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Remarks on the Maximum Principle for Nonlinear Elliptic PDEs with Quadratic Growth Conditions

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Abstract. In this article, we prove that the maximum principle holds for nonlinear second-order elliptic equations with quadratic growth conditions under general asumptions. We extend the results recently obtained by Fr. Murat and the first author by allowing a more general dependence in x in the growth condition, namely an L^N-dependence instead of a L^∞ one. Our framework is close to the recent existence results of Fr. Murat and V. Ferone and we provide the uniqueness of their solutions under slightly less general conditions. Our proofs consist in mixing the classical linear arguments of the weak maximum principle given in the book of Gilbarg and Trudinger with the nonlinear ones of G. Barles and Fr. Murat.

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Introduction

The aim of this work is to study the conditions under which equations like

$$-\mathrm{div}(a(x,u,Du)) + b(x,u,Du) = f \quad \text{in } \Omega \tag{1}$$

satisfy the maximum principle in $H^1(\Omega)$ or in $H^1(\Omega) \cap L^\infty(\Omega)$ where Ω is a bounded domain of $\mathbb{R}^N$. Here a and b are Caratheodory functions satisfying suitable ellipticity and growth conditions while f belongs a priori to $H^{-1}(\Omega)$. By maximum principle, we mean the following type of property

$$\begin{cases} \text{if } u_1, u_2 \in H^1(\Omega) \text{ or } H^1(\Omega) \cap L^\infty(\Omega) \text{ are respectively sub- and} \\ \text{supersolutions of (1), then} \\ \qquad u_1 \le u_2 \text{ on } \partial\Omega \Longrightarrow u_1 \le u_2 \text{ in } \Omega. \end{cases} \tag{2}$$

We recall here that the precise meaning of "$u_1 \le u_2$ on $\partial\Omega$" is $(u_1 - u_2)^+ \in H_0^1(\Omega)$ while the definition of sub- and supersolutions of (1) will be given

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in Section 1. It is a classical (and easy) remark that the maximum principle implies the uniqueness of the solution of (1) in $H_0^1(\Omega)$ or in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

We assume throughout this article that $N \geq 3$ since the main difficulties occur in this case. At the end of Section 1, we show how the statements of our results have to be modified in order to be valid in dimensions $N = 1$ or $N = 2$.

Roughly speaking, there are two main different types of approaches for proving properties like (2) in the literature. The first one (cf. D. Gilbarg and N.S. Trudinger [8], Theorem 8.1, Chapter 8) uses in an essential way the Sobolev embedding of $H^1(\Omega)$ into $L^q(\Omega)$ with $q = 2N/(N-2)$ and provides for linear equations (and in particular for the linearized equation coming from (1)) results with rather general assumptions on the coefficients.

But despite of this generality, this approach was unable to take in account problems with quadratic growth conditions since, in this case, the coefficients of the linearized equations are not in the right L^p – spaces. To solve this difficulty, F. Murat and the first author [1] used another approach which, in some sense, is more elementary since the conclusion is obtained through a classical eigenvalue argument.

In fact, the main idea in [1] consists in first finding a structure condition on a and b ensuring that a general equation like (1) satisfies the maximum principle and then, when considering a particular equation, to find a change of variable which allows to come down to an equation like (1) which satisfies such a structure condition. Of course this is possible only if the equation at hand has suitable properties but it is worth mentioning that this strategy allows to obtain rather general results, under natural conditions, for equations with quadratic growth conditions.

In this second approach, the coefficients of the linearized equation are not necessarely in the right L^p – spaces to apply the result of [8] but the idea is that the “large good terms” are able to compensate the “too large bad terms”. However this second approach is not more general than the first one since for other types of problems the first approach appears as being more efficient.

The main contribution of this article is to match together these two approaches and therefore to provide uniqueness and maximum principle type results for equations with quadratic growth conditions which are more general than the ones given in [1] and in [8].

In order to be more specific on our results, let us consider as an example the equation

$$-\Delta u + H(x, u, Du) = f \quad \text{in } \Omega, \tag{3}$$

where H is a Caratheodory function in $\Omega \times \mathbb{R} \times \mathbb{R}^N$ and $f \in L^{N/2}(\Omega)$.

We are able to prove that the maximum principle holds for (3) in $H^1(\Omega) \cap L^\infty(\Omega)$ if the function $H(x, u, p)$ is locally lipschitz in u and p for almost all $x \in \Omega$ and if we have, on one hand,

$$\left| \frac{\partial H}{\partial p}(x, u, p) \right| \leq C_0(|u|)(|p| + \overline{b}_1(x)) \quad \text{a.e. } x \in \Omega, u \in \mathbb{R}, p \in \mathbb{R}^N, \tag{4}$$

and

$$(5) \qquad |H(x,u,0)| \le C_1(|u|)\overline{b}_2(x) \quad \text{a.e. } x \in \Omega, u \in \mathbb{R},$$

where C_0 and C_1 are continuous functions of $|u|$, $\overline{b}_1 \in L^N(\Omega)$ and $\overline{b}_2 \in L^{N/2}(\Omega)$; the assumptions (4) and (5) have to be considered as the quadratic growth conditions on H.

On an other hand, we impose

$$(6) \qquad \frac{\partial H}{\partial u}(x,u,p) \ge \alpha_0 > 0 \quad \text{a.e. } x \in \Omega, u \in \mathbb{R}, p \in \mathbb{R}^N.$$

In [1], the analogous result was proved under the assumption that $\overline{b}_1, \overline{b}_2, f \in L^\infty(\Omega)$.

This type of examples are refered in [1] as being the *bounded case* since the solution belongs here to $H^1(\Omega) \cap L^\infty(\Omega)$. For nonlinear elliptic pdes with quadratic growth, existence results have been first proved by L. Boccardo, F. Murat & J.P Puel [3], [4] for the model case where H is a Caratheodory function which satisfies

$$H(x,u,p) = \alpha_0 u + \overline{H}(x,u,p) \text{ with } \alpha_0 > 0, \ |\overline{H}(x,u,p)| \le \overline{C}_0 + \overline{C}_1|p|^2.$$

Then V. Ferone and M.R. Posteraro [7] extended these results by allowing $\overline{C}_0$ to be a L^q – function of x with $q > N/2$. The results proved here completes in fact the existence results of [7].

It is worth noticing that our approach does not seem to provide, in general, far better results than the ones of [1] in the *unbounded case* i.e. when the solutions do not belong necessarily to $L^\infty(\Omega)$. We recall that, in this case, existence results have been first proved by A. Bensoussan, L. Boccardo and F. Murat [2] under some sign-condition on the nonlinearity H and more recently by V. Ferone and F. Murat [5], [6] who obtain the existence of a solution $u \in H_0^1(\Omega)$ such that $\exp(\gamma\,|u|) - 1 \in H_0^1(\Omega)$ for some $\gamma > 0$, when the source term f is "small". Our contribution to this "unbounded case" is at least, in Section 2, a maximum type result for (3) in a framework which is closely related to the one of [5], [6]; indeed the structure conditions on H and on the $L^{N/2}$ – norm of f we need in order to show that the maximum principle hold for equation (3) in the right class of solutions i.e. for solutions $u \in H^1(\Omega)$ such that $\exp(\gamma u) \in H^1(\Omega)$ for some $\gamma > 0$ are essentially the same (despite a bit stronger) as the ones used in [5], [6] for obtaining the existence of such a solution.

In order to prove the different results of this article, we follow the general strategy described in [1] for proving such maximum principle type results. The main new feature of the present work is the way we obtain what we call below the "basic result" i.e. the result we use after the change of variable we perform on the equation we are interested in. By "mixing" ideas coming from the proofs of the related results in [8] and in [1], we improve the general structure conditions under which one can prove that the maximum principle (2) holds for the equation (1).

The paper is organized as follows. The first Section is devoted to the statement and proof of three "basic results" while the second one is devoted to the model equation (3). Finally, in the third Section, we indicate a few extensions of the results of the second Section to quasilinear equations, to obstacle problems and to time-dependent problems.

1. – The basic result and its consequences

In this section, we present a maximum principle type result for linear equations which will be the corner-stone of all the results for quasilinear equations of this article. Then we describe its first consequences.

In the sequel, the space $H_0^1(\Omega)$ is equipped with the norm

$$||u||_{H_0^1(\Omega)} = \left(\int_\Omega |Du|^2\,dx\right)^{1/2},$$

for $u \in H_0^1(\Omega)$, and we denote by $K(N)$ the best constant in the Sobolev's embedding of $H_0^1(\Omega)$ equipped with this norm in $L^q(\Omega)$ with $q = 2N/(N-2)$ (recall that we assume $N \geq 3$) where $L^q(\Omega)$ is the equipped with the usual norm.

