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On Products of Non-Commuting Sectorial Operators

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Abstract. Products of non-commuting sectorial operators are investigated to provide functional-analytical tools for the treatment of multiplicative perturbations and degenerations in evolution problems. Using the operator sum method, combined with the theory of operators with bounded imaginary powers, the following result is shown: If sectorial operators A and B in a Banach space of class $\mathcal{HT}$ possess bounded imaginary powers, satisfy a parabolicity condition, and fulfil an appropriate commutator estimate, then $\nu + AB$ is sectorial as well for a sufficiently large $\nu \geq 0$. Examples show that the result can be applied to degenerate parabolic problems.

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Introduction

Aim of the present paper is to provide functional-analytical tools for the treatment of multiplicative perturbations and degenerations in evolution equations. Let us, for instance, consider the abstract Cauchy problem

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[b(t)u(t)] + \mathfrak{L}u(t) + \nu u(t) = f(t), t \in J_T = [0, T), T \leq \infty, \nu \geq 0 \\
&\lim_{t \to +0} (bu)(t) = 0
\end{aligned}
\tag{1}
$$

in a Banach space E, where $\mathfrak{L}$ is a closed linear operator, and b a complex-valued function which may have zeroes. Defining $(Au)(t) := u'(t)$, $(Bu)(t) := b(t)u(t)$, and $(Lu)(t) := \mathfrak{L}u(t)$ on suitably chosen domains (where $\mathcal{D}(A)$ incorporates the initial condition) we interpret this problem as the equation

$$(\nu + AB + L)u = f \tag{2}$$

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in some space $X = \mathfrak{F}(J_T, E)$ of functions $u : J_T \longrightarrow E$. In view of nonlinear applications it is quite useful to have *maximal regularity* for this linear problem, in the sense that the inverse $(\nu + AB + L)^{-1}$ exists as a bounded operator from X into $X_{AB+L} := (\mathcal{D}(AB) \cap \mathcal{D}(L) : \|\cdot\|_X + \|(AB + L)\cdot\|_X)$.

For the special case $B = I$, which corresponds to $b(t) \equiv 1$ in (1), a whole string of results is available. A fundamental theorem on maximal regularity in real interpolation spaces $(X, X_A)_{\gamma,p}$ of order $\gamma \in (0, 1)$ between the Banach spaces X and $X_A = (\mathcal{D}(A), \|\cdot\|_X + \|A\cdot\|_X)$ was proven by G. Da Prato and P. Grisvard in 1975 (cf. [DG75], Théorème 6.7). Their result, which is tailored to applications in Hölder spaces, essentially reads as follows:

Let A and L be sectorial operators in X, whose spectral angles ϕ_A, ϕ_L fulfil the parabolicity condition $\phi_A + \phi_L < \pi$ (cf. Definition 1.2). *If, in addition, the resolvents of A and L are commutative, or satisfy a certain commutator estimate in $(X, X_A)_{\gamma,p}$, then the $(X, X_A)_{\gamma,p}$-realization of $\nu + A + L$ (with $0 < \gamma < 1$) is invertible for some $\nu > 0$.*

The proof is based on the operator sum method, which consists of the construction of $(\nu + A + L)^{-1}$ by means of a functional calculus for sectorial operators.

Counterexamples, given e.g. in [BC91], [Dor93], Section 3, or [LeM97], show that maximal regularity for the equation $(\nu+A+L)u = f$ in the underlying Banach space X itself cannot be expected without additional assumptions. Such conditions, which apply to a wide class of parabolic problems in Lebesgue spaces $\mathfrak{F} \in \{L_p : 1 < p < +\infty\}$, were formulated by G. Dore and A. Venni in 1987 (cf. [DV87], Theorem 2.1):

Assume that X is a Banach space of class $\mathcal{HT}$ (cf. Definition 1.1). *Moreover, let A and L be sectorial operators with bounded imaginary powers in X, whose power angles θ_A, θ_L satisfy the strong parabolicity condition $\theta_A + \theta_L < \pi$* (cf. Definition 1.3). *The resolvents of A and L are supposed to commute. Then, the sum $\nu + A + L$ with $\nu > 0$ is invertible on X.*

The proof is based on a suitable representation of $(\nu+A+L)^{-1}$ by means of a functional calculus for operators with bounded imaginary powers (cf. [PS90], Section 3), combined with the operator sum method. A Dore-Venni type theorem, dealing with sums of operators whose (noncommutative) resolvents satisfy a certain commutator estimate, was recently proven by S. Monniaux and J. Prüss (cf. [MP97], Theorem 1).

In order to be in a position to apply the above mentioned results on operator sums to the perturbed equation (2), certain properties of the product AB, defined on $\mathcal{D}(AB) = \{x \in \mathcal{D}(B) : Bx \in \mathcal{D}(A)\}$, have to be established. A crucial problem consists of specifying assumptions on the Banach space X and on the sectorial operators A, B, which guarantee that $\nu + AB$ is sectorial in X as well.

Statements concerning the invertibility of $I+AB$ in real interpolation spaces between X and X_A were proven, for instance, by A. Favini. Modifying the operator sum method of G. Da Prato and P. Grisvard, he constructed explicit representations of $(I+AB)^{-1}$, which are based on a contour integral S_1 (of the form (8)). The derivation can be outlined as follows. Employing an extended version of Dunford's functional calculus it is shown that S_1 and ABS_1 are bounded linear maps on $(X, X_A)_{\gamma,p}$, provided that the involved sectorial operators fulfil a certain commutator estimate in X, and γ is strictly positive. For $x \in (X, X_A)_{\gamma,p}$ we moreover obtain $(I+AB)S_1x = (I+Q_1)x$, where the perturbation Q_1 is caused by the non-commutativity of A and B. A suitable commutator condition guarantees boundedness and invertibility of $I+Q_1$ on $(X, X_A)_{\gamma,p}$, so that $R_1 := S_1(I+Q_1)^{-1}$ is a right inverse map to $I+AB$ on this interpolation space. Using an additional commutator estimate, injectivity of $I+AB$ can be shown.

For the details of the construction we refer to [Fav85], or, for the case of interpolation spaces $(X, X_A)_{\gamma,\infty}$, to [FP88] and [Fav96], where the latter article is restricted to products of resolvent commuting operators. The considerations on $I+AB$ in these papers are motivated, for instance, by the degenerate Cauchy problem (1), whose abstract formulation (2) is equivalent to $(I+AB_L)v = f$ with $B_L := B(\nu+L)^{-1}$ and $u := (\nu+L)^{-1}v$, provided that $\nu+L$ is invertible.

The restriction of the described operator sum method to real interpolation spaces between X and X_A, however, does not admit the construction of resolvents to the product AB on Lebesgue spaces $\mathfrak{F} \in \{L_p : 1 \leq p \leq +\infty\}$. Therefore, especially the problem of maximal $L_p(J_T, E)$-regularity for the equation (2) gives rise to the question for additional assumptions, which enable us to extend A. Favini's technique to the underlying Banach space X. Motivated by the above mentioned result of G. Dore and A. Venni on operator sums, we essentially impose the following conditions. The underlying Banach space X is assumed to belong to the class $\mathcal{HT}$ (cf. Definition 1.1), which, for instance, contains $L_p(J_T, L_q)$ with $p, q \in (1, \infty)$. Moreover, we claim that the involved sectorial operators A, B possess bounded imaginary powers, where the corresponding power angles θ_A and θ_B satisfy the strong parabolicity condition $\theta_A + \theta_B < \pi$ (cf. Definition 1.3). These additional assumptions justify an alternative representation of the contour integral S_λ, on which the construction of the resolvent $(\lambda + AB)^{-1}$ is based. It can be shown that S_λ has a unique bounded extension S_λ on X. Considering the closedness of the involved, densely defined operators, this result enables us to extend the described operator sum method to the underlying Banach space X. We shall see that $\nu + AB$ is sectorial in X, where the constant $\nu \geq 0$ depends on a commutator estimate, which is assumed to be satisfied for A and B.

The paper is organized as follows: In Section 1 we provide the abstract theory. Our main result, stated in Theorem 1.1, specifies conditions on the underlying Banach space X and on the sectorial operators A, B, which guarantee that $\nu + AB$ is sectorial as well for a sufficiently large $\nu \geq 0$. As a consequence we deduce the Dore-Venni type Theorem 1.2 for sums of operators with noncommutative resolvents. Analogous results in Hilbert spaces on less restrictive assumptions are derived separately.

In Section 2 the abstract theory is applied to initial value problems. Besides the Cauchy problem (1), containing a degeneration of the time derivative, an evolution equation for a singularly perturbed Laplace operator shall be considered. We specify conditions under which the abstract Theorem 1.1 applies to these multiplicative perturbations. The resulting statements enable us to derive maximal regularity in Hilbert spaces.

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NOTATIONS. In this paragraph we collect some basic notations which shall be used throughout the paper.

Let $\varphi \in (0, \pi)$ be given. Then the open sector $\{\zeta \in \mathbb{C} \setminus \{0\} : |\arg \zeta| < \varphi\}$ is denoted by Σ_φ. The abbreviation $\Gamma_\varphi^{r,R}$ stands for the positively oriented boundary

$$\Gamma_\varphi^{r,R} := -(-R, -r)e^{i\varphi} \cup re^{-i[-\varphi,\varphi]} \cup (r, R)e^{-i\varphi} \cup Re^{i[-\varphi,\varphi]}, 0 \le r < R \le +\infty,$$

of the set $\Sigma_\varphi \cap \{\zeta \in \mathbb{C} : r < |\zeta| < R\}$. In particular, in case of $R = +\infty$ we shall write $\Gamma_\varphi^r := \Gamma_\varphi^{r,\infty}$, and $\Gamma_\varphi := \Gamma_\varphi^0 = \Gamma_\varphi^{0,\infty}$.

Let X and Y be Banach spaces. Then $\mathcal{L}(X, Y)$ denotes the Banach space of bounded linear operators $A : X \longrightarrow Y$, endowed with the usual uniform operator norm. $\mathcal{L}(X)$ stands for the Banach algebra $\mathcal{L}(X, X)$. The set of topological linear isomorphisms $\{A \in \mathcal{L}(X, Y) : A \text{ bijective}, A^{-1} \in \mathcal{L}(Y, X)\}$ shall be denoted by $\mathcal{L}\mathrm{is}(X, Y)$.

Now let $Y \hookrightarrow X$. Then $(X, Y)_{\gamma,p}$ is the standard real interpolation space of order $\gamma \in (0, 1)$ and exponent $p \in [1, \infty]$ between X and Y. Standard complex interpolation shall be denoted by $[X, Y]_\gamma$.

Let A be an operator in X. Then $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$ and $\rho(A)$ denote domain, range, kernel, spectrum or resolvent set of A, respectively. If A is linear and closed, $X_A := (\mathcal{D}(A), \|\cdot\|_X + \|A \cdot\|_X)$, i.e., the domain equipped with the graph norm, is a Banach space.

