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The Arithmetic-Geometric Mean and Isogenies for Curves of Higher Genus

RON DONAGI – RON LIVNÉ

Abstract. Computation of Gauss's arithmetic-geometric mean involves iteration of a simple step, whose algebro-geometric interpretation is the construction of an elliptic curve isogenous to a given one, specifically one whose period is double the original period. A higher genus analogue should involve the explicit construction of a curve whose jacobian is isogenous to the jacobian of a given curve. The doubling of the period matrix means that the kernel of the isogeny should be a Lagrangian subgroup of the group of points of order 2 in the jacobian. In genus 2 such a construction was given classically by Humbert and was studied more recently by Bost and Mestre. In this article we give such a construction for general curves of genus 3. We also give a similar but simpler construction for hyperelliptic curves of genus 3. We show that for $g \geq 4$ no similar construction exists, and we also reinterpret the genus 2 case in our setup. Our construction of these correspondences uses the bigonal and the trigonal constructions, familiar in the theory of Prym varieties

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1. – Introduction

It is well-known that computation of Gauss's arithmetic-geometric mean involves iteration of a simple step, whose algebro-geometric interpretation is the construction of an elliptic curve isogenous to a given one, specifically one whose period is double the original period (for a modern survey see [Cox]). A higher genus analogue should involve the explicit construction of a curve whose jacobian is isogenous to the jacobian of a given curve. The doubling of the period matrix means that the kernel of the isogeny should be a Lagrangian subgroup of the group of points of order 2 in the jacobian. In genus 2 such a construction was given classically by Humbert [Hum] and was studied more

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recently by Bost and Mestre [Bo-Me]. In this article we give such a construction for general curves of genus 3. We also give a similar but simpler construction for hyperelliptic curves of genus 3. We show that the hyperelliptic construction is a degeneration of the general one, and we prove that the kernel of the induced isogeny on jacobians is a Lagrangian subgroup of the points of order 2. We show that for $g \geq 4$ no similar construction exists, and we also reinterpret the genus 2 case in our setup.

To construct these correspondences we use the bigonal and the trigonal constructions, familiar in the theory of Prym varieties ([Don]). In genus 2 Bost and Mestre note that Humbert's construction induces on jacobians an isogeny whose kernel is of type $(\mathbf{Z}/2\mathbf{Z})^2$. We show that Humbert's construction is an instance of the bigonal construction, and prove that the above kernel is a Lagrangian subgroup of the points of order 2. In fact Bost and Mestre use Humbert's construction to give a variant of Richelot's genus 2 arithmetic-geometric mean. In light of the clear analogy, in particular the fact that a generic principally polarized abelian threefold is a jacobian, one might hope that our construction could be used in a similar way.

We work throughout over an algebraically closed field of characteristic 0. However, our methods clearly extend more generally. For example, the results of Section 4 hold if the characteristic is not 2, and those of Sections 5 and 6 if it is > 3 .

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2. – Preliminaries

POLARIZATIONS. For an abelian variety A denote by $A[n]$ the kernel of multiplication by n . In the sequel we will need the following standard facts and notation.

1. A polarization Θ on an abelian variety A induces by restriction a polarization Θ_B on any abelian subvariety B of A .
2. Recall that the type of a polarization Θ on a g -dimensional abelian variety is a g -tuple of positive integers $d_g | \dots | d_2 | d_1$. We say that Θ is a principal polarization if it is of type $1^g = (1, \dots, 1, 1)$ (g times). In that case, suppose that p is a prime, and that K is a subgroup of $A[p]$ isomorphic to $(\mathbf{Z}/p\mathbf{Z})^r$ and isotropic for the Weil pairing w_p . Then Θ induces a polarization on A/K , characterized by the property that its pull back to A is $p\Theta$. Its type is then $p^{g-r} \cdot 1^r$. In this situation we will say that K is a Lagrangian subgroup of $A[p]$ if $r = g$.
3. The type of a polarization is preserved under continuous deformations.

DOUBLE COVERS. Given a double cover, i.e. a finite morphism $\pi : \tilde{C} \rightarrow C$ of degree 2 between smooth projective curves, the Prym variety $\text{Prym}(\tilde{C}/C)$ is defined to be the connected component of the kernel of the norm map

$$\pi_* : \text{Jac}(\tilde{C}) \rightarrow \text{Jac}(C).$$

It is an abelian variety, and it has a natural principal polarization when π is unramified, namely one half of the polarization induced on it as an abelian subvariety of $\text{Jac}(\tilde{C})$ ([Mum2]). This definition extends to singular curves C, \tilde{C} , if we interpret Jac as the (not necessarily compact) generalized jacobian. This was studied by Beauville [Bea]. Particularly important for us will be the cases when 1. C, \tilde{C} have only ordinary double points, 2. $\pi^{-1}(C_{\text{sing}}) = \tilde{C}_{\text{sing}}$, and 3. for each $x \in C_{\text{sing}}$ the inverse image $\pi^{-1}(x)$ consists of a single point, and each branch of $\pi^{-1}(x)$ maps to a different branch of x and is ramified over it. (We shall then say that π is of Beauville type at x .) In such cases $\text{Prym}(\tilde{C}/C)$ is compact, and the following three conditions are equivalent:

1. π is unramified away from C_{sing} .
2. The arithmetic genera satisfy $g(\tilde{C}) = 2g(C) - 1$.
3. The cover \tilde{C}/C is a flat limit of smooth unramified double covers.

We shall call a cover satisfying these conditions allowable; from the third condition we see that the Prym is principally polarized in such a case.

Let C be a curve having only ordinary double points as singularities, and let $\nu_x : N_x \rightarrow C$ be the normalization map of exactly one such singular point x . We denote by $L(x)$ the line bundle of order 2 in $\text{Ker } \nu_x^*$. (It is obtained from the trivial line bundle on N_x by gluing the fibers over the two inverse images of x with a twist of -1 relative to the natural identification.)

