Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 28, nº 2 (1999), p. 341-356

<http://www.numdam.org/item?id=ASNSP_1999_4_28_2_341_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVIII (1999), pp. 341-356

On the Boundedness of Multipliers, Commutators and the Second Derivatives of Green's Operators on H^1 and BMO

DER-CHEN CHANG* – SONG-YING LI**

Abstract. Let Ω be a bounded domain in $\mathbb{R}^n$. Let $A(x) = (a_{jk}(x))$ be an $n \times n$ symmetric matrix such that $a_{jk} \in L^\infty(\Omega)$ and $\lambda I_n \le A(x) \le \Lambda I_n$ for a.e. $x \in \Omega$ with $0 < \lambda \le \Lambda < \infty$. Let the Green operator be the solving operator of the Dirichlet problem $\sum_{j,k=1}^n a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} = f$ in Ω with $u = 0$ on $\partial\Omega$. In this paper, we give sufficient, and almost necessary conditions on the smoothness of a_{jk} and $\partial\Omega$ so that the second derivatives of the Green's operator are bounded from $H_z^1(\Omega)$ into $H_r^1(\Omega)$ and bounded $BMO_z(\Omega)$ into $BMO_r(\Omega)$.

Mathematics Subject Classification (1991): 42B25, 46E15 (primary), 35J25 (secondary).

1. – Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. Let $A = (a_{jk})$ be an $n \times n$ symmetric matrix with coefficients $a_{jk} \in L^\infty(\Omega)$ satisfying the following condition:

$$\lambda I_n \le A(x) \le \Lambda I_n, \quad a.e.\ x \in \Omega, \quad \text{and} \quad 0 < \lambda \le \Lambda < \infty. \tag{1.1}$$

A fundamental question of whether the Dirichlet problem for the elliptic equation:

$$\sum_{j,k=1}^n a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} = f \quad \text{in} \quad \Omega; \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{1.2}$$

has a unique solution in $W^{2,p}(\Omega)$ when $f \in L^p(\Omega)$ has not been completely solved. For the case $p = 2$, G. Talenti [T] showed that (1.2) has a unique solution $u \in W^{2,2}(\Omega)$ if $f \in L^2(\Omega)$ provided $\partial\Omega$ satisfies a geometric condition.

* The author is supported in part by the NSF grant DMS9622249.
** The author is supported in part by the NSF grant DMS9705731.
Pervenuto alla Redazione l'8 giugno 1998 e in forma definitiva il 26 febbraio 1999.

It is also known that the answer to the question is negative when p is large. Recently, F. Chiarenza, M. Frasca and P. Longo made great contribution to this problem in their several published papers (see [CFL1, 2] and reference therein). They posed very natural conditions on a_{ij}. In particular, they proved that if $a_{ij} \in VMO(\Omega)$, then (1.2) has a unique solution $u \in W^{2,p}(\Omega)$ for all $f \in L^p(\Omega)$ and $1 < p < \infty$.

The main purpose of the present paper is to study the limiting cases of p in the above theorem. Following the definitions in [CKS] and [CDS], we let $H_z^1(\Omega)$ denote the local Hardy space with all local 1-atoms supported in Ω, and $H_r^1(\Omega)$ denote the Hardy space which is the restriction of the local Hardy space $h^1(\mathbb{R}^n)$ in Ω; and corresponding definitions for $BMO_z(\Omega)$ and $BMO_r(\Omega)$ (see definitions in Section 3). Let $\mathbf{G}[f]$ be the Green's operator of (1.2) if it exists. We search for a natural condition on a_{jk} so that the operators $\frac{\partial^2 \mathbf{G}[\cdot]}{\partial x_j \partial x_k}$, $j, k = 1, \ldots, n$, orginally defined on $C_0^\infty(\Omega)$, can be extended as bounded operators from $H_z^1(\Omega)$ into $H_r^1(\Omega)$ and $BMO_z(\Omega)$ into $BMO_r(\Omega)$. An example in Section 3 shows that a_{ij} satisfies at least logarithmic smoothness, more precisely, $a_{ij} \in LMO(\Omega)$ (see Definition 2.1). As we shall show in Section 3, a slightly stronger and well-known condition than logarithmic smoothness is the *Dini's* condition. One of our main results is that $\frac{\partial^2 \mathbf{G}[\cdot]}{\partial x_j \partial x_k}$, $j, k = 1, \ldots, n$, is bounded both from $H_z^1(\Omega)$ to $H_r^1(\Omega)$ and from $BMO_z(\Omega)$ to $BMO_r(\Omega)$ when a_{ij} satisfy the *Dini's* condition (Theorem 3.4). It is clear that the $C^{1,1}$ assumption on $\partial\Omega$ is the best condition so that $\frac{\partial^2 \mathbf{G}[\cdot]}{\partial x_i \partial x_j}$ is bounded from $BMO_z(\Omega)$ to $BMO_r(\Omega)$. However, it is not the minimum smoothness assumption for the H^1 case. In fact, in [CKS], a natural question was posed: what is the possible best smoothness assumption on $\partial\Omega$ so that $\frac{\partial^2 \mathbf{G}[\cdot]}{\partial x_j \partial x_k}$, $j, k = 1, \ldots, n$ is bounded from $H_z^1(\Omega)$ to $H_r^1(\Omega)$ when $\mathbf{G}$ is the Green's operator for the Laplacian? Various answers were given in [JK] (a counterexample for H^1) and [KL] (it is shown at least $C^{n(1/p-1)}$ for H^p, $p \le n/(n+1)$). It is known from [JK] that the above operator is not bounded from $H_z^1(\Omega)$ to $H_r^1(\Omega)$ if we merely assume that $\partial\Omega$ is C^1. A slightly stronger condition than C^1 which one can expect is that Ω has a defining function ρ so that $\nabla\rho$ satisfies the *Dini's* condition on $\partial\Omega$. The second purpose of this paper is to prove that $\frac{\partial^2 \mathbf{G}[\cdot]}{\partial x_j \partial x_k}$, $j, k = 1, \ldots, n$ is bounded from $H_z^1(\Omega)$ to $H_r^1(\Omega)$ with such an assumption on $\partial\Omega$ (Theorem 4.1). In order to prove Theorems 3.4 and 4.1, we first give some characterizations on f so that the multiplication operator M_f and commutator $[M_f, T_K]$ (of M_f and a Calderòn-Zygmund operator T_K induced by the kernel K) are bounded on H^1 and BMO on a space of homogeneous type. The results along this direction will be given in Section 2.