In our paper we shall employ various spaces $\mathfrak{F}(J, X)$ of X-valued functions on a perfect interval $J \subseteq \mathbb{R}$. Let $\mathfrak{F} \in \{C^0 = C, C^k : k \in \mathbb{N}\}$ be the set of continuous, k-times continuously differentiable maps. By $\mathfrak{F} \in \{C^\gamma : \gamma \in (0, 1)\}$ we denote the space of (locally) γ-Hölder continuous functions. Moreover,

$\mathfrak{F} \in \{C^{1+\gamma} : \gamma \in (0,1)\}$ contains all $u \in C^1(J, X)$ with a (locally) γ-Hölder continuous (Fréchet) derivative. Let $\mathrm{int}(J)$ denote the interior of J. Then we identify $\mathfrak{F}(J, X)$ with $\mathfrak{F}(\mathrm{int}(J), X)$ in the case of a Lebesgue or Sobolev space $\mathfrak{F} \in \{L_p, W_p^k : p \in [1, \infty], k \in \mathbb{N}\}$.

Now let Ω be an open set in $\mathbb{R}^N$, and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then $L_p(\Omega)$, $W_p^k(\Omega)$, and $B^s_{p,q}(\Omega)$ denote the usual Lebesgue, Sobolev or Besov spaces of $\mathbb{K}$-valued functions on Ω, respectively. Moreover, $\mathring{W}_p^1(\Omega)$ contains all $u \in W_p^1(\Omega)$ with $u|_{\partial\Omega} = 0$ in the sense of traces. $C^\gamma(\overline{\Omega})$ is the space of γ-Hölder continuous functions $\overline{\Omega} \longrightarrow \mathbb{K}$.

The letter c is often used to denote a constant, which may differ from occurrence to occurrence. If it depends upon additional parameters, say t, we sometimes indicate this by $c(t)$.

1. – Products of Sectorial Operators

1.1. – Basic Concepts

Banach spaces of class $\mathcal{HT}$ and sectorial operators with bounded imaginary powers play an essential role in our theory. Aim of this paragraph is, therefore, to provide these concepts. However, here we confine ourselves to those statements, properties, and examples, which are relevant for the comprehension of the present paper.

Definition 1.1. A Banach space X is said to belong to the class $\mathcal{HT}$, if the Hilbert transform

$$(\mathcal{H}u)(t) := \frac{1}{\pi} \lim_{\delta \to +0} \int_{|\tau|>\delta} u(t-\tau)\frac{d\tau}{\tau}$$

is bounded on $L_2(\mathbb{R}, X)$. Occasionally we shall write $X \in \mathcal{HT}$.

The following lemma provides essential properties and examples of the class $\mathcal{HT}$.

Lemma 1.1. (S_1) *Banach spaces of class $\mathcal{HT}$ are reflexive.*

(S_2) *Hilbert spaces belong to the class $\mathcal{HT}$.*

(S_3) *Finite-dimensional Banach spaces are in $\mathcal{HT}$.*

(S_4) *Let $X \in \mathcal{HT}$ and $p \in (1, \infty)$. Then, the Lebesgue space $L_p((\Omega, \mu), X)$ of X-valued functions on a σ-finite measure space (Ω, μ) belongs to $\mathcal{HT}$.*

(S_5) *Closed linear subspaces of $X \in \mathcal{HT}$ are in $\mathcal{HT}$.*

(S_6) *Assume that the Banach spaces X and Y with $Y \hookrightarrow X$ are of class $\mathcal{HT}$. Then, the interpolation spaces $[X, Y]_\gamma$, $(X, Y)_{\gamma,p}$ of order $\gamma \in (0,1)$ and exponent $p \in (1, \infty)$ belong to $\mathcal{HT}$ as well.*

For these statements and a more comprehensive treatment of the class $\mathcal{HT}$ we refer to the sections III.4.4 and III.4.5 of the monograph [Ama95].

Next we introduce the basic concept of a sectorial operator.

Definition 1.2. A closed linear operator A in a Banach space X is called *sectorial*, if the following conditions are satisfied:

(C_1) $\mathcal{N}(A) = \{0\}$, $\overline{\mathcal{D}(A)} = \overline{\mathcal{R}(A)} = X$.

(C_2) The positive real axis $(0, +\infty)$ is contained in $\rho(-A)$, and there exists some $M_A \geq 1$, such that $\|t(t+A)^{-1}\|_{\mathcal{L}(X)} \leq M_A$, $\forall t \in (0, +\infty)$.

Condition (C_2) implies that the *spectral angle* ϕ_A of A, defined by

$$\phi_A := \inf\left\{\phi : \Sigma_{\pi-\phi} \subseteq \rho(-A), \sup_{\zeta \in \Sigma_{\pi-\phi}} \left\|\zeta(\zeta + A)^{-1}\right\|_{\mathcal{L}(X)} \leq M_A(\phi) < +\infty\right\},$$

belongs to the interval $[0, \pi)$. The class of sectorial operators A, whose spectral angle ϕ_A satisfies the condition $\phi_A \leq \phi \in [0, \pi)$, is denoted by $\mathcal{S}(X, \phi)$. Moreover, we shall write $\mathcal{S}(X) := \bigcup_{\phi \in [0,\pi)} \mathcal{S}(X, \phi)$.

The assumption that a given operator A belongs to the class $\mathcal{S}(X)$ allows to introduce the complex powers A^z, $z \in \mathbb{C}$, (consistently) by means of several extended versions of Dunford's functional calculus (cf. e.g. [Kom66], [Prü93], Chapter 8, or [HP]). We are especially interested in the pure imaginary powers A^{is}, $s \in \mathbb{R}$, of A. Examples, given e.g. in [BC91] or [Ven93], show that there are sectorial operators (in Hilbert spaces) whose imaginary powers A^{is} are unbounded for some s in each neighbourhood $(-\delta, \delta)$, $\delta > 0$. This justifies the following definition.

Definition 1.3. A sectorial operator A in a Banach space X is said to possess *bounded imaginary powers*, if there are constants $\delta > 0$ and $K_A \geq 1$, such that

$$A^{is} \in \mathcal{L}(X), \left\|A^{is}\right\|_{\mathcal{L}(X)} \leq K_A, \forall s \in (-\delta, \delta).$$

This condition is satisfied, if and only if $\{A^{is}\}_{s \in \mathbb{R}}$ is a strongly continuous group of bounded linear operators on X (cf. [Ama95], III.4.7.1 Theorem, or [Prü93], Paragraph 8.1).

The type $\theta_A = \overline{\lim}_{|s| \to \infty} |s|^{-1} \log \|A^{is}\|_{\mathcal{L}(X)}$ of $\{A^{is}\}_{s \in \mathbb{R}}$ is called the *power angle* of A. It is related to the spectral angle ϕ_A by the inequality $\theta_A \geq \phi_A$ (cf. e.g. [PS90], Theorem 2).

The class of operators A with bounded imaginary powers on X, whose power angles satisfy $\theta_A \leq \theta$, shall be denoted by $\mathcal{BIP}(X, \theta)$. Moreover, we use the abbreviation $\mathcal{BIP}(X) := \bigcup_{\theta \in [0,\pi)} \mathcal{BIP}(X, \theta)$.

The following lemma provides some permanence properties of the class $\mathcal{BIP}(X)$.

LEMMA 1.2. *Let A be of class $\mathcal{BIP}(X, \theta_A)$ and $\xi \in \Sigma_{\pi-\theta_A}$. Then the following statements hold:*

(S_1) $A^{-1} \in \mathcal{BIP}(X, \theta_A)$, *and* $(A^{-1})^{is} = A^{-is}$, $\forall s \in \mathbb{R}$.
(S_2) $\xi A \in \mathcal{BIP}(X, \theta_A + |\arg \xi|)$, *and* $(\xi A)^{is} = \xi^{is} A^{is}$, $\forall s \in \mathbb{R}$.
(S_3) $\xi + A \in \mathcal{BIP}(X, \max\{\theta_A, |\arg \xi|\})$.

The statements (S_1) and (S_2) easily follow from the corresponding permanence properties of the class $\mathcal{S}(X)$ (cf. e.g. [HP]), and Definition 1.3. For the proof of (S_3) we refer to [Mon97], Theorem 2.4.

In order to illustrate Definition 1.3, we provide two examples of operators which possess bounded imaginary powers.

EXAMPLE 1.1. We consider the derivative $A = \partial = \mathrm{d}/\mathrm{d}t$ in $X = L_p(J_T, E)$, where E is a Banach space of class $\mathcal{HT}$, J_T the interval $[0, T)$, $T \leq \infty$, and $p \in (1, \infty)$. Namely, let A be defined by

$$(Au)(t) := \frac{\mathrm{d}u}{\mathrm{d}t}(t) \text{ on } \mathcal{D}(A) := \left\{u \in W_p^1(J_T, E) : u(0) = 0\right\} .$$

Then, the operator A belongs to the class $\mathcal{BIP}(X, \pi/2)$.

For the case of a bounded interval J_T this result is shown in [DV87], Theorem 3.1, or [Ama95], II.4.10.5 Lemma. In the event of $J_\infty = [0, +\infty)$ we refer to [PS90], Section 2, Example 4.

EXAMPLE 1.2. Let the differential operator $\mathfrak{L}$ be defined by

$$(\mathfrak{L}u)(x) := -\sum_{j,k=1}^{n} \partial_j[a_{jk}(x)\partial_k u(x)] + \kappa u(x)\,, \ \kappa \geq 0, \ x \in \Omega\,,$$

where $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, denotes either $\mathbb{R}^n$, the half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$, an exterior domain, or an interior domain. In the event of a nonempty boundary, $\partial\Omega$ is assumed to belong to the class C^2. The coefficients $\{a_{jk}(x)\}_{j,k=1}^n$ are supposed to satisfy the following conditions:

(C_1) For each $x \in \overline{\Omega}$, $\{a_{jk}(x)\}_{j,k=1}^n$ is a real symmetric matrix. Moreover, there is some $a_0 > 0$, such that $a_0 \leq \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \leq a_0^{-1}$ holds for all $x \in \overline{\Omega}$ and $\xi \in \{y \in \mathbb{R}^n : |y| = 1\}$.
(C_2) There are $\gamma \in (0,1)$ and $r \in (n, \infty]$, such that $a_{jk} \in C^\gamma(\overline{\Omega}) \cap W_r^1(\Omega)$.
(C_3) In the event of an unbounded Ω, the following additional conditions hold: The limits $a_{jk}^\infty := \lim_{|x|\to\infty} a_{jk}(x)$ exist and satisfy $|a_{jk}^\infty - a_{jk}(x)| \leq c|x|^{-\gamma}$ for all $x \in \{y \in \mathbb{R}^n : |y| \geq 1\}$. Moreover, κ is strictly positive.

We fix an arbitrary $p \in (1, \infty)$ with $p \leq r$. Then, the $L_p(\Omega)$-realization $\mathfrak{L}_p$ of $\mathfrak{L}$, supplemented by a Dirichlet boundary condition in case of $\partial\Omega \neq \emptyset$, i.e.,

$$(\mathfrak{L}_p u)(x) := (\mathfrak{L}u)(x) \text{ on } \mathcal{D}(\mathfrak{L}_p) := \left\{u \in W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)\right\}\,,$$

belongs to the class $\mathcal{BIP}(L_p(\Omega), 0)$, and $0 \in \rho(\mathfrak{L}_p)$.

