only on the behavior of f and g near 0. We emphasize that if $\alpha, \beta > 0$ are fixed, then $\mu_1(\lambda, \alpha) \to \infty$ faster than $\mu_2(\lambda, \beta)$ as $\lambda \to \infty$.

In order to motivate the results of this paper, let us recall some of the known facts concerning multiparameter problems. Linear multiparameter problems in ODE began with the study of Lamé's equation and have been extensively studied by many authors. We refer to Binding and Volkmer [5], Faierman [9], Rynne [18]. In particular, for the linear two-parameter ODE

$$u''(x) + \mu a(x)u(x) = \lambda b(x)u(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

where $\mu, \lambda > 0$ are parameters, there are many works for finding asymptotic directions $\lim_{\mu\to\infty} \lambda_n(\mu)/\mu$. Here $\lambda_n(\mu)$ is the n-th eigenvalue for a given $\mu > 0$. See Binding and Browne [3], [4] and the references therein. As for nonlinear problems, local bifurcation problems has been mainly interested in. We refer to Browne and Sleeman [6], Gómez [12], Rynne [17]. Concerning global behavior of eigenvalues corresponding to the works [3], [4] for the nonlinear problem (1.1), however, only a few results seems to have been given. Our problem can be interpreted as the nonlinear version of determining asymptotic directions of eigenvalues. Our problems are related to singular perturbation problem, also. In fact, if $f(u) = u^p$ $(p > 1), g(u) = u$ in (1.1), then we are led to the singular perturbation problem

$$-\epsilon\Delta v = v^p - v, \qquad v > 0 \text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega$$

by the transformation $v = (\lambda/\mu)^{-1/(p-1)}u$, $\epsilon = \lambda^{-1}$. Since the asymptotic behavior of the solution $v = v_\epsilon$ as $\epsilon \to 0$ is important to study, the asymptotic behavior of $\mu_1(\lambda, \alpha)$ and $\mu_2(\lambda, \beta)$ as $\lambda \to \infty$ seems meaningful to investigate. Motivated by these facts, Shibata [19] studied the asymptotic direction of the nonlinear two-parameter problems

$$u''(x) + \mu u(x)^p = \lambda u(x)^q, \quad u(x) > 0, \quad 0 < x < 1, \; u(0) = u(1) = 0.$$

Here $\mu, \lambda > 0$ are parameters and $1 \le q < p < q + 2$ are constants. In [19], by the variational method on the general level set

$$L_{\mu,\alpha} := \left\{ u \in W_0^{1,2}((0,1)) : \frac{1}{2}\int_0^1 u'(x)^2 dx - \frac{1}{p+1}\mu \int_0^1 |u(x)|^{p+1} dx = -\gamma \right\}$$

developed by Zeidler [21], the asymptotic formula for λ for a fixed $\gamma > 0$ was given:

$$\lambda(\mu, \gamma) = C_1 \mu^{\frac{q+3}{p+3}} + o(\mu^{\frac{q+3}{p+3}}) \quad \text{as } \mu \to \infty, \tag{1.5}$$

where $C_1 > 0$ is a constant represented explicitly by p, q, γ and the gamma function. Recently, by following the idea used in [19], Shibata [20] extended the asymptotic formula (1.5) to the multiparameter problems such as (1.1) including the nonlinearity $f(u)$ which behaves like $|u|^{p-1}u$ as $u \to \infty$, where $1 < p < 1 + 4/N$. However, unfortunately, the exponent $1 + 4/N \le p < (N+2)/(N-2)$ is not covered in [20]. This restriction comes from the structure of the manifold $L_{\mu,\gamma}$ for higher dimensions, since the Gagliard-Nirenberg inequality on $L_{\mu,\gamma}$ is used for the existence of eigenvalues. In order to remove such restriction of the exponent p as mentioned above, we apply here two kinds of variational framework (M.1) and (M.2) which are different from those of [19], [20] to obtain the variational eigenvalues $\mu_1(\lambda, \alpha)$ and $\mu_2(\lambda, \beta)$. In Theorem 2.1, under the suitable conditions on $\{(\lambda, \alpha)\} \subset \mathbb{R}^2_+$, we establish the asymptotic formula for $\mu_1(\lambda, \alpha)$ as $\lambda \to \infty$, whose top term depends only on the asymptotic behavior of $f(u), g(u)$ as $u \to \infty$, and does not depend on the properties of $f(u), g(u)$ near $u = 0$. It turns out that if $\alpha > 0$ *is fixed*, then the asymptotic behavior of $\mu_1(\lambda, \alpha)$ as $\lambda \to \infty$ depends mainly on the asymptotic behavior of $f(u), g(u)$ as $u \to \infty$. On the contrary, in Theorem 2.2, under the different conditions on $\{(\lambda, \alpha)\} \subset \mathbb{R}^2_+$, we also establish the asymptotic formula for $\mu_1(\lambda, \alpha)$ as $\lambda \to \infty$, which is influenced mainly by the asymptotics of $f(u), g(u)$ near $u = 0$. Similarly, we establish two asymptotic formulas for $\mu_2(\lambda, \beta)$ as $\lambda \to \infty$, which are dominated by the asymptotics of $f(u), g(u)$ near infinity and 0 in Theorem 2.3 and Theorem 2.4, respectively. As a consequence of Theorem 2.1 and Theorem 2.3, it turns out that if $\alpha, \beta > 0$ are fixed, then $\mu_1(\lambda, \alpha)/\mu_2(\lambda, \beta) \to \infty$ as $\lambda \to \infty$. This phenomenon occurs, since $\beta_{\lambda,\alpha} = \int_\Omega \left(\int_0^{u_{1,\lambda,\alpha}(x)} f(s)ds \right) dx$ varies accordingly as $\lambda \to \infty$. This will be explained precisely in Remark 2.5 (2) in Section 2 later.

2. – Main results

We begin with notation. For $u, v \in W_0^{1,2}(\Omega)$ and $t \in \mathbb{R}$, let

$$\|u\|_d^d := \int_\Omega |u(x)|^d dx \ (d \geq 1),\ \|u\|_\infty := \sup_{x \in \Omega} |u(x)|,\ (u, v) := \int_\Omega u(x)v(x)dx\,,$$

$$F(t) := \int_0^t f(s)ds,\ G(t) := \int_0^t g(s)ds,\ \Phi(u) := \int_\Omega F(u(x))dx\,,$$

$$\Psi(u) := \int_\Omega G(u(x))dx,\ \Lambda_\lambda(u) := \frac{1}{2}\|\nabla u\|_2^2 + \lambda\Psi(u).$$

Furthermore, for any domain $D \subset \mathbb{R}^N$ the norm of $L^d(D)$ will be denoted by $\|\cdot\|_d$ for simplicity. For a given $\lambda, \alpha, \beta > 0$, $\mu = \mu_1(\lambda, \alpha)$ and $\mu = \mu_2(\lambda, \beta)$ are defined as the Lagrange multipliers associated with the problem (M.1) and (M.2), respectively. Namely, $\mu_1(\lambda, \alpha)$ and $\mu_2(\lambda, \beta)$ are the Lagrange multipliers associated with the eigenfunctions $u_{1,\lambda,\alpha} \in N_{\lambda,\alpha}$ and $u_{2,\lambda,\beta} \in M_\beta$ which satisfy

$$\Phi(u_{1,\lambda,\alpha}) = \sup_{u \in N_{\lambda,\alpha}} \Phi(u). \tag{2.1}$$

$$\Lambda_\lambda(u_{2,\lambda,\beta}) = \inf_{u \in M_\beta} \Lambda_\lambda(u), \tag{2.2}$$

respectively. Then $(\lambda, \mu_1(\lambda, \alpha), u_{1,\lambda,\alpha})$ and $(\lambda, \mu_2(\lambda, \beta), u_{2,\lambda,\beta})$ satisfy (1.1) by the Lagrange multiplier theorem. Further, $\mu_1(\lambda, \alpha)$ and $\mu_2(\lambda, \beta)$ are represented as follows:

$$\mu_1(\lambda, \alpha) = \frac{2\alpha + \lambda\{(g(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha}) - 2\Psi(u_{1,\lambda,\alpha})\}}{(f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha})}, \tag{2.3}$$

$$\mu_2(\lambda, \beta) = \frac{\|\nabla u_{2,\lambda,\beta}\|_2^2 + \lambda(g(u_{2,\lambda,\beta}), u_{2,\lambda,\beta})}{(f(u_{2,\lambda,\beta}), u_{2,\lambda,\beta})}. \tag{2.4}$$

Indeed, if $(\lambda, \mu, u) \in \mathbb{R}_+^2 \times W_0^{1,2}(\Omega)$ satisfies (1.1), then multiply (1.1) by u. Then integration by parts yields

$$\|\nabla u\|_2^2 + \lambda(g(u), u) = \mu(f(u), u). \tag{2.5}$$

(2.5) implies (2.4). Since $u_{1,\lambda,\alpha} \in N_{\lambda,\alpha}$, (2.5) also yields (2.3). Let $w \in H^1(\mathbb{R}^N)$ be the unique solution of the following nonlinear scalar field equation:

$$-\Delta w = w^p - w, \quad w > 0 \quad \text{in } \mathbb{R}^N, \ w(0) = \max_{x\in\mathbb{R}^N} w(x). \tag{2.6}$$

Further, let W_1 be the unique solution of (2.6), in which the exponent p is replaced by q.

In order to state our results, we define the several conditions for (unindexed) sequences $\{(\lambda, \alpha)\} \subset \mathbb{R}_+^2$ and $\{(\lambda, \beta)\} \subset \mathbb{R}_+^2$:

(B.1) $\lambda \to \infty$;

(B.2) $\alpha^2\lambda^{N-2} \to \infty$;

(B.3) $\alpha^2\lambda^{N-2} \to 0$;

(B.4) $\beta^2\lambda^N \to \infty$;

(B.5) $\beta^2\lambda^N \to 0$.