LEMMA 1. *Let $\pi : \tilde{C} \rightarrow C$ be an allowable double cover, $\nu\pi : \nu\tilde{C} \rightarrow \nu C$ its (partial) normalization at $r \geq 1$ ordinary double points x_1, \dots, x_r . Let g be the (arithmetic) genus of the partial normalization νC , so the arithmetic genus of C is $g + r$. Then $\text{Prym}(\tilde{C}/C)$ has a principal polarization, $\text{Prym}(\nu\tilde{C}/\nu C)$ has a polarization of type $2^g 1^{r-1}$, and the pullback map $\nu^* : \text{Prym}(\tilde{C}/C) \rightarrow \text{Prym}(\nu\tilde{C}/\nu C)$ is an isogeny of degree 2^{r-1} . The kernel of ν^* is the subgroup of $\text{Prym}(\tilde{C}/C)[2]$ generated by the pairwise differences of the line bundles $L(x_i)$ defined above. This subgroup is isotropic for the mod 2 Weil pairing w_2 .*

PROOF. The generalized jacobians fit in short exact sequences

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbf{G}_m^r & \rightarrow & \text{Jac}(\tilde{C}) & \rightarrow & \text{Jac}(\nu\tilde{C}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathbf{G}_m^r & \rightarrow & \text{Jac}(C) & \rightarrow & \text{Jac}(\nu C) & \rightarrow & 0 \end{array}$$

where the vertical maps are the norm maps induced by π and by $\nu\pi$. We compare the kernels: to begin with, the kernel of the norm map is connected for ramified double covers (in particular for $\nu\tilde{C}/\nu C$), and has two components

for unramified covers. This is shown in [Mum2] in the nonsingular case, and so by continuity this holds also for allowable singular covers (in particular for \tilde{C}/C). The multiplicative groups parametrize extension data and the norm is the squaring map. So the short exact sequence of kernels gives

$$0 \rightarrow (\mathbf{Z}/2\mathbf{Z})^r \rightarrow \mathrm{Prym}(\tilde{C}/C) \times \mathbf{Z}/2\mathbf{Z} \xrightarrow{\nu^*} \mathrm{Prym}(\nu\tilde{C}/\nu C) \rightarrow 0,$$

and the first part of the lemma follows.

To prove that the subgroup $(\mathbf{Z}/2\mathbf{Z})^{r-1}$ of $\mathrm{Prym}(\tilde{C}/C)[2]$ is isotropic for w_2 , notice that its generators are reductions modulo 2 of the vanishing cycles for \tilde{C} , and vanishing cycles for distinct ordinary double points are disjoint. Therefore these vanishing cycles have 0 intersection number in \mathbf{Z} - (or \mathbf{Q} -) homology. By the definition of the polarization of $\mathrm{Prym}(\tilde{C}/C)$ in Section 2, and the well-known expression for the Weil pairing in terms of the intersection (or cup product) pairing (see e.g. [Mum, theorem 1, Ch. 23]), the rest of the lemma follows.

3. – The bigonal and the trigonal constructions

There are several elementary constructions which associate a double cover of some special kind with another cover (or curve) with related Prym (of Jacobian). We now review the bigonal and the trigonal constructions, following [Don]. Assume we are given smooth projective curves \tilde{C} , C and K and surjective maps $f : C \rightarrow K$ and $\pi : \tilde{C} \rightarrow C$, so that $\deg \pi = \deg f = 2$ over any component. The bigonal construction associates new curves and maps of the same type $\tilde{C}' \xrightarrow{\pi'} C' \xrightarrow{f'} K$ as follows. Let $U \subset K$ be the maximal open subset over which $f\pi$ is unramified. Then \tilde{C}' represents over U the sheaf of sections, in the complex or the étale topology, of $\pi : (f\pi)^{-1}U \rightarrow f^{-1}U$. It is a 4-sheeted cover of U . We then view $\tilde{C}'|_U$ as a locally closed subvariety of $\tilde{C} \times \tilde{C}$ and define \tilde{C}' as the closure. The projection to U extends to a morphism $\tilde{C}' \rightarrow K$, and the involution ι of $\tilde{C}'|_U$ which sends a section to the complementary section extends to \tilde{C}' . We define $C' = \tilde{C}'/\iota$ and f' and π' as the quotient maps.

We will need to extend this construction to allowable covers of curves with ordinary double points; however in a family acquiring a singularity of Beauville type the arithmetic genus of the resulting \tilde{C}' is not locally constant. More technically, the naive construction as the closure of $\tilde{C}'|_U$ in $\tilde{C} \times \tilde{C}$ is not flat in families, which is not adequate for our purposes: for example, we want the bigonal construction to be symmetric.

To achieve this, we define the bigonal construction for singular allowable covers by *choosing* a flat family of smooth covers whose limit is our allowable cover, and *defining* the construction to be the limit of the construction for the

nonsingular fibers. Beauville’s results imply that this is well defined, and does give a symmetric construction: this is more or less clear except at a singularity of Beauville type. There the problem reduces to a local calculation whose answer, which we record in 2. below, is visibly symmetric.

We will need a few properties of this construction (see [Don, Section 2.3])

1. As we said, the construction over U is symmetric: starting with \tilde{C}', \dots, f' gives back \tilde{C}, \dots, f .
2. Denote the type of \tilde{C}/C at a point $k \in K$ by
 - $c=/--$ if C is unramified over k and \tilde{C} is ramified over exactly one point in $f^{-1}(k)$;
 - ξ/c if C is ramified over k but \tilde{C} is unramified over the point $f^{-1}(k)$;
 - $cc/--$ if C is unramified over k and \tilde{C} is ramified over both branches of C over k ;
 - ∞/x if both C, \tilde{C} have ordinary double points above it, and \tilde{C}/C is of Beauville type there.

If \tilde{C}/C is of type $c=/--, \xi/c, cc/--, \infty/x$ at k then \tilde{C}'/C' is respectively of type $\xi/c, c=/--, \infty/x, cc/--$ there.

Notice that normalization takes type ∞/x to type $cc/--$.

3. The natural 2-2 correspondence between \tilde{C} and \tilde{C}' induces an isogeny $\text{Prym}(\tilde{C}/C) \rightarrow \text{Prym}(\tilde{C}'/C')$, whose kernel is the same as the kernel of the natural isogeny $\text{Prym}(\tilde{C}/C) \rightarrow \text{Prym}(\tilde{C}/C)^\vee$ induced by the polarization from $\text{Prym}(\tilde{C}/C)$ to its dual abelian variety $\text{Prym}(\tilde{C}/C)^\vee$. In other words we get an isomorphism $\text{Prym}(\tilde{C}/C)^\vee \xrightarrow{\sim} \text{Prym}(\tilde{C}'/C')$ (cf. Pantazis [Pan], at least when $K = \mathbf{P}^1$ which is all we need). As a check, let $a, b, c,$ and d be the numbers of points where \tilde{C}/C is of type $c=/--, \xi/c, cc/--,$ and ∞/x respectively. Then by Lemma 1 the polarization type for $\text{Prym}(\tilde{C}/C)$ is $1^{\frac{a+2c}{2}-1} 2^{\frac{b+2d}{2}-1}$. Similarly the polarization type for $\text{Prym}(\tilde{C}'/C')$ is obtained by interchanging a with b and c with d , and this gives exactly the type dual to the one of $\text{Prym}(\tilde{C}/C)$.