We remark that the method of the present paper works for the Neumann problem on $H_z^1(\Omega)$ either. We leave the details to interested readers. We would like to thank Galia Dafni for many inspiring conversations and valuable comments.

ACKNOWLEDGEMENTS. The authors would like to thank the referee for some nice suggestions for this revision by omitting some details of proofs from the previous version.

2. – Boundedness of Commutators on H^1 and BMO

Let X be a locally compact Hausdörff space, and let d be a quasi-metric on X. A ball $B(x;r)$ centered at $x \in X$ with radius $r > 0$ with respect to d is defined as follows: $B(x;r) = \{y \in X : d(x,y) < r\}$. Let μ be a positive regular measure on X. We say that (X, d, μ) is a *space of homogeneous type* if there is a constant $\gamma > 1$ such that

(i) $\mu(\{x\}) = 0$ and $0 < \mu(B(x;r)) < \infty$ for all $x \in X$ and all $r > 0$;

(ii) μ satisfies the "doubling property", *i.e.*, $\mu(B(x;2r)) \le \gamma\mu(B(x;r))$ for all $x \in X$ and all $r > 0$.

Let $f \in L^1_{\mathrm{loc}}(X)$. We define the maximum mean oscillation on balls with fixed radius r as follows:

$$M(r,f) = \sup_{x\in X}\left\{\frac{1}{\mu(B(x;r))}\int_{B(x;r)}\left|f - f_{B(x;r)}\right| d\mu\right\} \tag{2.1}$$

where $f_{B(x;r)} = \mu(B(x;r))^{-1}\int_{B(x;r)} f(y)d\mu(y)$.

DEFINITION 2.1. Let (X, d, μ) be a space of homogeneous type and let $f \in L^1_{\mathrm{loc}}(X)$. We define

(i) $f \in BMO(X)$ if

$$\|f\|_{BMO} \equiv \|f\|_* = \sup_{0<r<\infty} M(r,f) + \left(\int_X |f|^2 d\mu\right)^{\frac{1}{2}} < \infty.$$

(ii) $f \in VMO(X)$ if $f \in BMO(X)$ and $\lim_{r\to 0^+} M(r,f) = 0$.

(iii) $f \in LMO(X)$ if

$$\|f\|_{LMO} = \sup_{0<r<\infty,\, x\in X}\left\{\frac{|\log\mu(B(x;r))|}{\mu(B(x;r))}\int_{B(x;r)}\left|f - f_{B(x;r)}\right| d\mu < \infty\right\}. \tag{2.2}$$

Now we may define the atomic H^1 space as follows.

DEFINITION 2.2. Let (X, d, μ) be a space of homogeneous type. We say a is an *atom* if either $a \in L^2(X)$ with $\|a\|_{L^2} = 1$ or if there is a ball B such that $\mathrm{supp}(a) \subset B$ and

$$\text{(i)}\quad \|a(x)\|_{L^\infty} \le (\mu(B))^{-1}; \qquad \text{(ii)}\quad \int_B a(x)\, d\mu = 0.$$

Then the atomic Hardy space is defined as follows:

$$H^1(X) = \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j \text{ are atoms and } \{\lambda_j\}_{j=1}^{\infty} \in \ell^1, \ \lambda_j \geq 0 \right\}$$

with norm

$$\|u\|_{H^1} = \inf \left\{ \sum_{j=1}^{\infty} \lambda_j : u = \sum_{j=1}^{\infty} \lambda_j a_j \right\}. \tag{2.3}$$

Here the infimum is taken over all possible atomic decomposition for the function u. Then $(H^1)^* = BMO$ and $(VMO)^* = H^1$ (see Chapter IV in Stein [St]). Notice that we add L^2 functions to the definition of Hardy space H^1 and restrict BMO space as a subspace of L^2 which produces the same spaces when $\mu(X) < \infty$ (see Coifman and Weiss [CW]). When $\mu(X) = \infty$, our H^1 and BMO are different from the classical H^1 and BMO (see Nakai and Yabuta [NY]). However, we are mainly interested in applications of results in this section to the Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^n$ in this paper. Therefore, it is harmless for us to consider the case $\mu(X) < \infty$ here.

Let $M_\phi(f) = \phi \cdot f$ be the multiplication operator. Imitating the method in Stegenga [S], we have the following theorem.

THEOREM 2.3. *Let (X, d, μ) be a space of homogeneous type. Then the following assertions hold.*

(i) *M_ϕ is bounded on $H^1(X)$ if and only if $\phi \in LMO(X) \cap L^\infty(X)$.*
(ii) *M_ϕ is bounded on $BMO(X)$ if and only if $\phi \in LMO(X) \cap L^\infty(X)$.*
(iii) *If $f, g \in L^\infty(X) \cap LMO(X)$ then $|f|$, $fg \in LMO(X) \cap L^\infty(X)$; and if $|f| \geq a > 0$ on X then $1/f \in L^\infty(X) \cap LMO(X)$.*

DEFINITION 2.4. Let (X, d, μ) be a space of homogeneous type. A measurable function $K : X \times X \setminus \{x = y\} \to \mathbb{C}$ is said to be a *standard Calderón-Zygmund kernel* if there exist $\varepsilon > 0$ and $0 < C < \infty$ such that (2.4) and (2.5) hold. We say K is a *semi-standard Calderón-Zygmund kernel* if (2.4) and (2.6) hold. Here

$$|K(x, y)| \leq \frac{C}{\lambda(x, y)}, \quad \lambda(x, y) = \mu(B(x; d(x, y))), \quad x \neq y \in X \tag{2.4}$$

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq C_\varepsilon \frac{\mu(B(y; \delta))^\varepsilon}{\lambda(x, y)^{1+\varepsilon}} \tag{2.5}$$

and

$$|K(x, y) - K(x, z)| \leq C_\varepsilon \frac{\mu(B(y; \delta))^\varepsilon}{\lambda(x, y)^{1+\varepsilon}} \tag{2.6}$$

for all $z \in B(y; \delta)$ and $x \notin B(y; 2\delta)$.

Next, we will link the multiplication operator M_ϕ with singular integral operators. Let T_K be a singular integral operator with a kernel function $K(x, y)$. Then we denote by $C_\phi = [M_\phi, T_K] = M_\phi T_K - T_K M_\phi$, the commutator of M_ϕ and T_K.

