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Solutions of a Fourth Order Degenerate Parabolic Equation with Weak Initial Trace

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Abstract. We show that the nonlinear fourth order degenerate parabolic equation

$$u_t + \text{div}\,(u^n \nabla \Delta u) = 0, \qquad n > 0$$

admits nonnegative solutions to initial data which are a nonnegative Radon measure provided that $n < 2$. In addition, we prove that the equation has a regularizing effect in the sense that the solution we construct is in $H^1(\mathbb{R}^N)$ for all positive times and in $H^2_{\text{loc}}(\mathbb{R}^N)$ for almost all positive times. In particular, we give the first existence results to the Cauchy problem in the case that the initial data are not compactly supported. Hence, it is interesting to note that we can show that the solutions we construct preserve the initial mass. Our results depend on decay estimates in terms of the mass which are known for regularized problems. We also give a counterexample to a decay estimate for $2 < n < 3$ and show that the decay estimates are sharp for $0 < n < 2$.

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1. – Introduction

We study the Cauchy problem

$$\text{(CP)} \quad \begin{cases} u_t + \text{div}\,(u^n \nabla \Delta u) = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \quad (1) \\ u(0) = \mu_0 & \text{in } \mathbb{R}^N \quad (2) \end{cases}$$

where n denotes a positive real constant and u is a function depending on a space variable $x \in \mathbb{R}^N$, $N = 1, 2, 3$, and on the time $t \in [0, \infty)$. The initial data μ_0 are assumed to be a nonnegative Radon measure with finite mass. The above partial differential equation appears for example in lubrication theory for thin viscous films, but also many other physical phenomena are modeled by fourth order degenerate parabolic equations (see Bernis [B1] for an overview and Elliott and Garcke [EG] and Grün [G] for applications in materials science and plasticity). In applications, especially growth exponents $n \in (0, 3]$ appear.

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In a fundamental paper, Bernis and Friedman [BF] studied an initial boundary value problem to equation (1), in the case of space dimension one, and they showed that there exist nonnegative solutions provided the initial data were chosen nonnegative. This fact is remarkable not only because in general there is no comparison or maximum principle for fourth order equations, but also because the function u describes nonnegative quantities in applications.

Let us roughly describe some of the basic ideas which were used in the study of equation (1). The first a-priori estimate one can derive is obtained by differentiating the energy $\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2(t)$ with respect to t. We suppose μ_0 equals a function $u_0 \in H^1(\mathbb{R}^N)$ and assume appropriate conditions for $|x|$ large. A formal computation using equation (1) and integrating from 0 to t, gives

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2(t)+\int_0^t\int_{\mathbb{R}^N}u^n|\nabla\Delta u|^2=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_0|^2. \tag{3}$$

Bernis and Friedman [BF] used a variant of this energy identity for appropriate approximate problems to show existence of a Hölder continuous solution of the initial boundary value problem

$$\text{(IBP)}\qquad \begin{cases} u_t+\ \operatorname{div}\,(|u|^n\nabla\Delta u) & =0 \quad \text{in } \Omega\times(0,\infty)\\ \dfrac{\partial u}{\partial \nu}=u^n\dfrac{\partial\Delta u}{\partial\nu} & =0 \quad \text{on } \partial\Omega\times(0,\infty)\\ u(x,0)=u_0(x) & \quad \text{in } \Omega \end{cases}$$

where $\Omega\subset\mathbb{R}$ is an open interval, ν is the outer unit normal to $\partial\Omega$ and $u_0\in H^1(\Omega)$.

A solution which fulfills the energy identity (3) lies in $L^\infty((0,T);H^1(\Omega))$. We remark that this is only true if one assumes that the initial data belong to $H^1(\Omega)$. To be precise let us remark that in general it can be only shown that (3) is true if "=" is replaced by "$\le$" and that the term $\int_0^t\int_{\mathbb{R}^N}u^n|\nabla\Delta u|^2$ has to be given a proper interpretation because in general three spatial derivatives do not exist.

A second basic a-priori estimate can be obtained by differentiating the integral $\int_{\mathbb{R}^N}u^{\alpha+1}(t)$ with respect to t : it turns out that for $\alpha\in(\frac{1}{2}-n,2-n)$ integral (or "entropy") estimates can be obtained. In $\mathbb{R}^N$ they become

$$\begin{aligned}\frac{1}{\alpha(\alpha+1)}\int_{\mathbb{R}^N}u^{\alpha+1}(t_2)+C\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\left\{\left|D^2u^{\frac{\alpha+n+1}{2}}\right|^2+\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\right\}\\ \le\frac{1}{\alpha(\alpha+1)}\int_{\mathbb{R}^N}u^{\alpha+1}(t_1).\end{aligned}\tag{4}$$

This idea was used in the case of the initial boundary value problem (IBP) for the choice $\alpha=1-n$ to show existence of nonnegative solutions to nonnegative initial data (see [BF]). Later Beretta, Bertsch and Dal Passo [BBD] and Bertozzi and Pugh [BP] used the estimate for the full range of values of α to obtain existence of solutions to (IBP) with optimal regularity provided $n \in (0, 3)$ and $N = 1$. Their regularity result is optimal in the sense that they give exactly the regularity of the source type similarity solutions to (1), i.e., selfsimilar solutions to (CP) with $\mu_0 = \delta_0$ where δ_0 is the Dirac point measure. Existence of source type solutions was shown by Bernis, Peletier and Williams [BPW] in the case $N = 1, n \in (0, 3)$ and by Ferreira and Bernis [FB] for $N \geq 2,\ n \in (0, 3)$. Both papers also show that there is no similarity solution with finite mass if $n \geq 3$. This is one of the reasons to believe that $n = 3$ is a borderline value with respect to the qualitative behaviour of solutions to equation (1). Another reason is that only for $n \in (0, 3)$ there are estimates of the form (4) for arbitrary nonnegative initial data. This is due to the fact that for $n \geq 3$ there is no value of α possible such that $\alpha + 1 > 0$ and hence $\int_{\mathbb{R}^N} u_0^{\alpha+1}$ is unbounded for compactly supported initial data.

The first to construct solutions of problem (IBP) with the property of finite speed of propagation was Bernis [B2] (see also Yin and Gao [YG]). He used a local version of the integral estimate (4), which was first proved by [BBD], to show existence of solutions with finite speed of proposition provided $0 < n < 2, N = 1$. Using this property he could also establish existence of solutions to the Cauchy problem under the assumption that the initial data $u_0 \in H^1(\mathbb{R})$ are compactly supported. Here the value $n = 2$ was the critical value because only for $n < 2$ it is possible to choose an α out of the interval $\left(\frac{1}{2} - n, 2 - n\right)$ such that α is positive. Hence the factor $\frac{1}{\alpha(\alpha+1)}$ in (4) can be chosen positive and this makes it possible to use a local version of (4) to show finite speed of propagation (see [B2] for details). Recently Bernis [B4] could show existence of solutions with finite speed of propagation also for $2 \leq n < 3$. To establish this result, he used integral estimates obtained in [B3] and a localized version of the energy estimate (3) (see also Hulshof and Shiskov [HS]).

All results mentioned so far, were obtained in one space dimension only. In higher space dimensions new difficulties arise. First of all the norms one can control via energy and entropy estimates are not strong enough to obtain continuity of solutions via embedding theorems. In the arguments of Bernis and Friedman [BF], Beretta, Bertsch and Dal Passo [BBD] and Bertozzi and Pugh [BP] it was important to know continuity of solutions. This regularity property is so far not known in higher space dimensions. Elliott and Garcke [EG] and Grün [G] independently showed existence of solutions to degenerate parabolic equations of fourth order using a Faedo-Galerkin ansatz for regularized problems. A-priori estimates obtained through the energy estimate (3) and through the integral estimate (4), with $\alpha = 1 - n$, gave enough compactness to pass to the limit in the approximate problems and to prove existence of a nonnegative weak solution. In particular, they were able to show convergence of the approximate problems without using the continuity of solutions. By now it is also known that the integral estimates (4) hold in space dimensions two and three (see [DGG]). This new result was used by Dal Passo, Garcke and Grün [DGG] to show existence of nonnegative solutions to (IBP) for $\frac{1}{8} < n < 3$ and $N = 2, 3$

in the case that the initial data are nonnegative and in $H^1(\Omega)$ ($\Omega \subset \mathbb{R}^N$ an open, bounded domain with sufficiently smooth boundary).

Bertsch, Dal Passo, Garcke and Grün [BDGG] generalized the result of Bernis [B2] on finite speed of propagation to the case $\frac{1}{8} < n < 2, N = 2, 3$ using the techniques of Bernis [B2] and Dal Passo, Garcke and Grün [DGG]. The restriction $n > \frac{1}{8}$ in the above results is purely technical. In fact, also in the case $n \in (0, \frac{1}{8})$ there is a limit of solutions to sensible approximate problems which has the property of finite speed of propagation. The limit is just not regular enough to use the solution concepts available so far (see Definition 1). We remark that in [BDGG] a solution to (CP) was also constructed, provided the initial data are compactly supported and in $H^1(\mathbb{R}^N)$ (again one has to assume $\frac{1}{8} < n < 2, N = 2, 3$).

Bernis [B2] (for $0 < n < 2, N = 1$) and Bertsch et al. [BDGG] (for $\frac{1}{8} < n < 2, N = 2, 3$) also gave asymptotic estimates for certain integral norms and for the size of the support of the constructed solutions to (CP). Here it is remarkable that the estimates for the L^p–norm and for the H^1–semi-norm of the solution do only depend on the mass of the initial data. For example one obtains, for $0 < n < 2, N = 1$ and $\frac{1}{8} < n < 2, N = 2, 3,$ that the L^2–norm of the gradient decays as

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^N)} \leq C(n, N)\|u_0\|_1^{\frac{8+n(N-2)}{2(4+nN)}} t^{-\frac{1}{2}\frac{N+2}{4+nN}}.$$

So far all results on the equation (1) require the H^1–norm of the initial data to be bounded. But the above decay estimates gave the hope that it is possible to construct solutions to nonnegative initial data which lie in $L^1(\mathbb{R}^N)$.

The aim of this paper is to show existence of solutions to the Cauchy problem under the assumption that the initial data are a nonnegative Radon measure μ_0 having finite mass. Of course the case that the initial data are in $L^1(\mathbb{R}^N)$ is included and we remark that we do not assume that the initial data are compactly supported. The results we obtain are for $\frac{1}{8} < n < 2$ and $N = 1, 2, 3$ (see Section 3). In one space dimension we also establish results for $0 < n < \frac{1}{8}$ and $2 \leq n < 3$ under more restrictive assumptions on the initial data (Section 6). Furthermore, we show a smoothing property of equation (1). More precisely, we show existence of a solution having the property that $u(t) \in H^1(\mathbb{R}^N)$ for all $t \in (0, T)$.