We explain the meaning of these conditions. In the problem (M.1), $\|u_{1,\lambda,\alpha}\|_\infty$ behaves like $(\alpha^2\lambda^{N-2})^{1/4}$ for $\lambda \gg 1$. Therefore, if (B.2) (resp. (B.3)) is assumed, then $\|u_{1,\lambda,\alpha}\|_\infty \to \infty$ (resp. 0). Hence we see that the asymptotic behavior of $f(u), g(u)$ as $u \to \infty$ (resp. $u \to 0$) reflects mainly on the asymptotic formula for $\mu_1(\lambda, \alpha)$. Similarly, in the problem (M.2), the growth order of $\|u_{2,\lambda,\beta}\|_\infty$ is $(\beta^2\lambda^N)^{1/(2(p+1))}$. Hence the condition (B.4) (resp. (B.5)) implies $\|u_{2,\lambda,\beta}\|_\infty \to \infty$ (resp. 0). Therefore, the asymptotic behavior of $f(u), g(u)$ at $u = \infty$ (resp. $u = 0$) gives effect mainly on the asymptotic behavior of $\mu_2(\lambda, \beta)$.

Now we state our main results.

THEOREM 2.1. *Assume* (A.1). *If a sequence* $\{(\lambda, \alpha)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.2), *then the following asymptotic formula holds:*

$$\mu_1(\lambda, \alpha) = C_2\alpha^{\frac{1-p}{2}}\lambda^{\frac{N+2-p(N-2)}{4}} + o\left(\alpha^{\frac{1-p}{2}}\lambda^{\frac{N+2-p(N-2)}{4}}\right), \tag{2.7}$$

where $C_2 = K_1^{-1}K_0^{\frac{N+2-p(N-2)}{4}}(\|w\|_{p+1}^{p+1}/2)^{\frac{p-1}{2}}$.

We note that $\alpha > 0$ *may not be fixed* in Theorem 2.1. If $\alpha > 0$ is fixed, then (B.1) implies (B.2) immediately. However, if $\alpha > 0$ is *not fixed*, then (B.1) does not imply (B.2) in general.

THEOREM 2.2. *Assume* (A.1). *If a sequence* $\{(\lambda, \alpha)\} \subset \mathbb{R}^2_+$ *satisfies* (B.1) *and* (B.3), *then the following asymptotic formula holds:*

$$\mu_1(\lambda, \alpha) = C_3 \alpha^{\frac{1-q}{2}} \lambda^{\frac{N+2-q(N-2)}{4}} + o\left(\alpha^{\frac{1-q}{2}} \lambda^{\frac{N+2-q(N-2)}{4}}\right), \tag{2.8}$$

where $C_3 = J_1^{-1} J_0^{\frac{N+2-q(N-2)}{4}} (\|W_1\|_{q+1}^{q+1}/2)^{\frac{q-1}{2}}$.

We should notice that in the situation of Theorem 2.2, $\alpha > 0$ is *not fixed*. Clearly, if $\alpha > 0$ is fixed, then (B.1) contradicts (B.3). (B.1) and (B.3) are consistent, for example, if $\alpha = \lambda^{-m} (m > (N-2)/2)$.

THEOREM 2.3. *Assume* (A.1). *If a sequence* $\{(\lambda, \beta)\} \subset \mathbb{R}^2_+$ *satisfies* (B.1) *and* (B.4), *then the following asymptotic formula holds:*

$$\mu_2(\lambda, \beta) = C_4 \beta^{-\frac{p-1}{p+1}} \lambda^{\frac{N+2-p(N-2)}{2(p+1)}} + o\left(\beta^{-\frac{p-1}{p+1}} \lambda^{\frac{N+2-p(N-2)}{2(p+1)}}\right), \tag{2.9}$$

where $C_4 = K_1^{-\frac{2}{p+1}} K_0^{\frac{N+2-p(N-2)}{2(p+1)}} (p+1)^{-\frac{p-1}{p+1}} \|w\|_{p+1}^{p-1}$.

THEOREM 2.4. *Assume* (A.1). *If a sequence* $\{(\lambda, \beta)\} \subset \mathbb{R}^2_+$ *satisfies* (B.1) *and* (B.5), *then the following asymptotic formula holds:*

$$\mu_2(\lambda, \beta) = C_5 \beta^{-\frac{q-1}{q+1}} \lambda^{\frac{N+2-q(N-2)}{2(q+1)}} + o\left(\beta^{-\frac{q-1}{q+1}} \lambda^{\frac{N+2-q(N-2)}{2(q+1)}}\right), \tag{2.10}$$

where $C_5 = J_1^{-\frac{2}{q+1}} J_0^{\frac{N+2-q(N-2)}{2(q+1)}} (q+1)^{-\frac{q-1}{q+1}} \|W_1\|_{q+1}^{q-1}$.

REMARK 2.5. (1) Note that $\beta > 0$ *may not be fixed* in Theorem 2.3. If $\beta > 0$ is fixed, then (B.1) implies (B.4) immediately. However, if $\beta > 0$ is *not fixed*, then (B.1) does not imply (B.4) in general. Furthermore, in Theorem 2.4, $\beta > 0$ is *not fixed*. Clearly, if $\beta > 0$ is fixed, then (B.1) contradicts (B.5). (B.1) and (B.5) are consistent, for example, if $\beta = \lambda^{-m} (m > N/2)$.

(2) Theorem 2.1 and Theorem 2.3 imply that if $\alpha, \beta > 0$ *are fixed*, then

$$\frac{\mu_1(\lambda, \alpha)}{\mu_2(\lambda, \beta)} \to \infty \quad \text{as } \lambda \to \infty.$$

This phenomenon is explained as follows. We see that as $\lambda \to \infty$, $\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1}$ behaves like $\alpha^{(p+1)/2} \lambda^{-(N+2-p(N-2))/4}$ (cf. (3.15) in Section 3). Therefore, if $\alpha, \beta > 0$ are fixed, then $\Phi(u_{1,\lambda,\alpha}) \to 0$ and consequently, $u_{1,\lambda,\alpha} \in M_\beta$ is impossible. Hence if $\beta > 0$ behaves like $\alpha^{(p+1)/2} \lambda^{-(N+2-p(N-2))/4}$ as $\lambda \to \infty$, then the growth order of $\mu_2(\lambda, \beta)$ as $\lambda \to \infty$ is the same as that of $\mu_1(\lambda, \alpha)$. More precisely (let $K_0 = K_1 = 1$ for simplicity), if the top term of $\mu_1(\lambda, \alpha)$ coincides with that of $\mu_2(\lambda, \beta)$, then by Theorem 2.1 and Theorem 2.3, $\beta = \beta_{\lambda,\alpha}$ must satisfy $\beta_{\lambda,\alpha} = C_2^{-\frac{p+1}{p-1}} C_4^{\frac{p+1}{p-1}} \alpha^{\frac{p+1}{2}} \lambda^{-(N+2-p(N-2))/4}$. This corresponds to the fact that

$$\begin{aligned}\Phi(u_{1,\lambda,\alpha}) &= \frac{1}{p+1}(1+o(1))\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} \\ &= C_2^{-\frac{p+1}{p-1}} \frac{1}{p+1}(1+o(1))\|w\|_{p+1}^{p+1}\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}} \\ &= (1+o(1))\beta_{\lambda,\alpha}\,,\end{aligned}$$

which will be shown in Section 4.

Since the proof of Theorem 2.2 and Theorem 2.4 are similar to those of Theorem 2.1 and Theorem 2.3, respectively, we prove Theorem 2.1 and Theorem 2.3 in the rest of this paper.

3. – Lemmas for the proof of Theorem 2.1

Since (1.1) is autonomous, by translation, we may assume without loss of generality that $0 \in \Omega$. In Section 3 and Section 4, we consider the problem (M.1). For simplicity, C denotes various positive constants independent of (λ, α). In particular, the character C which may appear repeatedly in the same inequality sometimes denotes different constants independent of (λ, α). Further, a subsequence of a sequence will be denoted by the same notation as that of original sequence. Finally, for convenience, $K_0 = K_1 = J_0 = J_1 = 1$ in what follows. By (1.2) and (1.3), for $t \geq 0$ we have

$$C(t^p + t^q) \leq f(t) \leq C^{-1}(t^p + t^q)\,, \tag{3.1}$$

$$Ct \leq g(t) \leq C^{-1}t\,, \tag{3.2}$$

$$C(\|u\|_{p+1}^{p+1} + \|u\|_{q+1}^{q+1}) \leq (f(u), u) \leq C^{-1}(\|u\|_{p+1}^{p+1} + \|u\|_{q+1}^{q+1})\,, \tag{3.3}$$

$$C(\|u\|_{p+1}^{p+1} + \|u\|_{q+1}^{q+1}) \leq \Phi(u) \leq C^{-1}(\|u\|_{p+1}^{p+1} + \|u\|_{q+1}^{q+1})\,, \tag{3.4}$$

$$C\|u\|_2^2 \leq (g(u), u) \leq C^{-1}\|u\|_2^2\,, \tag{3.5}$$

$$C\|u\|_2^2 \leq \Psi(u) \leq C^{-1}\|u\|_2^2\,. \tag{3.6}$$

For a fixed $(\lambda, \alpha) \in \mathbb{R}_+^2$, the existence of $(\mu_1(\lambda, \alpha), u_{1,\lambda,\alpha}) \in \mathbb{R}_+ \times N_{\lambda,\alpha}$ follows from Zeidler [21, Proposition 2]. We can also prove the existence directly by choosing a maximizing sequence $\{u_n\} \subset N_{\lambda,\alpha}$ of (2.1), since $\sup_{u \in N_{\lambda,\alpha}} \Phi(u) < \infty$ for a fixed $(\lambda, \alpha) \in \mathbb{R}_+^2$. In fact, by (3.4) and the Gagliardo-Nirenberg inequality (cf. [7])

$$\|u\|_{\eta+1}^{\eta+1} \le C\|u\|_2^{\frac{N+2-\eta(N-2)}{2}} \|\nabla u\|_2^{\frac{N(\eta-1)}{2}} \quad (1 < \eta < (N+2)/(N-2)) \tag{3.7}$$

for $u \in W_0^{1,2}(\Omega)$, we obtain that $\sup_{u \in N_{\lambda,\alpha}} \Phi(u) < \infty$.