THEOREM 2.5. *Let (X, d, μ) be a space of homogeneous type with $\mu(X) < \infty$ and let K be a semi-standard Calderón-Zygmund kernel so that T_K is bounded on $L^2(X)$. Then*

(i) *If $\phi \in LMO(X)$ then C_ϕ is bounded from $H^1(X)$ into $L^1(X)$.*
(ii) *Assume that for any $x_0 \in X$ and $\delta > 0$ we have*

$$\int_{X \setminus B(x_0;2\delta)} |K(x, x_0)| d\mu(x) \geq \frac{1}{C} \Big| \log \mu(B(x_0; \delta)) \Big|$$

and if $\phi \in BMO(X)$ and C_ϕ is bounded from $H^1(X)$ to $L^1(X)$ then $\phi \in LMO(X)$.
(iii) *If K is a standard Calderòn-Zygmund kernel and $\phi \in LMO(X) \cap L^\infty(X)$ then $C_\phi : H^1(X) \to H^1(X)$ and $BMO(X) \to BMO(X)$ are bounded.*

This theorem was proved by Stegenga [S] for $X = \mathbf{T}$, the unit circle, and later generalized by Li [L] when X is the unit sphere. We can use the methods in [S] and [L] to obtain our result. We will leave the details to interested readers.

3. – Green's operator on $H^1(\Omega)$ and $BMO(\Omega)$

Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. In this section, we will consider the estimates for the second derivatives of the Green's operator for Dirichlet problem (1.2) in the Hardy class H^1. Since our operator has variable coefficients, we have to restrict to the local Hardy spaces. Let us first recall some definitions from [CKS]. Let $H^1_r(\Omega)$ be the set of all integrable functions in Ω which is a restriction of local Hardy in $\mathbb{R}^n$ with infimum norm of all possible extensions. This is the "biggest" Hardy space that can be defined on the domain Ω. We also have the "smallest" version of local Hardy space defined on Ω: $H^1_z(\Omega)$. This is the atomic Hardy space with all atoms supported in Ω. A natural problem related to the H^1 estimates is the BMO estimate. In order to round out the picture, we need a C. Fefferman type duality theorem for the Hardy space $H^1(\Omega)$. Corresponding to $H^1_r(\Omega)$ and $H^1_z(\Omega)$, we let $BMO_r(\Omega)$ be the space consisting of all functions f with

$$\begin{aligned} \|f\|_{BMO_r(\Omega)} = & \sup_{B(x;r)\subset\Omega, |B(x;r)|\leq 1} \left\{ \frac{1}{|B(x;r)|} \int_{B(x;r)} |f(y) - f_{B(x;r)}| \, d\mu(y) \right\} \\ & + \sup_{B(x;r)\subset\Omega, |B(x;r)|> 1} \left\{ \frac{1}{|B(x;r)|} \int_{B(x;r)} |f(y)| \, d\mu(y) \right\} < \infty; \end{aligned}$$

and let $BMO_z(\Omega) = \{f \in BMO(\mathbb{R}^n) : \operatorname{supp}(f) \subset \overline{\Omega}\}$. It was proved in [C] that: $H^1_r(\Omega)^* = BMO_z(\Omega)$ and $H^1_z(\Omega)^* = BMO_r(\Omega)$.

In order to prove the main theorem of this section, we need the function space LMO over Ω. We say $f \in LMO(\Omega)$ if $f \in L^1(\Omega)$ and

$$\|f\|_{LMO(\Omega)} = \sup_{r>0} \left\{ L(f; x, r) : B(x; r) \subset \Omega \right\} < \infty$$

where

$$L(f; x, r) = \left\{ \frac{|\log |B(x; r)||}{|B(x; r)|} \int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu(x) \right\}.$$

REMARK 1. The space $H_r^1(\Omega)$ defined above has an atomic decomposition also. In order to do this, following the definitions in [CKS], we consider two different kind of H_r^1 atoms. A bounded, measurable function $a(x)$ supported on a ball $B \subseteq \Omega$ is called a type (a) 1-atom if $4B \cap \partial\Omega = \emptyset$ with $\mathrm{diam}(B) \leq 1$, $\|a\|_{L^\infty(\mathbb{R}^n)} \leq |B|^{-1}$, and $\int_B a(x)dx = 0$.

We call $a(x)$ a type (b) 1-atom if either $\mathrm{diam}(B) > 1$ or $2B \cap \partial\Omega = \emptyset$ and $4B \cap \partial\Omega \neq \emptyset$ and the size condition $\|a\|_{L^\infty} \leq |B|^{-1}$ (but not necessary the moment condition) is satisfied.

For $f \in H_r^1(\Omega)$ has an atomic decomposition

$$f = \sum_{\text{type } (a) \text{ atoms}} \lambda_B a_B + \sum_{\text{type } (b) \text{ atoms}} \mu_B \tilde{a}_B$$

with

$$\sum_{\text{type } (a) \text{ atoms}} |\lambda_B| + \sum_{\text{type } (b) \text{ atoms}} |\mu_B| < \infty.$$

Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. Let $A(x) = (a_{jk})$ be an $n \times n$ matrix with coefficients in $L^\infty(\Omega)$ satisfy the condition (1.1). Assume that the Dirichlet problem of elliptic equation (1.2) has a uniqe solution $\mathbf{G}[f]$. We shall search for a natural condition on a_{jk} so that $\frac{\partial^2 \mathbf{G}[f]}{\partial x_j \partial x_k}$ is bounded on $H_z^1(\Omega)$ and $BMO_z(\Omega)$.

Let us start with the following simple example.

EXAMPLE 1. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary. Let $\phi \in L^\infty(\Omega)$ such that $0 < \lambda \leq \phi(x) \leq \Lambda$ for almost every $x \in \Omega$. Assume that $a_{ij}(z) = \phi(z)\delta_{ij}$. Let $\mathbf{G}[f]$ be the solution operator of (1.2). If $\frac{\partial^2 \mathbf{G}[f]}{\partial x_i \partial x_j} \in H_r^1(\Omega)$ for all $f \in H_z^1(\Omega)$ then $\phi \in LMO(\Omega)$.