This result is a special case of Theorem C, stated in [PS93]. Elliptic operators in non-divergence form are considered in the same paper. Moreover, we refer to [DS97]. Here elliptic operators on $\mathbb{R}^n$ (and on compact manifolds without boundary) are investigated under weaker assumptions on the regularity of the coefficients.

Both Example 1.1 and 1.2 are especially tailored to applications in Section 2. Further examples and a more comprehensive treatment of the class $\mathcal{BIP}(X)$ can be found e.g. in [Prü93], Chapter 8, [PS90], Section 2, or [HP].

We conclude this paragraph with the following statement.

LEMMA 1.3. *Let X be a Banach space of class $\mathcal{HT}$, and $\{\mathcal{A}(s)\}_{s\in\mathbb{R}}$ a strongly measurable family of operators in $\mathcal{L}(X)$ satisfying $\|\mathcal{A}(s)\|_{\mathcal{L}(X)} \leq K\mathrm{e}^{\varphi|s|}$, $\forall s \in \mathbb{R}$, for some $K > 0$ and $\varphi \in [0, \pi)$. Then,*

$$\mathfrak{A}x : t \longmapsto \lim_{\delta\to+0} \frac{1}{2i} \int_{|\tau|>\delta} \mathcal{A}(t-\tau)x \frac{\mathrm{d}\tau}{\sinh(\pi\tau)}$$

belongs to $L_2((-1/2, 1/2), X)$. Moreover, there is a $c = c(K, \varphi)$, such that

$$\|\mathfrak{A}x\|_{L_2((-1/2,1/2),X)} \leq c\|x\|_X, \quad \forall x \in X.$$

The basic idea of the proof, which is essentially due to G. Dore and A. Venni (cf. [DV87]), can be outlined as follows. Since $\|\mathcal{A}(s)\|_{\mathcal{L}(X)} \leq K\mathrm{e}^{\varphi|s|}$, $\forall s \in \mathbb{R}$, holds for some $\varphi < \pi$, the first summand of

$$\frac{1}{2i}\int_{|\tau|\geq 1} \mathcal{A}(t-\tau)\frac{\mathrm{d}\tau}{\sinh(\pi\tau)} + \frac{1}{2i}\int_{\delta<|\tau|<1} \mathcal{A}(t-\tau)\left[\left(\frac{1}{\sinh(\pi\tau)} - \frac{1}{\pi\tau}\right) + \frac{1}{\pi\tau}\right]\mathrm{d}\tau$$

is uniformly bounded in $\mathcal{L}(X)$ with respect to $t \in (-1/2, 1/2)$. Because the behaviour of the function $1/\sinh(\pi s)$ in a neighbourhood of $s = 0$ is described by $|1/\sinh(\pi s) - 1/(\pi s)| = \mathcal{O}(|s|)$, the assumption $X \in \mathcal{HT}$ enables us to take the limit $\delta \longrightarrow +0$. The asserted estimate follows from the boundedness of the Hilbert transform on $L_2(\mathbb{R}, X)$ as well. For details we refer to [MP97], Lemma 1.

Lemma 1.3 plays an essential role in our theory. In the proof of Lemma 1.9 it shall be applied to operators of the form $\mathcal{A}(s) = B^{is}A^{is}$, where A and B possess bounded imaginary powers on $X \in \mathcal{HT}$, and the corresponding power angles satisfy the strong parabolicitiy condition $\theta_A + \theta_B < \pi$.

1.2. – Formulation of the Main Results

Aim of this paragraph is to present the main statements of Section 1. First we impose some basic assumptions on the underlying Banach space X and on the involved operators A, B.

ASSUMPTION 1.1. The Banach space X belongs to the class $\mathcal{HT}$.

ASSUMPTION 1.2. The closed linear operators A and B are assumed to satisfy the following conditions:

(A_1) A belongs to the class $\mathcal{BIP}(X, \theta_A)$, and $0 \in \rho(A)$.
(A_2) $B \in \mathcal{BIP}(X, \theta_B)$.
(A_3) The strong parabolicity condition $\theta_A + \theta_B < \pi$ is satisfied.
(A_4) The inclusion $(\mu + B)^{-1}\mathcal{D}(A) \subseteq \mathcal{D}(A)$ holds for some $\mu \in \rho(-B)$.

The assumptions $A \in \mathcal{BIP}(X, \theta_A)$ and $B \in \mathcal{BIP}(X, \theta_B)$ (equivalently) state that $\{A^{is}\}_{s\in\mathbb{R}}$, $\{B^{is}\}_{s\in\mathbb{R}}$ are strongly continuous groups of bounded linear operators on X, whose types are given by the corresponding power angles. Therefore, the estimates

$$\|A^{is}\|_{\mathcal{L}(X)} \le K_A e^{\varphi_A|s|}\,, \quad \|B^{is}\|_{\mathcal{L}(X)} \le K_B e^{\varphi_B|s|}, \quad \forall s \in \mathbb{R}\,, \tag{3}$$

are satisfied for each $\varphi_A > \theta_A, \varphi_B > \theta_B$ with constants $K_A(\varphi_A), K_B(\varphi_B) \ge 1$. Moreover, it follows that A and B are sectorial with a spectral angle $\phi_A \le \theta_A$ or $\phi_B \le \theta_B$, respectively. Consequently, there exist some constants $M_A(\varphi_A)$, $M_B(\varphi_B)$, such that the resolvent inequalities

$$\begin{aligned} \left\|(\lambda + A)^{-1}\right\|_{\mathcal{L}(X)} &\le \frac{M_A}{1+|\lambda|}, \quad && \forall\lambda \in \Sigma_{\pi-\varphi_A}\,, \\ \left\|(\mu + B)^{-1}\right\|_{\mathcal{L}(X)} &\le \frac{M_B}{|\mu|}, && \forall\mu \in \Sigma_{\pi-\varphi_B}\,, \end{aligned} \tag{4}$$

hold.

Throughout the remainder of Section 1, let $\varphi_A > \theta_A$ and $\varphi_B > \theta_B$ be arbitrary, but fixed angles which satisfy the condition $\varphi_A + \varphi_B < \pi$.

In general, the resolvents of A and B are not supposed to commute. Nevertheless, basic steps of our considerations require a commutator estimate to be satisfied. Essentially, we shall make use of the following condition:

$$\begin{cases} Z_{A,B}(\lambda,\mu) := \left[A(\mu+B)^{-1} - (\mu+B)^{-1}A\right](\lambda+A)^{-1} \text{ satisfies} \\ \left\|Z_{A,B}(\lambda,\mu)\right\|_{\mathcal{L}(X)} \le \dfrac{c_{AB}}{(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}}, \forall(\lambda,\mu) \in \Sigma_{\pi-\varphi_A} \times \Sigma_{\pi-\varphi_B}\,, \\ \text{for some } c_{AB} \ge 0 \text{ and } \alpha, \beta \ge 0 \text{ with } \alpha+\beta < 1\,. \end{cases} \tag{5}$$

Our main result specifies conditions, which guarantee that $\nu + AB$, defined on $\mathcal{D}(AB) = \{x \in \mathcal{D}(B) : Bx \in \mathcal{D}(A)\}$, is sectorial for some $\nu \ge 0$. The statement reads as follows.

THEOREM 1.1. *Assume that the Banach space X and the operators A, B fulfil the Assumptions* 1.1 *and* 1.2. *Moreover, let the commutator estimate* (5) *be satisfied.*

Then, $\nu + AB$ with $\nu \ge 0$ is an operator of class $\mathcal{S}(X, \varphi_A + \varphi_B)$, provided $\sup_{\delta>\nu}\{c_{AB} \cdot \delta^{-\beta}\}$ is sufficiently small.

As an immediate consequence we obtain the following result.

COROLLARY 1.1. *The assumptions of Theorem 1.1 (which guarantee that $\nu + AB$ belongs to the class $\mathcal{S}(X, \varphi_A + \varphi_B)$ for a suitable $\nu \geq 0$), and the additional condition $\theta_A + \theta_B < \varphi_A + \varphi_B < \pi/2$ are supposed to be satisfied.*

Then, $-(\nu + AB)$ generates an analytic C_0-semigroup on X, which is holomorphic and uniformly bounded on each sector Σ_ω with $\omega < \pi/2 - (\varphi_A + \varphi_B)$.

For the proof we refer to basic results on the generation of analytic semigroups, stated e.g. in [Gol85], Section 1.5, or [Lun95], Section 2.1.

As a further consequence of Theorem 1.1 the following statement concerning the sum of operators (with noncommutative resolvents) can be shown.

THEOREM 1.2. *The Banach space X and the operators A, B are supposed to fulfil the Assumptions 1.1, 1.2. Moreover, let there be some constants $c_{A+B} \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$, such that the commutator estimate*

$$\left\| Z_{A,B}(\lambda, \mu) \right\|_{\mathcal{L}(X)} \leq \frac{c_{A+B}}{(1 + |\lambda|)^{1-\alpha} |\mu|^{1-\beta}} \tag{6}$$

holds for all $(\lambda, \mu) \in \Sigma_{\pi - \varphi_A} \times \Sigma_{\pi - \varphi_B}$ (where the fixed angles $\varphi_A > \theta_A$, $\varphi_B > \theta_B$ satisfy $\varphi_A + \varphi_B < \pi$). Recall that $(\mathcal{D}(A) \cap \mathcal{D}(B), \|\cdot\|_X + \|(A + B)\cdot\|_X)$ is denoted by X_{A+B}.

Then, $A + B$ is of class $\mathcal{L}\mathrm{is}(X_{A+B}, X)$, provided c_{A+B} is sufficiently small.

REMARK 1.1. An alternative Dore-Venni type theorem, dealing with sums of operators whose resolvents are not commutative, was proven by S. Monniaux and J. Prüss (cf. [MP97], Theorem 1). Their statement basically differs from Theorem1.2 in the assumed commutator estimate

$$\begin{aligned} &\left\| A(\lambda + A)^{-1} \left[A^{-1}(\mu + B)^{-1} - (\mu + B)^{-1} A^{-1} \right] \right\|_{\mathcal{L}(X)} \leq \frac{c_{A+B}}{(1 + |\lambda|)^{1-\alpha} |\mu|^{1+\beta}}, \\ &\forall (\lambda, \mu) \in \Sigma_{\pi - \varphi_A} \times \Sigma_{\pi - \varphi_B} \text{ with } 0 \leq \alpha < \beta \leq 1, \end{aligned} \tag{7}$$

which does not require the compatibility condition $(\mu + B)^{-1}\mathcal{D}(A) \subseteq \mathcal{D}(A)$.

It seems that the commutator conditions (6) and (7) are independent.