For Recillas’s trigonal construction start with $K, C, \tilde{C}, \pi,$ and f as before except that f now has degree 3. We get a cover $g : X \rightarrow K$ of degree 4 by making over the smooth unramified part U , defined as before, a construction analogous to what we previously did to get C' . Namely, let \tilde{X}/U represent the sheaf of sections of $\pi : (f\pi)^{-1}U \rightarrow f^{-1}U$, and define X/U as the quotient of \tilde{X} divided by ι (which is defined as before). In the nonsingular case we define X as the closure of X/U in $X \times X \times X$, and in the general allowable case by taking a flat limit of the construction for smooth, unramified covers. Here we have [Don, Section 2.4]

1. Over U the construction is reversible: $C|_U$ represents the sheaf of partitions of $X|_U$ to two pairs of sections, and \tilde{C}' represents the choice of one of these pairs.

2. Denote the type of \tilde{C}/C at a point $k \in K$ by

- $\xi_{=/C}$ – for C, \tilde{C} if C has exactly one simple branch point over K and π is unramified over $f^{-1}(k)$;
- $c_{C=/---$ if f is unramified at k and π is branched over two of the branches of f and unramified over the third;
- $\varkappa_{=/\times}$ – if two branches of C over K cross normally, the third is unramified, and moreover, if \tilde{C}/C is of Beauville type over the double point and unramified over the unramified branch.

Then X has exactly one simple branch point at a point $k \in K$ of type $\xi_{=/C}$ – for \tilde{C}/C , and we denote by c_{--} the type of X over k . Conversely, if X is of type c_{--} at k then \tilde{C}/C is of type $\xi_{=/C}$ – there. If \tilde{C}/C is of type $c_{C=/---$ at k then X has two simple branch points over k , which we denote by type c_C . Here the situation is not reversible: if X is of type c_C at k then \tilde{C}/C is of type $\varkappa_{=/\times}$ – there. Notice that normalization takes type $\varkappa_{=/\times}$ – to type $c_{C=/---$.

3. If $K \simeq \mathbf{P}^1$ and \tilde{C}/C is allowable, then X is smooth and $\text{Jac}(X) \simeq \text{Prym}(\tilde{C}/C)$. This is due to Recillas when \tilde{C}/C is smooth unramified, and again limiting arguments imply this in general.

4. – The genus 2 case

Humbert’s correspondence of curves of genus 2 was studied by Bost and Mestre (see [Hum], [Bo-Me]). We shall show how to make this correspondence via the bigonal construction, and use this to determine the type of the isogeny.

Humbert’s construction starts with a conic C in \mathbf{P}^2 with 6 general points on it (see Remark 3 below), which are given as 3 unordered pairs $\{P'_i, P''_i\}$, $i = 1, 2, 3$. It associates to these 3 new unordered pairs of points, all distinct, on C as follows. Let $\overline{P'_i P''_i}$ be the 3 lines joining paired points, and let l_k be the intersection of $\overline{P'_i P''_i}$ and $\overline{P'_j P''_j}$ if $\{i, j, k\} = \{1, 2, 3\}$. The new 3 unordered pairs of points on C are then the pairs of points of tangency to C from the l_k ’s.

For our purposes it is more convenient to view the new points as lying on the conic C^* dual to C in the dual plane \mathbf{P}^{2*} . A point of \mathbf{P}^{2*} is a line in \mathbf{P}^2 ; it is in C^* if and only if this line is tangent to C . Let $\phi : \mathbf{P}^2 \rightarrow \mathbf{P}^{2*}$ be the isomorphism defined by C ; namely, for $P \notin C$ there are two tangents to C through P , and $\phi(P)$ is the line joining their points of tangency. For $P \in C$, $\phi(P)$ is the tangent to C at P . Under the isomorphism $\phi|_C : C \xrightarrow{\sim} C^*$, Humbert’s new pairs go to the pairs L'_k, L''_k of tangents to C through l_k .

THEOREM 2. *Let $\pi : H \rightarrow C$ and $\pi^* : H^* \rightarrow C^*$ be double covers branched over the old and new sets of points respectively. Then there is an isogeny $\text{Jac}(H) \rightarrow \text{Jac}(H^*)$ whose kernel is a Lagrangian subgroup of $\text{Jac}(H)$ [2].*

PROOF. Choose some $k \in \{1, 2, 3\}$. The set $L^* = L_k^*$ of lines through l_k is the line in \mathbf{P}^{2*} dual to l_k . Let $f : C \rightarrow L^*$ be the “projection” sending each point of C to the line joining it to l_k . Dually, let $f^* : C^* \rightarrow L = \overline{P'_k P''_k}$ be the “projection” sending each tangent line of C to its intersection with L . Let $\psi : L \rightarrow L^*$ be the isomorphism sending a line through l_k to its intersection with L . The maps f and f^* have degree 2, and hence also $g = \psi f^*$ has degree 2. Both coverings $H \xrightarrow{\pi} C \xrightarrow{f} L^*$ and $H^* \xrightarrow{\pi^*} C^* \xrightarrow{g} L^*$ are unramified over the complement in L^* of the six points

- The tangents L'_k, L''_k to C through l_k ; there H/C is of type \overline{c}/c and H^*/C^* is of type $c=/--$.
- The lines $\overline{P'_i P''_i}$ and $\overline{P'_j P''_j}$ whose intersection defines l_k ; there both H/C and H^*/C^* are of type $c c / --$.
- The lines $\overline{P'_k l_k}$ and $\overline{P''_k l_k}$; there H/C is of type $c=/--$ and H^*/C^* is of type \overline{c}/c .