Our first (and main) result is the

THEOREM 1.1. *Let $(w_k)_{k\in\mathbb{N}}$ be a sequence of functions of $H^1(\Omega)\cap L^\infty(\Omega)$ such that, for any $k \in \mathbb{N}$, $w_k^+ \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and one has*

$$(7)\quad -\mathrm{div}\left(\alpha^k(x)Dw_k + \beta^k(x)w_k\right) + (\gamma^k(x), Dw_k) + \delta^k(x)w_k \leq 0 \quad \text{in } \mathcal{D}'(\Omega)\,,$$

where the measurable functions α^k, β^k, γ^k and δ^k satisfy the following properties: there exist $n > 0$, $0 < \theta \leq 1$, $\theta_1, \theta_2 > 0$ with $\frac{1}{2}(\theta_1+\theta_2) = \theta$, $0 < \underline{\eta} \leq \overline{\eta}$ and a function $\varsigma \in L^{N/2}(\Omega)$ such that, for all k, one has

(i) *for almost every $x \in \Omega$, $\alpha^k(x) = (\alpha^k_{i,j}(x))_{i,j}$ is a $N \times N$ matrix such that*

$$\underline{\eta}|\xi|^2 \leq \sum_{i,j=1}^N \alpha^k_{i,j}(x)\xi_i\xi_j \leq \overline{\eta}|\xi|^2 \quad a.e.\ x \in \Omega\,,\ \forall \xi \in \mathbb{R}^N\,.$$

(ii) $\delta^k \in L^1(\Omega)$, $\beta^k, \gamma^k \in [L^2(\Omega)]^N$

$$\delta^k(x) - \frac{n}{2\theta_1}(m_k^{-1}\beta^k(x), \beta^k(x)) - \frac{1}{2\theta_2 n}(m_k^{-1}\gamma^k(x), \gamma^k(x)) \geq \varsigma(x) \quad a.e.\ x \in \Omega\,,$$

with $m_k = \frac{1}{2}(\alpha^k(x) + [\alpha^k(x)]^t)$ *where* $[\alpha^k(x)]^t$ *is the adjoint matrix of* $\alpha^k(x)$.

(iii) $\limsup_k ||(\delta^k)^-||_{L^{N/2}(\Omega)} < (1-\theta)\frac{4n}{(n+1)^2}\underline{\eta}[K(N)]^{-2}$.

If $Dw_k^+ \to 0$ *in* $L^2(\Omega)$ *and almost everywhere, then for k large enough, we have*

$$w_k(x) \leq 0 \quad a.e.\ x \in \Omega .$$

It is natural that a result on linear equations plays a central role in the proof of a Maximum Principle type result for nonlinear equations since most of these proofs consists more or less in linearizing the equation. The justification of the rather strange statement of Theorem 1.1 – the formulation with sequences – is given in the proof of Theorem 2.1 below.

REMARK 1.1. Despite of its apparent generality, many variants of this results may be considered: we just want to point out here that one may take $\theta_1 = 0$ if $\beta^k \equiv 0$ (respectively $\theta_2 = 0$ if $\gamma^k \equiv 0$) and (ii) becomes

$$\delta^k(x) - \frac{1}{4\theta n}(m_k^{-1}\gamma^k(x), \gamma^k(x)) \geq \varsigma(x) \quad \text{a.e. } x \in \Omega$$

(respectively

$$\delta^k(x) - \frac{n}{4\theta}(m_k^{-1}\beta^k(x), \beta^k(x)) \geq \varsigma(x) \quad \text{a.e. } x \in \Omega\ ),.$$

To state the results on quasilinear equations, following [1], we first define what are sub- and supersolutions for the equation (1).

DEFINITION 1.1. The function $w \in H^1(\Omega)$ is a subsolution of (1) if

(a) $a(x, w, Dw) \in (L^2(\Omega))^N$,

(b) $b(x, w, Dw) \in L^1(\Omega)$,

(c) $\int_\Omega [a(x, w, Dw)D\psi + b(x, w, Dw)\psi]\,dx \leq \langle f, \psi\rangle$
$\forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \psi \geq 0$ in Ω.

The function $w \in H^1(\Omega)$ is a supersolution of (1) if (a), (b), (c) hold with the opposite inequality in (c).

The main consequence of Theorem 1.1 is the

THEOREM 1.2. *We assume that, for any* $k \in \mathbb{N}$, $u_k, v_k \in H^1(\Omega) \cap L^\infty(\Omega)$ *are respectively sub and supersolution of* (1), *that* $(u_k - v_k)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *and that when* $k \to \infty$, $D(u_k - v_k)^+ \to 0$ *in* $L^2(\Omega)$ *and almost everywhere. We also assume that* $f \in H^{-1}(\Omega)$ *and that a, b are Caratheodory functions satisfying the following properties: for almost all* $x \in \Omega$, $(u, p) \mapsto a(x, u, p), b(x, u, p)$ *are locally Lipschitz functions in* $\mathbb{R} \times \mathbb{R}^N$ *and there exists* $n > 0, 0 < \theta \leq 1, \theta_1, \theta_2 > 0$ *with* $\frac{1}{2}(\theta_1 + \theta_2) = \theta, 0 < \underline{\eta} \leq \overline{\eta}$ *and a function* $\zeta \in L^{N/2}(\Omega)$ *for* $R > 0$ *such that, one has*

1. *The equation is uniformly elliptic i.e., for almost every $x \in \Omega$, $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$, one has*

$$\underline{\eta}|\xi|^2 \leq \sum_{i,j=1}^{N} \frac{\partial a_i}{\partial p_j}(x, u, p)\xi_i\xi_j \leq \overline{\eta}|\xi|^2 \quad \forall \xi \in \mathbb{R}^N .$$

2. *If $R_k := \max(||u_k||_\infty, ||v_k||_\infty)$ and if $K_R^k := \{(u, p) \in \mathbb{R} \times \mathbb{R}^N ; \ |u| \leq R_k, |p| \leq R\}$, then for any $k \in \mathbb{N}$, $R > 0$ and for any $1 \leq i, j \leq N$*

$$\sup_{K_R^k} \left|\frac{\partial a_i}{\partial p_j}(\,.\,, u, p)\right| \in L^\infty(\Omega), \quad \sup_{K_R^k} \left|\frac{\partial a_i}{\partial u}(\,.\,, u, p)\right| \in L^2(\Omega),$$
$$\sup_{K_R^k} \left|\frac{\partial b}{\partial p_j}(\,.\,, u, p)\right| \in L^2(\Omega), \quad \sup_{K_R^k} \left|\frac{\partial b}{\partial u}(\,.\,, u, p)\right| \in L^1(\Omega).$$

3. *For any $k \in \mathbb{N}$ and for almost every $x \in \Omega$, $|u| \leq R_k$ and $p \in \mathbb{R}^N$, one has*

$$\frac{\partial b}{\partial u} - \frac{n}{2\theta_1}\left(m^{-1}\frac{\partial a}{\partial u}, \frac{\partial a}{\partial u}\right) - \frac{1}{2\theta_2 n}\left(m^{-1}\frac{\partial b}{\partial p}, \frac{\partial b}{\partial p}\right) \geq \zeta(x) \quad a.e.\ x \in \Omega ,$$

with $m = \frac{1}{2}(\frac{\partial a}{\partial p} + [\frac{\partial a}{\partial p}]^t)$.

4. *For any $k \in \mathbb{N}$, there exists a function $\delta_2^k \in L^{N/2}(\Omega)$ such that, for almost every $x \in \Omega$, $|u| \leq R_k$ and $p \in \mathbb{R}^N$, one has*

$$\frac{\partial b}{\partial u}(x, u, p) \geq \delta_2^k(x) ,$$

and

$$\limsup_k \left|\left|\left(\delta_2^k\right)^-\right|\right|_{L^{N/2}(\Omega)} < (1-\theta)\frac{4n}{(n+1)^2}\underline{\eta}[K(N)]^{-2} .$$

Then for k large enough, one has

$$u_k(x) \leq v_k(x) \quad a.e.\ x \in \Omega .$$

It is worth mentionning that assumptions 1. and 2. in Theorem 1.2 are basic assumptions on the ellipticity of the equation and on integrability properties for the nonlinearities: these assumptions will be clearly satisfied by all the equations we will consider (before and after changes of variables). The hypothesis 3. and 4. are the structure conditions which are required for having a Maximum Principle type result.