PROOF. Let $f \in H_z^1(\Omega)$; by assumption, we have $\frac{\partial^2}{\partial x_i \partial x_j}\mathbf{G}[f] \in H_r^1(\Omega)$. In particular, we have $\Delta\mathbf{G}[f] \in H_r^1(\Omega)$. Since

$$\mathcal{L}u = f \iff \phi\Delta u = f \iff \Delta u = \phi^{-1} f.$$

Therefore $\phi^{-1} f \in H_r^1(\Omega)$. This implies that the multiplication operator $M_{1/\phi}$ is bounded from $H_z^1(\Omega)$ to $H_r^1(\Omega)$. By Theorems 2.3 and 2.5, we have $1/\phi \in LMO(\Omega)$, and so is $\phi \in LMO(\Omega)$.

By the fact of Example 1, a necessary condition for $\frac{\partial^2 \mathbf{G}[\cdot]}{\partial x_i \partial x_j}$ is bounded from $H_z^1(\Omega)$ to $H_r^1(\Omega)$ or from $BMO_z(\Omega)$ to $BMO_r(\Omega)$ is $a_{ij} \in LMO(\Omega)$. We need a_{ij} satisfy a little bit better condition than $LMO(\Omega)$. One of them is $a_{ij} \in LMO_0(\Omega)$. *We say* $f \in LMO_0(\Omega)$ *if* $f \in LMO(\Omega)$ *and* $L(f; x, r) \to 0$ *as* $r \to 0$ *uniformly for* $x \in \Omega$. Another familiar function space $\mathcal{D}(\Omega)$ which is slightly smaller than $LMO_0(\Omega)$ is the space of all functions satisfying *Dini*'s condition:

$$\|f\|_{\mathcal{D}} = \int_0^1 \frac{\omega(f, t)}{t} dt < +\infty, \tag{3.3}$$

where

$$\omega(f, t) = \sup\{|f(x) - f(y)| : x, y \in \Omega,\ |x - y| < t\}. \tag{3.4}$$

REMARK 2. If we assume that Ω has a defining function ρ such that $\nabla\rho$ satisfies the *Dini's* condition on $\partial\Omega$, then we have a similar result as Theorem 2.3: If $\phi \in LMO(\Omega) \cap L^\infty(\Omega)$ then M_ϕ is bounded on $H_z^1(\Omega)$, $H_r^1(\Omega)$, $BMO_z(\Omega)$ and $BMO_r(\Omega)$, respectively. Conversely, if M_ϕ is bounded from $H_z^1(\Omega)$ to $H_r^1(\Omega)$ then $\phi \in LMO(\Omega)$.

The relation between $LMO_0(\Omega)$ and $\mathcal{D}(\Omega)$ is given by the following lemma. The proof is quite elementary, we omit the details here.

LEMMA 3.1. *If f satisfies Dini's condition, then* $f \in LMO_0(\Omega)$.

In order to solve (1.2), and derive the solution operator $\mathbf{G}[f]$ as an integral operator. We shall recall some arguments and fomulation given in [CFL2]. Let

$$\Gamma(x, t) = \frac{1}{(n-2)\omega_n (\det A(x))^{1/2}} \left(\sum_{i,j=1}^n a^{ij}(x) t_i t_j \right)^{(2-n)/2} \tag{3.5}$$

where $(a^{ij}(x)) = (A(x))^{-1}$ and $n > 2$. For $1 \le i, j \le n$, denote

$$\Gamma_i(x, t) = \frac{\partial}{\partial t_i} \Gamma(x, t), \quad \Gamma_{ij}(x, t) = \frac{\partial^2}{\partial t_i \partial t_j} \Gamma(x, t). \tag{3.6}$$

Let

$$\mathcal{L}(x) u(y) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial y_i \partial y_j}(y).$$

Then for any function $u \in C_0^\infty(\Omega)$ and $x_0 \in \Omega$, we have

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x_0, x - y) \mathcal{L}(x_0) u(y) dy.$$

Applying $\frac{\partial^2}{\partial x_i \partial x_j}$ to the above identity, we have

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \text{P.V.} \int_{\mathbb{R}^n} \Gamma_{ij}(x_0, x-y)\mathcal{L}(x_0)u(y)dy + \mathcal{L}(x_0)u(x_0)\int_{|t|=1} \Gamma_i(x_0, t)t_j d\sigma_t.$$

Since

$$\begin{aligned}\mathcal{L}(x)u(y) &= (\mathcal{L}(x) - \mathcal{L}(y))u(y) + \mathcal{L}(y)u(y)\\ &= \sum_{k,\ell=1}^{n} (a_{k\ell}(x) - a_{k\ell}(y))\frac{\partial^2 u(y)}{\partial y_k \partial y_\ell} + \mathcal{L}(y)u(y),\end{aligned}$$

replacing x_0 by x in the above identities, we have the following formula

$$\begin{aligned}\frac{\partial^2 u(x)}{\partial x_i \partial x_j} &= \text{P.V.} \int_{\mathbb{R}^n} \Gamma_{ij}(x, x-y) \sum_{k,\ell=1}^{n} (a_{k\ell}(x) - a_{k\ell}(y))\frac{\partial^2 u(y)}{\partial y_k \partial y_\ell} dy\\ &\quad + \text{P.V.} \int_{\mathbb{R}^n} \Gamma_{ij}(x, x-y)\mathcal{L}(y)u(y)dy + \mathcal{L}(x)u(x)\int_{|t|=1} \Gamma_i(x, t)t_j d\sigma_t\\ &= \sum_{k,\ell=1}^{n} \left[M_{a_{k\ell}}, T_{K_{ij}}\right]\left(\frac{\partial^2 u}{\partial x_k \partial x_\ell}\right)(x) + \text{P.V.} \int_{\mathbb{R}^n} \Gamma_{ij}(x, x-y)\mathcal{L}(y)u(y)dy\\ &\quad + \mathcal{L}(x)u(x)\int_{|t|=1} \Gamma_i(x, t)t_j d\sigma_t\end{aligned}$$

where

$$K_{ij}(x, y) = \Gamma_{ij}(x, x-y), \quad 1 \le i, j \le n.$$

A direct calculation shows that

$$\begin{aligned}\omega_n \det(A(x))^{1/2}\Gamma_{ij}(x, t) &= \frac{1}{n-2}\frac{\partial^2}{\partial t_k \partial t_\ell}\left(\sum_{k,\ell=1}^{n} a^{k\ell}(x)t_k t_\ell\right)^{(2-n)/2}\\ &= -\left(\sum_{k,\ell=1}^{n} a^{k\ell} t_k t_\ell\right)^{-n/2}\left[a^{ij}(x) - n\frac{(\sum_k a^{kj}t_k)(\sum_\ell a^{i\ell}t_\ell)}{\sum_{k,\ell=1}^{n} a^{k\ell}(x)t_k t_\ell}\right].\end{aligned}$$