REMARK 1.2. From the proof of Theorem 1.1, which is carried out in Paragraph 1.4, it follows that the involved commutator condition can be weakened in the following sense. Instead of (5), let the estimate

$$\left\| Z_{A,B}(\lambda, \mu) \right\|_{\mathcal{L}(X)} \leq \sum_{i=1}^{j} \frac{c_{AB}}{(1 + |\lambda|)^{1-\alpha_i} |\mu|^{1+\beta_i}}, \forall (\lambda, \mu) \in \Sigma_{\pi - \varphi_A} \times \Sigma_{\pi - \varphi_B},$$

be satisfied with $\alpha_i, \beta_i \geq 0, \alpha_i + \beta_i < 1, i \in \{1, ..., j\}, j \in \mathbb{N}$. For $\delta > 0$ we set $\delta^{-\beta} := \sum_{i=1}^{j} \delta^{-\beta_i}$. Then, Theorem 1.1 remains valid.

Note that the commutator condition (6) in Theorem 1.2 can be generalized in the same way.

1.3. – Preliminaries for the Proof

In the following discussion we provide some technical tools for the derivation of our abstract results.

In order to be in a position to apply Dunford's (classical) functional calculus, unbounded sectorial operators C shall occasionally be approximated by the bounded, invertible operators

$$C_\varepsilon := (\varepsilon + C)(I + \varepsilon C)^{-1}(\text{ or } C_\varepsilon := C(I+\varepsilon C)^{-1} \text{ if } C \text{ is invertible}) \, , \varepsilon > 0 .$$

Basic convergence properties of the Yosida approximations C_ε are given in the following lemma.

LEMMA 1.4. *Let C be an operator of class $\mathcal{S}(X, \phi_C)$ with $\phi_C \in [0, \pi)$.*
Then, the following convergence statements are valid:

(S_1) $\lim_{\varepsilon \to +0} C_\varepsilon x = Cx, \forall x \in \mathcal{D}(C)$.
(S_2) $\lim_{\varepsilon \to +0} (\lambda + C_\varepsilon)^{-1} = (\lambda + C)^{-1}$ *in* $\mathcal{L}(X)$ *for each* $\lambda \in \Sigma_{\pi - \varphi}$, $\varphi \in (\phi_C, \pi)$.

For the proof of Lemma 1.4 we refer to [Prü93], Section 8.1, or [HP].

LEMMA 1.5. *Let C be a sectorial operator in X with spectral angle ϕ_C. Then, all approximations C_ε with $\varepsilon > 0$ belong to $\mathcal{S}(X, \phi_C)$ as well. For each $\varphi \in (\phi_C, \pi)$ exists some $M'_C(\varphi)$ (which does not depend on $\varepsilon \in (0, 1)$), such that*

$$\sup_{\lambda \in \Sigma_{\pi - \varphi}} \left\| \lambda \, (\lambda + C_\varepsilon)^{-1} \right\|_{\mathcal{L}(X)} \leq M'_C(\varphi) .$$

Assume that C is an operator of class $\mathcal{BIP}(X, \theta_C)$, and $\varphi > \theta_C$. Then, there is some $K'_C(\varphi) \geq 1$, such that the estimate $\|C_\varepsilon^{is}\|_{\mathcal{L}(X)} \leq K'_C e^{\varphi|s|}$ holds uniformly for all $s \in \mathbb{R}$ and $\varepsilon \in (0, 1)$.

The first statement is obvious. For the second assertion we refer to [MP97], Section 4. Its proof is based on [PS90], Theorem 3, and the permanence properties of $\mathcal{BIP}(X)$, stated in Lemma 1.2.

Some of our considerations require a suitable estimate of $Z_{\cdot,\cdot}(\lambda, \mu)$, applied to the Yosida approximations $A_\varepsilon = A(I + \varepsilon A)^{-1}$, $B_\varepsilon = (\varepsilon + B)(I + \varepsilon B)^{-1}$. In those cases the following statement turns out to be useful.

LEMMA 1.6. *Let the commutator condition* (5) *be fulfilled.*
Then, there is a constant $c = c(\varphi_A, \varphi_B)$(which does not depend on $\varepsilon > 0$), such that the Yosida approximations of the involved operators satisfy

$$\left\| Z_{A_\varepsilon, B_\varepsilon}(\lambda, \mu) \right\|_{\mathcal{L}(X)} \leq \frac{c \cdot c_{AB}}{(1 + |\lambda|)^{1-\alpha} |\mu|^{1+\beta}}, \forall (\lambda, \mu) \in \Sigma_{\pi - \varphi_A} \times \Sigma_{\pi - \varphi_B}, \forall \varepsilon \in (0, 1/2) .$$

PROOF. Since $Z_{A_\varepsilon, B_\varepsilon}(\lambda, \mu)$ can be rewritten in the form of

$$Z_{A_\varepsilon, B_\varepsilon}(\lambda, \mu) = \frac{1}{1 + \varepsilon\lambda} \frac{1 - \varepsilon^2}{(1 + \varepsilon\mu)^2} \frac{1}{\varepsilon} \left(\frac{1}{\varepsilon} + A \right)^{-1} Z_{A,B} \left(\frac{\lambda}{1 + \varepsilon\lambda}, \frac{\varepsilon + \mu}{1 + \varepsilon\mu} \right) ,$$

our assertion follows from the commutator estimate (5) and the resolvent inequality for A. □

REMARK 1.3. In our proofs we employ contour integrals over $-\lambda \in \Gamma_A := \Gamma_{\omega_A}^{r_A}$, $\omega_A > \varphi_A$, $r_A > 0$, where, in particular, $r_A e^{-i[-\omega_A, \omega_A]} \subset \Gamma_A$ is not contained in $-\Sigma_{\pi-\varphi_A} = \mathbb{C} \setminus \overline{\Sigma}_{\varphi_A}$. However, the invertibility of A implies $-\lambda \in \rho(A)$ and

$$\left\|(\lambda + A)^{-1}\right\|_{\mathcal{L}(X)} \leq \left\|A^{-1}\right\|_{\mathcal{L}(X)} \left(1 - |\lambda| \left\|A^{-1}\right\|_{\mathcal{L}(X)}\right)^{-1} \text{ if } |\lambda| < \left\|A^{-1}\right\|_{\mathcal{L}(X)}^{-1},$$

so that the resolvent inequality (4) for A, and the commutator estimate (5) remain applicable on the whole path $-\Gamma_A$, provided that $r_A < \|A^{-1}\|_{\mathcal{L}(X)}^{-1}$.

The following statement shall be used in the proof of injectivity of $\lambda + AB$.

LEMMA 1.7. *Let $\Omega \subset \mathbb{C}$ be an open connected set, X a Banach space, and A a closed linear operator in X, which satisfies the following conditions:*

(A_1) $\mathcal{R}(\lambda + A) = X, \forall \lambda \in \Omega$.

(A_2) *There is some $\lambda_0 \in \Omega$, such that $\mathcal{N}(\lambda_0 + A) = \{0\}$.*

Then, $\mathcal{N}(\lambda + A) = \{0\}$ holds for all $\lambda \in \Omega$.

Lemma 1.7 is a consequence of the perturbation Theorem IV.5.22 for semi-Fredholm operators, proven in the monograph [Kat80].

1.4. – Proofs of the Main Results

Our proof of Theorem 1.1 can be outlined as follows. The construction of resolvents $(\lambda + AB)^{-1}$ for $\lambda \in \Sigma_\eta$, where η denotes an arbitrary, but fixed angle with $0 < \eta < \pi - (\varphi_A + \varphi_B)$, is based on the contour integral

$$\begin{aligned} S_\lambda x &:= \frac{1}{2\pi i} \int_{\Gamma_A} \frac{1}{\zeta} \left(\frac{\lambda}{\zeta} + B\right)^{-1} (\zeta - A)^{-1} x \, d\zeta \\ &= \frac{1}{2\pi i} \int_{\Gamma_A} \frac{1}{\zeta^2} \left(\frac{\lambda}{\zeta} + B\right)^{-1} A(\zeta - A)^{-1} x \, d\zeta \end{aligned} \tag{8}$$

$$\text{over } \Gamma_A := \Gamma_{\omega_A}^{r_A}, \ \varphi_A < \omega_A < \pi - (\varphi_B + \eta), 0 < r_A < \left\|A^{-1}\right\|_{\mathcal{L}(X)}^{-1},$$

which defines a bounded linear operator from X_A into X. Using our assumptions that the underlying Banach space X is of class $\mathcal{HT}$, and the involved operators A, B possess bounded imaginary powers, we are able to extend S_λ to an uniquely determined $\mathrm{S}_\lambda \in \mathcal{L}(X)$ (cf. Lemma 1.9 and 1.10). It turns out that S_λ maps X into $\mathcal{D}(AB) = \{x \in \mathcal{D}(B) : Bx \in \mathcal{D}(A)\}$, and solves $(\lambda + AB)\mathrm{S}_\lambda = I + Q_\lambda$ on X, where the perturbation Q_λ is due to the non-commutativity of A and B. By virtue of the commutator condition (5), we shall see that the linear operator Q_λ is bounded and satisfies the estimate $\|Q_\lambda\|_{\mathcal{L}(X)} < 1$, if $c_{AB} \cdot |\lambda|^{-\beta}$ is sufficiently small. In this case $R_\lambda := \mathrm{S}_\lambda(I + Q_\lambda)^{-1} \in \mathcal{L}(X)$ is a right inverse map to $\lambda + AB$. As a consequence of a uniqueness statement, the operator R_λ even turns out to be the resolvent $(\lambda + AB)^{-1}$.

Our first lemma provides two basic properties of the product AB.

LEMMA 1.8. *AB is a densely defined, closed linear operator.*

PROOF. First we show that AB is densely defined. Because of $\overline{\mathcal{D}(A)} = X$, an arbitrary $x \in X$ can be approximated by a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{D}(A)$. Then the assumption $(\mu + B)^{-1}\mathcal{D}(A) \subseteq \mathcal{D}(A)$, $\mu \in \rho(-B)$, implies that all members of $\{x_n^*\}_{n=1}^{\infty}$, defined by $x_n^* := n(n+B)^{-1}x_n$, belong to $\mathcal{D}(AB)$. Using the convergence property

$$\left\|t(t+B)^{-1}x - x\right\|_X \longrightarrow 0 \text{ as } t \longrightarrow \infty, \forall x \in X = \overline{\mathcal{D}(B)},$$

and the resolvent estimate $\|t(t+B)^{-1}\|_{\mathcal{L}(X)} \le M_B$, $\forall t > 0$, we consequently obtain

$$\|x_n^* - x\|_X \le \left\|n(n+B)^{-1}\right\|_{\mathcal{L}(X)} \|x_n - x\|_X + \left\|n(n+B)^{-1}x - x\right\|_X \longrightarrow 0$$

as $n \longrightarrow \infty$. This shows $\overline{\mathcal{D}(AB)} = X$.