It follows that if we perform the bigonal construction on $H \rightarrow C \rightarrow L^*$, the two points $\overline{P'_i P''_i}$ and $\overline{P'_j P''_j}$ of L^* are of Beauville type for the resulting cover $H' \rightarrow C' \rightarrow L^*$ and there are no other singularities (see Section 3). The preceding analysis of the ramification of H^*/C^* , combined with the one for the bigonal construction H'/C' in Section 3 shows that the normalization of H'/C' is isomorphic to H^*/C^* . It remains to determine the kernel of the induced isogeny on jacobians; by Pantazis’s result recalled above, it factors as

$$\begin{aligned} \text{Jac } H &\simeq \text{Prym}(H/C) \xrightarrow{\sim} \text{Prym}(H/C)^\vee \xrightarrow{\sim} \text{Prym}(H'/C') \\ &\xrightarrow{\nu^*} \text{Prym}(H^*/C^*) \simeq \text{Jac } H^*. \end{aligned}$$

To compute the kernel of ν^* we cannot use Lemma 1 directly, since H'/C' is not allowable, being ramified over two points $x' \in g^{-1}(L'_k), x'' \in g^{-1}(L''_k)$. Instead glue x' to x'' to obtain a curve with one more double point C'' and glue their inverse images in H' to get a curve H'' , which is now an allowable cover of C'' (H'' is obtained from H by gluing the Weierstrass points in pairs). We have maps of covers $H^*/C^* \rightarrow H'/C' \rightarrow H''/C''$ inducing maps of Prym varieties. Applying Lemma 1 twice now gives that the kernel of $\text{Prym}(H''/C'') \rightarrow \text{Prym}(H/C)$ is an isotropic subgroup isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$ and that $\text{Prym}(H''/C'')$ is isomorphic to $\text{Prym}(H^*/C^*)$. This implies that $\text{Ker } \nu^*$ is as asserted, completing the proof of the Theorem.

REMARK 3. The points $\{P'_i, P''_i\}$ are assumed general only to guarantee that they are distinct and that the resulting new 6 points are also distinct (for which it suffices that the tangents to C from l_k in the proof do not touch C at P'_k nor at P''_k). In the case considered in [Bo-Me] this holds, because they assume that C and the points are real and satisfy some ordering relations.

5. – The hyperelliptic genus 3 case

In this section we will solve our problem in the hyperelliptic case: we will construct a correspondence between the generic hyperelliptic curve of genus 3 and a certain non-generic curve of genus 3 (which is not hyperelliptic). Let H be a hyperelliptic curve of genus 3 and let $\pi_1 : H \rightarrow \mathbf{P}^1$ be the hyperelliptic double cover. Choose a grouping in pairs of the 8 branch points $w_1, \dots, w_8 \in \mathbf{P}^1$ of π_1 . We claim that there exists a map $g_1 : \mathbf{P}^1 \rightarrow \mathbf{P}^1$, of degree 3, which identifies paired points. This can be seen in several ways. Firstly, let T be the curve obtained from \mathbf{P}^1 by identifying paired points to ordinary double points. We think of T as a curve of genus 4 and take its canonical embedding to \mathbf{P}^3 . As in the nonsingular case, the canonical map is well behaved, and in particular the canonical image of T lies on a unique, generically nonsingular quadric by the Riemann-Roch theorem. Projecting via either of the two ruling of this quadric will give the desired map g_1 . Notice that by its construction g_1 factors as $\mathbf{P}^1 \xrightarrow{\nu} T \xrightarrow{g} \mathbf{P}^1$, where ν is a normalization map.

Another way to get g_1 is to embed \mathbf{P}^1 in \mathbf{P}^3 as a rational normal curve. We look for a projection from \mathbf{P}^3 to \mathbf{P}^1 which identifies paired points. The center of this projection is a line L which must meet the 4 lines joining the pairs. The grassmanian $G(1, \mathbf{P}^3)$ of lines in \mathbf{P}^3 is naturally a quadric in \mathbf{P}^5 and the condition to meet a line is a linear condition. We see again that there is always at least one such L , and generically two.

We now perform the trigonal construction. This gives a map of degree 4 $f : C \rightarrow \mathbf{P}^1$ sitting in a diagram

$$(1) \quad \begin{array}{ccc} & & H \\ & & \downarrow \pi_1 \\ & & \mathbf{P}^1 \\ C & \searrow f \quad \swarrow g_1 & \\ & \mathbf{P}^1 & \end{array}$$

Let w_{12}, \dots, w_{78} be the 4 images of the w_i 's under g_1 , with the indices indicating the grouping. By the Riemann-Hurwitz formula there are generically 4 points a_1, \dots, a_4 in \mathbf{P}^1 over which g_1 is branched, with a simple branch point over each. Hence H/\mathbf{P}^1 is of type $\overset{\subset}{\subset} = / \subset -$ at each a_i and of type $\subset \subset = / - - -$ at each $w_{2i-1, 2i}$. From the properties of the trigonal construction we get $2 - 2g(C) = 8 - 8 - 4$, so that C has genus 3. The trigonal construction gives a birational correspondence between

- The moduli of the data $(H \xrightarrow{\pi_1} \mathbf{P}^1 \xrightarrow{g_1} \mathbf{P}^1)$ with 4 points of type $\overset{\subset}{\subset} = / \subset -$ and 4 points of type $\subset \subset = / - - -$.
- A component of the Hurwitz scheme parametrizing 4-sheeted covers $f : C \rightarrow \mathbf{P}^1$ with 4 simple branch points and 4 double branch points.

Each of these moduli spaces is 5 dimensional. (Another component of this Hurwitz scheme parametrizes bielliptics, namely maps $f : C \rightarrow \mathbf{P}^1$ which factor through a double cover $E \rightarrow \mathbf{P}^1$ where E is elliptic. Curves in this latter

component are taken by the trigonal construction to towers $H \rightarrow \overline{H} \rightarrow \mathbf{P}^1$ where $\overline{H} = A \cup B$ is reducible, with A, B of degrees 1, 2 respectively over \mathbf{P}^1 . We shall not need this component in what follows.)

The key point for us is that the trigonal construction induces an isogeny $\text{Jac}(C) \rightarrow \text{Jac}(H)$ whose kernel is Lagrangian in $\text{Jac}(C)[2]$. More precisely we have the following

PROPOSITION 4. *Let $\mathbf{P}^1 \xrightarrow{\nu} T \xrightarrow{g} \mathbf{P}^1$ be as before, and let \tilde{T} be the curve obtained by identifying the Weierstrass points in H to ordinary double points with the same grouping as the one we chose to get T . Then*

1. *Diagram (1) extends to*

$$\begin{array}{ccccc}
 & & \tilde{T} & \xleftarrow{\tilde{\nu}} & H \\
 & & \pi \downarrow & & \downarrow \nu\pi \\
 C & & T & \xleftarrow{\nu} & \mathbf{P}^1 \\
 & f \searrow & & \swarrow g & \\
 & & \mathbf{P}^1 & &
 \end{array}$$

Here $\tilde{\nu} : H \rightarrow \tilde{T}$ is the normalization map, and we view $\nu\pi := \pi_1 : H \rightarrow \mathbf{P}^1$ as the normalization of $\pi : \tilde{T} \rightarrow T$.