The aim of this section is to estimate $\mu_1(\lambda, \alpha)$ from below and above by λ and α.

LEMMA 3.1. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.2). *Then*

$$\mu_1(\lambda, \alpha) \le C\alpha^{\frac{1-p}{2}} \lambda^{\frac{N+2-p(N-2)}{4}}. \tag{3.8}$$

To prove Lemma 3.1, we need some preparations.

LEMMA 3.2. *For* $\tau > 0$, *let* $w_\tau \in C^2(B_\tau)$ *be the unique solution of the equation*

$$\begin{aligned} &\Delta w_\tau + w_\tau^p - w_\tau = 0 \quad \text{in } B_\tau := \{x \in \mathbb{R}^N : |x| < \tau\}, \\ &w_\tau > 0 \quad \text{in } B_\tau, \qquad w_\tau = 0 \quad \text{on } \partial B_\tau. \end{aligned} \tag{3.9}$$

Then $w_\tau \to w$ *not only in* $H^1(\mathbb{R}^N)$, *but also uniformly on any compact subset in* $\mathbb{R}^N$ *as* $\tau \to \infty$.

The unique existence of w_τ follows from Kwong [13], and the latter assertion can be proved by the similar arguments as those of Lemmas 4.5, 4.7-4.8 in Section 4. Hence we omit the proof. By [10], w_τ is radially symmetric, that is, $w_\tau(x) = w_\tau(r)$ $(r = |x|)$.

LEMMA 3.3. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.2). *Let* $w_{\sqrt{\lambda}r_0}$ *be the solution of* (3.9) *for* $\tau = \sqrt{\lambda}r_0$, *where* $0 < r_0 \ll 1$ *is a constant. Put*

$$U_{\lambda,\alpha}(|x|) := \begin{cases} c_{\lambda,\alpha}\alpha^{1/2}\lambda^{(N-2)/4} w_{\sqrt{\lambda}r_0}(\sqrt{\lambda}|x|), & x \in B_{r_0} := \{x \in \mathbb{R}^N : |x| < r_0\} \subset \Omega, \\ 0, & x \in \Omega \setminus B_{r_0}, \end{cases}$$

where $c_{\lambda,\alpha} := \min\{c > 0 : c\alpha^{1/2}\lambda^{(N-2)/4} w_{\sqrt{\lambda}r_0}(\sqrt{\lambda}|x|) \in N_{\lambda,\alpha}\}$. *Then* $C \le c_{\lambda,\alpha} \le C^{-1}$.

PROOF. For $t \ge 0$, let $m_{\lambda,\alpha}(t) := \Lambda_\lambda(tU_{\lambda,\alpha}) = \frac{1}{2}\|\nabla(tU_{\lambda,\alpha})\|_2^2 + \lambda\Psi(tU_{\lambda,\alpha})$. Then clearly $m_{\lambda,\alpha}(0) = 0$ and $m_{\lambda,\alpha}(t) \to \infty$ as $t \to \infty$ for a fixed (λ, α). Hence $c_{\lambda,\alpha} > 0$ exists. Since

$$\|\nabla U_{\lambda,\alpha}\|_2^2 = c_{\lambda,\alpha}^2 \alpha \|\nabla w_{\sqrt{\lambda}r_0}\|_2^2, \quad \lambda\|U_{\lambda,\alpha}\|_2^2 = c_{\lambda,\alpha}^2 \alpha \|w_{\sqrt{\lambda}r_0}\|_2^2,$$

by (3.6), we obtain

$$\alpha = \Lambda_\lambda(U_{\lambda,\alpha}) \sim c_{\lambda,\alpha}^2 \alpha \left(\frac{1}{2}\|\nabla w_{\sqrt{\lambda}r_0}\|_2^2 + C^{-1}\|w_{\sqrt{\lambda}r_0}\|_2^2 \right). \tag{3.10}$$

By Lemma 3.2 and (3.10) we obtain our conclusion. □

PROOF OF LEMMA 3.1. By direct calculation we have

$$\|U_{\lambda,\alpha}\|_{p+1}^{p+1} = c_{\lambda,\alpha}^{p+1}\|w_{\sqrt{\lambda}r_0}\|_{p+1}^{p+1}\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}};$$

this along with (2.1), (3.3), (3.4) and Lemmas 3.2-3.3 implies

$$\begin{aligned}(f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha}) &\geq C\Phi(u_{1,\lambda,\alpha}) \geq C\Phi(U_{\lambda,\alpha}) \geq C\|U_{\lambda,\alpha}\|_{p+1}^{p+1} \\ &\geq C\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}}\end{aligned} \tag{3.11}$$

Furthermore, since $u_{1,\lambda,\alpha} \in N_{\lambda,\alpha}$, we have

$$\|\nabla u_{1,\lambda,\alpha}\|_2^2,\ \lambda\|u_{1,\lambda,\alpha}\|_2^2 \leq C\alpha\,. \tag{3.12}$$

Then, by (2.3), (3.6), (3.11) and (3.12)

$$\mu_1(\lambda,\alpha) \leq \frac{2\alpha + C\lambda\|u_{1,\lambda,\alpha}\|_2^2}{(f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha})} \leq C\alpha^{\frac{(1-p)}{2}}\lambda^{\frac{N+2-p(N-2)}{4}}\,.$$

Thus the proof is complete. □

LEMMA 3.4. *Assume that* $\{(\lambda,\alpha)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.2). *Then*

$$\mu_1(\lambda,\alpha) \geq C\alpha^{\frac{1-p}{2}}\lambda^{\frac{N+2-p(N-2)}{4}}\,. \tag{3.13}$$

PROOF. Since $u_{1,\lambda,\alpha} \in N_{\lambda,\alpha}$, we obtain by (3.6) that there exists a constant $\delta > 0$ such that

$$\begin{aligned}\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \lambda(g(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha}) &\geq \delta\left\{\frac{1}{2}\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \lambda\Psi(u_{1,\lambda,\alpha})\right\} \\ &= \delta\Lambda_\lambda(u_{1,\lambda,\alpha}) = \delta\alpha\,.\end{aligned} \tag{3.14}$$

Then we obtain by (B.2), (3.7) and (3.12) that

$$\begin{aligned}\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} &\leq C\|u_{1,\lambda,\alpha}\|_2^{\frac{N+2-p(N-2)}{2}}\|\nabla u_{1,\lambda,\alpha}\|_2^{\frac{N(p-1)}{2}} \\ &\leq C\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}}, \\ \|u_{1,\lambda,\alpha}\|_{q+1}^{q+1} &\leq C\|u_{1,\lambda,\alpha}\|_2^{\frac{N+2-q(N-2)}{2}}\|\nabla u_{1,\lambda,\alpha}\|_2^{\frac{N(q-1)}{2}} \\ &\leq C\alpha^{\frac{q+1}{2}}\lambda^{-\frac{N+2-q(N-2)}{4}} \\ &\leq C(\alpha^2\lambda^{N-2})^{\frac{q-p}{4}}\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}} \\ &\leq C\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}}\,.\end{aligned} \tag{3.15}$$

Then by (3.3) and (3.15), we obtain

$$(3.16)\qquad \begin{aligned}(f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha}) &\le C(\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} + \|u_{1,\lambda,\alpha}\|_{q+1}^{q+1})\\ &\le C\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}}.\end{aligned}$$

Then by (2.5), (3.14) and (3.16), we obtain

$$\begin{aligned}\mu_1(\lambda,\alpha) &= \frac{\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \lambda(g(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha})}{(f(u_{1,\lambda,\alpha}), u_{1,\lambda,\alpha})} \ge \frac{\delta\alpha}{C\alpha^{\frac{p+1}{2}}\lambda^{-\frac{N+2-p(N-2)}{4}}}\\ &\ge C\alpha^{\frac{1-p}{2}}\lambda^{\frac{N+2-p(N-2)}{4}}.\end{aligned}$$

□

4. – Proof of Theorem 2.1

We put

$$\begin{aligned}&\xi_{1,\lambda,\alpha} := (\lambda/\mu_1(\lambda,\alpha))^{1/(p-1)},\ v_{1,\lambda,\alpha}(x) := \xi_{1,\lambda,\alpha}^{-1}u_{1,\lambda,\alpha}(x),\\ &\Omega_\lambda := \{y \in \mathbb{R}^N : y = \sqrt{\lambda}x, x \in \Omega\},\ w_{1,\lambda,\alpha}(y) := \xi_{1,\lambda,\alpha}^{-1}u_{1,\lambda,\alpha}(x)\ (y := \sqrt{\lambda}x),\\ &h_0(t) := g(t) - t,\ H_0(t) := \int_0^t h_0(s)ds,\ h_1(t) := f(t) - |t|^{p-1}t,\ H_1(t) := \int_0^t h_1(s)ds.\end{aligned}$$

Then by (1.1), we see that $v_{1,\lambda,\alpha}$ and $w_{1,\lambda,\alpha}$ satisfy the following equations, respectively:

$$(4.1)\qquad \begin{aligned}-\frac{1}{\lambda}\Delta v_{1,\lambda,\alpha} &= v_{1,\lambda,\alpha}^p + \xi_{1,\lambda,\alpha}^{-p}h_1(\xi_{1,\lambda,\alpha}v_{1,\lambda,\alpha}) - v_{1,\lambda,\alpha}\\ &\quad - \xi_{1,\lambda,\alpha}^{-1}h_0(\xi_{1,\lambda,\alpha}v_{1,\lambda,\alpha}) \quad \text{in } \Omega,\\ v_{1,\lambda,\alpha} &> 0 \quad \text{in } \Omega, \qquad v_{1,\lambda,\alpha} = 0 \quad \text{on } \partial\Omega,\end{aligned}$$

$$(4.2)\qquad \begin{aligned}-\Delta w_{1,\lambda,\alpha} &= w_{1,\lambda,\alpha}^p - w_{1,\lambda,\alpha} + \xi_{1,\lambda,\alpha}^{-p}h_1(\xi_{1,\lambda,\alpha}w_{1,\lambda,\alpha})\\ &\quad - \xi_{1,\lambda,\alpha}^{-1}h_0(\xi_{1,\lambda,\alpha}w_{1,\lambda,\alpha}) \quad \text{in } \Omega_\lambda,\\ w_{1,\lambda,\alpha} &> 0 \quad \text{in } \Omega_\lambda, \qquad w_{1,\lambda,\alpha} = 0 \quad \text{on } \partial\Omega_\lambda.\end{aligned}$$