Theorem 1.2 is a generalization of Theorem 1.2 in [1]: the main difference with [1] is that the quantity which appears in the left-hand side of the inequality in 3. does not need to be "essentially positive" since ζ may be as large as we want. Instead of that, the $\frac{\partial b}{\partial u}$ – term has to be "essentially positive" in a way described by assumption 4. involving a measurement in norm $L^{N/2}$ and not in norm L^∞.

As in [1] we will generally use Theorem 1.2 after some change of variables. The interesting feature of this result is that the checking of its assumptions just consists in an estimation of the different derivatives of the nonlinearities a and b.

PROOF OF THEOREM 1.2. We sketch the proof since it is a straightforward consequence of arguments given in [1] and of the proof of Theorem 1.1 below.

The first step consists in following the arguments of the section III.1 of [1] which allow to reduce the proof to the case of non-linearities which are C^1 in u and p for a.e. $x \in \Omega$ and also to perform the computations for a fixed k on the set K_R for some $R > 0$ devoted to tend to $+\infty$. This is where the assumption 2. plays a role since it allows to justify these computations.

Then after these computations, we are (essentially) left with a linearized inequality satisfied by $w_k := u_k - v_k$. Applying the arguments of Theorem 1.1 (more than the result itself), we get an inequality analogous to (13) below but with a right-hand side which is a $o(1)$ as $R \to +\infty$. Keeping k fixed, we let R tend to $+\infty$. And the conclusion follows as in the proof of Theorem 1.1.

We leave the details to the reader. □

Now we turn to the first real maximum principle type result for (1).

THEOREM 1.3. *Assume that a and b satisfy the assumptions of Theorem 1.2 and assume in addition that*

$$(8) \quad a(x,u,p) \text{ does not depend on } u \text{ and } \frac{\partial b}{\partial u} \geq 0 \quad \text{a.e. } x \in \Omega, u \in \mathbb{R}, p \in \mathbb{R}^N,$$

then the Maximum Principle (2) *holds for sub and supersolutions in* $H^1(\Omega) \cap L^\infty(\Omega)$.

This result for subsolutions and supersolutions may be seen as the real analogue of the Theorem 1.2 in [1] but it is not since, on one hand, we have to impose (8) which was not the case in [1] and on the other hand, the role of Theorem 1.3 will not be the same as the one of Theorem 1.2 in [1] i.e. the result to be used after some change of variable. This role will be played by Theorem 1.2.

We have stated here Theorem 1.3 since its short proof that we provide now, justifies at least partially the admittedly strange statements of Theorem 1.1 and 1.2. A more complete justification is given in the next section where we will use Theorem 1.2 to provide results for the model case.

PROOF OF THEOREM 1.3. Let $u_1, u_2 \in H^1(\Omega) \cap L^\infty(\Omega)$ be respectively a sub and a supersolution of (1) such that $u_1 \leq u_2$ on $\partial\Omega$. We argue by contradiction assuming that $M := \left\|(u_1 - u_2)^+\right\|_\infty > 0$.

For $\varepsilon > 0$ small enough, we introduce the functions $u_\varepsilon := u_1 - M + \varepsilon$. Thanks to assumption (8), $u_\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega)$ is still a subsolution of (1). Moreover, if we set $w_\varepsilon = u_\varepsilon - u_2$, the definition of M and Stampacchia's theorem imply that $w_\varepsilon^+ \to 0$ in $H^1(\Omega)$ and we can extract a subsequence denoted by $(w_{\varepsilon_k})_{\varepsilon_k}$ such that $Dw_{\varepsilon_k}^+ \to 0$ a.e. in Ω.

We apply Theorem 1.2 to $w_k := u_{\varepsilon_k} - u_2$. This yields

$$u_{\varepsilon_k}(x) \leq u_2(x) \quad \text{a.e. } x \in \Omega,$$

for k large enough and therefore

$$u_1(x) - u_2(x) \leq M - \varepsilon_k \quad \text{a.e. } x \in \Omega.$$

This property is a contradiction with the definition of M and the proof is complete. □

We conclude this section by the

PROOF OF THEOREM 1.1. We first remark that, according to the assumptions we made on the data α^k, β^k, γ^k and δ^k, all the terms in (7) are actually in $\mathcal{D}'(\Omega)$ and therefore this inequality has a sense. Moreover, one can easily show that it implies the following: for any function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $\varphi \geq 0$ a.e. in Ω, one has

$$\text{(9)} \quad \int_\Omega \Big[(\alpha^k(x)Dw_k, D\varphi) + (\beta^k(x), D\varphi)w_k + (\gamma^k(x), Dw_k)\varphi + \delta^k(x)w_k\varphi\Big]\,dx \leq 0.$$

We consider a C^1–function $S : [0, +\infty[\to \mathbb{R}$ such that $S(0) = 0$ and $S'(t) > 0$ for $t > 0$. Since w_k^+ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, $S(w_k^+) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and therefore it can be used as a test-function in the inequality (9).

Following [1], we will choose $S(t) = t^n$ if the assumptions of Theorem 1.1 hold with $n \geq 1$ and $S(t) = (t^2 + \varepsilon^2)^{n/2} - \varepsilon^n$ if on the contrary $n < 1$. In the sequel, in order to simplify the computations and to point out the main arguments of the proof, we will assume that $n \geq 1$; the other case, a little bit more complicated, consists in performing the same computations with $\varepsilon > 0$ and then to let ε tend to 0.

The choice $\varphi = (w_k^+)^n$ in (9) yields

$$\text{(10)} \quad \begin{aligned} \int_{\{w_k>0\}} \Big[& n w_k^{n-1}(\alpha^k(x)Dw_k, Dw_k) + n w_k^n(\beta^k(x), Dw_k) + \\ & w_k^n(\gamma^k(x), Dw_k) + \delta^k(x) w_k^{n+1}\Big]\,dx \leq 0. \end{aligned}$$

We first follow the strategy of the proof of [1] and we apply Young's inequality to the β^k and γ^k–terms. We obtain

$$\begin{aligned} n w_k^n \left|(\beta^k(x), Dw_k)\right| \leq & \frac{\theta_1}{2} n w_k^{n-1}(\alpha^k Dw_k, Dw_k) \\ & + \frac{n}{2\theta_1}(m_k^{-1}\beta^k(x), \beta^k(x))\mathbb{1}_{\{|Dw_k|>0\}} w_k^{n+1}, \end{aligned}$$

and

$$w_k^n \left|(\gamma^k(x), Dw_k)\right| \le \frac{\theta_2}{2} n w_k^{n-1}(\alpha^k Dw_k, Dw_k) + \frac{1}{2\theta_2 n}(m_k^{-1}\gamma^k(x), \gamma^k(x)) \mathbb{1}_{\{|Dw_k|>0\}} w_k^{n+1}.$$

Plugging these estimates in (10), we get

$$\int_{\{w_k>0\}} \left[(1-\theta) n w_k^{n-1}(\alpha^k(x)Dw_k, Dw_k) + Q^k(x) w_k^{n+1}\right] dx \le 0, \tag{11}$$

where

$$Q^k(x) := \delta^k(x) - \left(\frac{n}{2\theta_1}(m_k^{-1}\beta^k(x), \beta^k(x)) + \frac{1}{2\theta_2 n}(m_k^{-1}\gamma^k(x), \gamma^k(x))\right) \mathbb{1}_{\{|Dw_k|>0\}}.$$

From now on we switch to the strategy of [8]. To do so, we set $\chi_k := (w_k^+)^{(n+1)/2}$ and $\Lambda = (1-\theta)\frac{4n}{(n+1)^2}$. With these new notations and after few elementary computations, (11) becomes

$$\Lambda \int_\Omega (\alpha^k(x) D\chi_k, D\chi_k) dx \le \int_\Omega [Q^k(x)]^- \chi_k^2 dx. \tag{12}$$

To proceed, we use the embedding of $H_0^1(\Omega)$ in $L^q(\Omega)$ which gives for the left-hand side

$$\int_\Omega (\alpha^k(x) D\chi_k, D\chi_k) dx \ge \underline{\eta}\, ||\chi_k||^2_{H_0^1(\Omega)} \ge \underline{\eta}\frac{1}{K(N)^2}\, ||\chi_k||^2_{L^q(\Omega)},$$

and the Hölder inequality for the right-hand side which gives

$$\int_\Omega [Q^k(x)]^- \chi_k^2 dx \le \left|\left|[Q^k(x)]^-\right|\right|_{L^{N/2}(\Omega)} ||\chi_k||^2_{L^q(\Omega)}.$$