Let us introduce the solution concept we use and which is appropriate also to higher space dimensions (see [BP], [DGG], [BDGG]). This solution concept differs from the concept of weak solutions (see [BF]) and strong solutions (see [B2]) which were used in one space dimension.

Definition 1. Let μ_0 be a Radon measure on $\mathbb{R}^N$ with finite mass, $n \in (\frac{1}{8}, 3)$ and $N = 1, 2, 3$. A nonnegative function $u \in L^\infty((0, \infty); L^1(\mathbb{R}^N)) \cap L^\infty_{\text{loc}}((0, \infty); H^1_{\text{loc}}(\mathbb{R}^N))$ is said to be a solution of the Cauchy problem (CP) if:

i) $\chi_{[u>0]}u^{n-2}|\nabla u|^3, \chi_{[u>0]}u^{n-1}|\nabla u|^2, u^n|\nabla u| \in L^1_{\text{loc}}(\mathbb{R}^N \times [0, \infty))$

and

ii)
$$-\int_0^\infty\int_{\mathbb{R}^N} u\zeta_t - \int_{\mathbb{R}^N}\zeta(0)d\mu_0 = \frac{1}{2}\int_{[u>0]} n(n-1)u^{n-2}|\nabla u|^2\nabla u\nabla\zeta$$
$$+\frac{1}{2}\int_{[u>0]} nu^{n-1}|\nabla u|^2\Delta\zeta + \int_{[u>0]} nu^{n-1}\langle\nabla u, D^2\zeta, \nabla u\rangle$$
$$+\int_0^\infty\int_{\mathbb{R}^N} u^n\nabla u\nabla\Delta\zeta$$

for all $\zeta \in C_0^\infty(\mathbb{R}^N \times [0,\infty))$.

We will show that the solutions we construct attain the initial data weak-$*$ in the sense of measures; i.e., for $t \searrow 0$ we have

$$\int_{\mathbb{R}^N} u(x,t)\varphi(x)\,dx \to \int_{\mathbb{R}^N} \varphi(x)d\mu_0(x) \tag{5}$$

for all test functions $\varphi \in C_0^0(\mathbb{R}^N)$ (see Theorem 4 vii)). It is not clear that such a solution preserves the initial mass. We will show that for $\frac{1}{8} < n < 2$ the solutions we construct also have the property of mass conservation (Section 4). If $2 < n < 3$ we give an example of initial data that lie in $L^1(\mathbb{R}^N)$ leading to a solution with the property that $u(t) \notin H^1(\mathbb{R}^N)$ for $t \in [0, T^*]$ where $T^* > 0$ (see Section 5). This implies that in the case $n \in (2,3)$ there are no decay rates for the the L^2-norm of the gradient which depend just one the mass of the initial data. In Section 6, we show some extensions valid in one space dimension. Finally, we give some estimates from below for the asymptotic rates of Sobolev- and Lebesgue–norms and for the spreading rate of the support. This shows the sharpness of the decay rates obtained in [BDGG] for $N = 2, 3$ and $\frac{1}{8} < n < 2$.

NOTATION.

By $B_\varepsilon(D)$ we denote the ε–ball around a subset D and $B_\varepsilon(x) := B_\varepsilon(\{x\})$ for points x. The characteristic function of a set D is denoted by χ_D and if $D \subset \mathbb{R}^N$ then $|D|$ is defined to be its Lebesgue measure. We define $[u > 0]$ to be the set of all points where the real valued function u attains positive values and supp u is the support of u. As usual $L^p(D)$ is the space of p-integrable Lebesgue functions ($1 \le p \le \infty$) and $W^{m,p}(D)$ is the space of Sobolev functions having p-integrable weak derivatives up to the order m ($m \in \mathbb{N}$). The norm in $L^p(D)$ is denoted by $\|.\|_p$. If $p = 2$ then we define $H^m(D) := W^{m,2}(D)$. The space L^{p-} is the space of all functions for which $u \in L^q(D)$ whenever $q < p$. The space $L^p_{\text{loc}}(D)$ consists of all measurable functions u for which $u \in L^p(D')$ for all compact $D' \subset D$. We will also use spaces of functions which depend on space and time like

$$L^p((0,T); W^{m,q}(D)), \qquad D \subset \mathbb{R}^N \quad \text{open}$$

which are defined as usual. By $\|\mu\|_1 := \mu(\mathbb{R}^N)$ we denote the total measure of a nonnegative Radon measure μ on $\mathbb{R}^N$. We say that μ has finite mass if $\mu(\mathbb{R}^N) < \infty$. A sequence of Radon measures $(\mu_n)_{n\in\mathbb{N}}$ is said to converge weak-$*$ in the sense of measures to μ if

$$\int_{\mathbb{R}^N} f\, d\mu_n \to \int_{\mathbb{R}^N} f\, d\mu$$

for all $f \in C_0^0(\mathbb{R}^N)$. In this case we write $\mu_n \overset{*}{\rightharpoonup} \mu$. For a definition and basic properties of Radon measures we refer to [EvG]. Finally we define $\langle a, A, b\rangle := \sum_{i,j=1}^N a_i A_{ij} b_j$ where A is a $(N \times N)$-matrix and $a, b \in \mathbb{R}^N$.

2. – Statement of the main results

In this section, we formulate our main results for the case $\frac{1}{8} < n < 2$. We construct solutions to (CP) with general nonnegative initial data as limits of solutions to (CP) having smooth initial data with compact support. Therefore, we cite results of Bernis [B2] ($N = 1$) and Bertsch et al. [BDGG] ($N = 2, 3$) on the existence of solutions to the Cauchy problem with H^1–initial data with compact support. We also state results on the regularity properties and the asymptotic behaviour of solutions.

THEOREM 2. *Let $u_0 \in H^1(\mathbb{R}^N)$ have compact support and let $n \in (\frac{1}{8}, 2)$, $N = 1, 2, 3$.*

Then there exists a solution to (CP) *in the sense of* Definition 1 *having the following properties:*

i) *$u \in H^1_{\text{loc}}([0,\infty), (W^{1,q}(\mathbb{R}^N))')$ for all $q > \frac{4N}{2N+(2-N)n}$ (or $q = 2$ if $N = 1$) and $u_t = -\operatorname{div} J$ in $L^2_{\text{loc}}([0,\infty); (W^{1,q}(\mathbb{R}^N))')$ where $J \in L^2_{\text{loc}}([0,\infty), L^{q'}(\mathbb{R}^N))$ for all $q' \in \left(1, \frac{4N}{2N+(N-2)n}\right)$ (or $q' = 2$ if $N = 1$). The flux J fulfills the following estimate*

$$\int_{t_1}^{t_2} \|J(t)\|^2_{q'} dt \le \|\nabla u(t_1)\|_2^2 \sup_{t\in(t_1,t_2)} \|u^n(t)\|_{\frac{q'}{2-q'}} \tag{6}$$

for almost every $0 < t_1 < t_2$;

ii) *for all $\alpha \in (\max(-1, \frac{1}{2} - n), 2 - n)$ with $\alpha \neq 0$ it holds*

$$u^{\frac{\alpha+n+1}{4}} \in L^4_{\text{loc}}([0,\infty), W^{1,4}(\mathbb{R}^N)),\; u^{\frac{\alpha+n+1}{2}} \in L^2_{\text{loc}}([0,\infty), H^2(\mathbb{R}^N))$$

and there exists a constant $C > 0$ depending on α and n such that

$$
\begin{aligned}
&\frac{1}{\alpha(\alpha+1)}\int_{\mathbb{R}^N}\zeta^4 u^{\alpha+1}(t_2) + C\left\{\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\zeta^4|D^2 u^{\frac{\alpha+n+1}{2}}|^2\right.\\
&\quad\left.+\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\zeta^4|\nabla u^{\frac{\alpha+n+1}{4}}|^4\right\} \le \frac{1}{\alpha(\alpha+1)}\int_{\mathbb{R}^N}\zeta^4 u^{\alpha+1}(t_1) \qquad (7)\\
&\quad + C\int_{t_1}^{t_2}\int_{\mathbb{R}^N}u^{\alpha+n+1}(|\nabla\zeta|^4+\zeta^2|\Delta\zeta|^2).
\end{aligned}
$$

for all $t_1, t_2 \in [0,\infty)$ $(t_1 < t_2)$ and all $\zeta \in C_0^2(\mathbb{R}^N)$;

iii) *the function $t \mapsto \int_{\mathbb{R}^N}|\nabla u|^2(t)$ is almost everywhere equal to a nonincreasing function;*

iv) *if $N = 1, 2$ and $p \in (1,\infty)$ or $N = 3$ and $p \in (1, 6)$, then there exists a constant C depending on p, n, N, such that for all $t > 0$*

$$\|u(t)\|_p \le C\|u_0\|_1^{\frac{4p+nN}{p(4+nN)}} t^{-\frac{p-1}{p}\frac{N}{4+nN}}; \qquad (8)$$

v) *there exists a constant $C > 0$ depending on n, N such that for all $t > 0$*

$$\|\nabla u(t)\|_2 \le C\|u_0\|_1^{\frac{8+n(N-2)}{2(4+nN)}} t^{-\frac{1}{2}\frac{N+2}{4+nN}};$$

vi) *if*

$$u_0(x) = 0 \text{ for all } x \in \mathbb{R}^N \text{ with } |x| > R_0$$

then

$$u(x,t) = 0 \text{ for almost all } x \in \mathbb{R}^N \text{ with } |x| > R_0 + B\|u_0\|_1^{\frac{n}{4+nN}} t^{\frac{1}{4+nN}},$$

where B is a constant depending on n and N;

vii) $\lim_{t\to 0} u(t) = u_0$ *in* $H^1(\mathbb{R}^N)$.