If $\{\lambda,\alpha)\} \subset \mathbb{R}_+^2$ satisfies (B.1) and (B.2), then by Lemma 3.1, we obtain

$$(4.3)\qquad \xi_{1,\lambda,\alpha}^{p-1} = \frac{\lambda}{\mu_1(\lambda,\alpha)} \ge C(\alpha^2\lambda^{N-2})^{\frac{p-1}{4}} \to \infty.$$

LEMMA 4.1. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.2). *Then*

$$\|\nabla w_{1,\lambda,\alpha}\|_2^2 \le C\,, \tag{4.4}$$

$$\|w_{1,\lambda,\alpha}\|_2^2 \le C\,, \tag{4.5}$$

$$\|w_{1,\lambda,\alpha}\|_{\eta+1}^{\eta+1} \le C \quad (1 \le \eta \le (N+2)/(N-2))\,. \tag{4.6}$$

PROOF. Since $u_{1,\lambda,\alpha} \in N_{\lambda,\alpha}$, by definition of $w_{1,\lambda,\alpha}$ and Lemma 3.1, we obtain

$$\|\nabla w_{1,\lambda,\alpha}\|_2^2 = \lambda^{\frac{N-2}{2}} \xi_{1,\lambda,\alpha}^{-2} \|\nabla u_{1,\lambda,\alpha}\|_2^2 \le \lambda^{-\frac{N+2-p(N-2)}{2(p-1)}} \mu_1(\lambda,\alpha)^{\frac{2}{p-1}} \alpha \le C\,,$$

$$\|w_{1,\lambda,\alpha}\|_2^2 = \xi_{1,\lambda,\alpha}^{-2} \lambda^{\frac{N}{2}} \|u_{1,\lambda,\alpha}\|_2^2 \le C\alpha\mu_1(\lambda,\alpha)^{\frac{2}{p-1}} \lambda^{-\frac{N+2-p(N-2)}{2(p-1)}} \le C\,.$$

Finally, (4.6) follows from (4.4), (4.5) and the Sobolev's embedding theorem. □

Now, we investigate the asymptotic location of the point $x_{1,\lambda,\alpha} \in \Omega$ at which the maximum of $u_{1,\lambda,\alpha}$ is attained. For this purpose, we study the behavior of $v_{1,\lambda,\alpha}$, since $v_{1,\lambda,\alpha}$ attains its maximum at the same point $x_{1,\lambda,\alpha}$ as $u_{1,\lambda,\alpha}$, and among other things, we can apply the same arguments as those used in Ni and Wei [16] to derive the properties of $v_{1,\lambda,\alpha}$. The following Lemma 4.2 corresponds to Ni and Wei [16, Lemma 3.2].

LEMMA 4.2. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.2). *Then*

(i) $\sup_{x\in\Omega} v_{1,\lambda,\alpha}(x) \le C$.

(ii) $c_\tau \lambda^{-N/2} \le \int_\Omega v_{1,\lambda,\alpha}^\tau dx \le C_\tau \lambda^{-N/2}$ *if* $1 < \tau < \infty$.

PROOF. By (4.4) and (4.5), we obtain

$$\begin{aligned}\int_\Omega \left(\frac{1}{\lambda}|\nabla v_{1,\lambda,\alpha}|^2 + v_{1,\lambda,\alpha}^2\right) dx &= \xi_{1,\lambda,\alpha}^{-2} \left(\frac{1}{\lambda}\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \|u_{1,\lambda,\alpha}\|_2^2\right) \\ &= (\|\nabla w_{1,\lambda,\alpha}\|^2 + \|w_{1,\lambda,\alpha}\|_2^2)\lambda^{-N/2} \le C\lambda^{-N/2}.\end{aligned} \tag{4.7}$$

Furthermore, by (3.6) and Lemma 3.4, we obtain

$$\begin{aligned}\int_\Omega \left(\frac{1}{\lambda}|\nabla v_{1,\lambda,\alpha}|^2 + v_{1,\lambda,\alpha}^2\right) dx &\ge C\xi_{1,\lambda,\alpha}^{-2}\lambda^{-1}\Lambda_\lambda(u_{1,\lambda,\alpha}) = C\xi_{1,\lambda,\alpha}^{-2}\lambda^{-1}\alpha \\ &= C\left\{\mu_1(\lambda,\alpha)^{\frac{2}{p-1}}\alpha\lambda^{-\frac{N+2-p(N-2)}{2(p-1)}}\right\}\lambda^{-N/2} \\ &\ge C\lambda^{-N/2}.\end{aligned} \tag{4.8}$$

Once (4.7) and (4.8) which correspond to Lin, Ni and Takagi [14, Corollary 2.1 (2.6), Proposition 2.2] are established, then (i) and (ii) follow from exactly the same arguments used in the proof of [14, Lemma 2.3 and Corollary 2.1 (2.7)] by using L^τ-estimate, (4.7) and (4.8). Hence the proof is complete. □

LEMMA 4.3. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.2). *Then* $\|v_{1,\lambda,\alpha}\|_\infty \ge C$.

PROOF. Since $\|v_{1,\lambda,\alpha}\|_\infty = \xi_{\lambda,\alpha}^{-1}\|u_{1,\lambda,\alpha}\|_\infty$, it is enough to show that $\|u_{1,\lambda,\alpha}\|_\infty \geq C\xi_{1,\lambda,\alpha}$. The proof is divided into two steps.

STEP 1. We show that $C\xi_{1,\lambda,\alpha} \leq s_{\lambda,\alpha} := \min\{s > 0 : \mu_1(\lambda, \alpha)f(s) = \lambda g(s)\}$. The existence of $s_{\lambda,\alpha} > 0$ follows from (1.2) and (1.3). We assume that there exists a subsequence of $\{s_{\lambda,\alpha}\}$ such that $s_{\lambda,\alpha}\xi_{1,\lambda,\alpha}^{-1} \to 0$ and derive a contradiction. Then there are three cases to consider:

CASE 1. Assume that there exists a subsequence of $\{s_{\lambda,\alpha}\}$ such that $s_{\lambda,\alpha} \to \infty$. Then by (1.2) we obtain

$$(1 + o(1))\mu_1(\lambda, \alpha)s_{\lambda,\alpha}^p = (1 + o(1))\lambda s_{\lambda,\alpha}\,. \tag{4.9}$$

(4.9) implies $s_{\lambda,\alpha}^{p-1} = (1 + o(1))\xi_{1,\lambda,\alpha}^{p-1}$. Then we obtain that $s_{\lambda,\alpha}\xi_{1,\lambda,\alpha}^{-1} \to 1$. This is a contradiction.

CASE 2. Assume that there exists a subsequence of $\{s_{\lambda,\alpha}\}$ such that $s_{\lambda,\alpha} \to 0$. Then by (1.3) we obtain

$$(1 + o(1))\mu_1(\lambda, \alpha)s_{\lambda,\alpha}^q = (1 + o(1))\lambda s_{\lambda,\alpha}\,;$$

this implies $s_{\lambda,\alpha}^{q-1} = (1 + o(1))\xi_{1,\lambda,\alpha}^{p-1}$. Hence we obtain $(1 + o(1))\xi_{1,\lambda,\alpha}^{p-q} = (s_{\lambda,\alpha}\xi_{1,\lambda,\alpha}^{-1})^{q-1} \to 0$. This contradicts (4.3).

CASE 3. Assume that there exists a subsequence of $\{s_{\lambda,\alpha}\}$ such that $C^{-1} \leq s_{\lambda,\alpha} \leq C$. Then by (4.3)

$$C_0 \geq \max_{C^{-1} \leq s \leq C} \frac{f(s)}{g(s)} \geq \frac{f(s_{\lambda,\alpha})}{g(s_{\lambda,\alpha})} = \frac{\lambda}{\mu_1(\lambda, \alpha)} = \xi_{1,\lambda,\alpha}^{p-1} \to \infty\,.$$

This is a contradiction. Thus the proof of Step 1 is complete.

STEP 2. We show that $\|u_{1,\lambda,\alpha}\|_\infty \geq s_{\lambda,\alpha}$. By (1.2), we obtain that $\mu_1(\lambda, \alpha)f(s) < \lambda g(s)$ for $0 < s < s_{\lambda,\alpha}$. If $u_{1,\lambda,\alpha}(x_{1,\lambda,\alpha}) = \|u_{1,\lambda,\alpha}\|_\infty < s_{\lambda,\alpha}$, then since $0 < \|u_{1,\lambda,\alpha}\|_\infty$, we obtain

$$\Delta u_{1,\lambda,\alpha}(x_{1,\lambda,\alpha}) = \lambda g(u_{1,\lambda,\alpha}(x_{1,\lambda,\alpha})) - \mu_1(\lambda, \alpha)f(u_{1,\lambda,\alpha}(x_{1,\lambda,\alpha})) > 0\,.$$

On the other hand, $\Delta u_{1,\lambda,\alpha}(x_{1,\lambda,\alpha}) \leq 0$, since $u_{1,\lambda,\alpha}(x_{1,\lambda,\alpha}) = \|u_{1,\lambda,\alpha}\|_\infty$. This is a contradiction. This along with Step 1 yields $\|u_{1,\lambda,\alpha}\|_\infty \geq s_{\lambda,\alpha} \geq C\xi_{1,\lambda,\alpha}$. Thus we obtain $\|v_{1,\lambda,\alpha}\|_\infty \geq C$. □

LEMMA 4.4. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.2). *Then*

$$\rho_{\lambda,\alpha} := \lambda^{1/2}\text{dist}(x_{1,\lambda,\alpha}, \partial\Omega) \to \infty\,.$$

Lemma 4.4 follows from Lemma 4.2, Lemma 4.3 and exactly the same arguments used in the proof of Ni and Wei [16, Step 1 (proof of (3.2)), p. 737-738]. Hence we omit the proof.