2. $\tilde{\nu}$ induces an isogeny of polarized abelian varieties $\tilde{\nu}^* : \text{Prym}(\tilde{T}/T) \rightarrow \text{Jac}(H)$ whose kernel is Lagrangian in $\text{Prym}(\tilde{T}/T)[2]$.
3. Let $\phi : \text{Jac}(C) \rightarrow \text{Jac}(H)$ be the isogeny obtained by composing $\tilde{\nu}^*$ with the isomorphism $\text{Jac}(C) \simeq \text{Prym}(\tilde{T}/T)$. Then the kernel of ϕ is Lagrangian in $\text{Jac}(C)[2]$, and the kernel of the dual isogeny $\phi^* : \text{Jac}(H) \rightarrow \text{Jac}(C)$ is the Lagrangian subgroup of $\text{Jac}(H)[2]$ generated by the differences of identified Weierstrass points.

PROOF. Part 1. holds because \tilde{T}/T is allowable. The pairs of points of H identified by ν lie over points of type $\infty = / \times -$ for T, \tilde{T} . Hence they are branch points for π , namely Weierstrass points. The rest follows from Lemma 1.

6. – The generic genus 3 case

Let C be a generic curve of genus 3. In this section we shall give a construction of a curve C' of genus 3 and an isomorphism $\text{Jac}(C)/L \simeq \text{Jac}(C')$ where L is a Lagrangian subgroup of $\text{Jac}(C)[2]$. Let $f : C \rightarrow \mathbf{P}^1$ be a map of degree 4, and let b_1, b_2 be points in \mathbf{P}^1 such that f has two simple branch points over each b_i . It is easy to show such f, b_1, b_2 exist, and in fact we will parametrize the space of such f 's in the end of this section.

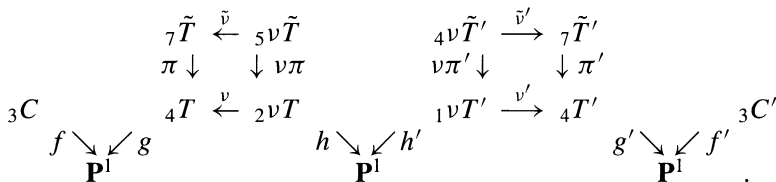
We perform the trigonal construction on f . This gives curves T, \tilde{T} and maps $g : T \rightarrow \mathbf{P}^1$ and $\pi : \tilde{T} \rightarrow T$, with $\deg g = 3$ and $\deg \pi = 2$. Let $\tilde{\nu} : \nu\tilde{T} \rightarrow \tilde{T}$ and $\nu : \nu T \rightarrow T$ be normalization maps and let $\nu\pi : \nu\tilde{T} \rightarrow \nu T$

be the map induced by π . The properties of the trigonal construction show the following. Firstly, T and \tilde{T} have each two ordinary double points, one over each b_i , and no other singularities. Next, the map $g\nu : \nu T \rightarrow \mathbf{P}^1$ has exactly 8 branch points, all simple, one over each a_i . It follows that the genus $g(\nu T)$ is 2 and therefore the arithmetic genus $g(T)$ is 4. The map $\nu\pi$ has exactly 4 ramification points P_i, Q_i , two over each b_i for $i = 1, 2$, and hence $g(\nu\tilde{T}) = 5$ and $g(\tilde{T}) = 7$.

Since νT has genus 2, it is hyperelliptic. Let $h : \nu T \rightarrow \mathbf{P}^1$ be the hyperelliptic double cover, and let $w_1, \dots, w_6 \in \mathbf{P}^1$ be the branch points of h . The bigonal construction gives curves and maps of degree 2 $\nu\tilde{T}' \xrightarrow{\nu\pi'} \nu T' \xrightarrow{h'} \mathbf{P}^1$. The points in \mathbf{P}^1 over which $h\nu\pi$ is not étale are the 6 w_i 's, which are of type \bar{c}/c for νT and $\nu\tilde{T}$, and the 4 points $h(P_i), h(Q_i), i = 1, 2$, which are of type $c=/--$. The types get reversed for $\nu T'$ and $\nu\tilde{T}'$, and in particular $\nu T'$ is ramified exactly over the $h(P_i)$'s and the $h(Q_i)$'s. It follows that $g(\nu T') = 1$. We also see that $\nu\pi'$ has 6 branch points, say w'_1, \dots, w'_6 , one over each of the w_i 's, and hence $g(\nu\tilde{T}') = 4$. The curves $\nu T'$ and $\nu\tilde{T}'$ are nonsingular.

Choose a grouping of the w_i 's in 3 pairs. Identify the corresponding w'_i 's in $\nu T'$ to get a curve T' with 3 ordinary double points, say $w'_{12}, w'_{34}, w'_{56}$, the indices indicating the groupings. T' has arithmetic genus 4. Likewise identify the corresponding points above the w'_i 's on $\nu\tilde{T}'$ to obtain a curve \tilde{T}' with 3 ordinary double points and arithmetic genus 7.

As in the nonsingular case, the canonical embedding sends T' to \mathbf{P}^3 and the image sits on a unique, generically smooth quadric. Choosing one of the two rulings of this quadric gives a map $g' : T' \rightarrow \mathbf{P}^1$. This map is of degree 3, because the canonical curve is a curve of type (3, 3) on the quadric. The map $g'\nu' : \nu T' \rightarrow \mathbf{P}^1$ is ramified over $n = 6$ points, since $2 - 2g(\nu T') = 0 = 3(2 - 2g(\mathbf{P}^1)) - n$. Over these the pair $\nu T', \nu\tilde{T}'$ is of type $\bar{c}c=/--$. There are also 3 points of type $cc=----$, the images under g' of the identified pairs $w'_{12}, w'_{34}, w'_{56}$. The trigonal construction performed on $\nu\tilde{T}' \xrightarrow{\nu\pi'} \nu T' \xrightarrow{g'\nu'} \mathbf{P}^1$ gives a curve C' and a map $f' : C' \rightarrow \mathbf{P}^1$ of degree 4. We readily see it has genus 3. The following diagram summarizes the procedure:



Before stating our main result we need to discuss the choices made in the construction. Writing $f^{-1}(b_i) = 2(P_i + Q_i)$, we obtain a point of order 2

$$\alpha = \alpha(f) = P_1 + Q_1 - P_2 - Q_2.$$

in $\text{Jac}(C)$. The trigonal isomorphism $\text{Jac}(C) \simeq \text{Prym}(\tilde{T}/T)$ maps α to the difference $L(b_1) - L(b_2)$ (defined in the discussion preceding Lemma 1), which

is the nontrivial element in $\text{Ker } \nu^*$. Moreover the only choice made other than f is the grouping of w_1, \dots, w_6 under ν' . The differences of the corresponding paired points in νT are the nonzero elements of a Lagrangian subgroup L_0 of $\text{Jac}(\nu T)[2]$.