Gathering all these informations, (12) reduces to

$$||\chi_k||^2_{L^q(\Omega)} \left(\underline{\eta}\frac{\Lambda}{K(N)^2} - \left|\left|[Q^k(x)]^-\right|\right|_{L^{N/2}(\Omega)}\right) \le 0. \tag{13}$$

Now we examine the term $||[Q^k(x)]^-||_{L^{N/2}(\Omega)}$. By assumption (ii), we know that

$$\delta_k^-,\ [Q^k(x)]^-(x) \le [\varsigma(x)]^- \quad \text{a.e. } x \in \Omega.$$

Moreover, since $Dw_k^+ \to 0$ a.e., we have $\mathbb{1}_{\{|D\chi_k|>0\}} \to 0$ a.e. and therefore

$$[Q^k(x)]^- - \delta_k^-(x) \to 0 \quad \text{a.e. } x \in \Omega.$$

By using the Lebesgue's dominated convergence theorem, we deduce from these properties and from (iii) that

$$\limsup_k \left|\left|[Q^k(x)]^-\right|\right|_{L^{N/2}(\Omega)} \leq \eta \frac{\Lambda}{K(N)^2}\,.$$

But using this in (13) implies that, for k large enough, $\chi_k \equiv 0$ and therefore that $w_k^+ \equiv 0$ and the proof is complete. □

To conclude this section, we turn to the discussion of the extensions of the results of this article to the cases when $N = 1$ and $N = 2$.

If $N = 1$ then $H_0^1(\Omega) \subset C^{0,1/2}(\Omega) \subset L^\infty(\Omega)$. It is clear that, in this case, the role played above and below by the space $L^{N/2}(\Omega)$ is played by $L^1(\Omega)$ and in the assumptions we use below the space $L^N(\Omega)$ has to be replaced by $L^2(\Omega)$. It is then easy to see that the growth conditions we impose on the nonlinearities are not restrictive since they coincide with the natural integrability conditions which are required for the equation to be set properly.

If $N = 2$ then $H_0^1(\Omega) \subset L^q(\Omega)$ for all $q \in [1, \infty)$. In the formulation of our results, the space $L^{N/2}(\Omega)$ can be replaced by $L^{1+\varepsilon}(\Omega)$ for any $\varepsilon > 0$ and the space $L^N(\Omega)$ simultaneously by $L^{2(1+\varepsilon)}(\Omega)$.

REMARK 1.2. For $N \geq 3$,the space $L^{N/2}(\Omega)$ appears naturally above in the Hölder-Sobolev's inequality

$$\int_\Omega f w^2 dx \leq [K(N)]^2\, ||f||_{L^{N/2}(\Omega)} \int_\Omega |Dw|^2\, dx\,.$$

It is well-known because of the injection of $H_0^1(\Omega)$ in the Lorentz space $L^{q,2}(\Omega)$ where $q = 2N/(N-2)$ that an analogous inequality is true when $f \in L^{N/2,\infty}(\Omega)$. It would be natural to think that all the results we prove above (and below) remain valid if the space $L^{N/2}(\Omega)$ is replaced by the space $L^{N/2,\infty}(\Omega)$ with straightforward adaptations of our assumptions. Unfortunately, we are only able to show that this is actually true for the Lorentz spaces $L^{N/2,m}(\Omega)$ for any $m < +\infty$ but not for $m = +\infty$. The problem for the extension to $L^{N/2,\infty}(\Omega)$ is the use of Lebesgue's dominated convergence theorem at the end of the proof of Theorem 1.1; this key argument is untracktable with $L^{N/2,\infty}(\Omega)$ and we were unable to turn around it.

2. – The maximum principle for the model equations

Our result is the following

THEOREM 2.1. *Assume that* (4), (5) *and* (6) *are satisfied with* $\alpha_0 > 0$. *Then* (2) *holds in* $H_0^1(\Omega) \cap L^\infty(\Omega)$. *In particular,* (3) *has at most one solution in* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Theorem 2.1 states the uniqueness of the solution of the equation (0.1) in $H_0^1(\Omega) \cap L^\infty(\Omega)$. It is worth noticing that this uniqueness property does not hold in the larger class $H_0^1(\Omega)$: we refer the reader of the counter-examples given in Kazdan and Kramer [9] (see also [4] or [1] where the counter-example is described).

An assumption of the type (6) with $\alpha_0 > 0$ is necessary for Theorem 2.1 to hold if no additional assumption on the dependence of H in p is made. One can however obtain a analogous result for $\alpha_0 = 0$ if one of the following hypotheses holds

$$\frac{\partial H}{\partial p} p - H \leq \overline{K}_1 |p|^2 \quad \text{a.e. } x \in \Omega, u \in \mathbb{R}, p \in \mathbb{R}^N, \tag{14}$$

for some constants $\overline{K}_1 \geq 0$, or there exist $k \in \mathbb{R}$, $n > 0$, $0 < \theta \leq 2$ and a function $\delta \in L^{N/2}(\Omega)$ such that

$$-k^2|p|^2 + k\left[\frac{\partial H}{\partial p} p - H\right] + \frac{\partial H}{\partial u} - \frac{1}{2\theta n}\left|\frac{\partial H}{\partial p} - 2kp\right|^2 \geq \delta(x), \tag{15}$$

for almost all $x \in \Omega$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$ and with

$$||\delta^-||_{L^{N/2}(\Omega)} < \left(1 - \frac{\theta}{2}\right) \frac{4n}{(n+1)^2} [K(N)]^{-2}.$$

Our result is

THEOREM 2.2. *Assume that* (4), (5) *and* (6) *are satisfied with* $\alpha_0 = 0$. *Then*

(i) *If we assume in addition that* (14) *is satisfied,* (2) *holds for sub- and super-solutions of equation* (3), *which are in* $H^1(\Omega) \cap L^\infty(\Omega)$. *In particular,* (3) *has at most one solution in* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

(ii) *If we assume in addition that* (15) *holds,* (2) *holds for sub- and supersolutions of equation* (3) *such that* $u_1, u_2 \in H^1(\Omega)$ *with* $\exp(nku_1), \exp(nku_2) \in H^1(\Omega)$ *and in particular if* $u_1, u_2 \in H^1(\Omega) \cap L^\infty(\Omega)$. *In particular,* (3) *has at most one solution* u *in* $H_0^1(\Omega)$ *such that* $\exp(nku) - 1 \in H_0^1(\Omega)$.

Before giving the proofs of the two above theorems, let us try to justify assumption (15) by considering the analogue in our context of the example of Kazdan-Kramer [9]; such argument was already used in [1] to justify an analogous hypothesis.

For $C_1 \neq 0$ and $f \in L^{N/2}(\Omega)$, $f \geq 0$ a.e. in Ω (f being not identically 0), we consider the equation

$$-\Delta u - C_1|Du|^2 = f \quad \text{in } \Omega. \tag{16}$$

If we test our condition (15) with $k = -C_1$ and the optimal value $n = 1$, we obtain the condition on f

$$C_1 ||f||_{L^{N/2}} < \frac{1}{K(N)^2} \tag{17}$$

in order to find a nonnegative θ fullfilling the second condition inside (15).

The condition (17) is the precise condition found by V. Ferone and F. Murat in [5], [6] required for this equation to have a positive solution $u \in H_0^1(\Omega)$ such that $e^{C_1 u} - 1 \in H_0^1(\Omega)$.