LEMMA 4.5. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbb{R}^2_+$ *satisfies* (B.1) *and* (B.2). *Furthermore, let* $y_{1,\lambda,\alpha} := \sqrt{\lambda} x_{1,\lambda,\alpha} \in \mathbb{R}^N$. *Then for any subsequence* $S \subset \{(\lambda, \alpha)\}$, *there exists a subsequence* $\{(\lambda_j, \alpha_j)_{j \in \mathbb{N}}\}$ *of* S *such that* $z_j(y) := w_{1,\lambda_j,\alpha_j}(y + y_{1,\lambda_j,\alpha_j}) \to w(y)$ *on any compact subset in* $\mathbb{R}^N$ *as* $j \to \infty$.

PROOF. Let $\rho_j := \rho_{\lambda_j,\alpha_j}$. Then by (4.2), we see that $z_j(y)$ satisfies the equation in (4.2) in the ball B_{ρ_j}. We note that $\{B_{\rho_j}\}_{j \in N} \subset \mathbb{R}^N$ is an exhaustion of $\mathbb{R}^N$ by Lemma 4.4. Then by using Lemmas 4.2-4.4 and the same arguments [15, (4.5)-(4.13), p.830-832], we see that we can extract a subsequence, again denoted by $\{z_j\}$ for simplicity, such that

$$z_j \to W \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^N)\,. \tag{4.10}$$

Then $W \geq 0$ in $\mathbb{R}^N$. Let $T_1 := \{y \in \mathbb{R}^N : W(y) > 0\}$, $T_2 := \{y \in \mathbb{R}^N : W(y) = 0\}$. By (4.2), we have

$$-\Delta z_j = \left(1 + \frac{h_1(\xi_{1,\lambda_j,\alpha_j} z_j)}{(\xi_{1,\lambda_j,\alpha_j} z_j)^p}\right) z_j^p - \left(1 + \frac{h_0(\xi_{1,\lambda_j,\alpha_j} z_j)}{\xi_{1,\lambda_j,\alpha_j} z_j}\right) z_j \quad \text{in } B_{\rho_j}\,. \tag{4.11}$$

Then we see form (1.2), (4.3) and (4.11) that W satisfies the equation in (2.6) on T_1. Next, let $y \in T_2$ be fixed. Then by (4.10), we see that $z_j(y) \to 0$ as $j \to \infty$. There are two possible cases:

CASE 1. If there exists a subsequence of $\{\xi_{1,\lambda_j,\alpha_j} z_j(y)\}$ such that $\xi_{1,\lambda_j,\alpha_j} z_j(y) \to \infty$ as $j \to \infty$, then (1.2) and the fact $z_j(y) \to 0$ imply that the right hand side of the equation (4.11) tends to 0 as $j \to \infty$. Hence, we obtain $-\Delta W(y) = 0$.

CASE 2. If there exists a subsequence of $\{\xi_{1,\lambda_j,\alpha_j} z_j(y)\}$ such that $\xi_{1,\lambda_j,\alpha_j} z_j(y) \leq C$, then since $|h_1(t)| \leq ct^q$, $|h_0(t)| \leq ct$ for $0 \leq t \leq C$ by (1.3), it follows from (4.3) and the fact $z_j(y) \to 0$ that as $j \to \infty$

$$\begin{aligned} &|z_j^p(y) - z_j(y) + \xi_{1,\lambda_j,\alpha_j}^{-p} h_1(\xi_{1,\lambda_j,\alpha_j} z_j(y)) - \xi_{1,\lambda_j,\alpha_j}^{-1} h_0(\xi_{1,\lambda_j,\alpha_j} z_j(y))| \\ &\leq |z_j(y)|^p + C|z_j(y)| + C\xi_{1,\lambda_j,\alpha_j}^{q-p} |z_j(y)|^q \to 0\,. \end{aligned} \tag{4.12}$$

Hence we see from (4.11) and (4.12) that $-\Delta W(y) = 0$ in this case, too.

Consequently, $-\Delta W = 0$ on T_2. This implies that W also satisfies the equation in (2.6) on T_2. Thus W satisfies the equation in (2.6) in $\mathbb{R}^N$. In addition, we obtain $W \not\equiv 0$. In fact, by Lemma 4.3, we obtain

$$W(0) = \lim_{j \to \infty} z_j(0) = \max_{x \in \Omega} v_{1,\lambda,\alpha}(x) \geq C\,. \tag{4.13}$$

Further, $W \in H^1(\mathbb{R}^N)$, since we obtain by (4.4), (4.5), (4.10) and Fatou's lemma that

$$\|\nabla W\|_2^2 \leq \liminf_{j \to \infty} \|\nabla z_j\|_2^2 = \liminf_{j \to \infty} \|\nabla w_{1,\lambda_j,\alpha_j}\|_2^2 \leq C\,, \tag{4.14}$$

$$\|W\|_2^2 \leq \liminf_{j \to \infty} \|z_j\|_2^2 = \liminf_{j \to \infty} \|w_{1,\lambda_j,\alpha_j}\|_2^2 \leq C\,. \tag{4.15}$$

Hence it follows from the results of Kwong [13] that $W \equiv w$. □

LEMMA 4.6. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbb{R}^2_+$ *satisfies* (B.1) *and* (B.2). *Then*

$$\xi_{1,\lambda,\alpha}^{-p} \int_{\Omega_\lambda} h_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) w_{1,\lambda,\alpha}(y) dy \to 0\,, \tag{4.16}$$
$$\xi_{1,\lambda,\alpha}^{-(p+1)} \int_{\Omega_\lambda} H_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) dy \to 0\,,$$

$$\xi_{1,\lambda,\alpha}^{-1} \int_{\Omega_\lambda} h_0(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) w_{1,\lambda,\alpha}(y) dy \to 0\,, \tag{4.17}$$
$$\xi_{1,\lambda,\alpha}^{-2} \int_{\Omega_\lambda} H_0(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) dy \to 0\,.$$

PROOF. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. For $0 < \delta \ll 1$, let

$$\Omega_{\lambda,\alpha,1} := \{y \in \Omega_\lambda : \xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y) \ge \delta^{-1}\},$$
$$\Omega_{\lambda,\alpha,2} := \{y \in \Omega_\lambda : \xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y) \le \delta\}\,,$$
$$\Omega_{\lambda,\alpha,3} := \Omega_\lambda \setminus (\Omega_{\lambda,\alpha,1} \bigcup \Omega_{\lambda,\alpha,2})\,.$$

Then by (1.2), we obtain that $h_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}) \le \epsilon(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha})^p$ for $y \in \Omega_{1,\lambda,\alpha}$ if we choose $\delta > 0$ sufficiently small. Therefore, by (4.6)

$$\xi_{1,\lambda,\alpha}^{-p} \int_{\Omega_{\lambda,\alpha,1}} h_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) w_{1,\lambda,\alpha}(y) dy \le \epsilon \int_{\Omega_{\lambda,\alpha,1}} w_{1,\lambda,\alpha}(y)^{p+1} dy \le C\epsilon\,.$$

Next, for $y \in \Omega_{\lambda,\alpha,2}$,

$$h_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}) \le C(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha})^q - (\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha})^p \le C\delta^\sigma(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha})^{q-\sigma}\,,$$

where $0 < \sigma \ll 1$ is a constant. Hence by (4.3) and (4.6),

$$\begin{aligned}
&\xi_{1,\lambda,\alpha}^{-p} \int_{\Omega_{\lambda,\alpha,2}} h_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) w_{1,\lambda,\alpha}(y) dy \\
&\le C\delta^\sigma \xi_{1,\lambda,\alpha}^{-p} \int_{\Omega_{\lambda,\alpha,2}} \xi_{1,\lambda,\alpha}^{q-\sigma} w_{1,\lambda,\alpha}(y)^{q+1-\sigma} dy \\
&= C\delta^\sigma \xi_{1,\lambda,\alpha}^{q-p-\sigma} \|w_{1,\lambda,\alpha}\|_{q-\sigma+1}^{q-\sigma+1} \le C\xi_{1,\lambda,\alpha}^{q-p-\sigma} \to 0\,.
\end{aligned}$$

Since $|h_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y))| \le C_\delta\ (\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y))^{q-\sigma}$ for $y \in \Omega_{\lambda,\alpha,3}$, by (4.3) and Lemma 4.2 (ii), we obtain

$$\begin{aligned}
\xi_{1,\lambda,\alpha}^{-p} \int_{\Omega_{\lambda,\alpha,3}} h_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) w_{1,\lambda,\alpha}(y) dy &\le C_\delta \xi_{1,\lambda,\alpha}^{q-p-\sigma} \int_{\Omega_{\lambda,\alpha,3}} w_{1,\lambda,\alpha}(y)^{q+1-\sigma} dy \\
&= C_\delta \xi_{1,\lambda,\alpha}^{q-p-\sigma} \lambda^{N/2} \|v_{1,\lambda,\alpha}\|_{q+1-\sigma}^{q+1-\sigma} \le CC_\delta \xi_{1,\lambda,\alpha}^{q-p-\sigma} \to 0\,.
\end{aligned}$$

Hence we obtain the first inequality of (4.16). By the same arguments as those just above, we also obtain the other inequalities. Thus the proof is complete. □

LEMMA 4.7. *Assume* $\{(\lambda,\alpha)\} \subset \mathbb{R}^2_+$ *satisfies* (B.1) *and* (B.2). *Then*

$$\|w\|_{p+1} \le \liminf \|w_{1,\lambda,\alpha}\|_{p+1} \le \limsup \|w_{1,\lambda,\alpha}\|_{p+1} \le \|w\|_{p+1}\,. \tag{4.18}$$