The closedness of AB can be proven as follows. Assume that the sequences $\{x_n\}_{n=1}^{\infty} \subset \mathcal{D}(AB)$ and $\{w_n\}_{n=1}^{\infty} := \{ABx_n\}_{n=1}^{\infty}$ are convergent in X, namely

$$x_n \longrightarrow x \in X, \quad w_n = ABx_n \longrightarrow w \in X \text{ as } n \longrightarrow \infty.$$

Since A is invertible, we obtain $Bx_n = A^{-1}w_n \longrightarrow A^{-1}w$ as $n \longrightarrow \infty$, so that the closedness of B implies $x \in \mathcal{D}(B)$ and $Bx = A^{-1}w$. Consequently, x even belongs to $\mathcal{D}(AB)$ and satisfies $ABx = w$. □

The aim of our following considerations is to prove that $S_\lambda \in \mathcal{L}(X_A, X)$, $\lambda \in \Sigma_\eta$, has a unique extension $\mathrm{S}_\lambda \in \mathcal{L}(X)$. For that purpose we introduce

$$\begin{aligned} S_\lambda^{(\varepsilon)} &= \frac{1}{2\pi i}\int_{\Gamma_{A_\varepsilon}} \frac{1}{\zeta}\left(\frac{\lambda}{\zeta} + B_\varepsilon\right)^{-1} (\zeta - A_\varepsilon)^{-1}\, \mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{\Gamma_{A_\varepsilon}} \frac{1}{\zeta^2}\left(\frac{\lambda}{\zeta} + B_\varepsilon\right)^{-1} A_\varepsilon\,(\zeta - A_\varepsilon)^{-1}\, \mathrm{d}\zeta \in \mathcal{L}(X), \varepsilon > 0 \end{aligned} \tag{9}$$

with $A_\varepsilon = A(I + \varepsilon A)^{-1}$, $B_\varepsilon = (\varepsilon + B)(I + \varepsilon B)^{-1}$,

where the positively oriented, closed contour $\Gamma_{A_\varepsilon} \subset \Sigma_{\varphi_A} \setminus \sigma(A_\varepsilon)$ surrounds the compact spectrum of the bounded, invertible approximation A_ε. Now the basic idea consists of showing that the integrals $S_\lambda^{(\varepsilon)} \in \mathcal{L}(X)$ are uniformly bounded with respect to $\varepsilon > 0$, and approach S_λ pointwise on the dense domain $\mathcal{D}(A)$. In order to obtain the desired uniform estimate, we first derive a suitable representation of $S_\lambda^{(\varepsilon)}$. Using Dunford's functional calculus, (9) can be rewritten as

$$S_\lambda^{(\varepsilon)} = \frac{1}{(2\pi i)^2}\int_{\Gamma_{A_\varepsilon}}\int_{\Gamma_{B_\varepsilon}} \frac{1}{\lambda + \xi\zeta}\,(\xi - B_\varepsilon)^{-1}\,(\zeta - A_\varepsilon)^{-1}\, \mathrm{d}\xi\mathrm{d}\zeta\,,$$

where the positively oriented, closed contour $\Gamma_{B_\varepsilon} \subset \Sigma_{\varphi_B} \setminus \sigma(B_\varepsilon)$ surrounds the spectrum of B_ε. The application of the identity

$$\frac{\lambda}{\lambda + \xi\zeta} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi}{\sin(\pi z)}\lambda^z \xi^{-z}\zeta^{-z} \mathrm{d}z\,, \quad \forall(\xi,\zeta) \in \mathbb{C}^2 : |\arg\xi| + |\arg\zeta| + \eta < \pi\,,\ \gamma \in (0,1)\,, \tag{10}$$

which is based on the inverse Mellin transform, consequently yields

$$S_\lambda^{(\varepsilon)} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi}{\sin(\pi z)}\lambda^{z-1} B_\varepsilon^{-z} A_\varepsilon^{-z} \mathrm{d}z, \gamma \in (0,1), \varepsilon > 0\,. \tag{11}$$

Using this representation and the assumptions $X \in \mathcal{HT}$, $A, B \in \mathcal{BIP}(X)$, we are able now to prove the following statement.

LEMMA 1.9. *There is a positive constant c, such that $S_\lambda^{(\varepsilon)} \in \mathcal{L}(X)$ can uniformly be estimated by*

$$\left\| S_\lambda^{(\varepsilon)} \right\|_{\mathcal{L}(X)} \le \frac{c}{|\lambda|}\left(1 + \frac{c_{AB}}{|\lambda|^\beta}\right), \forall\varepsilon > 0, \forall\lambda \in \Sigma_\eta\,.$$

PROOF. Our derivation is based on a method, which was similarly applied to the case of operator sums by S. Monniaux and J. Prüss (cf. [MP97], Lemma 3). First we rewrite the integral $\lambda S_\lambda^{(\varepsilon)}$, given by (11), in the form of

$$\begin{aligned} \lambda S_\lambda^{(\varepsilon)} &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi}{\sin(\pi z)}\left(\lambda^{-1}A_\varepsilon\right)^{-it} B_\varepsilon^{-it} B_\varepsilon^{it-z}\left(\lambda^{-1}A_\varepsilon\right)^{it-z} \mathrm{d}z \\ &\quad + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi}{\sin(\pi z)}\lambda^z\left[B_\varepsilon^{-z}A_\varepsilon^{-it} - A_\varepsilon^{-it}B_\varepsilon^{-z}\right]A_\varepsilon^{it-z}\mathrm{d}z\,. \end{aligned} \tag{12}$$

According to Lemma 1.5, all approximations A_ε and B_ε with $\varepsilon > 0$ belong to $\mathcal{BIP}(X,\theta_A)$ and $\mathcal{BIP}(X,\theta_B)$, respectively. Moreover, there is a constant c (which does not depend on ε), such that

$$\left\| B_\varepsilon^{it}\left(\lambda^{-1}A_\varepsilon\right)^{it} \right\|_{\mathcal{L}(X)} \le c e^{\varphi|t|}, \quad \forall\varepsilon > 0, \quad \forall\lambda \in \Sigma_\eta, \quad \forall t \in \mathbb{R}$$

with $\varphi := \varphi_A + \varphi_B + \eta < \pi$. Consequently, Lemma 1.3 justifies the representation

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi}{\sin(\pi z)}\left(\lambda^{-1}A_\varepsilon\right)^{-it} B_\varepsilon^{-it} B_\varepsilon^{it-z}\left(\lambda^{-1}A_\varepsilon\right)^{it-z} x\mathrm{d}z \\ &= \frac{1}{2}x + \left(\lambda^{-1}A_\varepsilon\right)^{-it} B_\varepsilon^{-it}\mathfrak{A}(t)x, \forall x \in X, \varepsilon > 0, \lambda \in \Sigma_\eta, t \in (-1/2, 1/2), \\ &\quad \text{with } \mathfrak{A}(t)x := \lim_{\delta\to+0}\frac{1}{2i}\int_{|\tau|>\delta} B_\varepsilon^{i(t-\tau)}\left(\lambda^{-1}A_\varepsilon\right)^{i(t-\tau)} x\frac{\mathrm{d}\tau}{\sinh(\pi\tau)}\,, \end{aligned}$$

which is obtained by deforming the path $\gamma + i\mathbb{R} = \gamma + \Gamma_{\pi/2}$ into $\Gamma^{\delta}_{\pi/2}$, and taking the limit $\delta \longrightarrow +0$. Moreover, it leads to the uniform estimate

$$\int_{-1/2}^{1/2} \left\| \left(\lambda^{-1}A_\varepsilon\right)^{-it} B_\varepsilon^{-it} \left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi}{\sin(\pi z)} B_\varepsilon^{it-z}\left(\lambda^{-1}A_\varepsilon\right)^{it-z} x \mathrm{d}z\right]\right\|_X \mathrm{d}t$$
$$\leq \frac{1}{2}\|x\|_X + \left\{\int_{-1/2}^{1/2} \mathrm{e}^{-2t\arg\lambda}\left\|A_\varepsilon^{-it}\right\|^2_{\mathcal{L}(X)}\left\|B_\varepsilon^{-it}\right\|^2_{\mathcal{L}(X)}\mathrm{d}t\right\}^{1/2}\|\mathfrak{A}x\|_{L_2((-1/2,1/2),X)}$$
$$\leq c\|x\|_X, \forall x \in X, \forall \varepsilon > 0, \forall \lambda \in \Sigma_\eta .$$

Using identity (10) the second summand on the right hand side of (12) can be rewritten as

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi}{\sin(\pi z)}\lambda^z\left[B_\varepsilon^{-z}A_\varepsilon^{-it} - A_\varepsilon^{-it}B_\varepsilon^{-z}\right]A_\varepsilon^{it-z}\mathrm{d}z$$
$$= \frac{\lambda}{(2\pi i)^2}\int_{\Gamma_{A_\varepsilon}}\int_{\Gamma_{B_\varepsilon}} \frac{1}{\lambda+\xi\zeta}\left[(\xi-B_\varepsilon)^{-1}A_\varepsilon^{-it} - A_\varepsilon^{-it}(\xi-B_\varepsilon)^{-1}\right](\zeta-A_\varepsilon)^{-1}A_\varepsilon^{it}\mathrm{d}\xi\mathrm{d}\zeta$$
$$= \frac{\lambda}{2\pi i}\int_{\Gamma_{A_\varepsilon}}\left[(\lambda+\zeta B_\varepsilon)^{-1}A_\varepsilon^{-it} - A_\varepsilon^{-it}(\lambda+\zeta B_\varepsilon)^{-1}\right](\zeta-A_\varepsilon)^{-1}A_\varepsilon^{it}\mathrm{d}\zeta .$$

Now we employ the Dunford integral

$$A_\varepsilon^{-it} := \frac{1}{2\pi i}\int_{\Gamma^*_{A_\varepsilon}} \xi^{-it}(\xi - A_\varepsilon)^{-1}\mathrm{d}\xi ,$$

where the positively oriented, closed contour $\Gamma^*_{A_\varepsilon} \subset \Sigma_{\varphi_A}$ surrounds $\sigma(A_\varepsilon)$. Without loss of generality $\Gamma^*_{A_\varepsilon}$ can be chosen in such a manner that $\Gamma^*_{A_\varepsilon}\cap\Gamma_{A_\varepsilon} = \emptyset$ and $\mathrm{diam}(\Gamma^*_{A_\varepsilon}) > \mathrm{diam}(\Gamma_{A_\varepsilon})$ are satisfied. This implies

$$\Psi_\varepsilon(\xi) := \frac{1}{2\pi i}\int_{\Gamma_{A_\varepsilon}} \frac{1}{\xi-\zeta}(\lambda+\zeta B_\varepsilon)^{-1}\mathrm{d}\zeta = 0, \forall \xi \in \Gamma^*_{A_\varepsilon}, \forall\varepsilon > 0, \forall\lambda \in \Sigma_\eta ,$$

by holomorphy of the integrand on a simply connected, open set containing Γ_{A_ε}. Using the resolvent equation we consequently obtain