Now it is worth noticing that one may construct (see again [5], [6]) for any $\varepsilon > 0$, a ball $B(0, R_\varepsilon)$ and a constant C_1 such that:

$$\frac{1}{K(N)^2} < C_1 \, ||f_0||_{L^{N/2}} < \frac{1}{K(N)^2} + \varepsilon$$

with $f_0(x) = \frac{N(N-2)}{(1+|x|^2)^2}$ such that the problem finding a $u \in H_0^1(B(0, R_\varepsilon))$ such that $e^{C_1 u} - 1 \in H_0^1(B(0, R_\varepsilon))$ satisfying (16) with right-hand side f_0 has no solution at all. This recall a Fredlholm-like alternative and is related with an eigenvalue problem allready observed when f is constant in the paper [1] and in Kazdan-Kramer [9]: the change of function $v = e^{C_1 u} - 1$ in (16) leads to the transformed equation

$$-\Delta v = C_1(v+1)f \quad \text{in } \Omega ,$$

and $v \in H_0^1(\Omega)$. Consider the eigenvalue problem

$$\tilde{\lambda}_1 = \min\left\{ \int_\Omega |Dw|^2 \, dx \; ; \; w \in H_0^1(\Omega), \; \int_\Omega f w^2 dx = 1 \right\} .$$

Since $f \in L^{N/2}(\Omega)$ and because of the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ with $q = 2N/(N-2)$, this problem is well-posed, $\tilde{\lambda}_1 > 0$ and there exists a positive function $\tilde{e}_1 \in H_0^1(\Omega)$, such that

$$-\Delta \tilde{e}_1 = \tilde{\lambda}_1 f \tilde{e}_1 \quad \text{in } \Omega .$$

Choosing $\tilde{e}_1$ as test-function in the v equation, we obtain

$$\tilde{\lambda}_1 \int_\Omega f \tilde{e}_1 v dx = C_1 \int_\Omega (v+1) f \tilde{e}_1 dx ,$$

and therefore

$$(\tilde{\lambda}_1 - C_1) \int_\Omega f \tilde{e}_1 v = \int_\Omega f \tilde{e}_1 dx \geq 0 .$$

Since $v \geq 0$ a.e. in Ω, this implies that $\tilde{\lambda}_1 \geq C_1$.

But on an other hand, if $w \in H_0^1(\Omega)$ satisfies $\int_\Omega f w^2 dx = 1$, we have by Hölder inequality and by the Sobolev embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ with $q = 2N/(N-2)$

$$1 = \int_\Omega f w^2 dx \leq ||f||_{L^{N/2}(\Omega)} \, ||w||^2_{L^q(\Omega)} \leq [K(N)]^2 \, ||f||_{L^{N/2}(\Omega)} \int_\Omega |Dw|^2 \, dx .$$

We deduce from that

$$1 \leq [K(N)]^2 \, ||f||_{L^{N/2}(\Omega)} \, \tilde{\lambda}_1 \,,$$

and the condition (17) insures automatically that $\tilde{\lambda}_1 \geq C_1$.

This means that this condition is in some sense sharp for equation (16).

PROOF OF THEOREM 2.1. the idea of the proof follows the one given in [1] and consists essentially in using a change of function $u = \varphi(v)$, where φ is a C^3 function with $\varphi' > 0$ in $\mathbb{R}$ in order to get a transformed equation to which we can apply Theorem 1.2. But, and this is the main difference with [1], the transformed equation which is of the type (1) will not satisfy the assumption (8) of Theorem 1.3 and in particular the term $\frac{\partial b}{\partial u}$ will not be positive.

To take care of this difficulty, we argue in the following way: if $u_1, u_2 \in H^1(\Omega) \cap L^\infty(\Omega)$ are respectively sub and supersolution of (3) such that $u_1 \leq u_2$ on $\partial\Omega$, we argue by contradiction assuming that $M := \left|\left|(u_1 - u_2)^+\right|\right|_\infty > 0$.

By the same arguments as in the proof of Theorem 1.3 and using (6) in an essential way, there exists a sequence $(\varepsilon_k)_k$ of non-negative real numbers, converging to 0 such that $u_1^k := u_1 - M + \varepsilon_k$ is still a subsolutions of (3), $u_1^k \in H^1(\Omega) \cap L^\infty(\Omega)$ and $u_1^k \leq u_2$ on $\partial\Omega$ if k is large enough since $M > 0$. Finally, $(u_1^k - u_2)^+ \to 0$ in $H^1(\Omega)$ and we have $D(u_1^k - u_2)^+ \to 0$ a.e. in Ω.

The new point here is really to perform the change of variable on u_1^k and u_2 and not on u_1 and u_2. If v_1^k and v_2 are define through $u_1^k = \varphi(v_1^k)$ and $u_2 = \varphi(v_2)$ then one checks easily that $(v_1^k - v_2)^+ \to 0$ in $H^1(\Omega)$ and we have $D(v_1^k - v_2)^+ \to 0$ a.e. in Ω.

In order to apply Theorem 1.2, it is therefore enough to check the assumptions on the non-linearities. Here the computations are analogous to the one of [1] but, of course, the requirements on these non-linearities are different.

Using $\frac{y}{\varphi'(v)}$ where $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in the variational formulation of (3) shows that the transformed equation is

$$-\Delta v - \frac{\varphi''(v)}{\varphi'(v)}|Dv|^2 + \frac{1}{\varphi'(v)} H(x, \varphi(v), \varphi'(v)Dv) = 0 \quad \text{in } \Omega \,. \tag{18}$$

Therefore we have

$$a(x, v, \xi) = \xi \quad \text{and} \quad b(x, v, \xi) = -\frac{\varphi''(v)}{\varphi'(v)}|\xi|^2 + \frac{1}{\varphi'(v)} H(x, \varphi(v), \varphi'(v)\xi) \,.$$

Thanks to Remark 1.1, we can take $\theta_1 = 0$. In order to check the assumptions 2., 3., 4. of Theorem 1.2, we have essentially to estimate $\frac{\partial b}{\partial v}(x, v, \xi)$ and $\frac{\partial b}{\partial \xi}(x, v, \xi)$.

As in [1], we use the "old variable" $u = \varphi(v)$ and $p = \varphi'(v)\xi$ to examine these quantities and the change of variable φ defined by

$$\varphi(v) = -\frac{1}{A} \log \left(e^{-KAv} + \frac{1}{K} \right)$$

where we will first fix $A > 0$ and then choose $K > 0$ large enough.

Recall that u_1 and u_2 are assumed to be bounded. We thus only need the range of φ to cover $[-\tilde{M}, +\tilde{M}]$ with $\tilde{M} = 3\max(\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Omega)})$ (the constant "3" is here to take in account the fact that we deal with u_1^k instead of u_1) . This will be the case if K is large enough, and more precisely if $K > e^{\tilde{M}A}$.

If the function ω defined by

$$\omega = \varphi' \circ \varphi^{-1}, \text{ i.e. } \omega(u) = \omega(\varphi(v)) = \varphi'(v), \quad \text{and } p = w(u)\xi\,,$$

one has

$$\frac{\partial b}{\partial v}(x, v, \xi) = \frac{1}{\omega(u)} \left\{ -\omega''(u)|p|^2 + \omega'(u) \left[\frac{\partial H}{\partial p} p - H \right] (x, u, p) \right\} + \frac{\partial H}{\partial u}(x, u, p)\,.$$

An analogous computation yields

$$\frac{\partial b}{\partial \xi}(x, v, \xi) = \frac{\partial H}{\partial p}(x, u, p) - 2\frac{\omega'(u)}{\omega(u)} p\,.$$

Since $u = \varphi(v) = -\frac{1}{A}\log(e^{-KAv} + \frac{1}{K})$ and $\omega(u) = \varphi'(v)$, we have

$$\omega(u) = K - e^{Au}\,.$$

Since we want to compare u_1^k and u_2 which both belong to $L^\infty(\Omega)$ with $\|u_i\|_{L^\infty(\Omega)} \leq \tilde{M}$ for $i = 1, 2$, it is enough to prove the estimates in this range of values. Thanks to this fact, we can replace the functions C_0 and C_1 in (4) and (5) by some constants . We get the estimate

$$\left| \left[\frac{\partial H}{\partial p} p - H \right] (x, u, p) \right| \leq K_2|p|^2 + K_1([\bar{b}_1(x)]^2 + [\bar{b}_2(x)] + |f(x)|)\,,$$

a.e. for $x \in \Omega$, $u \in [-3\tilde{M}, +3\tilde{M}]$ and $p \in \mathbb{R}^N$ for some nonnegative constants K_1, K_2.

Because of (6), this implies

$$\frac{\partial b}{\partial v}(x, v, \xi) \geq \frac{e^{Au}}{K - e^{Au}} \left\{ (A^2 - K_2 A)|p|^2 - AK_1([\bar{b}_1(x)]^2 + [\bar{b}_2(x)] + |f(x)|) \right\} + \alpha_0\,.$$

We choose $A = K_2 + 1$ in this expression. It is then clear that, for K large enough, we have

$$\begin{aligned} \frac{\partial b}{\partial v}(x, v, \xi) &\geq \eta(K)|p|^2 + \alpha_0 - \frac{AK_1 e^{Au}}{K - e^{Au}} \left([\bar{b}_1(x)]^2 + [\bar{b}_2(x)] + |f(x)| \right), \\ &\geq \alpha_0 - \frac{AK_1 e^{Au}}{K - e^{Au}} \left([\bar{b}_1(x)]^2 + [\bar{b}_2(x)] + |f(x)| \right)\,, \end{aligned}$$

where

$$\eta(K) := \frac{1}{2} \min_{[-3\tilde{M},3\tilde{M}]} \frac{Ae^{Au}}{K - e^{Au}} .$$

We notice that as $K \to +\infty$, $\eta(K)$ behaves like $c_1 K^{-1}$.