It is easy to see that for any $y \in B(x_0, \delta)$ and $x \in \mathbb{R}^n \setminus B(x_0, 2\delta)$, we have

$$\text{(3.7)} \quad |K_{ij}(x, y) - K_{ij}(x, x_0)| = \left|\Gamma_{ij}(x, x-y) - \Gamma_{ij}(x, x-x_0)\right| \le \frac{C|y-x_0|}{|x-x_0|^{n+1}}$$

where C depending only on $\|a^{ij}\|_\infty$, λ and Λ.

Thus K_{ij} is a *semi-standard Calderòn-Zygmund* kernel. In order that Theorem 2.5 (iii) holds for $T_{K_{ij}}$, we need the following lemma.

LEMMA 3.2. *Let $a_{ij} \in \mathcal{D}(\Omega) \cap L^\infty(\Omega)$ and $\partial\Omega \in C^{1,1}$. Then*
(i) *$T_{K_{ij}}$ and $T^*_{K_{ij}}$: $H^1_z(\Omega) \to H^1_r(\Omega)$ are bounded;*
(ii) *$T_{K_{ij}}$ and $T^*_{K_{ij}}$: $BMO_z(\Omega) \to BMO_r(\Omega)$ are bounded;*
(iii) *$[M_{a_{k\ell}}, T_{K_{ij}}]$ is bounded on both $H^1_r(\Omega)$ and $BMO_r(\Omega)$. Moreover,*

$$\|[M_{a_{k\ell}}, T_{K_{ij}}]\|_{op} \leq C\|a_{k\ell}\|_{\mathcal{D}}.$$

The proof of the lemma can be obtained by using the smoothness of K_{ij} (in particular, (3.7)), the atomic decomposition for $H^1_r(\Omega)$ (see Remark 1), Theorems 2.3 and 2.5. We omit the details here.

As a corollary of Lemma 3.2 and Remark 2, we have

COROLLARY 3.3. *If $a_{ij} \in \mathcal{D}(\Omega) \cap L^\infty(\Omega)$, and if $\phi \in LMO(\Omega) \cap L^\infty(\Omega)$ then $C_\phi = [M_\phi, T_{K_{ij}}]$ is bounded from $BMO_z(\Omega)$ to $BMO_r(\Omega)$, and on $H^1_z(\Omega)$ to $H^1_r(\Omega)$. Moreover, let B_0 be a ball in Ω such that*

$$\eta_1 = C(\lambda, \Lambda, \Omega) \sum_{k,\ell=1}^{n} \left(2\omega(a_{k\ell}, B_0) + \|a_{k\ell}\|_{LMO(B_0)}\right) < 1. \tag{3.8}$$

and

$$\eta_2 = C(\lambda, \Lambda, \Omega) \sum_{k,\ell=1}^{n} \left(2\omega(a_{k\ell}, B_0) + \|a_{k\ell}\|_{LMO(B_0)}\right) < 1. \tag{3.9}$$

Then for any $\operatorname{supp}(u) \subset B_0$ *we have*

$$\sum_{i,j=1}^{n} \left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{H^1_r(\Omega)} \leq \frac{C}{1-\eta_1}\|\mathcal{L}u\|_{H^1_z(\Omega)} \tag{3.10}$$

and

$$\sum_{i,j=1}^{n} \left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{BMO_r(\Omega)} \leq \frac{C}{1-\eta_2}\|\mathcal{L}u\|_{BMO_z(\Omega)}. \tag{3.11}$$

The main purpose of this section is to prove the following theorem.

THEOREM 3.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. Let $A = (a_{jk}(x))_{n\times n}$ be an $n \times n$ symmetric matrix satisfying* (1.1), *and let a_{jk} satisfy the Dini's condition* (3.3). *Then the equation* (1.2) *has a unique solution* $\mathbf{G}[f]$ *such that $\frac{\partial^2}{\partial x_i \partial x_j}\mathbf{G}[\,\cdot\,]$ is bounded from $H^1_z(\Omega)$ to $H^1_r(\Omega)$ and as well as from $BMO_z(\Omega)$ to $BMO_r(\Omega)$.*

PROOF. The existence and uniqueness of the solution $\mathbf{G}[f]$ can be found in [GT]. We need only to prove

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} \mathbf{G}[f] \right\|_{H_r^1(\Omega)} \leq C \|f\|_{H_z^1(\Omega)} \tag{3.12}$$

and

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} \mathbf{G}[f] \right\|_{BMO_r(\Omega)} \leq C \|f\|_{BMO_z(\Omega)}. \tag{3.13}$$

We shall prove (3.12) while (3.13) follows similarly.

In order to prove (3.12), we use the partition of unity. Let $\psi_k \in C_0^\infty(\mathbb{R}^n)$ with support in $B(x_k; \varepsilon)$ for some $\varepsilon > 0$ small (will be chosen later) so that

(i) $0 \leq \psi_k \leq 1$, $\sum_{k=1}^m \psi_k \equiv 1$ on $\overline{\Omega}$ and $|\nabla \psi_k| \leq C\varepsilon^{-1} \psi_k$ for all $1 \leq k \leq m$;

(ii) either $x_k \in \partial\Omega$ or $\operatorname{dist}(B(x_k; \varepsilon), \partial\Omega) \geq \frac{\varepsilon}{100}$.

Thus $\mathbf{G}[f] = \sum_{k=1}^m \mathbf{G}[f]\psi_k = \sum_{k=1}^m \mathbf{G}_k[f]$ for all $x \in \overline{\Omega}$ where $\mathbf{G}_k[f] = \mathbf{G}[f]\psi_k$. Therefore, we separate the estimation into the following two cases.