Proof. (see [B2], [BDGG]). The results ii) - vi) were obtained by Beretta, Bertsch and Dal Passo [BBD] and Bernis [B2] ($N = 1$) and Dal Passo, Garcke and Grün [DGG] and Bertsch et al. [BDGG] ($N = 2, 3$). To prove i) it only remains to show the estimate for the flux. All the other results in i) follow from the above papers. To obtain the estimate on the flux we consider how the solution in [BDGG] was constructed (see the proof of Theorem 5.1 and Section 2 of [BDGG]). There the authors studied the initial boundary value problem for

$$u_t + \text{ div }(m_{\sigma\delta}(u)\nabla\Delta u) = 0$$

with $m_{\sigma\delta}(\tau) := \frac{\tau^{n+s}}{\delta\tau^n+\tau^s+\sigma\tau^{n+s}}$, where s is a sufficiently large real number, and initial data $u_{0\sigma\delta} := u_0 + \delta^{\theta_1} + \sigma^{\theta_2}$ $(\theta_1, \theta_2 > 0)$. It was shown in [DGG] that this problem admits a solution, $u_{\sigma\delta}$, for which $\nabla\Delta u(t)$ exists for almost all t in a weak sense. Hence $J_{\sigma\delta}(t) := m_{\sigma\delta}(u_{\sigma\delta})\nabla\Delta u_{\sigma\delta}(t)$ exists for almost all t. Next an application of Hölder's inequality and an appeal to the energy estimate gives

$$\begin{aligned}\int_{t_1}^{t_2}\left(\int_\Omega |J_{\sigma\delta}|^{q'}\right)^{\frac{2}{q'}} &= \int_{t_1}^{t_2}\left(\int_\Omega |m_{\sigma\delta}(u_{\sigma\delta})\nabla\Delta u_{\sigma\delta}|^{q'}\right)^{\frac{2}{q'}} \\ &\leq \int_{t_1}^{t_2}\left(\int_\Omega |m_{\sigma\delta}(u_{\sigma\delta})||\nabla\Delta u_{\sigma\delta}|^2\right)\left(\int_\Omega m_{\sigma\delta}(u_{\sigma\delta})^{\frac{q'}{2-q'}}\right)^{\frac{2-q'}{2}\frac{2}{q'}} \\ &\leq \|\nabla u_{\sigma\delta}(t_1)\|_2^2 \sup_{t\in(t_1,t_2)}\left(\int_\Omega m_{\sigma\delta}(u_{\sigma\delta})^{\frac{q'}{2-q'}}\right)^{\frac{2-q'}{q'}}.\end{aligned}$$

Now the result follows by letting σ, δ tend to zero.

The convergence vii) follows because $u(t) \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$ and $\|\nabla u(t)\|_2 \leq \|\nabla u_0\|_2$. This implies $\|\nabla u(t)\|_2 \to \|\nabla u_0\|_2$ and hence $\nabla u(t) \to \nabla u_0$ in $L^2(\mathbb{R}^N)$. This completes the proof of the theorem. □

REMARK 3. If $N = 1, n \in (0,2)$ and if $\mu_0 = u_0 \in H^1(\mathbb{R})$ has compact support then there exists a solution $u \in C^{\frac{1}{2},\frac{1}{8}}(\mathbb{R}\times[0,\infty))$ of (CP) which solves (1) in the sense that

$$\int_0^\infty \int_{\mathbb{R}} u\psi_t + \int_{\mathcal{P}} u^n u_{xxx}\psi_x = 0 \tag{9}$$

where $\mathcal{P} = \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ : u(x,t) > 0\}$ and ψ is a Lipschitz continuous function having compact support in $\mathbb{R}\times(0,\infty)$. In addition, $u \in C^{4,1}(\mathcal{P})$ and $u^{n/2}u_{xxx} \in L^2(\mathcal{P})$. This solution satisfies the properties i) - vi) in Theorem 2. We refer to Bernis [B2] for the precise statement of the result.

Let us now state our main results.

THEOREM 4. *Let* $N \in \{1,2,3\}, n \in (\frac{1}{8}, 2)$ *and let* μ_0 *be a nonnegative Radon measure with finite mass. Then there exists a solution u of the Cauchy problem* (CP) *in the sense of* Definition 1.

In addition u has the following properties:

i)
$$u_t = -\ \operatorname{div}\ J \qquad \textit{in the sense of distributions,}$$
where $J \in L^2_{\mathrm{loc}}((0,\infty); L^{q'}(\mathbb{R}^N))$ *for all* $q' \in (\max(1, \frac{2}{n+1}), \frac{4N}{2N+(N-2)n})$ $(q' = 2$ *if* $N = 1)$;

ii) *for all* $\alpha \in (\max(-1, \frac{1}{2} - n), 2-n)$ *with* $\alpha \neq 0$ *it holds*

$$u^{\frac{\alpha+n+1}{4}} \in L^4_{\mathrm{loc}}((0,\infty), W^{1,4}_{\mathrm{loc}}(\mathbb{R}^N)),$$
$$u^{\frac{\alpha+n+1}{2}} \in L^2_{\mathrm{loc}}((0,\infty), H^2_{\mathrm{loc}}(\mathbb{R}^N)),$$

and if in addition $\alpha > 0$ *we have the above regularity properties with* $W^{1,4}_{\mathrm{loc}}(\mathbb{R}^N)$ *and* $H^2_{\mathrm{loc}}(\mathbb{R}^N)$ *replaced by* $W^{1,4}(\mathbb{R}^N)$ *and* $H^2(\mathbb{R}^N)$ *respectively;*

iii) *the function* $t \mapsto \int_{\mathbb{R}^N} |\nabla u|^2(t)$ $(t > 0)$ *is almost everywhere equal to a nonincreasing function;*

iv) *if* $N = 1, 2$ *and* $p \in (1, \infty)$ *or* $N = 3$ *and* $p \in (1, 6)$, *then there exists a constant* C, *depending on* p, n, N, *such that for all* $t > 0$

$$\|u(t)\|_p \leq C \|\mu_0\|_1^{\frac{4p+nN}{p(4+nN)}} t^{-\frac{p-1}{p}\frac{N}{4+nN}}; \tag{10}$$

v) *there exists a constant* $C > 0$, *depending on* n, N *such that for all* $t > 0$

$$\|\nabla u(t)\|_2 \leq C \|\mu_0\|_1^{\frac{8+n(N-2)}{2(4+nN)}} t^{-\frac{1}{2}\frac{N+2}{4+nN}};$$

vi) *for solutions with compactly supported initial data it holds: if* $\operatorname{supp} \mu_0 \subset B_{R_0}(0)$ *then* $\operatorname{supp} u(t) \subset B_{R(t)}(0)$ *with* $R(t) := R_0 + B\|\mu_0\|^{\frac{n}{4+nN}} t^{\frac{1}{4+nN}}$ *and where* B *is a constant depending on* n *and* N;

vii) $u(t) \stackrel{*}{\rightharpoonup} \mu_0$ *as* $t \searrow 0$ *in the sense of Radon measures.*

The next theorem states that the solution we construct preserves its initial mass.

THEOREM 5. *Let the assumptions of* Theorem 4 *be satisfied. Then for all* $p \in [\max(1, \frac{8}{8n+5}), \frac{4}{3})$, *we have*

$$\chi_{[u>0]} u^{n-2} |\nabla u|^3,\ \chi_{[u>0]} u^{n-1} |\nabla u|^2,\ u^n |\nabla u| \in L^p\left(\mathbb{R}^N \times (0, T)\right) \text{ for all } T > 0$$

and

$$\int_{\mathbb{R}^N} u(t) = \mu_0(\mathbb{R}^N) \text{ for all } t > 0.$$

3. – Construction of solutions

We approximate the nonnegative Radon-measure μ_0, which is assumed to have the property $\mu_0\left(\mathbb{R}^N\right) < \infty$, by nonnegative compactly supported functions $u_{0\varepsilon} \in H^1(\mathbb{R}^N)$ such that:

(H1) $u_{0\varepsilon} \stackrel{*}{\rightharpoonup} \mu_0$ in the sense of Radon measures,

(H2) $\int_{\mathbb{R}^N} u_{0\varepsilon} = \mu_0\left(\mathbb{R}^N\right)$,

(H3) $\operatorname{supp} u_{0\varepsilon} \subset B_\varepsilon(\operatorname{supp} \mu_0)$.

The results of Theorem 2 give the existence of solutions $u_\varepsilon \in L^\infty((0,\infty); H^1(\mathbb{R}^N))$ to the Cauchy problem (CP) with initial data $u_{0\varepsilon}$ fulfilling the properties i) - vii) of Theorem 2. Now we state a compactness property.

LEMMA 6. *Let u_ε be the solution of* (CP) *constructed as in the proof of* Theorem 2 *and let J_ε be the corresponding fluxes. Then there exists a subsequence $(u_\varepsilon)_{\varepsilon>0}$ and fluxes $(J_\varepsilon)_{\varepsilon>0}$ such that*

$$\begin{aligned} u_\varepsilon &\to u && \text{in } L^1_{\text{loc}}(\mathbb{R}^N \times [0,\infty)), \\ J_\varepsilon &\rightharpoonup J && \text{in } L^2_{\text{loc}}\left((0,\infty), L^{q'}(\mathbb{R}^N)\right), \\ \nabla u_\varepsilon &\to \nabla u && \text{in } L^2_{\text{loc}}(\mathbb{R}^N \times (0,\infty)), \end{aligned}$$

for all $q' \in (\max(1, \frac{2}{n+1}), \frac{4N}{2N+(N-2)n})$ *and* ($q' = 2$ *if* $N = 1$). *For all* $\alpha \in (\max(-1, \frac{1}{2}-n), 2-n)$ *with* $\alpha \neq 0$ *it holds*

$$\begin{aligned} \nabla u_\varepsilon^{\frac{\alpha+n+1}{4}} &\rightharpoonup \nabla u^{\frac{\alpha+n+1}{4}} && \text{in } L^4_{\text{loc}}(\mathbb{R}^N \times (0,\infty)), \\ \nabla u_\varepsilon^{\frac{\alpha+n+1}{4}} &\to \nabla u^{\frac{\alpha+n+1}{4}} && \text{in } L^{4-}_{\text{loc}}([u>0] \cap \mathbb{R}^N \times (0,\infty)). \end{aligned}$$

Furthermore, u fulfills iii)-vi) *of* Theorem 4 *and the inequality* (7) *for all* $t_1, t_2 \in (0,\infty)$, $t_1 < t_2$, $\alpha \in (\max(-1, \frac{1}{2}-n), 2-n)$ *with* $\alpha \neq 0$ *and all* $\zeta \in C_0^2(\mathbb{R}^N)$.

PROOF. Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of solutions to (CP) with initial data $(u_{0\varepsilon})_{\varepsilon>0}$ constructed as in Theorem 2. Since for u_ε mass is preserved we get that $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty([0,\infty); L^1(\mathbb{R}^N))$.