PROOF. The first inequality in (4.18) follows from (4.6), Lemma 4.5 and Fatou's lemma. We show the last inequality. First, multiply (2.6) by w. Then integration by parts yields

$$\|\nabla w\|_2^2 + \|w\|_2^2 = \|w\|_{p+1}^{p+1}\,. \tag{4.19}$$

Let $B_{r_0} \subset \Omega$. Furthermore, let $\chi_\lambda \in C^2(\mathbb{R}^N)$ satisfy

$$\chi_\lambda(y) = \begin{cases} 1, & |y| \le \sqrt{\lambda}r_0 - 1, \\ 0, & |y| \ge \sqrt{\lambda}r_0\,, \end{cases}$$

and

$$0 \le \chi_\lambda(y) \le 1, \quad |\nabla\chi_\lambda(y)| \le C \quad \text{for } y \in \mathbb{R}^N,\ \lambda \gg 1\,.$$

Let $V_\lambda(y) = w(y)\chi_\lambda(y)$ for $y \in \mathbb{R}^N$. Then for $\lambda \gg 1$, clearly, we have

$$\begin{aligned} \|\nabla V_\lambda\|_2 &= (1+o(1))\|\nabla w\|_2, \\ \|V_\lambda\|_2 &= (1+o(1))\|w\|_2\,, \\ \|V_\lambda\|_{p+1} &= (1+o(1))\|w\|_{p+1}\,. \end{aligned} \tag{4.20}$$

Let $c_\lambda := \inf\{c > 0 : cV_\lambda(\sqrt{\lambda}x) \in N_{\lambda,\alpha}\}$ and $e_\lambda(x) := c_\lambda V_\lambda(\sqrt{\lambda}x)$. Now we study the asymptotic behavior of c_λ. The arguments are divided into several steps.

STEP 1. We first show that $c_\lambda \to \infty$ as $\lambda \to \infty$. By (3.6) and (4.20), we obtain

$$\begin{aligned} \alpha = \Lambda_\lambda(e_\lambda) &\le \frac{1}{2}(\|\nabla e_\lambda\|_2^2 + \lambda C^{-1}\|e_\lambda\|_2^2) \\ &= \frac{1}{2}c_\lambda^2\lambda^{\frac{2-N}{2}}(\|\nabla V_\lambda\|_2^2 + C^{-1}\|V_\lambda\|_2^2) \le Cc_\lambda^2\lambda^{\frac{2-N}{2}}\,. \end{aligned} \tag{4.21}$$

Then we obtain by (B.2) that $c_\lambda^2 \ge C\alpha\lambda^{\frac{N-2}{2}} \to \infty$.

STEP 2. We show

$$c_\lambda^2 = 2(1+o(1))\alpha\lambda^{\frac{N-2}{2}}\|w\|_{p+1}^{-(p+1)}\,. \tag{4.22}$$

To do this, we first show that

$$\left|\int_\Omega H_0(e_\lambda(x))dx\right| = o(1)\|e_\lambda\|_2^2\,. \tag{4.23}$$

To show (4.23), since we have

$$\|e_\lambda\|_2^2 = c_\lambda^2 \lambda^{-N/2} \|V_\lambda\|_2^2 ,$$

$$\left|\int_\Omega H_0(e_\lambda(x))dx\right| = c_\lambda^2 \lambda^{-N/2} \left|\int_{\Omega_\lambda} c_\lambda^{-2} \left(\int_0^{c_\lambda V_\lambda(y)} h_0(s)ds\right) dy\right| ,$$

it is enough to show that

$$\left|\int_{\Omega_\lambda} c_\lambda^{-2} \left(\int_0^{c_\lambda V_\lambda(y)} h_0(s)ds\right) dy\right| \to 0 . \tag{4.24}$$

By (3.2), $|h_0(s)| \le C|s|$ for $s \in \mathbb{R}$. Hence

$$\left|c_\lambda^{-2} \int_0^{c_\lambda V_\lambda(y)} h_0(s)ds\right| \le C V_\lambda(y)^2 \le C w(y)^2 . \tag{4.25}$$

By (1.2), for an arbitrary $0 < \epsilon \ll 1$, there exists a constant $s_\epsilon > 0$ such that $|h_0(s)| \le \epsilon s$ for $s \ge s_\epsilon$. Since $\|V_\lambda\|_\infty = \|w\|_\infty$ and $c_\lambda \to \infty$, we obtain by Step 1 that for $\lambda \gg 1$

$$\begin{aligned} \left|c_\lambda^{-2} \int_0^{c_\lambda V_\lambda(y)} h_0(s)ds\right| &\le c_\lambda^{-2} \int_{s_\epsilon}^{c_\lambda V_\lambda(y)} |h_0(s)|ds + c_\lambda^{-2} \int_0^{s_\epsilon} |h_0(s)|ds \\ &\le c_\lambda^{-2} \left(\frac{1}{2}\epsilon(c_\lambda^2 V_\lambda^2(y) - s_\epsilon^2) + \frac{1}{2} C s_\epsilon^2\right) \le C\epsilon . \end{aligned} \tag{4.26}$$

Therefore, by (4.25), (4.26) and Lebesgue's convergence theorem, we obtain (4.24). This implies (4.23). Then we obtain by (4.19), (4.20) and (4.23) that

$$\begin{aligned} \alpha = \Lambda_\lambda(e_\lambda) &= \frac{1}{2}\|\nabla e_\lambda\|_2^2 + \frac{1}{2}\lambda \left(\|e_\lambda\|_2^2 + \int_\Omega H_0(e_\lambda(x))dx\right) \\ &= \frac{1}{2} c_\lambda^2 \lambda^{\frac{2-N}{2}} (\|\nabla V_\lambda\|_2^2 + (1+o(1))\|V_\lambda\|_2^2) = \frac{1}{2} c_\lambda^2 \lambda^{\frac{2-N}{2}} (\|\nabla w\|_2^2 + (1+o(1))\|w\|_2^2) \\ &= \frac{1}{2} c_\lambda^2 \lambda^{\frac{2-N}{2}} (1 + o(1))\|w\|_{p+1}^{p+1} . \end{aligned}$$

This implies (4.22).

STEP 3. By using the calculation to obtain (4.23) just above, we also obtain

$$\int_\Omega H_1(e_\lambda(x))dx = o(1)\|e_\lambda\|_{p+1}^{p+1} = o(1) c_\lambda^{p+1} \lambda^{-N/2} \|V_\lambda\|_{p+1}^{p+1} . \tag{4.27}$$

By (2.1) we have $\Phi(u_{1,\lambda,\alpha}) \ge \Phi(e_\lambda)$, namely,

$$\frac{1}{p+1}\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} + \int_\Omega H_1(u_{1,\lambda,\alpha}(x))dx \ge \frac{1}{p+1}\|e_\lambda\|_{p+1}^{p+1} + \int_\Omega H_1(e_\lambda(x))dx .$$

This along with (4.17), (4.20) and (4.27) yields

$$\begin{aligned}(1+o(1))\xi_{\lambda,\alpha}^{p+1}\lambda^{-N/2}\|w_{1,\lambda,\alpha}\|_{p+1}^{p+1} &= (1+o(1))\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1} \geq (1+o(1))\|e_\lambda\|_{p+1}^{p+1} \\ &= (1+o(1))c_\lambda^{p+1}\lambda^{-N/2}\|V_\lambda\|_{p+1}^{p+1} \\ &= (1+o(1))c_\lambda^{p+1}\lambda^{-N/2}\|w\|_{p+1}^{p+1}.\end{aligned}$$

This along with (4.22) implies that

$$\begin{aligned}&\|w_{1,\lambda,\alpha}\|_{p+1}^{p+1} \\ &\geq (1+o(1))(2\alpha)^{\frac{p+1}{2}}\lambda^{-\frac{(p+1)(N+2-p(N-2))}{4(p-1)}}\mu_1(\lambda,\alpha)^{\frac{p+1}{p-1}}\|w\|_{p+1}^{-\frac{(p+1)(p-1)}{2}}.\end{aligned} \tag{4.28}$$

Finally, by Lemma 3.4, (4.16) and (4.17), we obtain

$$\begin{aligned}&\lambda\{(g(u_{1,\lambda,\alpha}),u_{1,\lambda,\alpha}) - 2\Psi(u_{1,\lambda,\alpha})\} \\ &= \lambda\left\{\int_\Omega h_0(u_{1,\lambda,\alpha}(x))u_{1,\lambda,\alpha}(x)dx - 2\int_\Omega H_0(u_{1,\lambda,\alpha}(x))dx\right\} \\ &= \xi_{1,\lambda,\alpha}\lambda^{\frac{2-N}{2}}\int_{\Omega_\lambda} h_0(\xi_{1,\lambda,\alpha}w_{1,\lambda,\alpha}(y))w_{1,\lambda,\alpha}(y)dy \\ &\quad - 2\lambda^{\frac{2-N}{2}}\int_{\Omega_\lambda} H_0(\xi_{1,\lambda,\alpha}w_{1,\lambda,\alpha}(y))dy \\ &= o(1)\xi_{1,\lambda,\alpha}^2\lambda^{\frac{2-N}{2}} = o(1)\mu_1(\lambda,\alpha)^{-\frac{2}{p-1}}\lambda^{\frac{N+2-p(N-2)}{2(p-1)}} = o(1)\alpha.\end{aligned}$$

This along with (2.3) and (4.16) yields

$$\mu_1(\lambda,\alpha) = \frac{2(1+o(1))\alpha}{(1+o(1))\|u_{1,\lambda,\alpha}\|_{p+1}^{p+1}} = \frac{2(1+o(1))\alpha}{(1+o(1))\xi_{\lambda,\alpha}^{p+1}\lambda^{-N/2}\|w_{1,\lambda,\alpha}\|_{p+1}^{p+1}}.$$