Then we set, for $x \in \Omega$,

$$\delta_2(x) = \alpha_0 - \ell(K)\chi(x) ,$$

where

$$\ell(K) := \max_{[-3\tilde{M},3\tilde{M}]} \frac{AK_1 e^{Au}}{K - e^{Au}} ,$$

and

$$\chi(x) := \left([\overline{b}_1(x)]^2 + [\overline{b}_2(x)] + |f(x)|\right) .$$

From the above computations, we have

$$\frac{\partial b}{\partial v}(x, v, \xi) \geq \delta_2(x) ,$$

for almost all $x \in \Omega$, $v \in \varphi^{-1}([-3\tilde{M}, 3\tilde{M}])$ and $\xi \in \mathbb{R}^N$. Moreover, because of the assumptions on $\overline{b}_1$, $\overline{b}_2$ and f, $\delta_2 \in L^{N/2}(\Omega)$ for any choice of K large enough.

Now we are going to choose simultaneously K and n in order to have assumptions 3. and 4. of Theorem 1.2 being satisfied. To do so, we remark that

$$||(\delta_2)^-||^{N/2}_{L^{N/2}(\Omega)} \leq [\ell(K)]^{N/2} \int_\Omega 1\!\!1_{\{\chi \geq [\ell(K)]^{-1}\}} \, |\chi(x)|^{N/2} \, dx .$$

But since $\chi \in L^{N/2}(\Omega)$ and since $\ell(K)$ behaves like $c_2 K^{-1}$ for some constant $c_2 > 0$ as $K \to +\infty$, it is clear that the right-hand side of this inequality is a $o(1)K^{-1}$ as $K \to \infty$.

On an other hand, by easy computations, we obtain

$$\left|\frac{\partial b}{\partial \xi}(x, v, \xi)\right|^2 \leq L_1 |p|^2 + L_2 \left|\overline{b}_1(x)\right|^2 ,$$

for some constants L_1, L_2 depending on A and K; but since A has already be chosen, these constants may be considered as independent of K provided it is taken large enough (here the use of $\omega(u)$ and $\omega'(u)$ instead of v is the keystone of the proof).

We recall that we choose $\theta_2 = 1$ which yields $\theta = 1/2$ and thanks to the above estimates we have

$$\frac{\partial b}{\partial v}(x, v, \xi) - \frac{1}{2n}\left|\frac{\partial b}{\partial \xi}(x, v, \xi)\right|^2 \geq \left(\eta(K) - \frac{1}{2n}L_1\right) |p|^2 + \delta_2(x) - \frac{1}{2n} L_2 \left|\overline{b}_1(x)\right|^2 .$$

In order to satisfy the third assumption of Theorem 1.2, we first choose n such that $\eta(K) - \frac{1}{2n}L_1 = 0$ i.e.

$$n := \frac{1}{2\eta(K)}L_1\,,$$

we have

$$\frac{\partial b}{\partial v}(x,v,\xi) - \frac{1}{2n}\left|\frac{\partial b}{\partial \xi}(x,v,\xi)\right|^2 \geq \delta_2(x) - \frac{1}{2n}L_2\left|\overline{b}_1(x)\right|^2\,,$$

and since the right-hand side is in $L^{N/2}(\Omega)$, assumption 3. holds for any choice of the parameter n satisfying the above condition.

It remains to check assumption 4. and this will be done by a suitable choice of K. Indeed, as shown above

$$||(\delta_2)^-||_{L^{N/2}(\Omega)} \leq o(1)K^{-1}\,,$$

and we need the property

$$||(\delta_2)^-||_{L^{N/2}(\Omega)} < \frac{2n}{(n+1)^2}[K(N)]^{-2}\,.$$

But by the choice of n we made above, n behaves like c_3K for some constant $c_3 > 0$ as $K \to \infty$ and therefore $\frac{2n}{(n+1)^2}[K(N)]^{-2}$ behaves like c_4K^{-1} for some constant $c_4 > 0$ as $K \to \infty$.

It is then clear that, for a choice of K large enough, the assumption 4. is fullfilled. Therefore Theorem 1.2 applies and the proof is complete. □

Now we turn to the proof of Theorem 2.2.

In the case where (14) holds, we choose again

$$\varphi(v) = -\frac{1}{A}\log\left(e^{-KAv} + \frac{1}{K}\right),\quad \omega(u) = K - e^{Au}\,,$$

for some $A > 0$ and K large enough to be fixed later. Here we have

$$\frac{\partial b}{\partial v}(x,v,\xi) \geq \frac{e^{Au}}{K - e^{Au}}\left\{\left(A^2 - \overline{K}_1A\right)|p|^2\right\}\,,$$

where $\overline{K}_1 \geq 0$ is given by (14). We choose $A = K_1$ and since with this choice we have

$$\frac{\partial b}{\partial v}(x,v,\xi) \geq 0\,,$$

we conclude easily as in the situation of the proof of Theorem 2.1. .

Consider now the case where (15) holds. If $k = 0$, there is nothing to do since the assumptions of Theorem 1.3 hold.

If $k \neq 0$, we choose $\omega(u) = e^{ku}$. The function φ is nothing but

$$\varphi(v) = -\frac{1}{k}\log(1 - kv) \quad \text{for } v < \frac{1}{k},$$

which is equivalent to $v = e^{ku} - 1$.

According to the computations done in the proof of Theorem 2.1, we have

$$\begin{aligned}\frac{\partial b}{\partial v}(x, v, \xi) - \frac{1}{2\theta n}\left|\frac{\partial b}{\partial \xi}(x, v, \xi)\right|^2 =& -\frac{\omega''(u)}{\omega(u)}|p|^2 \\ &+\frac{\omega'(u)}{\omega(u)}\left[\frac{\partial H}{\partial p}p - H\right](x, u, p) + \frac{\partial H}{\partial u}(x, u, p) \\ &-\frac{1}{2\theta n}\left|\frac{\partial H}{\partial p}(x, u, p) - 2\frac{\omega'(u)}{\omega(u)}p\right|^2.\end{aligned}$$

Since $\omega(u) = e^{ku}$, the right hand side of this equality is nothing but

$$-k^2|p|^2 + k\left[\frac{\partial H}{\partial p}p - H\right] + \frac{\partial H}{\partial u} - \frac{1}{2\theta n}\left|\frac{\partial H}{\partial p} - 2kp\right|^2,$$

which, in view of hypothesis (15), is greater than $\delta(x)$. And one easily completes the proof. □

3. – Extensions

3.1. – Extensions to more general equations

We consider here extensions of the results of Section 2 to more general quasilinear elliptic equations of the form

$$-\text{div}(d(x, Du)) + h(x, u, Du) = 0 \quad \text{in } \Omega,$$

where d_i $(1 \leq i \leq N)$ and h are Caratheodory functions in $\Omega \times \mathbb{R} \times \mathbb{R}^N$ which are locally Lipschitz in (u, p) for almost all x in Ω.

To state our result, we introduce the following assumptions: for almost every $x \in \Omega$, $u \in \mathbb{R}$ and p in $\mathbb{R}^N$, we assume that

$$(20)\qquad \begin{cases} \displaystyle\sum_{i,j=1}^{N} \frac{\partial d_i}{\partial p_j}(x, p)\eta_i \eta_j \geq \gamma|\eta|^2 \quad \forall \eta \in \mathbb{R}^N \quad (\gamma > 0) \\ \displaystyle\left|\frac{\partial d}{\partial p}(x, p)\right| \leq C_0 \quad \text{and} \quad d(x, 0) = 0 \end{cases}$$

and

$$
\left\{
\begin{array}{l}
\dfrac{\partial h}{\partial u}(x,u,p) \geq \gamma_0, \quad (\gamma_0 > 0) \\
\left|\dfrac{\partial h}{\partial p}(x,u,p)\right| \leq C_1(|u|)(|p| + \overline{b}_1(x)) \\
|h(x,u,0)| \leq C_2(|u|)\overline{b}_2(x)
\end{array}
\right. \tag{21}
$$

where C_0 is a constant, C_1, C_2 are continuous functions of $|u|$ and $\overline{b}_1 \in L^N(\Omega)$, $\overline{b}_2 \in L^{N/2}(\Omega)$.