Now observe that the pullback to $\nu\tilde{T}$ by $\nu\pi$ of a line bundle of order 2 on νT is in the kernel of the norm map to νT . This gives a symplectic embedding

$$\iota : \text{Jac}(\nu T)[2] \hookrightarrow \text{Prym}(\nu\tilde{T}/\nu T)[2].$$

The image $\text{Im}(\iota)$ of ι can be described in two ways. On the one hand, it is the kernel of the polarization map $\text{Prym}(\nu\tilde{T}/\nu T) \rightarrow \text{Prym}(\nu\tilde{T}/\nu T)^\vee$. (Observe that $\text{Prym}(\nu\tilde{T}/\nu T)$ has a polarization of type 221 by Lemma 1, whose kernel is then isomorphic to $(\mathbf{Z}/2\mathbf{Z})^4$.) On the other hand, let α^\perp denote the orthogonal complement to (the image of) α in $\text{Prym}(\tilde{T}/T)$ for the Weil pairing w_2 . Then $\text{Im}(\iota)$ is also the pullback of α^\perp by the normalization map. Indeed, for $u \in \text{Jac}(\nu T)[2]$ we have $\iota(u) \in \alpha^\perp$ because

$$w_2(\iota(u), \alpha) = w_2(u, \text{Nm}_{\tilde{T}/T}(\alpha)) = 0,$$

and as both groups have cardinality 16 they coincide. Hence this image is isomorphic to $\alpha^\perp/\langle\alpha\rangle$. In particular, the inverse image L of L_0 in $\text{Jac}(C)$ is a Lagrangian subgroup of $\text{Jac}(C)[2]$ containing α .

Conversely, let $L \subset \text{Jac}(C)[2]$ be a Lagrangian subgroup. We will say that the choices f, ν' made in the course of the construction are compatible with L if $\alpha = \alpha(f)$ is in L and ν' corresponds to $L/\langle\alpha\rangle$ as above. We can now formulate our main theorem, to which we shall give two proofs:

THEOREM 5. *Let C' be the result of the construction applied to a curve C of genus 3 compatibly with a Lagrangian subgroup $L \subset \text{Jac}(C)[2]$. Then there is an induced isomorphism $\text{Jac}(C)/L \xrightarrow{\sim} \text{Jac}(C')$. In particular C' is independent of the (compatible) choices made in the construction.*

PROOF. The construction induces isogenies whose degrees are marked below:

$$\begin{array}{ccccc} \text{Jac}(C) \simeq & \text{Prym}(\tilde{T}/T) & \xrightarrow{\nu^*} & \text{Prym}(\nu\tilde{T}/\nu T) & \xleftarrow{\delta} & \text{Prym}(\nu\tilde{T}'/\nu T') \\ & \nu'^* & & & & \\ & \xleftarrow{4} & & & & \\ & & & \text{Prym}(\tilde{T}'/T') & \simeq & \text{Jac}(C'). \end{array}$$

Here the middle step δ is identified with the polarization map from an abelian variety of polarization type 211 to its dual. As before we identify ν^* with the quotient by α , so to construct our isomorphism $\text{Jac}(C)/L \simeq \text{Jac}(C')$ it would suffice to produce a natural map $\epsilon : \text{Prym}(\nu\tilde{T}/\nu T) \rightarrow \text{Prym}(\tilde{T}'/T')$ whose kernel is the subgroup $L/\langle\alpha\rangle$ of $\text{Prym}(\nu\tilde{T}/\nu T)$.

One way to do this is to define ϵ as the dual map of ν'^* , using δ to identify the dual of $\text{Prym}(\nu\tilde{T}'/\nu T')$ with $\text{Prym}(\nu\tilde{T}/\nu T)$, and using the principal

polarization on $\text{Prym}(\tilde{T}'/T')$ to view it as its own dual. Tracing through the definitions one verifies that $\text{Ker}(\epsilon)$ is indeed $L/\langle\alpha\rangle$ as asserted.

An alternative, and more geometric approach, is to show that the hyperelliptic case treated in Section 5 is a specialization of our present general construction. In fact the hyperelliptic case is obtained when the 4-sheeted cover $f : C \rightarrow \mathbf{P}^1$ happens to have 4, rather than the generic 2, double branch points. We shall see that this determines a preferred gluing ν' . In going to this special case we have to note that the limits of the curves νT , $\nu\tilde{T}$ (which we continue to denote with the same symbols) are no longer non-singular: they are now only partial normalizations of T , \tilde{T} , and the map $\nu\tilde{T} \rightarrow \nu T$ now has 2 points of Beauville type, at the singularities which were not normalized. The full normalizations, say $\nu\nu T$ and $\nu\nu\tilde{T}$, now have genera 0 and 3 respectively, and the resulting diagram

$$\begin{array}{ccc}
 & & \nu\nu\tilde{T} \\
 & & \pi \downarrow \\
 & & \nu\nu T \\
 C & & \\
 & f \searrow & \swarrow g' \\
 & & \mathbf{P}^1
 \end{array}$$

clearly coincides with diagram (1).

Regardless of the singularities of the intermediate curves, we will see that each of the abelian varieties in the diagram specializes to an abelian variety. In particular, the limit of $\text{Prym}(\nu\tilde{T}/\nu T)$ is, by Lemma 1, a 4-sheeted cover of $\text{Prym}(\nu\nu\tilde{T}/\nu\nu T) \simeq \text{Jac}(H)$, whose kernel is $L/\langle\alpha\rangle$. Below we will also identify the limit of $\text{Prym}(\tilde{T}'/T')$ with $\text{Jac}(H)$. This will produce the desired map ϵ in this special case, and hence in general.

For this, we note that the bigonal data $\nu\tilde{T} \rightarrow \nu T \rightarrow \mathbf{P}^1$ has 4, 2, 0 and 2 points of types $c=/--$, \bar{c}/c , $cc/--$, and ∞/\times respectively, which turn into points of types \bar{c}/c , $c=/--$, ∞/\times and $cc/--$, respectively, for $\nu\tilde{T}' \rightarrow \nu T' \rightarrow \mathbf{P}^1$. To obtain $\tilde{T}' \rightarrow T'$ we need to pair the 6 ramification points of $\nu\tilde{T}' \rightarrow \nu T'$. There are 15 ways to do this, of which one is distinguished: each pair of Beauville branches gets paired, as do the remaining two ramification points. Let T'_h be the intermediate object, obtained by gluing only the Beauville branches but not the remaining pair. It is a singular hyperelliptic curve of genus 3, and is a partial normalization of T' at the double point p . Let \tilde{T}'_h be the corresponding 1-point partial normalization of \tilde{T}' , of arithmetic genus 6.