This implies

$$\mu_1(\lambda,\alpha)^{\frac{2}{p-1}} = \frac{(1+o(1))\lambda^{\frac{N+2-p(N-2)}{2(p-1)}}\|w_{1,\lambda,\alpha}\|_{p+1}^{p+1}}{2(1+o(1))\alpha}. \tag{4.29}$$

By substituting (4.29) into (4.28), we obtain

$$\|w\|_{p+1}^{\frac{(p+1)(p-1)}{2}} \geq (1-o(1))\|w_{1,\lambda,\alpha}\|_{p+1}^{\frac{(p+1)(p-1)}{2}}.$$

Thus we obtain the last inequality in (4.18). □

LEMMA 4.8. *Assume that* $\{(\lambda, \alpha)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.2). *Then*

$$\|w_{1,\lambda,\alpha}\|_2 \to \|w\|_2, \quad \|\nabla w_{1,\lambda,\alpha}\|_2 \to \|\nabla w\|_2 . \tag{4.30}$$

PROOF. Assume that $\|\nabla w\|_2 \neq \lim \|\nabla w_{1,\lambda,\alpha}\|_2$. Then we see by (4.14) that there exists a constant $\delta_1 > 0$ and a subsequence of $\{(\lambda, \alpha)\}$ such that

$$\|\nabla w\|_2 + \delta_1 < \|\nabla w_{1,\lambda,\alpha}\|_2 . \tag{4.31}$$

Multiplying (4.2) by $w_{1,\lambda,\alpha}$ and using Lemma 4.6, we obtain by integration by parts that

$$\begin{aligned} \|\nabla w_{1,\lambda,\alpha}\|_2^2 + \|w_{1,\lambda,\alpha}\|_2^2 &= \|w_{1,\lambda,\alpha}\|_{p+1}^{p+1} \\ &\quad + \xi_{1,\lambda,\alpha}^{-p} \int_{\Omega_\lambda} h_1(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) w_{1,\lambda,\alpha}(y) dy \\ &\quad - \xi_{1,\lambda,\alpha}^{-1} \int_{\Omega_\lambda} h_0(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) w_{1,\lambda,\alpha}(y) dy \\ &= \|w_{1,\lambda,\alpha}\|_{p+1}^{p+1} + o(1) . \end{aligned} \tag{4.32}$$

Then by (4.31) and (4.32),

$$\|\nabla w\|_2^2 + \delta_1 + \|w_{1,\lambda,\alpha}\|_2^2 < \|\nabla w_{1,\lambda,\alpha}\|_2^2 + \|w_{1,\lambda,\alpha}\|_2^2 = \|w_{1,\lambda,\alpha}\|_{p+1}^{p+1} + o(1) . \tag{4.33}$$

By taking liminf in (4.33), by (4.15), (4.19) and Lemma 4.7 we obtain

$$\begin{aligned} \|w\|_{p+1}^{p+1} + \delta_1 = \|\nabla w\|_2^2 + \delta_1 + \|w\|_2^2 &\le \|\nabla w\|_2^2 \\ &+ \delta_1 + \liminf \|w_{1,\lambda,\alpha}\|_2^2 \le \|w\|_{p+1}^{p+1} . \end{aligned} \tag{4.34}$$

This is a contradiction. Thus, $\|\nabla w\|_2 = \lim \|\nabla w_{1,\lambda,\alpha}\|_2$. By the same arguments as above, we also obtain that $\|w\|_2 = \lim \|w_{1,\lambda,\alpha}\|_2$. Thus the proof is complete. □

Now we are ready to prove Theorem 2.1.

PROOF OF THEOREM 2.1. By Lemma 4.6 and Lemma 4.8, we obtain

$$\begin{aligned} \Psi(u_{1,\lambda,\alpha}) &= \frac{1}{2}\|u_{1,\lambda,\alpha}\|_2^2 + \int_\Omega H_0(u_{1,\lambda,\alpha}(x)) dx \\ &= \frac{1}{2}\lambda^{-N/2}\xi_{1,\lambda,\alpha}^2 \|w_{1,\lambda,\alpha}\|_2^2 + \lambda^{-N/2} \int_{\Omega_\lambda} H_0(\xi_{1,\lambda,\alpha} w_{1,\lambda,\alpha}(y)) dy \\ &= \frac{1}{2}\lambda^{-N/2}\xi_{1,\lambda,\alpha}^2 (\|w\|_2^2 + o(1)). \end{aligned} \tag{4.35}$$

Then by (4.19) and (4.35)

$$\begin{aligned}\alpha = \Lambda_\lambda(u_{1,\lambda,\alpha}) &= \frac{1}{2}\|\nabla u_{1,\lambda,\alpha}\|_2^2 + \lambda\Psi(u_{1,\lambda,\alpha})\\ &= \frac{1}{2}\xi_{1,\lambda,\alpha}^2\lambda^{(2-N)/2}\left((1+o(1))\|\nabla w\|_2^2 + (1+o(1))\|w\|_2^2\right)\\ &= \frac{1}{2}(1+o(1))\xi_{1,\lambda,\alpha}^2\lambda^{(2-N)/2}\|w\|_{p+1}^{p+1};\end{aligned}$$

this implies

$$\mu_1(\lambda,\alpha)^{-\frac{2}{p-1}}\lambda^{\frac{N+2-p(N-2)}{2(p-1)}} = \frac{2\alpha}{(1+o(1))\|w\|_{p+1}^{p+1}}. \tag{4.36}$$

Now, Theorem 2.1 is a direct consequence of (4.36). For the case where $K_0 \neq 1$, $K_1 \neq 1$, we have only to replace $\lambda, \mu_1(\lambda,\alpha)$ by $K_0\lambda, K_1\mu_1(\lambda,\alpha)$, respectively. □

5. – Proof of Theorem 2.3

The proof of Theorem 2.3 is a variant of that of Theorem 2.1.

LEMMA 5.1. *Assume that* $\{(\lambda,\beta)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.4). *Then*

$$\mu_2(\lambda,\beta) \geq C\beta^{-\frac{p-1}{p+1}}\lambda^{\frac{N+2-p(N-2)}{2(p+1)}}. \tag{5.1}$$

PROOF. For $u \in M_\beta$, by (3.4) and (3.7) we obtain

$$\begin{aligned}\lambda^{\frac{N+2-p(N-2)}{2(p+1)}}\beta^{\frac{2}{p+1}} &= \lambda^{\frac{N+2-p(N-2)}{2(p+1)}}\Phi(u)^{\frac{2}{p+1}}\\ &\leq C\lambda^{\frac{N+2-p(N-2)}{2(p+1)}}(\|u\|_{p+1}^{p+1} + \|u\|_{q+1}^{q+1})^{\frac{2}{p+1}}\\ &\leq C\lambda^{\frac{N+2-p(N-2)}{2(p+1)}}\Big(\|\nabla u\|_2^{\frac{N(p-1)}{p+1}}\|u\|_2^{\frac{N+2-p(N-2)}{p+1}}\\ &\qquad + \|\nabla u\|_2^{\frac{N(q-1)}{p+1}}\|u\|_2^{\frac{N+2-q(N-2)}{p+1}}\Big).\end{aligned} \tag{5.2}$$

By Young's inequality,

$$\begin{aligned}\lambda^{\frac{N+2-p(N-2)}{2(p+1)}}\|\nabla u\|_2^{\frac{N(p-1)}{p+1}}\|u\|_2^{\frac{N+2-p(N-2)}{p+1}} &\leq C(\|\nabla u\|_2^2 + \lambda\|u\|_2^2),\\ \lambda^{\frac{N+2-p(N-2)}{2(p+1)}}\|\nabla u\|_2^{\frac{N(q-1)}{p+1}}\|u\|_2^{\frac{N+2-q(N-2)}{p+1}} &\leq C\lambda^{\frac{(q-p)(N-2)}{2(p+1)}}(\|\nabla u\|_2^2+\lambda\|u\|_2^2)^{\frac{q+1}{p+1}}\\ &\leq C\{\lambda^{-\frac{N-2}{2}} + (\|\nabla u\|_2^2 + \lambda\|u\|_2^2)\}.\end{aligned} \tag{5.3}$$

By (B.1) and (B.4),

$$\frac{\lambda^{-\frac{N-2}{2}}}{\beta^{\frac{2}{p+1}}\lambda^{\frac{N+2-p(N-2)}{2(p+1)}}} = \frac{1}{(\beta^2\lambda^N)^{1/(p+1)}} \to 0. \tag{5.4}$$

Thus by (5.2)-(5.4),

$$\beta^{\frac{2}{p+1}}\lambda^{\frac{N+2-p(N-2)}{2(p+1)}} \le C(\|\nabla u\|_2^2 + \lambda\|u\|_2^2). \tag{5.5}$$

Now, by (2.4), (3.3)-(3.7) and (5.5) we have

$$\begin{aligned}\mu_2(\lambda,\beta) &\ge C\frac{\|\nabla u_{2,\lambda,\alpha}\|_2^2 + \lambda\|u_{2,\lambda,\beta}\|_2^2}{\Phi(u_{2,\lambda,\beta})} \ge C\frac{\beta^{\frac{2}{p+1}}\lambda^{\frac{N+2-p(N-2)}{2(p+1)}}}{\beta}\\ &\ge C\beta^{-\frac{p-1}{p+1}}\lambda^{\frac{N+2-p(N-2)}{2(p+1)}}.\end{aligned}$$

□

LEMMA 5.2. *Assume that* $\{(\lambda,\beta)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.4). *Let* $w_{\sqrt{\lambda}r_0}$ *be the solution of* (3.9) *for* $\tau = \sqrt{\lambda}r_0$, *where* $0 < r_0 \ll 1$ *is a constant. Put*

$$V_{\lambda,\beta}(|x|) := \begin{cases} c_{\lambda,\beta}\beta^{\frac{1}{p+1}}\lambda^{\frac{N}{2(p+1)}} w_{\sqrt{\lambda}r_0}(\sqrt{\lambda}|x|), & x \in B_{r_0} := \{x \in \mathbb{R}^N : |x| < r_0\} \subset \Omega,\\ 0, & x \in \Omega \setminus B_{r_0},\end{cases}$$

where $c_{\lambda,\beta} := \min\{c > 0 : c\beta^{1/(p+1)}\lambda^{N/2(p+1)}w_{\sqrt{\lambda}r_0}(\sqrt{\lambda}|x|) \in M_\beta\}$. *Then* $C \le c_{\lambda,\beta} \le C^{-1}$.