CASE 1. If $\operatorname{dist}(B(x_k; \varepsilon), \partial\Omega) > 0$. We choose ε small so that

$$\eta_1 = C(\lambda, \Lambda, \Omega) \sum_{k,\ell=1}^n \left\{ 2\omega(a_{k\ell}, B(x_k; \varepsilon)) + \|a_{k\ell}\|_{LMO(B(x_k;\varepsilon))} \right\} \leq \tfrac{1}{2}.$$

By Corollary 3.3 and the following calculation:

$$\mathcal{L}(y)\mathbf{G}_k[f] = \psi_k \mathcal{L}(y)\mathbf{G}[f] + \mathbf{G}[f](y)\mathcal{L}(y)\psi_k(y) + 2\sum_{i,j=1}^n a_{ij} \frac{\partial \mathbf{G}[f]}{\partial y_i} \frac{\partial \psi_k}{\partial y_j}.$$

One can easily see that

$$\begin{aligned} &\sum_{i,j=1}^n \left\| \frac{\partial^2 \mathbf{G}_k[f]}{\partial x_i \partial x_j} \right\|_{H_r^1(\Omega)} \\ &\quad \leq \frac{C}{\varepsilon^2} \|a_{ij}\|_{\mathcal{D}} \left\{ \|\psi_k f\|_{H_z^1(\Omega)} + \|\mathbf{G}_k[f]\|_{H_z^1(\Omega)} + \|\psi_k \nabla \mathbf{G}[f]\|_{H_z^1(\Omega)} \right\}. \end{aligned} \tag{3.14}$$

CASE 2. If $x_k \in \partial\Omega$. As usual, we try to flatten the boudary of $\partial\Omega \cap B(x_k; \varepsilon)$. Let $\Phi : B_\varepsilon^+ \to B(x_k; \varepsilon) \cap \Omega$ is a $C^{1,1}$ diffeomorphsim so that $\Phi(y', 0) \in \partial\Omega \cap B(x_k; \varepsilon)$ where $B_\varepsilon^+ = \{(y', y_n) \in B(0; \varepsilon) : y_n > 0\}$. If we let $(b_{ij}(y)) = B(y) = \Phi'(y)^t A(\Phi(y))\Phi'(y)$. Then

$$\lambda_1 I_n \leq B(y) \leq \Lambda_1 I_n, \qquad \text{for a.e.} \quad y \in B_\varepsilon^+(0)$$

where $\lambda_1 = \lambda \cdot \lambda(\Phi)$ and $\Lambda_1 = \Lambda \cdot \Lambda(\Phi)$ where $\lambda(\Phi)$ and $\Lambda(\Phi)$ denote the smallest and greatest eigenvalues of $\Phi'(y)^t\Phi'(y)$ for all $y \in B_\varepsilon^+(0)$. Let

$$\tilde{\mathcal{L}} = \sum_{i,j=1}^{n} b_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j}.$$

Then, a simple calculation shows that

$$\tilde{\mathcal{L}}(\mathbf{G}[f] \circ \Phi(y)) = (\mathcal{L}\mathbf{G}[f]) \circ \Phi(y) + \sum_{k=1}^{n} \frac{\partial \mathbf{G}[f]}{\partial x_k} \circ \Phi(y)\tilde{\mathcal{L}}\Phi_k(y).$$

Since $\Phi_k \in C^{1,1}$, the linear term above is easy to handle. This leads us to consider the solution operator $\tilde{\mathbf{G}}[f]$ for the Dirichlet problem:

$$\tilde{\mathcal{L}}(y)v(y) = f \circ \Phi(y), \quad y \in B_\varepsilon^+(0); \quad v = 0 \quad \text{on} \quad \partial B_\varepsilon^+(0).$$

For this case, the expression for $\frac{\partial^2 \tilde{\mathbf{G}}[f]}{\partial y_i \partial y_j}$ was given by Chiarenza, Frasca and Longo [CFL2]. Let us recall their formula as follows. First we need some notations. Let $b(x)$ the n-th row vector of $B(x)$, and let

$$T(x, y) = x - \frac{2x_n}{b_{nn}(x)}b(y) = (T_1, \cdots, T_n), \quad T_{nj}(x) = \delta_{nj} - \frac{2}{b_{nn}(x)}b_{nj}(x).$$

Let $\tilde{\Gamma}$, $\tilde{\Gamma}_i$ and $\tilde{\Gamma}_{ij}$ be defined as (3.5) and (3.6) by replacing a^{ij} by b^{ij}. We first find an integral representation for the second derivative of the solution u for problem (1.2). Imitating notations in [CFL2], we have the following: If $u \in C_0^2(B_\varepsilon)$ and $u(x', 0) = 0$, then

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i \partial x_j} &= \text{P.V.} \int_{B_\varepsilon^+} \tilde{\Gamma}_{ij}(x, x - y)(\tilde{\mathcal{L}}(x) - \tilde{\mathcal{L}}(y))(u(y))dy \\ &\quad + \text{P.V.} \int_{B_\varepsilon^+} \tilde{\Gamma}_{ij}(x, x - y)\tilde{\mathcal{L}}(y)(u(y))dy \\ &\quad + \tilde{\mathcal{L}}(x)u(x) \int_{|t|=1} \tilde{\Gamma}_i(x, t)t_j d\sigma_t + I_{ij}(u)(x)\end{aligned}$$

where

$$\begin{aligned}I_{ij}(u)(x) = &\int_{B_\varepsilon^+} \tilde{\Gamma}_{ij}(x, T(x) - y)\left(\sum_{k,\ell=1}^{n}(b_{k\ell}(x) - b_{k\ell}(y))\frac{\partial^2 u}{\partial x_k \partial x_\ell}(y)\right)dy \\ &+ \int_{B_\varepsilon^+} \tilde{\Gamma}_{ij}(x, T(x) - y)\tilde{\mathcal{L}}(y)(u(y))dy\end{aligned}$$

when $1 \le i, j \le n-1$,

$$
\begin{aligned}
I_{in}(u)(x) &= I_{ni}(u)(x) \\
&= \int_{B_\varepsilon^+} \Big[\sum_{j=1}^{n} \tilde{\Gamma}_{ij}(x, T(x)-y)T_{nj}(x)\Big](\tilde{\mathcal{L}}(x)-\tilde{\mathcal{L}}(y))(u(y))dy \\
&\quad + \int_{B_\varepsilon^+} \Big[\sum_{j=1}^{n} \tilde{\Gamma}_{ij}(x, T(x)-y)T_{nj}(x)\Big]\tilde{\mathcal{L}}(y)(u(y))dy
\end{aligned}
$$

when $1 \le i < n$, and

$$
\begin{aligned}
I_{nn}(u)(x) &= \int_{B_\varepsilon^+} \Big[\sum_{i,j=1}^{n} \tilde{\Gamma}_{ij}(x, T(x)-y)T_{ni}(x)\, T_{nj}(x)\Big](\tilde{\mathcal{L}}(x)-\tilde{\mathcal{L}}(y))(u(y))dy \\
&\quad + \int_{B_\varepsilon^+} \Big[\sum_{i,j=1}^{n} \tilde{\Gamma}_{ij}(x, T(x)-y)T_{ni}(x)\, T_{nj}(x)\Big]\tilde{\mathcal{L}}(y)u(y)dy\,.
\end{aligned}
$$