In addition to these "natural" assumptions, we need, as this was the case in [1], the following condition

$$
\left\{
\begin{array}{l}
\text{for each } \varepsilon > 0, \text{ there exists } C(\varepsilon) \text{ such that} \\
\left|\dfrac{\partial d}{\partial p}(x,p)p - d(x,p)\right| \leq \varepsilon |p| + C(\varepsilon) \quad \text{a.e. } x \in \Omega,\, p \in \mathbb{R}^N .
\end{array}
\right. \tag{22}
$$

Our result is the following

THEOREM 3.1. *Assume that* (20), (21) *and* (22) *hold then the maximum principle holds for* (19) *in* $H^1(\Omega) \cap L^\infty(\Omega)$. *In particular,* (19) *has at most one solution in* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

We leave the proof of this result to the reader since it is a routine adaptation of the arguments of the analogous result in [1] and of the proof of Theorem 2.1 above.

3.2. – Some results in the case of non Lipschitz continuous nonlinearities

The aim of this section is to provide several results for equations involving nonlinearities which are not assumed to be Lipschitz continuous. In order to simplify the exposure and to point out the main ideas, we will only focus on the case of the model equation (3).

3.2.1. – The case of uniformly continuous nonlinearities

Again to simplify, we assume that H has the form

$$
H(x,t,p) := \tilde{H}(x,p) + \alpha_0 t \quad \text{a.e. } x \in \Omega,\ u \in \mathbb{R},\ p \in \mathbb{R}^N .
$$

with $\tilde{H}(x,0) = 0$.

We introduce the following assumption which is the analogue of (4).

There exist $m : \mathbb{R}^+ \to \mathbb{R}^+$ with $m(0+) = 0$ and a function $\overline{b}_1 \in L^N(\Omega)$ such that

$$
\left|\tilde{H}(x,p) - \tilde{H}(x,q)\right| \leq m\left(|p-q|\left[|p| + |q| + \overline{b}_1(x)\right]\right) , \tag{23}
$$

for almost every $x \in \Omega$ and every $p, q \in \mathbb{R}^N$.

Our result is the

THEOREM 3.2. *Assume that* $\alpha_0 > 0$, *that* $f \in L^{N/2}(\Omega)$ *and that* (23) *holds then the conclusion of Theorem* 2.1 *remains valid.*

PROOF OF THEOREM 3.2. Assume that $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ are respectively sub and supersolution of (3) and assume that $(u - v)^+ \in H_0^1(\Omega)$.

We introduce the sequence of functions $\tilde{H}_\varepsilon$ which are defined for a.e. $x \in \Omega$ and all $p \in \mathbb{R}^N$ by

$$\tilde{H}_\varepsilon(x, p) := \inf_{q \in \mathbb{R}^N} \left(\tilde{H}(x, q) + \frac{1}{\varepsilon} |p - q| (|p| + |q| + \bar{b}_1(x)) \right) .$$

This regularization procedure is called Inf-convolution (see Lasry and Lions [11]) and because of assumption (23) one can easily prove that the functions $\tilde{H}_\varepsilon$ are Caratheodory functions such that

(i) $\tilde{H}_\varepsilon(x, p) \le \tilde{H}(x, p) \le \tilde{H}_\varepsilon(x, p) + \rho(\varepsilon)$ for almost all $x \in \Omega$ and all $p \in \mathbb{R}^N$ where $\rho(\varepsilon) \to 0$ when $\varepsilon \to 0$.

(ii) $\tilde{H}_\varepsilon$ is Lipschitz continuous in p for almost every $x \in \Omega$ and (4) holds with $C_0(|u|)$ replaced by $C_0^\varepsilon > 0$.

Because of these properties, u is still a subsolution of the equation where $\tilde{H}$ is replaced by $\tilde{H}_\varepsilon$ while $v + \frac{\rho(\varepsilon)}{\alpha_0}$ is a supersolution of the new equation. Moreover because of property (ii), we may apply Theorem 2.1 to the $\tilde{H}_\varepsilon$ – equation and therefore

$$u \le v + \frac{\rho(\varepsilon)}{\alpha_0} \quad \text{a.e. in } \Omega$$

and the conclusion follows by letting ε tend to 0. □

3.2.2. – Two results in the case of non uniformly continuous nonlinearities

In this section we present two results in the case when one part of the nonlinearity is only continuous but with a strict subquadratic behavior. Again in order to simplify the exposure we consider only the case of (3) and we assume that H has the following form

$$H(x, u, p) := H_1(x, u, p) + H_2(x, p) \quad \text{a.e. } x \in \Omega, \ u \in \mathbb{R}, \ p \in \mathbb{R}^N .$$

where H_1, H_2 are Caratheodory functions and where $H_2(x, 0) = 0$ for $x \in \Omega$.

We introduce the following assumptions: we suppose that, for almost all $x \in \Omega$, $(u, p) \mapsto H_1(x, u, p)$ is locally Lipschitz continuous in $\mathbb{R} \times \mathbb{R}^N$ and there exists $\alpha_0 > 0$ such that

$$\frac{\partial H_1}{\partial u}(x, u, p) \ge \alpha_0(1 + |p|^2) \quad \text{a.e. } x \in \Omega, \ u \in \mathbb{R}, \ p \in \mathbb{R}^N . \tag{24}$$

For H_2, we first introduce the *subquadratic* assumption:

$$\frac{H_2(x,p)}{1+|p|^2} \to 0 \quad \text{as} \quad |p| \to +\infty\,, \tag{25}$$

for almost all $x \in \Omega$, uniformly with respect to $x \in \Omega$ and the *continuity* assumption.

For any $R > 0$, there exists a function $m_R : \mathbb{R}^+ \to \mathbb{R}^+$ with $m_R(0+) = 0$ and satisfying $m_R(t+s) \le m_R(t) + m_R(s)$ for any $t, s \ge 0$ such that

$$|H_2(x,p) - H_2(x,q)| \le m_R(|p-q|) \quad \text{a.e. } x \in \Omega\,, \forall\, |p|\,, |q| \le R\,. \tag{26}$$

This assumption means that the function H_2 is continuous in p on each compact subset of $\mathbb{R}^N$ uniformly with respect to $x \in \Omega$.

Our result is the

THEOREM 3.3. *Assume that* $f \in H^{-1}(\Omega)$, *that* H_1 *satisfies* (4), (5), (24) *and that* H_2 *satisfies* (25), (26). *Then the conclusion of Theorem* 2.1 *remains valid for* (3).

Before giving the proof of Theorem 3.3, we remark that the continuity properties we impose on H_2 are rather weak: this assumption holds typically if $H_2(x,p) = F(p) - g(x)$ for any continuous over $\mathbb{R}^N$ function F with subquadratic growth at infinity and any $g \in L^\infty$. This is compensated in the proof below on one hand by the subquadratic assumption on H_2 and on another hand by (24) which is a rather strong requirement. It is worth mentioning that this assumption is exactly the property which is indeed satisfied by the nonlinearity we obtain after the change of variable we make in the proof of Theorem 2.1 but, as in [1], when $\bar{b}_1, \bar{b}_2$ and f are in $L^\infty(\Omega)$. We will come back on this remark later on.

PROOF OF THEOREM 3.3. Assume that $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ are respectively sub and supersolution of (3) and assume that $(u-v)^+ \in H_0^1(\Omega)$. We argue by contradiction, following the proof of [8], by assuming that $M := \|(u-v)^+\|_{L^\infty(\Omega)} > 0$, we choose $0 < M/2 \le k < M$ and we introduce the functions $w_k := (u - v - k)^+$.