To continue our construction, we need to identify the two g_3^1 's on T' : these two turn out to coincide, and the unique g_3^1 is in fact given by the g_2^1 on T'_h plus a base point at the double point p . To see this we examine what happens to our general construction of the g_3^1 in this case. The unique quadric surface through the canonical model of T' is now a quadric cone, with vertex at (the image of) p , because projection from p gives the canonical image of the hyperelliptic T'_h , which is the double cover of a conic. Therefore the two rulings, hence the two g_3^1 's, coincide and have a base point at p , as asserted.

At this point we need to turn the g_3^1 into a morphism, which requires us to blow up the point p . This results in a reducible trigonal curve $T'_i := T'_h \cup P$, where P is a copy of \mathbf{P}^1 intersecting T'_h in the two inverse images p_1, p_2 of p in T'_h . The trigonal map has degrees 2 and 1 respectively on the two components T'_h and P . This curve is indeed a flat limit, in the family of triple covers of \mathbf{P}^1 , of the trigonal curves encountered in the non-hyperelliptic situation. The corresponding double cover $\tilde{T}'_i \rightarrow T'_i$ is of Beauville type at all 4 of the singular points (the two singularities of T'_h plus p_1, p_2). Here $\tilde{T}'_i = \tilde{T}'_h \cup \tilde{P}$, where \tilde{P} is another copy of \mathbf{P}^1 , double cover of P branched at the points glued to p_1 and p_2 .

Now that we have identified the trigonal data, we can complete the construction. By example 2.10(iii) of [Don], or by inspection, we see that the result C' of applying the trigonal construction to the reducible trigonal data $(\tilde{T}'_h \cup \tilde{P}) \rightarrow (T'_h \cup P) \rightarrow \mathbf{P}^1$ is the 4-sheeted cover of \mathbf{P}^1 obtained by applying the bigonal construction to $\tilde{T}'_h \rightarrow T'_h \rightarrow \mathbf{P}^1$. But since the bigonal construction is reversible this is nothing but the hyperelliptic curve $H = \nu\nu\tilde{T}$ which resulted from the construction of Section 5, as claimed.

The degeneration just described involves a flat family of abelian varieties, so the polarization type and the type of the kernel of the isogeny on jacobians remain constant. From the hyperelliptic case we now see that L is the kernel of our isogeny in the general case. By Torelli's theorem, C' is determined by its polarized jacobian, which is $\text{Jac}(C)/L$. Hence C' is indeed independent of the choices (compatible with L) made during the construction. This concludes the proof of Theorem 5.

We now make some further comments on the choices we made in the course of the construction. Starting on the left, we fix the curve C , the Lagrangian subgroup L and an element $\alpha \in L$. Our g_4^1 's $f : C \rightarrow \mathbf{P}^1$ with $\alpha = \alpha(f)$ are determined by a divisor class in the intersection

$$Z = Z_\alpha = \Theta_C \cap (\alpha + \Theta_C) \subset \text{Pic}^2(C).$$

More accurately Z parametrizes the family of g_4^1 's (with the specified α), together with a marking of the two singular points b_1, b_2 . Interchanging these two points gives an involution i of Z induced by the involution $x \mapsto x + \alpha$ of $\text{Pic}^2(C)$, and it is the quotient of Z by i which parametrizes the g_4^1 's alone. Z is also invariant under the involution $j : x \mapsto K_C - x$ (where K_C is the canonical class) and i and j commute. In addition, since Θ_C is an ample divisor, Z is connected. Counting fixed points shows that the respective quotients of Z by $i, j, k = ij$ have genera 4, 1, 4, and that the common quotient $\bar{Z} := Z/(i, j)$ has genus 1.

These quotients clearly have the following interpretations as parameter spaces:

1. Z parametrizes the g_4^1 's $f : C \rightarrow \mathbf{P}^1$ (equivalently, via the trigonal construction, towers of double covers $\tilde{T} \rightarrow T \rightarrow \mathbf{P}^1$ of the indicated type), with a choice of a double branch point b_1 .
2. Z/i parametrizes the g_4^1 's $f : C \rightarrow \mathbf{P}^1$ (equivalently, towers of double covers $\tilde{T} \rightarrow T \rightarrow \mathbf{P}^1$ of the indicated type).
3. Z/j parametrizes the double covers $\tilde{T} \rightarrow T$ of the indicated type together with a singular point of T .
4. \overline{Z} parametrizes the double covers $\tilde{T} \rightarrow T$ of the indicated type, hence it also parametrizes their normalizations, as well as the maps $\nu\pi' : \nu\tilde{T}' \rightarrow \nu T'$.

We will now discuss what choices we make when we perform the construction in reverse order, and how the choices from the two directions are related.

Starting with the genus 3 curve C' , we now assume given a Lagrangian subgroup $L' (= \text{Jac}(C)[2]/L)$ of $\text{Jac}(C')[2]$, and a subgroup $G \subset L'$ of order 4 (which corresponds to α^\perp/L). A marking of the three double branch points b'_i of f' is equivalent to a choice of a basis $\beta' = P'_2 + Q'_2 - P'_1 - Q'_1$ and $\gamma' = P'_3 + Q'_3 - P'_1 - Q'_1$ of G . Let $\Theta' \subset \text{Pic}^2(C')$ be the theta divisor of C' , and for a class $u \in \text{Jac}(C')$ let Θ'_u denote the translation of Θ' by u . Consider a line bundle L in the intersection $S = \Theta \cap \Theta_{\beta'} \cap \Theta_{\gamma'}$. Since the canonical bundle is the only degree 4 bundle on C' with $h^0 > 2$, there are only two possibilities: either $L^{\otimes 2}$ gives a $g_4^1 f' : C' \rightarrow \mathbf{P}^1$ with three marked double branch points $b'_i \in \mathbf{P}^1$, $i = 1, \dots, 3$, or else L must be a theta characteristic on C' . We claim the following:

- (1) S consists of six points
- (2) S is closed under $v \rightarrow K_{C'} - v$.
- (3) Four of the points of S are theta characteristics, and two are not.

PROOF. (1) holds because $6 = g!$. For (2), suppose that $f', f'' : C' \rightarrow \mathbf{P}^1$ correspond to $2v, 2K_{C'} - 2v$ respectively. Then for each double ramification point P'_i, Q'_i of f' we get a unique double ramification point P''_i, Q''_i for f'' by imposing the condition $P''_i + Q''_i + P'_i + Q'_i = K_{C'}$.