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi}{\sin(\pi z)}\lambda^z\left[B_\varepsilon^{-z}A_\varepsilon^{-it} - A_\varepsilon^{-it}B_\varepsilon^{-z}\right]A_\varepsilon^{it-z}\mathrm{d}z$$
$$= \frac{\lambda}{(2\pi i)^2}\int_{\Gamma_{A_\varepsilon}}\int_{\Gamma^*_{A_\varepsilon}} \xi^{-it}(\lambda+\zeta B_\varepsilon)^{-1}(\xi-A_\varepsilon)^{-1}(\zeta-A_\varepsilon)^{-1}A_\varepsilon^{it}\mathrm{d}\xi\mathrm{d}\zeta$$
$$- \frac{\lambda}{(2\pi i)^2}\int_{\Gamma_{A_\varepsilon}}\int_{\Gamma^*_{A_\varepsilon}} \xi^{-it}(\xi-A_\varepsilon)^{-1}(\lambda+\zeta B_\varepsilon)^{-1}(\zeta-A_\varepsilon)^{-1}A_\varepsilon^{it}\mathrm{d}\xi\mathrm{d}\zeta$$
$$= \frac{\lambda}{(2\pi i)^2}\int_{\Gamma_{A_\varepsilon}}\int_{\Gamma^*_{A_\varepsilon}} \frac{\xi^{-it}}{\zeta(\xi-\zeta)}(\xi-A_\varepsilon)^{-1}Z_{A_\varepsilon,B_\varepsilon}(-\zeta,\lambda/\zeta)A_\varepsilon^{it}\mathrm{d}\xi\mathrm{d}\zeta$$
$$- \frac{\lambda}{2\pi i}\int_{\Gamma^*_{A_\varepsilon}} \xi^{-it}(\xi-A_\varepsilon)^{-1}\left[\Psi_\varepsilon(\xi)A_\varepsilon - A_\varepsilon\Psi_\varepsilon(\xi)\right](\xi-A_\varepsilon)^{-1}A_\varepsilon^{it}\mathrm{d}\xi$$
$$= -\frac{\lambda}{2\pi i}\int_{\Gamma_{A_\varepsilon}} \zeta^{-1}A_\varepsilon^{-it}(\zeta-A_\varepsilon)^{-1}Z_{A_\varepsilon,B_\varepsilon}(-\zeta,\lambda/\zeta)A_\varepsilon^{it}\mathrm{d}\zeta, \forall\varepsilon > 0, \forall\lambda \in \Sigma_\eta .$$

In view of Lemma 1.6, commutator condition (5) leads to the estimate

$$\int_{\Gamma_A} \left\| \zeta^{-1} A_\varepsilon^{-it} (\zeta - A_\varepsilon)^{-1} Z_{A_\varepsilon, B_\varepsilon}(-\zeta, \lambda/\zeta) A_\varepsilon^{it} \right\|_{\mathcal{L}(X)} |\mathrm{d}\zeta|$$

$$\leq \frac{c \cdot c_{AB}}{|\lambda|^{1+\beta}} \int_{\Gamma_A} \frac{|\zeta|^\beta |\mathrm{d}\zeta|}{(1+|\zeta|)^{2-\alpha}} \leq \frac{c \cdot c_{AB}}{|\lambda|^{1+\beta}}, \forall \varepsilon > 0, \forall \lambda \in \Sigma_\eta, \forall t \in (-1/2, 1/2),$$

so that the closed contour Γ_{A_ε} can be extended to Γ_A by holomorphy of the integrand, and

$$\left\| \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\pi}{\sin(\pi z)} \lambda^z \left[B_\varepsilon^{-z} A_\varepsilon^{-it} - A_\varepsilon^{-it} B_\varepsilon^{-z} \right] A_\varepsilon^{it-z} \mathrm{d}z \right\|_{\mathcal{L}(X)} \leq \frac{c \cdot c_{AB}}{|\lambda|^\beta}$$

holds uniformly for all $\varepsilon > 0$, $\lambda \in \Sigma_\eta$ and $t \in (-1/2, 1/2)$. The integrals $S_\lambda^{(\varepsilon)}$, represented by (12), consequently satisfy

$$\left\| S_\lambda^{(\varepsilon)} \right\|_{\mathcal{L}(X)} = \int_{-1/2}^{1/2} \left\| S_\lambda^{(\varepsilon)} \right\|_{\mathcal{L}(X)} \mathrm{d}t \leq \frac{c}{|\lambda|} \left(1 + \frac{c_{AB}}{|\lambda|^\beta} \right), \forall \varepsilon > 0, \forall \lambda \in \Sigma_\eta,$$

so that our proof is complete. □

We are in a position now to show the following result.

LEMMA 1.10. *The operators $S_\lambda^{(\varepsilon)}$ with $\lambda \in \Sigma_\eta$ approximate $S_\lambda \in \mathcal{L}(X_A, X)$ in the sense that $S_\lambda^{(\varepsilon)} x \longrightarrow S_\lambda x$ as $\varepsilon \longrightarrow +0$, $\forall x \in \mathcal{D}(A)$.*

S_λ has a unique extension $\mathrm{S}_\lambda \in \mathcal{L}(X, X_B)$, which satisfies the estimate

$$\|\mathrm{S}_\lambda\|_{\mathcal{L}(X)} \leq \frac{c}{|\lambda|} \left(1 + \frac{c_{AB}}{|\lambda|^\beta} \right), \forall \lambda \in \Sigma_\eta.$$

PROOF. Let us consider the bounded linear operator $T_\lambda^{(\varepsilon)} := S_\lambda^{(\varepsilon)} A_\varepsilon^{-1}$, where $S_\lambda^{(\varepsilon)}$ is applied in form of the contour integral (9). Since Γ_{A_ε} can be extended to Γ_A by holomorphy of the integrand, we have the representation

$$T_\lambda^{(\varepsilon)} = \frac{1}{2\pi i} \int_{\Gamma_A} \frac{1}{\zeta^2} \left(\frac{\lambda}{\zeta} + B_\varepsilon \right)^{-1} (\zeta - A_\varepsilon)^{-1} \mathrm{d}\zeta, \forall \varepsilon > 0, \lambda \in \Sigma_\eta.$$

Now the resolvent estimates (4) and Lemma 1.4 (S_2) enable us to take the limit $\varepsilon \longrightarrow +0$: applying Lebesgue's theorem on majorized convergence we obtain

$$S_\lambda^{(\varepsilon)} A_\varepsilon^{-1} = T_\lambda^{(\varepsilon)} \longrightarrow \frac{1}{2\pi i} \int_{\Gamma_A} \frac{1}{\zeta^2} \left(\frac{\lambda}{\zeta} + B \right)^{-1} (\zeta - A)^{-1} \mathrm{d}\zeta =: T_\lambda \text{ in } \mathcal{L}(X), \forall \lambda \in \Sigma_\eta.$$

Since $A_\varepsilon x$ approximates Ax on $\mathcal{D}(A)$ (cf. Lemma 1.4 (S_1)), this implies

$$\left\| T_\lambda^{(\varepsilon)} A_\varepsilon x - T_\lambda A x \right\|_X \longrightarrow 0 \text{ as } \varepsilon \longrightarrow +0$$

for $x \in \mathcal{D}(A)$ and $\lambda \in \Sigma_\eta$, so that

$$S_\lambda^{(\varepsilon)} x \longrightarrow T_\lambda A x = S_\lambda x \text{ as } \varepsilon \longrightarrow +0, \quad \forall x \in \mathcal{D}(A), \forall \lambda \in \Sigma_\eta .$$

From Lemma 1.9 we deduce the estimate

$$\|S_\lambda x\|_X \le \frac{c}{|\lambda|}\left(1 + \frac{c_{AB}}{|\lambda|^\beta}\right)\|x\|_X, \quad \forall x \in \mathcal{D}(A), \forall \lambda \in \Sigma_\eta .$$

As a consequence, the densely defined S_λ has a unique extension $\mathrm{S}_\lambda \in \mathcal{L}(X)$, which satisfies the asserted inequality (cf. e.g. [HP57], Theorem 2.11.2).

In order to show that S_λ belongs to $\mathcal{L}(X, X_B)$, we apply B to S_λ. This yields

$$\begin{aligned} BS_\lambda x &= \frac{1}{2\pi i}\int_{\Gamma_A} \frac{1}{\zeta^2}(\zeta - A)^{-1} A x \, d\zeta \\ &\quad - \frac{\lambda}{2\pi i}\int_{\Gamma_A} \frac{1}{\zeta^3}\left(\frac{\lambda}{\zeta} + B\right)^{-1}(\zeta - A)^{-1} A x \, d\zeta \\ &= A^{-1}x - \frac{\lambda}{2\pi i}\int_{\Gamma_A}\frac{1}{\zeta^2}\left(\frac{\lambda}{\zeta} + B\right)^{-1}(\zeta - A)^{-1} x \, d\zeta, \forall x \in \mathcal{D}(A) \cap \mathcal{R}(A) . \end{aligned} \tag{13}$$

Since the right hand side of the equation can be extended to a bounded linear operator on X, the closedness of B implies that $B\mathrm{S}_\lambda$ belongs to $\mathcal{L}(X)$, which proves our assertion. □

We are able now to derive our abstract main result, stated in Theorem 1.1.

Proof of Theorem 1.1. Our proof is subdivided into three parts. First we construct a right inverse mapping to $\lambda + AB$. Then injectivity shall be proven. Summing up these results we deduce the asserted statements in the concluding part.

a) *Existence of a solution.* The construction of a right inverse to $\lambda + AB$ is based on the contour integral S_λ and its unique extension $\mathrm{S}_\lambda \in \mathcal{L}(X, X_B)$ (cf. Lemma 1.10). In view of formula (13), the application of B to S_λ yields

$$\begin{aligned} BS_\lambda x &= A^{-1}\left[x - \frac{\lambda}{2\pi i}\int_{\Gamma_A}\frac{1}{\zeta^2} A\left(\frac{\lambda}{\zeta} + B\right)^{-1}(\zeta - A)^{-1} x \, d\zeta\right] \\ &= A^{-1}\left[(I + Q_\lambda)x - \lambda S_\lambda x\right], \forall x \in \mathcal{D}(A) \cap \mathcal{R}(A), \forall \lambda \in \Sigma_\eta , \end{aligned}$$

where Q_λ denotes the perturbation

$$\begin{aligned} Q_\lambda &= \frac{\lambda}{2\pi i}\int_{\Gamma_A}\frac{1}{\zeta^2}\left[\left(\frac{\lambda}{\zeta} + B\right)^{-1} A - A\left(\frac{\lambda}{\zeta} + B\right)^{-1}\right](\zeta - A)^{-1} d\zeta \\ &= \frac{\lambda}{2\pi i}\int_{\Gamma_A}\frac{1}{\zeta^2} Z_{A,B}(-\zeta, \lambda/\zeta) d\zeta , \end{aligned} \tag{14}$$

which vanishes if A and B are resolvent commuting. In general, commutator condition (5) guarantees that Q_λ is a bounded linear operator on X, and satisfies

$$\|Q_\lambda\|_{\mathcal{L}(X)} \leq \frac{c_{AB}}{2\pi|\lambda|^\beta}\int_{\Gamma_A}\frac{|\mathrm{d}\zeta|}{|\zeta|^{1-\beta}(1+|\zeta|)^{1-\alpha}} \leq \frac{c\cdot c_{AB}}{|\lambda|^\beta}, \quad \forall\lambda\in\Sigma_\eta\,. \tag{15}$$