Since the sequence $(u_\varepsilon)_{\varepsilon>0}$ fulfills the estimate v) in Theorem 2 with a right hand side independent of ε we conclude

$$(\nabla u_\varepsilon)_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^\infty_{\text{loc}}((0,\infty); L^2(\mathbb{R}^N)).$$

The estimate of Gagliardo-Nirenberg then implies that

$$(u_\varepsilon)_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^\infty_{\text{loc}}((0,\infty); L^p(\mathbb{R}^N))$$

for all $p \in [1,\infty]$ if $N = 1$, all $p \in [1,\infty)$ if $N = 2$ and all $p \in [1,6)$ if $N = 3$. From the above and estimate (7) we want to deduce that: for all $\alpha \in \left(\max(-1, \frac{1}{2}-n), 2-n\right)$ with $\alpha \neq 0$

$$\left(u_\varepsilon^{\frac{\alpha+n+1}{4}}\right)_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^4_{\text{loc}}((0,\infty); H^{1,4}_{\text{loc}}(\mathbb{R}^N))$$

and

$$\left(u_\varepsilon^{\frac{\alpha+n+1}{2}}\right)_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^2_{\text{loc}}((0,\infty); H^2_{\text{loc}}(\mathbb{R}^N)).$$

This follows for $\alpha \in \left(\max\left\{\frac{1}{2}-n, 0\right\}, 2-n\right)$ if we choose $\zeta \equiv 1$ in the estimate (7). The right hand side is uniformly bounded for t_1, t_2 lying in a compact interval of $(0, \infty)$. For these α the estimate gives bounds uniform in space. In the case that $\alpha \in (-1, 0)$ we have

$$\int_{B_R(0)} u_\varepsilon^{\alpha+1}(t) \leq \left(\int_{\mathbb{R}^N} u_\varepsilon(t) \right)^{\alpha+1} \left(\int_{B_R(0)} 1 \right)^{-\alpha} \leq C(R, \alpha, \|\mu_0\|_1).$$

Hence, for $\alpha \in (-1, 0)$ the bounds in $L^4_{\text{loc}}((0, \infty); H^{1,4}_{\text{loc}}(\mathbb{R}^N))$ and $L^2_{\text{loc}}((0, \infty); H^2_{\text{loc}}(\mathbb{R}^N))$ follow from estimate (7) if we choose a ζ with supp $\zeta \subset B_R(0)$ with $\zeta \equiv 1$ in $B_{R/2}(0)$.

By i) of Theorem 2 we get

$$(J_\varepsilon)_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^2_{\text{loc}}\left((0, \infty), L^{q'}(\mathbb{R}^N)\right),$$

for all $q' \in \left(\max\left(1, \frac{2}{n+1}\right), \frac{4N}{2N+(N-2)n}\right)$ and ($q' = 2$ if $N = 1$). Here we used that the right hand side of inequality (6) is bounded in terms of the initial mass (see (iv) and (v) of Theorem 2). Then, by Proposition 1.6 and Corollary 1.7 in [DGG] and by using a standard diagonal procedure we can claim all the convergence results for the gradient locally in $\mathbb{R}^N \times (0, \infty)$ up to a subsequence. Obviously properties iv)–vi) continue to be valid in the limit and inequality (7) is verified for all $t_1, t_2 \in (0, \infty)$, $t_1 < t_2$ and all $\zeta \in C_0^2(\mathbb{R}^N)$. Having in mind the approximation procedure we have used property iii) can be shown similar as in the proof of Theorem 2.5 of [DGG]. It remains to show the convergence

$$u_\varepsilon \to u \qquad \text{in } L^1_{\text{loc}}(\mathbb{R}^N \times [0, \infty)).$$

First we choose the subsequence such that $u_\varepsilon \to u$ almost everywhere in $\mathbb{R}^N \times (0, \infty)$. This is possible because u_ε converges locally in L^1 on the set $\mathbb{R}^N \times (0, \infty)$. By integrating inequality (8) with respect to time, it follows that

$$(u_\varepsilon)_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^p_{\text{loc}}([0, \infty); L^p(\mathbb{R}^N))$$

for all $1 < p < 1 + \frac{4+nN}{N}$. This gives the uniform integrability of the sequence up to the initial time and hence Vitali's theorem gives the convergence in $L^1_{\text{loc}}(\mathbb{R}^N \times [0, \infty))$. □

The results stated in Lemma 6 are not enough to pass to the limit in the equation: we need more precise information on the behaviour of the solution for $t \searrow 0$. Therefore, it is necessary to estimate the terms $\chi_{[u>0]} u^{n-2} |\nabla u|^3$, $\chi_{[u>0]} u^{n-1} |\nabla u|^2$ and $u^n |\nabla u|$ which appear in the weak formulation of the flux term.

LEMMA 7. *Let u_ε be the solution of* (CP) *with initial data $u_{0\varepsilon}$ satisfying* (H1)-(H3) *constructed as in the proof of* Theorem 2. *Suppose $n \in (\frac{1}{8}, 2)$, $N = 1, 2, 3$ and $p \in [\max(1, \frac{8}{8n+5}), \frac{4}{3})$ then there exists a constant $C > 0$ depending on p, n, N and $\|u_{0\varepsilon}\|_1$ such that*

i) $$\int_0^\tau \| \chi_{[u_\varepsilon>0]} u_\varepsilon^{n-2} |\nabla u_\varepsilon|^3 \|_p \le C\tau^{\beta_1},$$

ii) $$\int_0^\tau \| \chi_{[u_\varepsilon>0]} u_\varepsilon^{n-1} |\nabla u_\varepsilon|^2 \|_p \le C\tau^{\beta_2},$$

iii) $$\int_0^\tau \| u_\varepsilon^{n} |\nabla u_\varepsilon| \|_p \le C\tau^{\beta_3}.$$

Where $\beta_k = (k + N(\frac{1}{p} - 1))/(4 + nN)$, $k = 1, 2, 3$.

PROOF. In the proof of the lemma we will omit the index ε. Applying Hölder's inequality twice and using estimates (7) and (8) we obtain for $\alpha \in (\max\{\frac{1}{2} - n, 0\}, 2 - n)$:

$$\int_{\tau_1}^{\tau_2} \left[\int_{\mathbb{R}^N} \chi_{[u>0]} \left(u^{n-2} |\nabla u|^3 \right)^p \right]^{\frac{1}{p}} \tag{11}$$

$$\le \int_{\tau_1}^{\tau_2} \left[\int_{\mathbb{R}^N} u^{\left[-(\alpha+n-3)\frac{3p}{4}+(n-2)p\right]\frac{4}{(4-3p)}} \right]^{\frac{4-3p}{4p}} \left[\int_{\mathbb{R}^N} \chi_{[u>0]} u^{\alpha+n-3} |\nabla u|^4 \right]^{3/4}$$

$$\le C \left(\int_{\tau_1}^{\tau_2} \left[\int_{\mathbb{R}^N} u^{\left[-(\alpha+n-3)\frac{3p}{4}+(n-2)p\right]\frac{4}{(4-3p)}} \right]^{\frac{4-3p}{p}} \right)^{\frac{1}{4}} \left(\int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^N} \chi_{[u>0]} u^{\alpha+n-3} |\nabla u|^4 \right)^{3/4}$$

$$\le C \left(\int_{\tau_1}^{\tau_2} t^{-\left(\left[-(\alpha+n-3)\frac{3p}{4}+(n-2)p\right]\frac{4}{(4-3p)}-1\right)\frac{N}{4+nN}\frac{4-3p}{p}} dt \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}^N} u^{\alpha+1}(\tau_1) \right)^{3/4}$$

$$\le C \left(\int_{\tau_1}^{\tau_2} t^{-\left(\left[-(\alpha+n-3)\frac{3p}{4}+(n-2)p\right]\frac{4}{(4-3p)}-1\right)\frac{N}{4+nN}\frac{4-3p}{p}} dt \right)^{\frac{1}{4}} \tau_1^{-\frac{\alpha N}{4+nN}\cdot\frac{3}{4}}$$

$$\le C \left[\tau_2^{-\left(4+n-3\alpha-\frac{4}{p}\right)\frac{N}{4+nN}+1} - \tau_1^{-\left(4+n-3\alpha-\frac{4}{p}\right)\frac{N}{4+nN}+1} \right]^{\frac{1}{4}} \tau_1^{-\frac{\alpha N}{4+nN}\cdot\frac{3}{4}}$$

where C denotes a constant which depends on α, n, N and $\|\mu_0\|_1$. To use estimate (8) we had to make sure that α and p are such that

$$\left[-(\alpha + n - 3)\frac{3p}{4} + (n-2)p \right] \frac{4}{(4-3p)} > 1$$

which implies that p has to be chosen such that $p > \frac{8}{8n+5}$. Now a generalization of an estimate proved by Bernis (see Lemma 17 in the Appendix) can be used to show that (11) implies estimate i).

Using the inequalities i) and (8) we can derive the estimate ii):

$$\begin{aligned}
&\int_0^\tau \left[\int_{\mathbb{R}^N} \chi_{[u>0]} \left(u^{n-1}|\nabla u|^2\right)^p\right]^{\frac{1}{p}} \\
&\leq \int_0^\tau \left[\int_{\mathbb{R}^N} u^{3(-(n-2)\frac{2p}{3}+(n-1)p)}\right]^{\frac{1}{3p}} \left[\int_{\mathbb{R}^N} \chi_{[u>0]} \left(u^{n-2}|\nabla u|^3\right)^p\right]^{\frac{2}{3p}} \\
&\leq \left(\int_0^\tau \left[\int_{\mathbb{R}^N} u^{3(-(n-2)\frac{2p}{3}+(n-1)p)}\right]^{\frac{1}{p}}\right)^{\frac{1}{3}} \left(\int_0^\tau \left[\int_{\mathbb{R}^N} \chi_{[u>0]} \left(u^{n-2}|\nabla u|^3\right)^p\right]^{\frac{1}{p}}\right)^{\frac{2}{3}} \\
&\leq \left(\int_0^\tau t^{-(n+1-\frac{1}{p})\frac{N}{4+nN}} dt\right)^{\frac{1}{3}} \tau^{\frac{2}{3}\frac{1+\frac{N}{p}-N}{4+nN}}.
\end{aligned}$$

Integrating the first factor gives the result. The last inequality can be obtained in a similar way. □

Proof of Theorem 4. Let $(u_\varepsilon)_{\varepsilon>0}$ be a subsequence of solutions of problem (CP) which converges as stated in Lemma 6. For each ε, u_ε satisfies

$$\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}^N} u_\varepsilon \zeta_t - \int_{\mathbb{R}^N} \zeta(0) u_{0\varepsilon} dx = \frac{1}{2}\int_{[u_\varepsilon>0]} n(n-1) u_\varepsilon^{n-2}|\nabla u_\varepsilon|^2 \nabla u_\varepsilon \nabla\zeta \\
&+\frac{1}{2}\int_{[u_\varepsilon>0]} n u_\varepsilon^{n-1}|\nabla u_\varepsilon|^2 \Delta\zeta + \int_{[u_\varepsilon>0]} n u_\varepsilon^{n-1}\langle\nabla u_\varepsilon, D^2\zeta, \nabla u_\varepsilon\rangle \qquad (12)\\
&+\int_0^\infty \int_{\mathbb{R}^N} u_\varepsilon^n \nabla u_\varepsilon \nabla\Delta\zeta,
\end{aligned}$$

for all $\zeta \in C_0^\infty\left(\mathbb{R}^N \times [0,\infty)\right)$.