For (3), one checks that there is a unique coset G' of G in the set of odd theta characteristics on C' : indeed, in coordinates we may take the set of theta characteristics to be $V = (\mathbf{Z}/2\mathbf{Z})^6$ with coordinates x_1, \dots, x_6 , and we may suppose that $h^0(C', \mathcal{O}_{C'}(x)) \bmod 2$ for $x = (x_1, \dots, x_6)$ is given by $q(x) = x_1x_2 + x_3x_4 + x_5x_6$. Also we may simultaneously identify $\text{Jac}(C')[2]$ with V , with the Weil pairing given by $w_2(x, y) = q(x + y) - q(x) - q(y)$. Without loss of generality we can also take $\beta' = e_1$ and $\gamma' = e_3$, with e_i the standard i th unit vector. Then $G' = \{(a, 0, b, 0, 1, 1)\}$.

Part (3) is now clear: G' is contained in S , and no other theta characteristics appear in S . This establishes our claim.

We can now describe all the choices made when we start from the right side. Our data C', L', G determines a complementary pair of maps f', f'' .

These determine the data $\tilde{T}' \xrightarrow{\pi'} T'$ uniquely (the two resulting maps g', g'' are the usual two g_3^1 's on the genus 4 curve T'). The normalization $\nu\pi' : \nu\tilde{T}' \rightarrow \nu T'$ is therefore also uniquely determined. So the *only* choice made is that of h' , given by an arbitrary point of $Pic^2(\nu T') \approx \nu T'$. Comparing with what we found starting from the left, we discover that $\nu T'$ is precisely identified with the double quotient $\overline{\overline{Z}}$.

7. – The case of genus ≥ 4

One might try to generalize our construction to higher genus by finding, for a generic curve C of genus g , a correspondence with another generic curve C' of genus g such that $Jac(C') \simeq Jac(C)/K$, with K a Lagrangian subgroup of $Jac(C)[2]$. We shall show that this is not possible.

THEOREM 6. *Let K be a Lagrangian subgroup in $Jac(C)[p]$, where C is a generic curve of genus $g \geq 4$ and p is a prime. Then $Jac(C)/K$ with its induced principal polarization is not a jacobian.*

PROOF. Let $\mathcal{T}, \mathcal{S}, \mathcal{M}$ and \mathcal{A} denote respectively the Teichmüller space, the Siegel space, the moduli space of curves and the moduli space of principally polarized abelian varieties, all of genus g . The mapping class group $M = M(g)$ acts on \mathcal{T} with quotient \mathcal{M} and the modular group $\Gamma = Sp(2g, \mathbf{Z})$ acts on \mathcal{S} with quotient \mathcal{A} . Moreover Γ is naturally a quotient of M , because M acts on symplectic bases for $H_1(C, \mathbf{Z})$ through its action on $\pi_1(C)$, and the period map $\tau : \mathcal{T} \rightarrow \mathcal{S}$ is M -equivariant for these actions. Passing to the quotient, we get Torelli's map $\bar{\tau} : \mathcal{M} \rightarrow \mathcal{A}$, which is injective (Torelli's theorem) and exhibits \mathcal{M} as a locally closed subvariety of \mathcal{A} . Since \mathcal{T} is irreducible it follows that the *Torelli space* $\overline{\mathcal{T}} = \tau(\mathcal{T})$ is a locally closed irreducible analytic subvariety of \mathcal{S} .

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\tau} & \mathcal{S} \supset \overline{\mathcal{T}} = \tau(\mathcal{T}) \\ M \downarrow & & \downarrow \Gamma \\ \mathcal{M} & \xrightarrow{\bar{\tau}} & \mathcal{A} \end{array}$$

Let W be the finite cover of \mathcal{S} obtained by taking over each marked abelian variety A the Lagrangian subgroups of $A[p]$. Since W is unramified over the contractible space \mathcal{S} , it is in fact a union of copies of \mathcal{S} . Our generic isogeny $Jac(C) \rightarrow Jac(C)/K$ translates to the following data. The curve C lives over an open subset of \mathcal{M} , hence of $\overline{\mathcal{T}}$. The subgroup K corresponds to a sheet of $W|_{\overline{\mathcal{T}}}$. Therefore our isogeny extends to the quotient map by the subgroup, still denoted K , corresponding to the “same” sheet over all of \mathcal{S} . Now recall that \mathcal{S} is the space of symmetric $g \times g$ complex matrices Ω with positive imaginary part, and the abelian variety over Ω is $A_\Omega = \mathbf{C}^g / (\mathbf{Z}^g + \Omega \mathbf{Z}^g)$. Since monodromy (i.e. Γ) acts transitively on the Lagrangian subgroups of A_Ω , we

may take $K = (\frac{1}{p}\mathbf{Z}/\mathbf{Z})^g$ for convenience. Then $A_\Omega/K \simeq A_{p\Omega} = A_{s\Omega}$, with

$$s = \begin{bmatrix} pI_{g \times g} & 0 \\ 0 & I_{g \times g} \end{bmatrix} \in \mathrm{Sp}(2g, \mathbf{R}).$$

If $\mathrm{Jac}(C)/K$, with its principal polarization, were a jacobian, it would follow that the Torelli locus $\overline{\mathcal{T}}$ was invariant under the subgroup Δ of $\mathrm{Sp}(2g, \mathbf{R})$ generated by Γ and by s . We claim that Δ is dense in $\mathrm{Sp}(2g, \mathbf{R})$. Indeed, consider the subgroup N_+ of $\mathrm{Sp}(2g, \mathbf{R})$ consisting of the matrices $n(x) = \begin{bmatrix} I_{g \times g} & x \\ 0 & I_{g \times g} \end{bmatrix}$, where x runs over the real symmetric $g \times g$ matrices. Then Δ contains $s^i n(x) s^{-i} = n(p^{-i}x)$ for all integral symmetric matrices x and integers i . These are dense in N_+ , and Δ likewise contains a dense subgroup of $N_- = {}^t N_+$. It is well-known (and easy) that N_+ and N_- generate $\mathrm{Sp}(2g, \mathbf{R})$, so Δ is indeed dense in $\mathrm{Sp}(2g, \mathbf{R})$.

Therefore, under our assumption, $\overline{\mathcal{T}}$ would be dense in \mathcal{S} (in the complex topology), so that \mathcal{M} would be dense in \mathcal{A} . This is a contradiction when $g > 3$, because for dimension reasons \mathcal{M} is not dense in \mathcal{A} even for the Zariski topology then, and the theorem follows.

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