Consequently, the closedness of A^{-1} and B implies

$$BS_\lambda x = A^{-1}[(I+Q_\lambda)-\lambda S_\lambda]x \in \mathcal{D}(A), \quad \forall x\in X, \forall\lambda\in\Sigma_\eta\,.$$

Thus, S_λ maps X into $\mathcal{D}(AB)$ and solves the equation $(\lambda+AB)S_\lambda = I+Q_\lambda$. From (15) we deduce $\|Q_\lambda\|_{\mathcal{L}(X)} < 1$, and therefore $(I+Q_\lambda)^{-1}\in\mathcal{L}(X)$, provided $c_{AB}\cdot|\lambda|^{-\beta}$ is sufficiently small. In this case, $R_\lambda := S_\lambda\,(I+Q_\lambda)^{-1}$ is a well-defined operator in $\mathcal{L}(X, X_{AB})$, and a right inverse to $\lambda+AB$.

b) *Uniqueness.* Let us first consider the bounded operators $\lambda + A_\varepsilon B_\varepsilon$. Analogously to the previous part of our proof we deduce

$$(\lambda + A_\varepsilon B_\varepsilon)\, S_\lambda^{(\varepsilon)} = I + Q_\lambda^{(\varepsilon)}, \forall\varepsilon>0, \forall\lambda\in\Sigma_\eta\,,$$

where the perturbation

$$Q_\lambda^{(\varepsilon)} = \frac{\lambda}{2\pi i}\int_{\Gamma_{A_\varepsilon}}\frac{1}{\zeta^2}Z_{A_\varepsilon,B_\varepsilon}(-\zeta,\lambda/\zeta)\mathrm{d}\zeta \in \mathcal{L}(X)$$

is due to the non-commutativity of A_ε and B_ε. Commutator condition (5) and Lemma 1.6 enable us to extend the contour Γ_{A_ε} in $Q_\lambda^{(\varepsilon)}$ to Γ_A (by holomorphy of the integrand), and lead to the uniform estimate

$$\left\|Q_\lambda^{(\varepsilon)}\right\|_{\mathcal{L}(X)} \leq \frac{c\cdot c_{AB}}{|\lambda|^\beta}, \forall\varepsilon>0, \forall\lambda\in\Sigma_\eta\,.$$

Thus, $R_\lambda^{(\varepsilon)} := S_\lambda^{(\varepsilon)}(I+Q_\lambda^{(\varepsilon)})^{-1}\in\mathcal{L}(X)$ is a right inverse to $\lambda+A_\varepsilon B_\varepsilon$, provided $c_{AB}\cdot|\lambda|^{-\beta}$ is sufficiently small. Accordingly, we can fix a $\nu_0\geq 0$ (which does not depend on ε), such that

$$\mathcal{R}\,(\lambda+A_\varepsilon B_\varepsilon) = X \text{ and } \left\|R_\lambda^{(\varepsilon)}\right\|_{\mathcal{L}(X)} \leq \frac{c}{|\lambda|}$$

hold for all $\varepsilon>0$ and $\lambda\in\Sigma_\eta^{\nu_0} := \{\lambda\in\Sigma_\eta : |\lambda|>\nu_0\}$. Moreover, $A_\varepsilon B_\varepsilon\in\mathcal{L}(X)$ implies

$$\left\{\lambda\in\mathbb{C} : |\lambda| > \nu_\varepsilon := \|A_\varepsilon B_\varepsilon\|_{\mathcal{L}(X)}\right\} \subset \rho\,(-A_\varepsilon B_\varepsilon)\ .$$

By Lemma 1.7 we consequently obtain $\rho(-A_\varepsilon B_\varepsilon)\supseteq\Sigma_\eta^{\nu_0}$, where the resolvents, given by $R_\lambda^{(\varepsilon)}$, satisfy the estimate

$$\left\|(\lambda+A_\varepsilon B_\varepsilon)^{-1}\right\|_{\mathcal{L}(X)} = \left\|R_\lambda^{(\varepsilon)}\right\|_{\mathcal{L}(X)} \leq \frac{c}{|\lambda|}, \forall\varepsilon>0, \forall\lambda\in\Sigma_\eta^{\nu_0}\,.$$

Now let $x \in \mathcal{N}(\lambda + AB) \subset \mathcal{D}(AB)$ for some $\lambda \in \Sigma_\eta^{\nu_0}$. It is easily seen that x satisfies the equation $x = (\lambda + A_\varepsilon B_\varepsilon)^{-1}(A_\varepsilon B_\varepsilon - AB)x$, which leads to

$$\|x\|_X \leq \frac{c}{|\lambda|} \left(\|A_\varepsilon (B_\varepsilon x - Bx)\|_X + \|(A_\varepsilon - A)\, Bx\|_X\right), \forall \varepsilon > 0 .$$

In order to estimate the norm of

$$A_\varepsilon (B_\varepsilon x - Bx) = \varepsilon A_\varepsilon (I + \varepsilon B)^{-1} x + A_\varepsilon \left[(I + \varepsilon B)^{-1} Bx - Bx\right] ,$$

the identities

$$\begin{aligned} &\varepsilon A_\varepsilon (I + \varepsilon B)^{-1} x = (I + \varepsilon B)^{-1} x - (I + \varepsilon A)^{-1} (I + \varepsilon B)^{-1} x \\ &= \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon} + B\right)^{-1} \left[x - (I + \varepsilon A)^{-1} x\right] + \frac{1}{\varepsilon^2}\left(\frac{1}{\varepsilon} + A\right)^{-1} Z_{A,B}(1/\varepsilon, 1/\varepsilon)x \end{aligned}$$

and

$$\begin{aligned} &A_\varepsilon\left[(I+\varepsilon B)^{-1} Bx - Bx\right] = \left[A(I + \varepsilon A)^{-1}(I+\varepsilon B)^{-1}A^{-1} - (I+\varepsilon A)^{-1}\right] ABx \\ &= (I + \varepsilon A)^{-1} \left[(I + \varepsilon B)^{-1}ABx - ABx\right] + \frac{1}{\varepsilon}(I + \varepsilon A)^{-1}Z_{A,B}(0, 1/\varepsilon)ABx \\ &= \frac{\lambda}{\varepsilon}\left(\frac{1}{\varepsilon} + A\right)^{-1} \left[x - (I + \varepsilon B)^{-1}x\right] - \frac{\lambda}{\varepsilon^2}\left(\frac{1}{\varepsilon} + A\right)^{-1} Z_{A,B}(0, 1/\varepsilon)x \end{aligned}$$

are derived. Using the resolvent inequalities (4) and the commutator condition (5) we consequently obtain

$$\begin{aligned} \|A_\varepsilon (B_\varepsilon x - Bx)\|_X \leq M_B \left\|x - (I + \varepsilon A)^{-1} x\right\|_X + M_A|\lambda| \left\|x - (I + \varepsilon B)^{-1} x\right\|_X \\ + M_A \cdot c_{AB} \cdot \varepsilon^\beta \left(\varepsilon^{1-\alpha} + |\lambda|\right) \|x\|_X, \forall \varepsilon > 0 , \end{aligned}$$

so that

$$\begin{aligned} \|x\|_X \leq \frac{c}{|\lambda|} \left\|x - (I + \varepsilon A)^{-1} x\right\|_X + c \left\|x - (I + \varepsilon B)^{-1} x\right\|_X \\ + \frac{c}{|\lambda|} \|(A_\varepsilon - A)\, Bx\|_X + c \cdot c_{AB} \cdot \varepsilon^\beta \|x\|_X, \forall \varepsilon \in (0, 1) . \end{aligned}$$

The convergence properties $\lim_{\varepsilon\to+0} A_\varepsilon Bx = ABx$ (cf. Lemma 1.4 (S_1)) and $\lim_{\varepsilon\to+0}(I + \varepsilon A)^{-1}x = \lim_{\varepsilon\to+0}(I + \varepsilon B)^{-1}x = x$ (cf. e.g. [HP]) enable us to take the limit $\varepsilon \longrightarrow +0$. This yields the estimate

$$\|x\|_X \leq c \cdot c_{AB} \lim_{\varepsilon\to+0} \varepsilon^\beta \|x\|_X ,$$

which implies $x = 0$ for $\beta > 0$ or a small constant c_{AB}. To sum up, we obtain $\mathcal{N}(\lambda + AB) = \{0\}$ for $\lambda \in \Sigma_\eta$, provided $c_{AB} \cdot |\lambda|^{-\beta}$ is sufficiently small.

c) The previous parts of our proof lead to the following statement.

A complex number λ in the sector Σ_η belongs to $\rho(-AB)$, and the corresponding resolvent $(\lambda + AB)^{-1} = R_\lambda = S_\lambda(I + Q_\lambda)^{-1}$ satisfies the uniform estimate

$$\left\|(\lambda + AB)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{c}{|\lambda|}\left(1 + \frac{c_{AB}}{|\lambda|^\beta}\right)\frac{1}{1 - c \cdot c_{AB}|\lambda|^{-\beta}} \leq \frac{c}{|\lambda|},$$

provided $c_{AB} \cdot |\lambda|^{-\beta}$ is sufficiently small, say $c_{AB} \cdot |\lambda|^{-\beta} < c_*$.

If the commutator condition (5) is fulfilled with $\beta > 0$, we consequently obtain $\nu + AB \in \mathcal{S}(X)$ for an arbitrary $\nu > 0$ with $c_{AB} \cdot \nu^{-\beta} < c_*$.

In case of $\{c_{AB} = 0\}$ or $\{\beta = 0$ and $c_{AB} < c_*\}$ the resolvent set $\rho(-AB)$ contains the sector Σ_η, and $\|(\lambda + AB)^{-1}\|_{\mathcal{L}(X)} \leq c/|\lambda|$ holds for all $\lambda \in \Sigma_\eta$. This implies $X = \mathcal{N}(AB) \oplus \overline{\mathcal{R}(AB)}$ by reflexivity of $X \in \mathcal{HT}$ (cf. e.g. [HP]). Since $\mathcal{N}(AB) = \{0\}$ immediately follows from $\mathcal{N}(A) = \mathcal{N}(B) = \{0\}$, we consequently obtain $\overline{\mathcal{R}(AB)} = X$. Thus, AB is an operator of class $\mathcal{S}(X)$.

Since $\eta \in (0, \pi - (\varphi_A + \varphi_B))$ was fixed arbitrarily, the spectral angle of $\nu + AB$ satisfies the asserted estimate. Here the above chosen constant $\nu \geq 0$ has to be substituted by $\nu/\sin(\varphi_A + \varphi_B)$, if, in particular, $\varphi_A + \varphi_B < \pi/2$. □

Proof of Theorem 1.2. Our proof consists of showing that Theorem 1.1 applies to AB^{-1}, and justifies the representation $(A+B)^{-1} = B^{-1}(I+AB^{-1})^{-1} = A^{-1} - A^{-1}(I + AB^{-1})^{-1}$.