PROOF. The existence of $c_{\lambda,\beta} > 0$ follows from the fact that $\Phi(0) = 0$ and $\Phi(tU_{\lambda,\beta}) \to \infty$ as $t \to \infty$ for a fixed $(\lambda,\beta) \in \mathbb{R}_+^2$. Then by direct calculations,

$$\begin{aligned}\|V_{\lambda,\beta}\|_{p+1}^{p+1} &= c_{\lambda,\beta}^{p+1}\beta\|w_{\sqrt{\lambda}r_0}\|_{p+1}^{p+1},\\ \|V_{\lambda,\beta}\|_{q+1}^{q+1} &= c_{\lambda,\beta}^{q+1}\beta(\beta^2\lambda^N)^{\frac{q-p}{2(p+1)}}\|w_{\sqrt{\lambda}r_0}\|_{q+1}^{q+1}.\end{aligned} \tag{5.6}$$

Then by (3.4) and (5.6),

$$\begin{aligned}&C\beta\left(c_{\lambda,\beta}^{p+1}\|w_{\sqrt{\lambda}r_0}\|_{p+1}^{p+1} + c_{\lambda,\beta}^{q+1}(\beta^2\lambda^N)^{\frac{q-p}{2(p+1)}}\|w_{\sqrt{\lambda}r_0}\|_{q+1}^{q+1}\right)\\ &\le \Phi(V_{\lambda,\beta}) = \beta \le C^{-1}\beta\left(c_{\lambda,\beta}^{p+1}\|w_{\sqrt{\lambda}r_0}\|_{p+1}^{p+1} + c_{\lambda,\beta}^{q+1}(\beta^2\lambda^N)^{\frac{q-p}{2(p+1)}}\|w_{\sqrt{\lambda}r_0}\|_{q+1}^{q+1}\right).\end{aligned}$$

This along with (B.4) and Lemma 3.2 implies our conclusion. □

LEMMA 5.3. *Assume that* $\{(\lambda, \beta)\} \subset \mathbb{R}_+^2$ *satisfies* (B.1) *and* (B.4). *Then*

$$\mu_2(\lambda, \beta) \le C\beta^{-\frac{p-1}{p+1}} \lambda^{\frac{N+2-p(N-2)}{2(p+1)}}. \tag{5.7}$$

PROOF. By direct calculation we have

$$\|\nabla V_{\lambda,\beta}\|_2^2 = c_{\lambda,\beta}^2 \|\nabla w_{\sqrt{\lambda} r_0}\|_2^2 \beta^{\frac{2}{p+1}} \lambda^{\frac{N+2-p(N-2)}{2(p+1)}},$$
$$\lambda \|V_{\lambda,\beta}\|_2^2 = c_{\lambda,\beta}^2 \|w_{\sqrt{\lambda} r_0}\|_2^2 \beta^{\frac{2}{p+1}} \lambda^{\frac{N+2-p(N-2)}{2(p+1)}};$$

this along with (2.2), (3.5), (3.6) and Lemma 5.2 implies that

$$\Lambda_\lambda(u_{2,\lambda,\beta}) \le \Lambda_\lambda(V_{\lambda,\beta}) \le C(\|\nabla V_{\lambda,\beta}\|_2^2 + \lambda\|V_{\lambda,\beta}\|_2^2) \le C\beta^{\frac{2}{p+1}} \lambda^{\frac{N+2-p(N-2)}{2(p+1)}}. \tag{5.8}$$

Hence by (2.4), (3.3)-(3.6) and (5.8), we obtain

$$\begin{aligned}\mu_2(\lambda, \beta) &= \frac{\|\nabla u_{2,\lambda,\beta}\|_2^2 + \lambda(g(u_{2,\lambda,\beta}), u_{2,\lambda,\beta})}{(f(u_{2,\lambda,\beta}), u_{2,\lambda,\beta})} \le C \frac{\Lambda_\lambda(u_{2,\lambda,\beta})}{\Phi(u_{2,\lambda,\beta})} \\ &\le C \frac{\Lambda_\lambda(V_{\lambda,\beta})}{\beta} \le C\beta^{\frac{-(p-1)}{p+1}} \lambda^{\frac{N+2-p(N-2)}{2(p+1)}}.\end{aligned}$$

□

Now we are ready to prove Theorem 2.3.

PROOF OF THEOREM 2.3. We define $\xi_{2,\lambda,\beta}, w_{2,\lambda,\beta}$ by the same manner as those in Section 4 (replacing α and the subscript 1 by β and 2, respectively). Then by (B.4) and Lemma 5.2,

$$\xi_{2,\lambda,\beta}^{p-1} = \frac{\lambda}{\mu_2(\lambda, \beta)} \ge C(\beta^2 \lambda^N)^{\frac{p-1}{2(p+1)}} \to \infty. \tag{5.9}$$

By the same arguments as those used in Section 4, the lemmas in Section 4 are valid in this case, too. Hence,

$$\begin{aligned}\Phi(u_{2,\lambda,\beta}) = \beta &= \frac{1}{p+1}(1 + o(1))\|u_{2,\lambda,\beta}\|_{p+1}^{p+1}, \\ (f(u_{2,\lambda,\beta}), u_{2,\lambda,\beta}) &= (1 + o(1))\|u_{2,\lambda,\beta}\|_{p+1}^{p+1} = (1 + o(1))(p + 1)\beta.\end{aligned} \tag{5.10}$$

Then by (2.4), Lemma 4.6-Lemma 4.8, and (5.10),

$$\mu_2(\lambda, \beta) = \frac{\|\nabla u_{2,\lambda,\beta}\|_2^2 + \lambda(g(u_{2,\lambda,\beta}), u_{2,\lambda,\beta})}{(f(u_{2,\lambda,\beta}), u_{2,\lambda,\beta})} = \frac{\lambda^{\frac{2-N}{2}} \xi_{2,\lambda,\beta}^2 (\|\nabla w\|_2^2 + \|w\|_2^2 + o(1))}{(p+1)(1+o(1))\beta}.$$

this along with (4.19) implies

$$\mu_2(\lambda,\beta)^{\frac{p+1}{p-1}} = \frac{1}{p+1}(1+o(1))\|w\|_{p+1}^{p+1}\beta^{-1}\lambda^{\frac{N+2-p(N-2)}{2(p-1)}}. \tag{5.11}$$

Now, we get Theorem 2.3 by (5.11). For the general case $K_0 \neq 1$, $K_1 \neq 1$, we have only to replace $\beta, \lambda, \mu_2(\lambda,\beta)$ by $K_1^{-1}\beta, K_0\lambda, K_1\mu_2(\lambda,\beta)$, respectively. □

REFERENCES

[1] C. BANDLE, "Isoperimetric Inequalities and Applications", Pitman, Boston,1980.

[2] H. BERESTYCKI – P. L. LIONS, *Nonlinear scalar field equations, I existence of a ground state*, Arch. Rat. Mech. Anal. **82** (1983), 313-345.

[3] P. BINDING – P. J. BROWNE, *Asymptotics of eigencurves for second order ordinary differential equations, I*, J. Differential Equations **88** (1990), 30-45.

[4] P. BINDING – P. J. BROWNE, *Asymptotics of eigencurves for second order ordinary differential equations, II*, J. Differential Equations **89** (1991), 224-243.

[5] P. BINDING – H. VOLKMER, *Eigencurves for two-parameter Sturm-Liouville equations*, SIAM Rev. **38** (1996), 27-48.

[6] P. J. BROWNE – B. D. SLEEMAN, *Nonlinear multiparameter Sturm-Liouville problems*, J. Differential Equations **34** (1979), 139-146.

[7] R. CHIAPPINELLI, *Remarks on bifurcation for elliptic operators with odd nonlinearity*, Israel J. Math. **65** (1989), 285-292.

[8] E. N. DANCER, *On the uniqueness of the positive solution of a singularly perturbed problem*, Rocky Mountain J. Math. **25** (1995), 957-975.

[9] M. FAIERMAN, "Two-parameter eigenvalue problems in ordinary differential equations", Longman House, Essex, UK, 1991.

[10] B. GIDAS – W. M. NI – L. NIRENBERG, *Symmetry and related properties via the maximum principle*, Commn. Math. Phys. **68** (1979), 209-243.

[11] D. GILBERG – N. S. TRUDINGER, "Elliptic partial differential equations of second order", Springer, New York, 1983.

[12] J. L. GÓMEZ, *Multiparameter local bifurcation based on the linear part*, J. Math. Anal. Appl. **138** (1989), 358-370.

[13] M. K. KWONG, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$*, Arch. Rat. Mech. Anal. **105** (1989), 243-266.

[14] C. S. LIN – W. M. NI – I. TAKAGI, *Large amplitude stationary solutions to a chemotaxis system*, J. Differential Equations **72** (1988), 1-27.

[15] W. M. NI – I. TAKAGI, *On the shape of least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **44** (1991), 819-851.

[16] W. M. NI – J. WEI, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math. **48** (1995), 731-768.

[17] B. P. RYNNE, *Bifurcation from eigenvalues in nonlinear multiparameter problems*, Nonlinear Anal. **15** (1990), 185-198.

[18] B. P. RYNNE, *Uniform convergence of multiparameter eigenfunction expansions*, J. Math. Anal. Appl. **147** (1990), 340-350.

[19] T. SHIBATA, *Asymptotic behavior of eigenvalues of two-parameter nonlinear Sturm-Liouville problems*, J. Analyse Math. **66** (1995), 277-294.

[20] T. SHIBATA, *Spectral asymptotics of nonlinear elliptic multiparameter problems*, (preprint).

[21] E. ZEIDLER, *Ljusternik-Schnirelman theory on general level sets*, Math. Nachr. **129** (1986), 235-259.

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