In order to prove Theorem 4, we need to pass to the limit as $\varepsilon \to 0$ in (12). Using Lemma 6 and the Assumption (H1) on $u_{0\varepsilon}$, the left hand side in (12) converges to

$$-\int_0^\infty \int_{\mathbb{R}^N} u\zeta_t - \int_{\mathbb{R}^N} \zeta(0) d\mu_0.$$

To handle the other terms, we observe that

$$\begin{aligned}
\int_{[u_\varepsilon>0]} u_\varepsilon^{n-2}|\nabla u_\varepsilon|^2 \nabla u_\varepsilon \nabla\zeta &= \int_\sigma^\infty \int_{\mathbb{R}^N} \chi_{[u_\varepsilon>0]} u_\varepsilon^{n-2}|\nabla u_\varepsilon|^2 \nabla u_\varepsilon \nabla\zeta \\
&\quad + \int_0^\sigma \int_{\mathbb{R}^N} \chi_{[u_\varepsilon>0]} u_\varepsilon^{n-2}|\nabla u_\varepsilon|^2 \nabla u_\varepsilon \nabla\zeta \\
&= I_1(\sigma) + I_2(\sigma).
\end{aligned}$$

Applying Lemma 7. i) we conclude that $I_2(\sigma)$ converges to zero as σ converges to zero uniformly in ε. For a fixed σ, using the local convergence properties stated in Lemma 6, we can argue as in the proof of Theorem 2.1 in [DGG] obtaining:

$$I_1(\sigma) \to \int_\sigma^\infty \int_{\mathbb{R}^N} \chi_{[u>0]} u^{n-2} |\nabla u|^2 \nabla u \nabla \zeta \quad \text{as } \varepsilon \to 0.$$

Letting $\sigma \to 0$, we get

$$\lim_{\varepsilon \to 0} \int_{[u_\varepsilon > 0]} u_\varepsilon^{n-2} |\nabla u_\varepsilon|^2 \nabla u_\varepsilon \nabla \zeta = \int_{[u>0]} u^{n-2} |\nabla u|^2 \nabla u \nabla \zeta.$$

In a similar way all the remaining terms on the right hand side of (12) can be treated. Hence u solves (CP) in the sense of Definition 1. Properties i)-vi) easily follow from Lemmas 6 and 7.

It remains to show vii). To this end, we choose for h and t positive

$$\zeta^{h,t}(x, \tau) = \eta(x) \nu^{h,t}(\tau),$$

with $\eta \in C_0^\infty(\mathbb{R}^N)$ and

$$\nu^{h,t}(\tau) = \begin{cases} 1 & \tau \le t \\ 1 - \dfrac{\tau - t}{h} & t < \tau < t + h \\ 0 & \tau \ge t + h \end{cases},$$

as a test function in ii) of Definition 1. Since $u \in C^0_{\text{loc}}((0, \infty); L^1_{\text{loc}}(\mathbb{R}^N))$ (see [BDGG] and [DGG]), we have

$$\lim_{h \to 0} - \int_0^t \int_{\mathbb{R}^N} u(x, \tau) \zeta_t^{h,t}(x, \tau) = \int_{\mathbb{R}^N} u(x, t) \eta(x) dx$$

for all t. Recalling that

$$\chi_{[u>0]} u^{n-2} |\nabla u|^3,\ \chi_{[u>0]} u^{n-1} |\nabla u|^2,\ u^n |\nabla u| \in L^1_{\text{loc}}(\mathbb{R}^N \times [0, \infty)),$$

we can conclude from (12) that

$$\int_{\mathbb{R}^N} u(x, t) \eta(x) dx \to \int_{\mathbb{R}^N} \eta(x) d\mu_0 \text{ as } t \to 0 \tag{13}$$

for all $\eta \in C_0^\infty(\mathbb{R}^N)$. Then, since $u(t)$ is uniformly bounded in $L^1(\mathbb{R}^N)$, the limit (13) is still valid for all $\eta \in C_0^0(\mathbb{R}^N)$, which completes the proof. □

4. – Conservation of mass

In this section we show that the solution we constructed in Theorem 4 has the property that its initial mass is preserved during the evolution.

PROOF OF THEOREM 5. The first assertion follows from Lemma 7.a) in the limit as $\varepsilon \to 0$.

To prove the conservation of mass property, we consider ii) of Definition 1 which is the weak formulation of the equation and choose test functions of the following form:

$$\zeta_R(x,s) := \chi_{[0,t]}(s)\eta_R(x), \quad R \geq 1,$$

where η_R denote smooth functions such that $\eta_R = 1$ on $B_R(0), \eta_R = 0$ on $\mathbb{R}^N \setminus B_{2R}(0)$ and $0 \leq \eta_R \leq 1$ on $B_{2R}(0)\setminus B_R(0)$, which have the property that

$$|\nabla\eta_R|(x), |D^2\eta_R|(x), |D^3\eta_R|(x) \leq \frac{C}{R}$$

where C is a constant that does not depend on R and x.

In fact ζ_R is not allowed as a test function, but after a standard regularizing procedure (see the proof of Theorem 4. vii)) we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^N} u(x,t)\eta_R(x) - \int_{\mathbb{R}^N} \eta_R d\mu_0 \\
&= \int_0^t \int_{\mathbb{R}^N\setminus B_{2R}(0)} \left\{ \chi_{[u>0]} \left[\frac{1}{2}(n(n-1)u^{n-2}|\nabla u|^2 \nabla u \nabla \eta_R \right.\right. \\
&\quad \left.\left. + nu^{n-1}|\nabla u|^2 \Delta\eta_R) + nu^{n-1}\langle\nabla u, D^2\eta_R, \nabla u\rangle \right] + u^n \nabla u \nabla\Delta\eta_R \right\} \\
&=: RHS\,.
\end{aligned} \tag{14}$$

We choose p as in Lemma 7. Then Hölder's inequality (defining $q := \frac{p}{p-1}$) and Lemma 7 imply

$$\begin{aligned}
|RHS| &\leq C(t)\left(\int_{B_{2R}(0)\setminus B_R(0)} \left(|\nabla\eta_R|^q + |D^2\eta_R|^q + |D^3\eta_R|^q\right)\right)^{\frac{1}{q}} \\
&\leq C(t)\left(\left[(2R)^N - R^N\right]\left(\frac{C}{R}\right)^q\right)^{\frac{1}{q}} \\
&\leq C(t)R^{\frac{N}{q}-1}.
\end{aligned}$$

For N, p and q as above $R^{\frac{N}{q}-1}$ converges to zero for R tending to infinity. Hence we can conclude that for all $t > 0$

$$\left|\int_{\mathbb{R}^N} u(x,t) - \mu_0(\mathbb{R}^N)\right| = \lim_{R\to\infty}\left|\int_{\mathbb{R}^N} u(x,t)\eta_R(x) - \int_{\mathbb{R}^N}\eta_R d\mu_0\right| = 0.$$

This proves the theorem. □

5. – A counterexample to a decay estimate for $2 < n < 3$.

In Bernis [B2] it was shown in one space dimension and for $0 < n < 2$ that there exists a solution u to the Cauchy problem (CP) with a decay estimate in time for the L^2–norm of the gradient. Later Bertsch et al. [BDGG] generalized this result to space dimensions two and three provided that $\frac{1}{8} < n < 2$. More precisely, it was shown that to all compactly supported initial data $u_0 \in H^1(\mathbb{R}^N)$ there exists a solution u to (CP) such that

$$\| \nabla u(t) \|_2 \leq C(n, N) \; \| u_0 \|_1^{\frac{8+n(N-2)}{2(4+nN)}} \; t^{-\frac{1}{2}\frac{N+2}{4+nN}}$$

for all $t > 0$. In this paper, we generalized the above result to initial data which are a nonnegative Radon measure with finite mass. In particular, we did not need to assume that the initial data are compactly supported.

In the following we demonstrate that such an estimate cannot be expected for $2 < n < 3$.

THEOREM 8. *For all $n \in (2, 3)$ there exist nonnegative initial data $u_0 \in L^1(\mathbb{R}^N)$ and a time $T > 0$ such that the Cauchy problem* (CP) *with initial data u_0 has a C^1 solution u in the sense of* Definition 1 *with the property that $\| \nabla u(t) \|_2$ is unbounded for $t \in (0, T)$.*

PROOF. Let $V(x, t) = t^{-\frac{N}{4+nN}} f(|x| t^{-\frac{1}{4+nN}})$ be the selfsimilar source type solution with mass one centered at the origin; i.e. V solves the Cauchy problem

$$\begin{aligned} u_t + \operatorname{div}(u^n \nabla \Delta u) &= 0, \\ u(x, 0) &= \delta_0, \end{aligned}$$

where δ_0 is the Dirac delta distribution centered at the point $x = 0$. Existence and uniqueness of a C^1 selfsimilar source type solution has been established by Bernis, Peletier and Williams [BPW] for $N = 1$ and by Ferreira and Bernis [FB] for $N \geq 2$.

A straightforward calculation shows that

$$\overline{V}(x, t) = M^{\frac{4}{4+nN}} t^{-\frac{N}{4+nN}} f\left(M^{-\frac{n}{4+nN}} |x - x_0| \, t^{-\frac{1}{4+nN}} \right)$$

is a solution with initial data $u_0 = M \delta_{x_0}$. Now we choose

i) a sequence $(M_j)_{j=1}^{\infty}$ of positive real numbers such that

$$\sum_{j=1}^{\infty} M_j < \infty,$$

ii) a time $T > 0$ and

iii) a sequence of points $x_j \in \mathbb{R}^N$ ($j \in \mathbb{N}$) such that the self similar solutions

$$u_j(x,t) = M_j^{\frac{4}{4+nN}} t^{-\frac{N}{4+nN}} f\left(M_j^{-\frac{n}{4+nN}} |x-x_j| t^{-\frac{1}{4+nN}}\right)$$

have mutually disjoint support until the time T. This is possible, because it is known that f has compact support (see [BPW],[FB]).