By using the estimation [CFL2, Lemma 3.1]: $|(x', -x_n)-y| \le C|T(x)-y|$ for all $y \in \mathbb{R}^n_+$, we can deal with kernels: $\tilde{\Gamma}_{ij}(x, T(x)-y)$,

$$
\left(\sum_{j=1}^{n} \tilde{\Gamma}_{ij}(x, T(x)-y)T_{nj}(x)\right) \quad \text{and} \quad \left(\sum_{i,j=1}^{n} \tilde{\Gamma}_{ij}(x, T(x)-y)T_{ni}(x)T_{nj}(x)\right)
$$

similarily to $\Gamma_{ij}(x, x-y)$ in Lemma 3.2 and Corollary 3.3. Therefore, with similar arguments of Corollary 3.3 and the argument of Case 1, we have the following result by choosing small ε depending only on $\|a_{ij}\|_{\mathcal{D}}$ so that

$$
\begin{aligned}
&\sum_{i,j=1}^{n} \left\| \frac{\partial^2 \mathbf{G}_k[f]}{\partial x_i \partial x_j} \right\|_{H_r^1(\Omega)} \\
&\quad \le \frac{C}{\varepsilon^3}\|a_{ij}\|_{\mathcal{D}}\Big(\|\psi_k f\|_{H_z^1(\Omega)} + \|\psi_k \mathbf{G}[f]\|_{H_r^1(\Omega)} + \|\psi_k \nabla \mathbf{G}[f]\|_{H_r^1(\Omega)}\Big).
\end{aligned}
\tag{3.15}
$$

Combining (3.14) and (3.15), we have

$$
\begin{aligned}
&\left\| \frac{\partial^2 \mathbf{G}[f]}{\partial x_i \partial x_j} \right\|_{H_r^1(\Omega)} \\
&\quad \le \sum_{k=1}^{m}\sum_{i,j=1}^{n} \left\| \frac{\partial^2 \mathbf{G}_k[f]}{\partial x_i \partial x_j} \right\|_{H_r^1(\Omega)} \\
&\quad \le \frac{Cm}{\varepsilon^4}\|a_{ij}\|_{\mathcal{D}}\Big(\|f\|_{H_z^1(\Omega)} + \|\mathbf{G}[f]\|_{H_r^1(\Omega)} + \|\nabla \mathbf{G}[f]\|_{H_r^1(\Omega)}\Big).
\end{aligned}
\tag{3.16}
$$

Notice that $\mathbf{G}[f]$ can be approximated by smooth functions in $C_0^\infty(\overline{\Omega})$ in $W^{2,1}$-norm, embedding $W^{2,1}(\Omega) \subset H^{1,1}(\Omega)$ is compact, and $m \le C|\Omega|/\varepsilon^n$. We have

$$\sum_{i,j=1}^n \left\| \frac{\partial^2 \mathbf{G}[f]}{\partial x_i \partial x_j} \right\|_{H_r^1(\Omega)} \le C(\varepsilon, \Omega)\|a_{ij}\|_{\mathcal{D}} \|f\|_{H_z^1(\Omega)}$$

where $C(\varepsilon, \Omega)$ is a constant depending only on ε, Ω and $\|a_{ij}\|_{\mathcal{D}}$. Here $H^{1,1}(\Omega)$ denotes the space of all functions with themself and their gradients belong to $H^1(\Omega)$. Therefore, the proof of Theorem 3.4 is complete.

4. – Sharper condition of $\partial\Omega$

It is clear that $\partial\Omega \in C^{1,1}$ is the minimum condition so that $\frac{\partial^2 \mathbf{G}[\cdot]}{\partial x_i \partial x_j}$ is bounded from $BMO_z(\Omega)$ to $BMO_r(\Omega)$. However, the condition $\partial\Omega \in C^{1,1}$ can be weakened so that $\frac{\partial^2 \mathbf{G}[\cdot]}{\partial x_i \partial x_j}$ is bounded from $H_z^1(\Omega)$ to $H_r^1(\Omega)$. The main purpose of this section is to prove the following theorem.

THEOREM 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a defining function ρ such that $\nabla\rho$ satisfying the Dini's condition on $\partial\Omega$. Let $A = (a_{jk})$ be an $n \times n$ matrix satisfying* (1.1), *and let $a_{jk} \in C^1(\overline{\Omega})$. Then the equation* (1.2) *has a unique solution $\mathbf{G}[f]$ such that $\frac{\partial^2}{\partial x_i \partial x_j}\mathbf{G}[\cdot]$ is bounded on $H_z^1(\Omega)$ to $H_r^1(\Omega)$.*

PROOF. The proof of Theorem 4.1 is similar to the proof of Theorem 3.4. But in the case $B(x_k; \varepsilon) \cap \partial\Omega \ne \emptyset$ is much subtle. Therefore, we shall give a detail argument for this case. In fact, the difficulty is from the linear term by when we make a change of coordinates. As we did in Section 3, since $\partial\Omega$ has a defining function ρ so that $\nabla\rho \in \mathcal{D}(\Omega)$, the Dini's class, it follows that there is a map $\Phi^0 : B_\varepsilon^+ \to B(x_k; \varepsilon) \cap \partial\Omega$ with $\nabla\Phi \in \mathcal{D}(\Omega)$. Let Ψ^0 be the inverse map of Φ^0. Denote $\Omega_1 = \Phi(B_\varepsilon^+)$ and let Ψ_p be the solution of

$$\sum_{k,\ell=1}^n a_{k,\ell} \frac{\partial^2 \Psi_p}{\partial x_k \partial x_\ell} = 0 \quad \text{in } \Omega_1, \quad \Psi_p = \Psi_p^0 \quad \text{on } \partial\Omega_1.$$