Subtracting the inequalities satisfied respectively by u and v and multiplying by w_k^n where n is a large integer, we obtain by linearizing equation (3)

$$\begin{aligned} \int_\Omega n w_k^{n-1} |Dw_k|^2 dx + \int_\Omega w_k^n(\beta(x), Dw_k) dx + \int_\Omega \alpha(x) w_k^n (u-v) dx \\ + \int_\Omega (H_2(x,Du) - H_2(x,Dv))\, w_k^n dx \le 0\,, \end{aligned} \tag{27}$$

where, by assumptions (4) and (24) on H_1, the functions α and β satisfy, for some constants $\eta > 0$ and $C > 0$

$$\begin{aligned} \alpha(x) &\ge \eta(1 + |Du|^2 + |Dv|^2) && \text{a.e. } x \in \Omega\,, \\ |\beta(x)| &\le C(|Du| + |Dv| + \bar{b}_1(x)) && \text{a.e. } x \in \Omega\,. \end{aligned}$$

Easy computations then lead to

$$
\begin{aligned}
&\int_\Omega n w_k^{n-1} |Dw_k|^2 dx + \int_\Omega w_k^n(\beta(x), Dw_k)dx + \int_\Omega \alpha(x) w_k^{n+1} dx \\
&+ \int_\Omega \Big(H_2(x, Du) - H_2(x, Dv) + \eta k(1 + |Du|^2 + |Dv|^2)\Big) w_k^n dx \le 0 \ .
\end{aligned} \tag{28}
$$

The main point is to estimate the last integral of the left-hand side of this inequality. We claim that there exists $R > 0$ such that

$$
\begin{gathered}
H_2(x, p) - H_2(x, q) + \eta k(1 + |p|^2 + |q|^2) \ge 0 \\
\text{a.e. } x \in \Omega, \text{ if } |p| \ge R \text{ or } |q| \ge R\,.
\end{gathered} \tag{29}
$$

Indeed to build R, we first use (25): there exists $R' > 0$ such that if $|p| \ge R'$ then

$$
\frac{|H_2(x, p)|}{1 + |p|^2} \le \eta k/2 \quad \text{a.e. } x \in \Omega\,.
$$

Then we choose $R \ge R'$ such that for any $|p| \le R'$, one has

$$
\frac{|H_2(x, p)|}{1 + R^2} \le \eta k/2 \quad \text{a.e.} \quad x \in \Omega\,.
$$

This is possible because of (26). Finally a case by case analysis ($|p| \le R'$, $R' \le |p| \le R$ or $|p| \ge R$) shows that the claim (29) is true.

The next step consists in proving that, for $k \ge M/2$, there exists $K = K(R) > 0$ such that

$$
H_2(x, p) - H_2(x, q) + \eta k(1 + |p|^2 + |q|^2) \ge -K\,|p - q| \quad \text{a.e. } x \in \Omega\,,
$$

if $|p| \le R$ and $|q| \le R$. This property is a consequence of (26) since the function m_R clearly satisfies for some $K > 0$

$$
m_R(t) \le \eta M/2 + Kt \quad \text{for all } t > 0\,,
$$

and by (26) we have

$$
H_2(x, p) - H_2(x, q) \ge -m_R(|p - q|) \ge -\eta M/2 - K\,|p - q| \quad \text{a.e. } x \in \Omega\,.
$$

We denote by Ω_R the set $\{x \in \Omega;\ |Du(x)| \le R \text{ and } normDv(x) \le R\}$. Using the above properties in (28) yields

$$
\begin{aligned}
&\int_\Omega n w_k^{n-1} |Dw_k|^2 dx + \int_\Omega w_k^n(\beta(x), Dw_k)dx \\
&\qquad + \int_\Omega \alpha(x) w_k^{n+1} dx - \int_{\Omega_R} K\,|Du - Dv|\, w_k^n dx \le 0\,.
\end{aligned} \tag{30}
$$

This inequality remains valid if we replace Ω_R by Ω and the remainder of the proof consists in following the computations and the arguments of the proof of Theorem 1.1, choosing in particular n large enough. We leave the details to the reader. □

As we already mentionned it, the assumptions we require on H_1 above are exactly the ones which are satisfied by the nonlinearity obtained after the change of variable we made in the proof of Theorem 2.1 when $\overline{b}_1, \overline{b}_2$ and f are in $L^\infty(\Omega)$; indeed, for K large enough $\delta_2(x) \geq \alpha_0/2$ in Ω, and by reading carefully our successive estimates on $\frac{\partial b}{\partial v}$ shows that b satisfies (24).

This suggests that one can certainly extend Theorem 3.3 to cases when H_1 satisfies (6) instead of (24). Unfortunately, it is not so easy to obtain interesting extensions because a change of variable will affect also the function H_2 which is only *continuous* and this leads to difficult estimates on the v-derivative of the transformed equation.

The following result describes a first tentative of extension in this direction.

THEOREM 3.4. *Assume that* $f \in L^\infty(\Omega)$, *that* $H = \tilde{H}_1 + \tilde{H}_2$ *and that the following conditions hold*

1. $\tilde{H}_1$ *satisfies the conditions* (4), (5), (6) *with* $\overline{b}_1, \overline{b}_2, f \in L^\infty(\Omega)$.
2. $\tilde{H}_2 = \sum_{i=1}^{I} \tilde{H}_2^i$ *where the* $\tilde{H}_2^i$ *are Caratheodory functions of x and p which satisfies* (25), (26) *and which are respectively homogeneous of degree* m_i *with* $0 < m_i < 2$.

Then the conclusion of Theorem 2.1 *remains valid for* (3).

PROOF OF THEOREM 3.4. We just sketch the proof since it can be obtained by combining in a suitable way the arguments we used in the proofs of Theorem 2.1 and Theorem 3.3. Therefore, we only indicate how to combine them.

The first step consists in performing the same change of variable $u = \varphi(v)$ as in the proof of Theorem 2.1; if we set

$$b(x, v, \xi) = -\frac{\varphi''(v)}{\varphi'(v)}|\xi|^2 + \frac{1}{\varphi'(v)}\tilde{H}_1(x, \varphi(v), \varphi'(v)\xi) + \frac{1}{\varphi'(v)}\tilde{H}_2(x, \varphi'(v)\xi),$$

then the homogeneity properties of the $\tilde{H}_2^i$ allow us to compute $\frac{\partial b}{\partial v}(x, v, \xi)$; following readily the proof of Theorem 2.1, one easily shows that, for any $R > 0$ and for a suitable choice of $A(R) > 0$ and of $K(R) > 0$, one has

$$\frac{\partial b}{\partial v}(x, v, \xi) \geq \eta(R)(1 + |\xi|^2) \quad \text{a.e. } x \in \Omega, \ |v| \leq R, \ \xi \in \mathbb{R}^N .$$

and this is a (24)-type property. This step uses in an essential way the homogeneity assumption on the $\tilde{H}_2^i$.

The second step consists in reproducing the proof of Theorem 3.3: if v_1, $v_2 \in H^1(\Omega) \cap L^\infty(\Omega)$ are respectively sub and supersolution of the transformed equation, the main point is the following computation

$$\begin{aligned} b(x, v_1, Dv_1) - b(x, v_2, Dv_2) &= b(x, v_1, Dv_1) - b\left(x, \frac{v_1+v_2}{2}, Dv_1\right) \\ &\quad + b\left(x, \frac{v_1+v_2}{2}, Dv_1\right) - b\left(x, \frac{v_1+v_2}{2}, Dv_2\right) \\ &\quad + b\left(x, \frac{v_1+v_2}{2}, Dv_2\right) - b(x, v_2, Dv_2)\,. \end{aligned}$$

According to the first step the sum of the first and third terms give the analogue of the $\alpha(x)$ term in the proof of Theorem 3.3. Indeed, according to the above computation of $\frac{\partial b}{\partial v}(x, v, \xi)$, one has

$$b(x, v_1, Dv_1) - b\left(x, \frac{v_1+v_2}{2}, Dv_1\right) \geq \eta(R)(1+|Dv_1|^2)\frac{v_1-v_2}{2}$$
$$\text{a.e. on the set } \{v_1 \geq v_2\}\,,$$

and

$$b\left(x, \frac{v_1+v_2}{2}, Dv_2\right) - b(x, v_2, Dv_2) \geq \eta(R)(1+|Dv_2|^2)\frac{v_1-v_2}{2}$$
$$\text{a.e. on the set} \{v_1 \geq v_2\}\,.$$

In the same way, the second term gives the analogue of the β and the $H_2(x, Du)$ $-H_2(x, Dv)$ terms since

$$\begin{aligned} |b|\left(x, \frac{v_1+v_2}{2}, Dv_1\right) - b\left(x, \frac{v_1+v_2}{2}, Dv_2\right) &\leq C(1+|Dv_1|+|Dv_2|)\,|Dv_1 - Dv_2| \\ &\quad + C\sum_{i=1}^{I}\left|\tilde{H}_2^i(x, Dv_1) - \tilde{H}_2^i(x, Dv_2)\right|\,, \end{aligned}$$

for some constant $C > 0$. With this parallel, the rest of the proof is a routine adaptation of the one of Theorem 3.3 and we leave the details to the reader. □

REMARK 3.1. Of course, the same result is valid if $\tilde{H}_2$ depends of u provided that $\tilde{H}_2$ is an increasing function of u. The modifications in the above proof in order to handle this case are straightforward and left to the reader.

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