A solution to the Cauchy problem with initial data

$$u_0 = \sum_{j=1}^{\infty} M_j \delta_{x_j}$$

on the time interval $[0,T]$ is then given by

$$u(x,t) = \sum_{j=1}^{\infty} M_j^{\frac{4}{4+nN}} t^{-\frac{N}{4+nN}} f\left(M_j^{-\frac{n}{4+nN}} |x-x_j| t^{-\frac{1}{4+nN}}\right)$$

for $t \in [0,T]$ and $x \in \mathbb{R}^N$.

Hence

$$\begin{aligned} &\nabla u(x,t) \\ &= \sum_{j=1}^{\infty} M_j^{\frac{4}{4+nN}} t^{-\frac{N}{4+nN}} f'\left(M_j^{-\frac{n}{4+nN}} |x-x_j| t^{-\frac{1}{4+nN}}\right) M_j^{-\frac{n}{4+nN}} t^{-\frac{1}{4+nN}} \frac{x-x_j}{|x-x_j|} \\ &= \sum_{j=1}^{\infty} M_j^{\frac{4-n}{4+nN}} t^{\frac{-N-1}{4+nN}} f'\left(M_j^{-\frac{n}{4+nN}} |x-x_j| t^{-\frac{1}{4+nN}}\right) \frac{x-x_j}{|x-x_j|} \end{aligned}$$

and therefore

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u|^2(x,t)dx \\ &= \sum_{j=1}^{\infty} M_j^{\frac{8-2n}{4+nN}} t^{-\frac{2(N+1)}{4+nN}} \int_{\mathbb{R}^N} (f')^2 \left(M_j^{-\frac{n}{4+nN}} |x-x_j| t^{-\frac{1}{4+nN}}\right) dx \\ &= \left(\sum_{j=1}^{\infty} M_j^{\frac{8+n(N-2)}{4+nN}}\right) t^{-\frac{N+2}{4+nN}} \int_{\mathbb{R}^N} (f')^2(|y|)dy. \end{aligned}$$

If $\frac{8+n(N-2)}{4+nN} < 1$ we can choose a sequence $(M_j)_{j=1}^{\infty}$ such that i) is fulfilled and $\sum_{j=1}^{\infty} M_j^{\frac{8+n(N-2)}{4+nN}}$ diverges. Since

$$\frac{8+n(N-2)}{nN+4} < 1 \quad \Leftrightarrow \quad 8+nN-2n < nN+4 \quad \Leftrightarrow \quad 2 < n,$$

this is possible if $n > 2$. The proof of the theorem is hence completed since we can take $u(t)$ $(0 < t < T)$ as initial data. □

REMARK 9.
a) The above example shows that for $2 < n < 3$ the operator $u_0 \mapsto u(t)$ does not map $L^1(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N)$. This is in contrast to the case $0 < n < 2$.

b) Later we will demonstrate that a decay estimate is still possible if one considers compactly supported solutions in the case $N = 1$, $2 \leq n < 3$. This is possible if one allows the constant in the decay estimate to depend on the support of the initial data.

c) A similar calculation as in the proof of the theorem shows that the solution in the above example fulfills

$$\|u(t)\|_p^p = \left(\sum_{j=1}^{\infty} M_j^{\frac{4p+nN}{4+nN}}\right) t^{-\frac{(p-1)N}{4+nN}} \cdot \int_{\mathbb{R}^N} |f|^p(|y|)dy,$$

$$\int_{\mathbb{R}^N} u^{n-2}|\nabla u|^3(t) = \sum_{j=1}^{\infty} (M_j)^{\frac{n+4+nN}{4+nN}} t^{-\frac{nN-3}{4+nN}} \int_{\mathbb{R}^N} |f|^{n-2}|f'|^3(|y|)dy$$

which are both bounded for $t \in (0, T]$. Furthermore, we calculate

$$\int_{[u>0]} u^n|\nabla \Delta u|^p(t) = C \sum_{j=1}^{\infty} (M_j)^{\frac{4n+nN+4p-3np}{4+nN}} t^{-\frac{nN+pN+3p-N}{4+nN}}$$

which is bounded for $p = 1$ and unbounded for $p = 2$. The terms relevant for the definition of a weak formulation of the flux, i.e., $u^{n-2}|\nabla u|^3$, $u^{n-1}|\nabla u|^2$, $u^n|\nabla u|$ and $\chi_{[u>0]}u^n\nabla\Delta u$ are bounded in L^1 globally in space, whereas the terms appearing in the energy estimate, i.e.

$$\int_{\mathbb{R}^N} |\nabla u|^2(t) \quad \text{and} \quad \int_{t_1}^{t_2}\int_{\mathbb{R}^N} u^n|\nabla \Delta u|^2$$

are unbounded for the example in Theorem 8.

6. – Some further results in one space dimension

In one space dimension it is known that for all values of $n \in (0, 3)$ and all compactly supported initial data in $H^1(\mathbb{R}^N)$ a compactly supported solution to the Cauchy problem (CP) exists. This was shown by Bernis ([B2],[B4]).

Let us show that for a compactly supported Radon measure as initial data, it is still possible to show the existence of a solution in the sense of Definition 1 also in the case $2 \leq n < 3$. We suppose that μ_0 is a compactly supported nonnegative Radon measure defined on $\mathbb{R}$.

As in Section 3 we approximate μ_0 by compactly supported functions $u_{0\varepsilon} \in H^1(\mathbb{R})$ satisfying (H1)-(H3). Let u_ε be the solution of the Cauchy problem (CP) to initial data $u_{0\varepsilon} \in H^1(\mathbb{R})$ constructed as in the paper of Beretta, Bertsch and Dal Passo [BBD]. Recently Hulshof and Shiskov [HS] showed that the so constructed solution has the following property:

$$\text{if supp } u_{0\varepsilon} \subset [-R_{0\varepsilon}, R_{0\varepsilon}] \text{ then } \quad \text{supp } u_\varepsilon(t) \subset [-R_\varepsilon(t), R_\varepsilon(t)]$$

with

$$R_\varepsilon(t) \leq R_{0\varepsilon} + \bar{C} t^{\frac{1}{n+4}},$$

where $\bar{C}$ is a constant only depending on n and $\|u_{0\varepsilon}\|_1$. Bernis [B4] was the first to show finite speed of propagation in the case $2 \leq n < 3$. The later work of Hulshof and Shiskov [HS] makes it possible not only to show finite speed of propagation but also to give an asymptotic rate for the spreading of the support which is just in terms of the mass which doesn' t seem to be possible with the methods of [B4].

Now the following lemma holds.

LEMMA 10. *Let* $2 \leq n < 3$, $N = 1$ *and let* u_ε *be the solution to the Cauchy problem* (CP) *with initial data* $u_{0\varepsilon}$ *constructed as in* [BBD] *and* [HS]. *Then*

i) *u_ε fulfills the weak formulation ii) of the equation $u_t + (u^n u_{xxx})_x = 0$ of Definition 1;*

ii) *there exists a constant $C(\alpha, n) > 0$ such that for all $\alpha \in (\frac{1}{2} - n, 2 - n)$ with $\alpha + 1 > 0$*

$$\int_{\mathbb{R}} u_\varepsilon^{\alpha+1}(t_1) + C(\alpha, n) \int_{t_1}^{t_2} \int_{\mathbb{R}} u_\varepsilon^{\alpha+n-3} |u_{\varepsilon,x}|^4 \leq \int_{\mathbb{R}} u_\varepsilon^{\alpha+1}(t_2)$$

for all $0 < t_1 < t_2$;

iii) *there exists a constant $C(n) > 0$ such that*

$$\int_{\mathbb{R}} u_{\varepsilon,x}^2(t) \leq \frac{C(n)}{\sqrt{t}} \cdot \|u_{0\varepsilon}\|_1^{\frac{4-n}{2}} \left(2R_0 + \bar{C} t^{\frac{1}{n+4}}\right)^{\frac{n-2}{2}};$$

iv) *for all $p \in (1, \infty)$ there exists a constant $C(p, \|u_{0\varepsilon}\|_1, n) > 0$ such that it holds*

$$\|u_\varepsilon(t)\|_p \leq C(p, \|u_{0\varepsilon}\|_1, n)\, t^{\frac{1-p}{6p}} \left|R_0 + \bar{C} t^{\frac{1}{n+4}}\right|^{\frac{(p-1)(n-2)}{6p}}; \tag{15}$$

v) *there exists a constant* $\beta > 0$ *such that for all* $T > 0$ *a constant* $C = C(T, R_0, \|u_{0\varepsilon}\|_1, n) > 0$ *can be found such that*

$$\int_0^t \int_{\mathbb{R}} u_\varepsilon^{n-2} |u_{\varepsilon,x}|^3 \le C t^{\beta},$$

$$\int_0^t \int_{\mathbb{R}} u_\varepsilon^{n-1} u_{\varepsilon,x}^2 \le C t^{2\beta},$$

$$\int_0^t \int_{\mathbb{R}} u_\varepsilon^{n} \left|u_{\varepsilon,x}\right| \le C t^{3\beta}$$

for all $t \in [0, T]$.