Then $\Psi'(x)$ is non-degenerate in Ω_1 when ε is small enough. Let $b_{ij}(y)$ be defined by

$$a_{k\ell}(\Phi(y)) = \sum_{i,j=1}^n b_{ij}(y) \frac{\partial \Phi_k(y)}{\partial y_i} \frac{\partial \Phi_\ell}{\partial y_j}.$$

Then

$$\begin{aligned}
0 &= \sum_{k,\ell=1}^{n} a_{k,\ell}\frac{\partial^2\Psi_p}{\partial x_k\partial x_\ell}\\
&= \sum_{k,\ell=1}^{n}\sum_{i,j=1}^{n} b_{ij}(y)\frac{\partial\Phi_k(y)}{\partial y_i}\frac{\partial\Phi_\ell}{\partial y_j}\frac{\partial^2\Psi_p}{\partial x_k\partial x_\ell}\\
&= \sum_{i,j=1}^{n} b_{ij}(y)\frac{\partial^2}{\partial y_i\partial y_j}(\Psi_p(\Phi)) - \sum_{k=1}^{n}\frac{\partial\Psi_p}{\partial x_k}\sum_{i,j=1}^{n} b_{ij}\frac{\partial\Phi_k}{\partial y_i\partial y_j}\\
&= \sum_{i,j=1}^{n} b_{ij}(y)\frac{\partial^2}{\partial y_i\partial y_j}(y_p) - \sum_{k=1}^{n}\frac{\partial\Psi_p}{\partial x_k}\sum_{i,j=1}^{n} b_{ij}\frac{\partial\Phi_k}{\partial y_i\partial y_j}\\
&= -\sum_{k=1}^{n}\frac{\partial\Psi_p}{\partial x_k}\sum_{i,j=1}^{n} b_{ij}\frac{\partial\Phi_k}{\partial y_i\partial y_j}\,.
\end{aligned}$$

Since $(\frac{\partial\Psi_p}{\partial x_k})$ is non-singular matrix, this implies that

$$\sum_{i,j=1}^{n} b_{ij}\frac{\partial\Phi_k}{\partial y_i\partial y_j} = 0, \quad k = 1, 2, \cdots, n.$$

Assume $\mathbf{G}[f]$ is the unique solution of

$$\sum_{ij=1}^{n} a_{ij}(x)\frac{\partial^2\mathbf{G}[f]}{\partial x_i\partial x_j} = f, \quad x\in\Omega; \quad \mathbf{G}[f] = 0 \quad \text{on} \quad \partial\Omega. \tag{4.1}$$

Let $v(y) = \mathbf{G}[f]\circ\Phi(y)$. Then

$$\frac{\partial^2 v}{\partial y_i\partial y_j} = \sum_{k,\ell=1}^{n}\frac{\partial^2\mathbf{G}[f]}{\partial x_k\partial x_\ell}\circ\Phi(y)\frac{\partial\Phi_k}{\partial y_i}\frac{\partial\Phi_\ell}{\partial y_j} + \sum_{k=1}^{n}\frac{\partial\mathbf{G}[f]}{\partial x_k}\circ\Phi(y)\frac{\partial^2\Phi_k}{\partial y_i\partial y_j}\,. \tag{4.2}$$

Therefore,

$$\sum_{i,j=1}^{n} b_{ij}\frac{\partial^2 v}{\partial y_i\partial y_j} = f\circ\Phi(y) + \sum_{k=1}^{n}\frac{\partial G[f]}{\partial x_k}\circ\Phi\sum_{i,j=1}^{n} b_{ij}\frac{\partial\Phi_k}{\partial y_i\partial y_j} = f\circ\Phi(y)\,.$$

Let $\eta\in C_0^\infty(B(0,\varepsilon))$ so that $\eta = 1$ on $B(0,\varepsilon/2)$. Then

$$\sum_{i,j=1}^{n} b_{ij}\frac{\partial^2(\eta v)}{\partial y_i\partial y_j}(y) = \eta(y)f(\Phi(y)) + v(y)\sum_{i,j=1}^{n} b_{ij}\frac{\partial^2\eta}{\partial y_i\partial y_j} + 2\sum_{i,j=1}^{n} b_{ij}\frac{\partial\eta}{\partial y_i}\frac{\partial v}{\partial y_j}\,.$$

Since a_{ij} is C^1 and $\nabla\Psi$ is *Dini*, we have b_{ij} is *Dini*. Applying Theorem 3.4, we have

$$\left\| \frac{\partial^2(\eta v)}{\partial y_i \partial y_j} \right\|_{H_r^1(B^+_{\varepsilon/2})} \leq C\left\{ \|f\|_{H_z^1} + \|v\|_{H_z^1(B^+_{\varepsilon/2})} + |\nabla v\|_{H_z^1(B^+_{\varepsilon/2})} \right\}.$$

By the fact $W^{2,1} \subset H^{1,1}$ is compact, one can see that

$$\left\| \frac{\partial^2(\eta v)}{\partial y_i \partial y_j} \right\|_{H_r^1(B^+_{\varepsilon/2})} \leq C_\varepsilon \|f\|_{H_z^1}$$

where C is constant depending on ε and *Dini* norm of b_{ij}, as we explained in Section 3. By partition of the unity, we have completed the proof of Theorem 4.1.

Finally, we give the following remark.

REMARK 3. The argument of the proof of Theorem 4.1 can be used to prove the asssertion: With the assumption of Theorem 4.1, the second derivative of the Green's operator for the Dirichlet problem (1.2) is bounded from $H_r^1(\Omega)$ to $H_r^1(\Omega)$. As we have seen in Remark 1, there are two kinds of H_r^1 atoms. For a type (a) atom, the proof follows exactly like Theorem 4.1 since they are also H^1 atoms in $\mathbb{R}^n$. For a type (b) atom $a(x)$, since the support of $a(x)$ is contained in a ball B whose diameter is comparable to distance $\operatorname{dist}(B, \partial\Omega)$, we may extend $a(x)$ oddly as an H^1 atom. Similarly, we need to extend the Green's operator to a neighborhood of $\overline{\Omega}$ with reflection as what was done in [CKS] and [CDS], one can prove the above assertion, we leave the details to interested readers.

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