Proof. The results i) and ii) follow from the results of [BBD]. Moreover, for $\alpha \in (-1, 0)$

$$\int_{\mathbb{R}} u_\varepsilon^{\alpha+1}(t) \le \|u_{0\varepsilon}\|_1^{\alpha+1} |\ \text{supp}\ u_\varepsilon(t)|^{-\alpha} \le \|u_{0\varepsilon}\|_1^{\alpha+1} \left| R_0 + \bar{C} t^{\frac{1}{n+4}} \right|^{-\alpha} . \tag{16}$$

In addition, we have for almost all $t > 0$

$$\int_{\mathbb{R}} u_{\varepsilon,x}^2(t) = - \int_{[u_\varepsilon > 0]} u_\varepsilon u_{\varepsilon,xx}(t) \le \|u_{0\varepsilon}\|_1^{\frac{1}{2}} \left(\int_{[u_\varepsilon > 0]} u_\varepsilon\ u_{\varepsilon,xx}^2(t) \right)^{\frac{1}{2}} .$$

In one space dimension it can be shown that a version of the integral estimates (4) also holds for $\alpha = 2 - n$ (see Bernis [B2], Remark 3.2). In this case, the gradient term drops out. Using this and the fact that $\int_{\mathbb{R}} u_{\varepsilon,x}^2(t)$ is nonincreasing in time gives for $n \in (2, 3)$

$$\begin{aligned} t \left(\int_{\mathbb{R}} u_{\varepsilon,x}^2(t) \right)^2 &\le \int_0^t \left(\int_{\mathbb{R}} u_{\varepsilon,x}^2(t) dt \right)^2 \\ &\le \|u_{0\varepsilon}\|_1 \int_0^t \int_{[u_\varepsilon > 0]} u_\varepsilon u_{\varepsilon,xx}^2 \\ &\le C(n) \|u_{0\varepsilon}\|_1 \int_{\mathbb{R}} u_\varepsilon^{3-n}(t) \\ &\le C(n) \|u_{0\varepsilon}\|_1 \|u_{0\varepsilon}\|_1^{3-n} |R_0 + \bar{C} t^{\frac{1}{n+4}}|^{n-2}, \end{aligned}$$

which implies

$$\int_{\mathbb{R}} u_{\varepsilon,x}^2(t) \le C(n)\, t^{-\frac{1}{2}} \|u_{0\varepsilon}\|_1^{\frac{4-n}{2}} \left| R_0 + \bar{C} t^{\frac{1}{n+4}} \right|^{\frac{n-2}{2}} .$$

Applying the Gagliardo-Nirenberg inequality (see Appendix) we obtain for all $p \in (1, \infty)$

$$\begin{aligned} \|u_\varepsilon(t)\|_p &\le C_p \|u_{\varepsilon,x}(t)\|_2^{\frac{2p-2}{3p}} \|u_{0\varepsilon}\|_1^{\frac{p+2}{3p}} \\ &\le C_p\, t^{\frac{1-p}{6p}} \|u_{0\varepsilon}\|_1^{\frac{4-n}{4}\frac{2p-2}{3p}} |R_0 + \bar{C} t^{\frac{1}{n+4}}|^{\frac{n-2}{4}\frac{2p-2}{3p}} \|u_{0\varepsilon}\|_1^{\frac{p+2}{3p}} . \end{aligned}$$

It remains to prove v). We take $\alpha \in (\frac{1}{2} - n, 2 - n)$ with $\alpha + 1 > 0$. Then for all $T > 0$ there exists a constant $C = C(T, R_0, \|u_{0\varepsilon}\|_1, \alpha, n) > 0$ such that for all $t \in [0, T]$

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} u_\varepsilon^{n-2} |u_{\varepsilon,x}|^3 &\le \left(\int_0^t \int_{\mathbb{R}} u_\varepsilon^{n+1-3\alpha} \right)^{\frac{1}{4}} \left(\int_0^t \int_{\mathbb{R}} \chi_{[u_\varepsilon > 0]} u_\varepsilon^{\alpha+n-3} |u_{\varepsilon,x}|^4 \right)^{\frac{3}{4}} \\ &\le C \left(\int_0^t C \tau^{\frac{-n+3\alpha}{6}} d\tau \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} u_\varepsilon^{\alpha+1}(t) \right)^{\frac{3}{4}} \\ &\le C \left[t^{\frac{-n+3\alpha+6}{6}} \right]^{\frac{1}{4}} \\ &= C t^{\frac{-n+3\alpha+6}{24}} , \end{aligned}$$

where we have used estimate iv) of this lemma and (16). Since for $n \in (2, 3)$ it is always possible to choose α such that $-n + 3\alpha + 6 > 0$, this proves the first statement in (v). The two other estimates in v) follow from the first one with similar arguments as in the proof of Lemma 8.

The case $n = 2$ needs to be considered slightly different because the integral estimates of [BBD] have a logarithmic correction. Since the modifications are straightforward we do not go into details. □

REMARK 11. The decay rates in iii) and iv) of Lemma 10 give the asymptotic rates for the source type solutions for large t.

THEOREM 12. *Let $N = 1, n \in [2, 3)$ and let μ_0 be a nonnegative Radon measure with* supp $\mu_0 \subset [-R_0, R_0]$. *Then there exists a solution u of the Cauchy problem* (CP), *in the sense of* Definition 1, *and u has the following properties:*

i) $u^{\frac{\alpha+n+1}{4}} \in L^4_{\text{loc}}([0, \infty); W^{1,4}(\mathbb{R}))$, $u^{\frac{\alpha+n+1}{2}} \in L^2_{\text{loc}}([0, \infty); H^2(\mathbb{R}))$ *whenever* $\alpha \in (-1, 2 - n)$;

ii) *it holds that* supp $u(t) \subset [-R(t), R(t)]$ *with* $R(t) \leq R_0 + Ct^{\frac{1}{n+4}}$ *and a constant C depending on n and* $\|\mu_0\|_1$. *In addition, the estimates* iii), iv) *and* v) *of* Lemma 10 *hold for u*;

iii) *the initial mass in preserved, i.e.*

$$\int_{\mathbb{R}} u(t) = \mu_0(\mathbb{R}) \quad \textit{for all} \quad t > 0.$$

Proof. The statements i) and ii) follow from Lemma 10 with the same techniques as in Section 3. Mass preservation follows directly from the weak formulation of the equation using the fact that the solution is compactly supported. □

Corollary 13. *The solution constructed in* Theorem 12 *has a decay behaviour for large t similar as in the case* $n < 2$. *Precisely we get: for all* R_0 *and* $p > 1$ *there exists a* $T(R_0)$ *such that for all* $t > T(R_0)$

$$\|u(t)\|_p \leq C(p, \|\mu_0\|_1, n)\, t^{-\frac{p-1}{p}\frac{1}{n+4}}$$

and

$$\|u_x(t)\|_2 \leq C(\|\mu_0\|_1, n)]\, t^{-\frac{1}{2}\frac{3}{4+n}}.$$

Here R_0 *is the same as in* Theorem 12.

To obtain results on solutions with unbounded support in one space dimension for $n \in [2, 3)$ we now consider the weak formulation of Bernis and Friedman [BF]. In their formulation a function u is a weak solution of the equation $u_t + (u^n u_{xxx})_x = 0$ if

$$\int_0^\infty \int_{\mathbb{R}} u\psi_t dxdt + \int_{[u>0]} u^n u_{xxx} \psi_x dxdt = 0 \tag{17}$$

for all Lipschitz continuous ψ having compact support in $\mathbb{R} \times (0, \infty)$. Bernis [B2] introduced the notion of a strong solution which is a weak solution in the sense of Bernis and Friedman having the property that $u(\cdot, t) \in C^1(\mathbb{R})$ for almost every $t > 0$. This definition was motivated by the regularity results of [BBD] and [BP] who showed existence of strong solutions in one space dimensions.

To establish the mass preserving property for a weak solution in the sense of [BF] it is necessary to estimate $\chi_{[u>0]} u^n u_{xxx}$ in $L^1_{\text{loc}}([0, \infty); L^1(\mathbb{R}))$ in terms of the initial mass.

Let us now consider the case $N = 1, 1 < n < 3$ with initial data which do not necessarily have compact support. Here we can establish a result if $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$.

THEOREM 14. *Let* $N = 1, 1 < n < 3$ *and let* $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ *be nonnegative. Then there exists a strong solution to the Cauchy Problem* (CP) *with initial trace* u_0 *which preserves its initial mass.*

Moreover, the properties i)-iii), vi), vii) *of* Theorem 4 *hold.*

PROOF. As in Section 3 we approximate u_0 by nonnegative compactly supported functions $u_{0\varepsilon} \in H^1(\mathbb{R})$ which fulfill (H1)–(H3). In addition we require $u_{0,\varepsilon} \to u_0$ in $H^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Then we can estimate the flux by using the energy inequality and the inequality of Gagliardo-Nirenberg:

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} \chi_{[u_\varepsilon>0]} u_\varepsilon^n |u_{\varepsilon,xxx}| &\le \left(\int_0^t \int_{\mathbb{R}} u_\varepsilon^n\right)^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}} \chi_{[u_\varepsilon>0]} u_\varepsilon^n u_{\varepsilon,xxx}^2\right)^{\frac{1}{2}} \\
&\le C(\|u_0\|_1) \left(\int_0^t \left(\int_{\mathbb{R}} u_{\varepsilon,x}^2(\tau)\right)^{\frac{n-1}{3}} d\tau\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} u_{\varepsilon,x}^2(0)\right)^{\frac{1}{2}} \\
&\le C\left(\|u_0\|_1, \|u_{0,x}\|_2\right) \left(\int_0^t \left(\int_{\mathbb{R}} u_{\varepsilon,x}^2(0)\right)^{\frac{n-1}{3}}\right)^{\frac{1}{2}} \\
&\le C\left(\|u_0\|_1, \|u_{0,x}\|_2\right) t^{\frac{1}{2}}.
\end{aligned}$$

To pass to the limit in ε and to show that a limit u of a subsequence of the u_ε is a strong solution follows with standard techniques (see [B2] for similar arguments). The mass conservation property follows similar as in the proof of Theorem 5. One just has to replace the role of the terms $\chi_{[u>0]} u^{n-2} |\nabla u|^3$, $\chi_{[u>0]} u^{n-1} |\nabla u|^2$ and $u^n |\nabla u|$ by $\chi_{[u>0]} u^n |u_{xxx}|$. □

In one space dimension the formulation of Bernis and Friedman (BF) can also be used to show existence of solutions with μ_0 being a finite Radon measure when $n \in (0, \frac{1}{8}]$. Let us briefly describe, how this can be done. We choose the same regularization of μ_0 as in Section 3.

Then it holds for the solutions u_ε, for $D \subset \mathbb{R}$ with finite Lebesgue measure $|D|$ and for all $t_1 < t_2$

$$\begin{aligned}
\int_{t_1}^{t_2} \int_D \chi_{[u_\varepsilon>0]} u_\varepsilon^n |u_{\varepsilon,xxx}| &\le \left(\int_{t_1}^{t_2} \int_D u_\varepsilon^n\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \int_D \chi_{[u_\varepsilon>0]} u_\varepsilon^n u_{\varepsilon,xxx}^2\right)^{\frac{1}{2}} \\
&\le \left(\int_{t_1}^{t_2} \left(\int_{\mathbb{R}} u_\varepsilon(t)\right)^n |D|^{1-n}\right)^{\frac{1}{2}} \left(\int_D u_{\varepsilon,x}^2(t_1)\right)^{\frac{1}{2}} \\
&\le C\left(\|u_\varepsilon\|_1, n, |D|\right) (t_2 - t_1)^{\frac{1}{2}} t_1^{-\frac{1}{2}\frac{3}{4+n}}.
\end{aligned}$$

Hence, Lemma 17 (see Appendix) gives

$$\int_0^t \int_D \chi_{[u_\varepsilon>0]} u_\varepsilon^n |u_{\varepsilon,xxx}| \leq C\left(\|u_0\|_1, n, |D|\right) t^{\frac{1}{2}\left(1-\frac{3}{4+n}\right)}. \tag{18}$$

In fact, this estimate is true for all $0 < n \leq 1$. This gives a local estimate for the flux term.

To show that a limit u fulfills the identity (17) follows similar as in the work of Bernis and Friedman [BF]. Note that we only need test functions which are zero close to the initial time and that u is therefore more regular on the support of the test function. This guarantees enough regularity to pass to the limit in the formulation (17).

Since we have the local estimate on the flux we can show

$$u(t) \overset{*}{\rightharpoonup} \mu_0 \quad \text{as} \quad t \searrow 0$$

in the sense of Radon measures. This follows because we control the flux $\chi_{[u>0]} u^n u_{xxx}$ locally in L^1 and hence we follow the lines of the proof of Theorem 4. As in the proof of Theorem 14, $u^n|u_{xxx}|$ replaces the terms $u^{n-2}|\nabla u|^3, u^{n-1}|\nabla u|^2$ and $u^n|\nabla u|$ in the arguments.

Hence, we proved the following theorem

THEOREM 15. *Let $N = 1$, $0 < n \leq 1$ and assume that μ_0 is a nonnegative Radon measure with finite mass.*

Then there exists a solution u to the Cauchy problem (CP) *which solves the equation $u_t + (u^n u_{xxx})_x = 0$ in the sense of* (17) *and for which $u(t) \overset{*}{\rightharpoonup} \mu_0$ as $t \searrow 0$ in the sense of measures. In addition u has the properties* i)-vii) *of* Theorem 4.

7. – On the sharpness of the decay estimates

In this section we want to study whether the decay estimates established in the preceding sections are sharp. More precisely we give estimates from below for the behaviour of the L^p-norms of the solution and the L^2-norm of its gradient. In addition we give an estimate from below for the Lebesgue measure of the support of the solution. All these estimates are valid for large times and they show that the asymptotic rates of the solution we construct are the same as the rates for the selfsimilar source type solutions. For one space dimension and for $0 < n < 2$ these results were established by Bernis [B2]. The following theorem generalizes his results to space dimensions two and three (with $n \in (\frac{1}{8}, 2)$) and for values $n \in [2, 3)$ (for $N = 1$).

THEOREM 16. *Let μ_0 be a nonnegative Radon measure with finite mass and let u be a solution constructed as described in the* Theorems 4, 12, 14 *and* 15.

1) *If $N = 1, n \in (0, 2)$ or $N = 2, 3, n \in (\frac{1}{8}, 2)$ then there exists a constant $C > 0$ depending on n and N such that*

$$|\{u(t) > 0\}| \geq C^{-1} \|\mu_0\|_1^{\frac{nN}{4+nN}} t^{\frac{N}{4+nN}}.$$

2) *If $N = 1, n \in [2, 3)$ then for all $p \in (0, 3 - n)$ there exists a constant C depending on p, n, N such that*

$$\int_{\mathbb{R}} u^p \geq C^{-1} \|\mu_0\|_1^{\frac{4p+n}{4+n}} t^{-\frac{p-1}{4+n}}.$$

Assume now in addition, that μ_0 has compact support.

3) *If $N = 1, n \in (0, 3)$ or $N = 2, 3, n \in \left(\frac{1}{8}, 2\right)$ then it holds*
   a) *for all $p \in (1, \infty)$ there exists a time t_0 depending on $\|\mu_0\|_1, n, p$ and N such that for all $t > t_0$*

$$\|u(t)\|_p \geq C^{-1} \|\mu_0\|_1^{\frac{4p+nN}{p(4+nN)}} t^{-\frac{p-1}{p} \frac{N}{4+nN}};$$

   b) *there exist a time t_0 depending on $\|\mu_0\|_1, n,$ and N and a constant C depending on n, N such that for all $t > t_0$*

$$\|\nabla u(t)\|_2 \geq C^{-1} \|\mu_0\|_1^{\frac{8+n(N-2)}{2(4+nN)}} t^{-\frac{1}{2} \frac{N+2}{4+nN}}.$$

4) *If $N = 1, n \in (2, 3)$ and $p \in (0, 1)$ then there exist constants R_0 and B such that*

$$\int_{\mathbb{R}} u^p \leq \|\mu_0\|_1^p \left(R_0 + B t^{\frac{1}{4+n}} \right)^{1-p}.$$

PROOF.

1) The case $N = 1$ was established by Bernis [B2] for solutions with initial data in $H^1(\mathbb{R})$ but his proof carries over also to the case with more general initial data considered here. For $N = 2, 3$ we can use the decay estimates as formulated in Theorem 4 to get

$$\begin{aligned} \|\mu_0\|_1 = \|u(t)\|_1 &\leq \|u(t)\|_p |\{u(t) > 0\}|^{\frac{p-1}{p}} \\ &\leq C \|\mu_0\|_1^{\frac{4p+nN}{p(4+nN)}} t^{-\frac{p-1}{p} \frac{N}{4+nN}} |\{u(t) > 0\}|^{\frac{p-1}{p}} \end{aligned}$$

which proves the claim.

2) Choosing $\alpha \in (-1, 2-n)$ we can deduce as in Remark 3.2 of [BDGG] that there exists a constant C depending on α and n such that for all $t_1, t_2 > 0$ with $t_1 < t_2$:

$$\int_{\mathbb{R}} u^{\alpha+1}(t_1) + C^{-1} \int_{t_1}^{t_2} \int_{\mathbb{R}} |\nabla u^{\frac{\alpha+n+1}{4}}|^4 \leq \int_{\mathbb{R}} u^{\alpha+1}(t_2). \tag{19}$$

The inequality of Gagliardo-Nirenberg for the function $w = u^{\frac{\alpha+n+1}{4}}$ with $a = 1 - (\alpha+1)\frac{4+(\alpha+n)}{4(\alpha+1)+n}$ gives

$$\left(\int_{\mathbb{R}} u(t)\right)^{\frac{\alpha+n+1}{4}} \leq \left(\int_{\mathbb{R}} |\nabla u^{\frac{\alpha+n+1}{4}}|^4\right)^{\frac{a}{4}} \left(\int_{\mathbb{R}} u^{\alpha+1}\right)^{(1-a)\frac{\alpha+n+1}{4(\alpha+1)}} \tag{20}$$

Defining $Y(t) := \int_{\mathbb{R}} u^{\alpha+1}(x,t)dx$ and $b := \frac{1-a}{a}\frac{\alpha+n+1}{\alpha+1}$ we can deduce from (19) and (20) that

$$\|u_0\|_1^{\frac{\alpha+n+1}{a}} Y^{-b} \leq CY'$$

in the sense of distributions. This differential inequality implies (see [BDGG] Sections 5 and 6)

$$\left(C^{-1}(1+b)\|u_0\|_1^{\frac{\alpha+n+1}{a}} t\right)^{\frac{1}{1+b}} \leq Y(t)$$

which proves 2).

3) We define a radius $R(t)$ such that $B_{R(t)}(0)$ is the smallest ball which contains the support of $u(t)$. Then we know

$$R(t) \leq R_0 + B\|\mu_0\|_1^{\frac{n}{4+nN}} t^{\frac{1}{4+nN}}. \tag{21}$$

Assertion a) now follows from

$$\|\mu_0\|_1 \leq \|u(t)\|_p | \text{ supp } u(t)|^{\frac{p-1}{p}} \leq \|u(t)\|_p |R_0 + B\|\mu_0\|_1^{\frac{n}{4+nN}} t^{\frac{1}{4+nN}}|^{\frac{p-1}{p}N}$$

which holds for all $p \in (1, \infty)$. The estimate for the gradient can be proven with the help of the inequality of Gagliardo-Nirenberg.

4) This estimate follows from the estimate (21) for the size of the support which was established by Hulshof and Shiskov [HS] for the case $N = 1, n \in [2, 3)$, and Hölders inequality.

8. – Appendix

We frequently use the following lemma which is based on a result by Bernis [B2].

LEMMA 17 (Bernis). *Let* $g : [0, \infty) \to \mathbb{R}$ *be a function such that* g *is continuous at* 0 *and*

$$g(\tau) - g(\sigma) \leq M(\tau^\beta - \sigma^\beta)^a \sigma^{-b}, \quad \textit{for all} \quad \tau > \sigma > 0,$$

where M, a, b *and* β *are positive real numbers satisfying* $\beta a - b > 0$. *Then*

$$g(\tau) - g(0) \leq M(1 - 2^{b-a})^{-1} \tau^{\beta a - b}$$

for all $\tau > 0$.

PROOF. Define $f(t) := g(t^{1/\beta})$. Then it holds

$$f(t) - f(s) = g(t^{1/\beta}) - g(s^{1/\beta}) \leq M(t - s)^a s^{-b/\beta}.$$

Now Lemma 7.6 of Bernis [B2] gives

$$f(T) - f(0) \leq M(1 - 2^{b-a})^{-1} T^{a-b/\beta}$$

which proves the lemma. □

Also we applied the inequality of Gagliardo–Nirenberg (see [Ga], [Ni]) in the following form:

THEOREM 18 (Gagliardo-Nirenberg). *For* $1 \leq p, q, r \leq \infty$ *and* $m, N \in \mathbb{N}$ *suppose that the real number* a *defined by the relation*

$$-\frac{N}{p} = a\left(m - \frac{N}{r}\right) + (1 - a)\left(-\frac{N}{q}\right).$$

lies in the interval $[0, 1)$. *Then there exists a constant* C *depending only on* m, N, p, q *and* r *such that for* $u \in L^q(\mathbb{R}^N)$ *with* $D^m u \in L^r(\mathbb{R}^N)$ *the inequality*

$$\|u\|_p \leq C \|D^m u\|_r^a \|u\|_q^{1-a}$$

holds.

The Gagliardo-Nirenberg inequality also holds for $q \in (0, 1)$ (see for example [B2], Lemma 10.3).

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