According to Lemma 1.2 (S_1), the property $B^{-1} \in \mathcal{BIP}(X, \theta_B)$ immediately follows from the assumption $B \in \mathcal{BIP}(X, \theta_B)$. Using the representation

$$\left(\mu + B^{-1}\right)^{-1} = \frac{1}{\mu} - \frac{1}{\mu^2}\left(\frac{1}{\mu} + B\right)^{-1}, \mu \in \Sigma_{\pi-\varphi_B},$$

we obtain $(\mu + B^{-1})^{-1}\mathcal{D}(A) \subseteq \mathcal{D}(A)$ for $\mu \in \Sigma_{\pi-\varphi_B} \subseteq \rho(-B^{-1})$. Moreover, $Z_{A,B^{-1}}(\lambda, \mu)$ can be rewritten in the form of

$$Z_{A,B^{-1}}(\lambda, \mu) = -\mu^{-2} Z_{A,B}(\lambda, 1/\mu), (\lambda, \mu) \in \Sigma_{\pi-\varphi_A} \times \Sigma_{\pi-\varphi_B},$$

so that the commutator estimate (6) leads to

$$\left\|Z_{A,B^{-1}}(\lambda, \mu)\right\|_{\mathcal{L}(X)} \leq \frac{c \cdot c_{A+B}}{(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}}, \forall(\lambda, \mu) \in \Sigma_{\pi-\varphi_A} \times \Sigma_{\pi-\varphi_B}.$$

Consequently, Theorem 1.1 applies to the product AB^{-1} and ensures the existence of the resolvent $(I + AB^{-1})^{-1} \in \mathcal{L}(X)$ which maps X into $\mathcal{D}(AB^{-1})$, provided $c_{A+B} \geq 0$ is sufficiently small. It is easily seen now that

$$B^{-1}\left(I + AB^{-1}\right)^{-1} = A^{-1} - A^{-1}\left(I + AB^{-1}\right)^{-1} \in \mathcal{L}(X, X_{A+B})$$

is the inverse operator to $A + B \in \mathcal{L}(X_{A+B}, X)$. □

Products of sectorial operators in Hilbert spaces X are considered separately in Paragraph 1.5. The approach used there requires the contour integral S_λ to be bounded on a real interpolation space $(X, X_A)_{\gamma,2}$ of order $\gamma \in (0, 1)$, and to satisfy a suitable estimate in $\mathcal{L}((X, X_A)_{\gamma,2})$ (cf. Proof of Theorem 1.3). For this reason we provide the following statement, which, in particular, contains the desired properties.

PROPOSITION 1.1. *Let the operators A, B in a Banach space X fulfil the conditions $A \in \mathcal{S}(X, \phi_A)$, $0 \in \rho(A)$, and $B \in \mathcal{S}(X, \phi_B)$ with $\phi_A + \phi_B < \pi$. Moreover, assume that the commutator condition* (5) *holds for $(\lambda, \mu) \in \Sigma_{\pi-\varphi_A} \times \Sigma_{\pi-\varphi_B}$, where $\varphi_A > \phi_A$ and $\varphi_B > \phi_B$ satisfy $\varphi_A + \varphi_B < \pi$.*

Then, the contour integral S_λ, defined by (8), *is a bounded linear map on the real interpolation space $(X, X_A)_{\gamma,p}$ of order $\gamma \in (0,1)$ and exponent $p \in (1, \infty)$. Moreover, we have the estimate*

$$\|S_\lambda\|_{\mathcal{L}((X,X_A)_{\gamma,p})} \le \frac{c}{|\lambda|}\left(1 + \frac{c_{AB}}{|\lambda|^\beta}\right), \forall \lambda \in \Sigma_\eta .$$

PROOF. Using Hölder's inequality we see that

$$\|S_\lambda x\|_X \le \frac{c}{|\lambda|}\int_{\Gamma_A}\left\|A(\zeta - A)^{-1}x\right\|_X \frac{|\mathrm{d}\zeta|}{|\zeta|} \le \frac{c}{|\lambda|}\left\{\int_{\Gamma_A}\left\|\zeta^\gamma A(\zeta - A)^{-1}x\right\|_X^p \frac{|\mathrm{d}\zeta|}{|\zeta|}\right\}^{\frac{1}{p}}$$

holds for all $\lambda \in \Sigma_\eta$ and $x \in (X, X_A)_{\gamma,p}$. For this reason, the $L_p(\mathbb{R}_+)$-norm of $t^{\gamma - 1/p}\|A(t + A)^{-1}S_\lambda x\|_X$ remains to be estimated. To this end we derive the identity

$$\begin{aligned}
&A(t + A)^{-1}S_\lambda x = S_\lambda x - t(t + A)^{-1}S_\lambda x \\
&= S_\lambda x - \frac{1}{2\pi i}\int_{\Gamma_A}\frac{t}{\zeta^2}\left(\frac{\lambda}{\zeta} + B\right)^{-1} A(t + A)^{-1}(\zeta - A)^{-1}x\mathrm{d}\zeta \\
&\quad - \frac{1}{2\pi i}\int_{\Gamma_A}\frac{t}{\zeta^2}\left[(t+A)^{-1}\left(\frac{\lambda}{\zeta}+B\right)^{-1} - \left(\frac{\lambda}{\zeta}+B\right)^{-1}(t+A)^{-1}\right] A(\zeta - A)^{-1}x\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_{\Gamma_A}\frac{1}{\zeta(t+\zeta)}\left(\frac{\lambda}{\zeta} + B\right)^{-1} A(\zeta - A)^{-1}x\mathrm{d}\zeta \\
&\quad - t(t + A)^{-1}\frac{1}{2\pi i}\int_{\Gamma_A}\frac{1}{\zeta^2}Z_{A,B}(-\zeta, \lambda/\zeta)\, A(t + A)^{-1}x\mathrm{d}\zeta, x \in (X, X_A)_{\gamma,p} ,
\end{aligned}$$

which, in view of the resolvent inequalities (4) and the commutator condition (5), leads to

$$\begin{aligned}
\left\|A(t + A)^{-1}S_\lambda x\right\|_X &\le \frac{c}{|\lambda|}\int_{\Gamma_A}\frac{1}{|\zeta|^\gamma(t + |\zeta|)}|\zeta|^\gamma\left\|A(\zeta - A)^{-1}x\right\|_X |\mathrm{d}\zeta| \\
&\quad + \frac{c\cdot c_{AB}}{|\lambda|^{1+\beta}}\left\|A(t + A)^{-1}x\right\|_X, \forall\lambda \in \Sigma_\eta, \forall x \in (X, X_A)_{\gamma,p} .
\end{aligned}$$

Since the first summand on the right hand side, multiplied by $t^{\gamma-1/p}$, satisfies

$$\begin{aligned}
&\int_0^\infty \left\{ \int_{\Gamma_A} \frac{t^\gamma}{|\zeta|^\gamma (t+|\zeta|)} |\zeta|^\gamma \left\| A(\zeta - A)^{-1} x \right\|_X |\mathrm{d}\zeta| \right\}^p \frac{\mathrm{d}t}{t} \\
&\leq \int_0^\infty \left\{ \int_{\Gamma_A} \frac{t^\gamma |\mathrm{d}\zeta|}{|\zeta|^\gamma (t+|\zeta|)} \right\}^{p-1} \left\{ \int_{\Gamma_A} \frac{t^\gamma |\zeta|^{\gamma p}}{|\zeta|^\gamma (t+|\zeta|)} \left\| A(\zeta - A)^{-1} x \right\|_X^p |\mathrm{d}\zeta| \right\} \frac{\mathrm{d}t}{t} \\
&\leq c \int_{\Gamma_A} \left\{ \int_0^\infty \frac{|\zeta|^{1-\gamma} \mathrm{d}t}{t^{1-\gamma}(t+|\zeta|)} \right\} |\zeta|^{\gamma p - 1} \left\| A(\zeta - A)^{-1} x \right\|_X^p |\mathrm{d}\zeta| \\
&\leq c \int_{\Gamma_A} \left\| \zeta^\gamma A(\zeta - A)^{-1} x \right\|_X^p \frac{|\mathrm{d}\zeta|}{|\zeta|}, \forall \lambda \in \Sigma_\eta, \forall x \in (X, X_A)_{\gamma,p} ,
\end{aligned}$$

the desired estimate

$$\begin{aligned}
\|S_\lambda x\|_{(X,X_A)_{\gamma,p}} &= \|S_\lambda x\|_X + \left\{ \int_0^\infty \left\| t^\gamma A(t+A)^{-1} S_\lambda x \right\|_X^p \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{p}} \\
&\leq \frac{c}{|\lambda|} \left(1 + \frac{c_{AB}}{|\lambda|^\beta} \right) \|x\|_{(X,X_A)_{\gamma,p}}
\end{aligned}$$

holds for all $\lambda \in \Sigma_\eta$ and $x \in (X, X_A)_{\gamma,p}$. Consequently, the contour integral S_λ defines a bounded linear operator on $(X, X_A)_{\gamma,p}$. □

REMARK 1.4. The technique that we employed in the proof of Proposition 1.1 is essentially due to G. Da Prato and P. Grisvard. It was originally applied to the representations of resolvents of operator sums (cf. [DG75], Lemma 6.6). Using this technique A. Favini proved the boundedness of contour integrals of the form S_1 and derived maximal regularity results for the equation $(I + AB)u = f$ on $(X, X_A)_{\gamma,p}$ (cf. [Fav85], Theorem 3.5).

1.5. – Products in Hilbert Spaces

In the proof of Theorem 1.1 resolvents of the product AB, $\mathcal{D}(AB) \subseteq X$, were constructed on the basis of the contour integral S_λ given by formula (8). An essential step of our derivation consisted of showing that S_λ has a unique extension $\mathbf{S}_\lambda \in \mathcal{L}(X)$. This could be done in Banach spaces X of class $\mathcal{HT}$ by means of a method, whose application requires that both A and B possess bounded imaginary powers (cf. Lemma 1.9 and 1.10).

In Hilbert spaces X we may use a different approach. The alternative method relies on a similarity argument, which requires only A to be of class $\mathcal{BIP}(X)$. Namely we obtain the following modification of Theorem 1.1.

THEOREM 1.3. *Let X be a Hilbert space. Assume that A is an invertible operator of class $\mathcal{BIP}(X, \theta_A)$, and B belongs to $\mathcal{S}(X, \phi_B)$. The corresponding angles are supposed to satisfy the (strong) parabolicity condition $\theta_A + \phi_B < \pi$. We fix $\varphi_A > \theta_A$ and $\varphi_B > \phi_B$ with $\varphi_A + \varphi_B < \pi$. Let the commutator estimate* (5) *be satisfied.*

Then, $\nu + AB$ with $\nu \geq 0$ is an operator of class $\mathcal{S}(X, \varphi_A + \varphi_B)$, provided $\sup_{\delta > \nu}\{c_{AB} \cdot \delta^{-\beta}\}$ is sufficiently small.

PROOF. Our derivation differs from the proof of Theorem 1.1 in the way that we show the existence of a unique extension $\mathrm{S}_\lambda \in \mathcal{L}(X)$ of the contour integral S_λ. The alternative method is based on the representation

$$S_\lambda = A^\gamma S_\lambda A^{-\gamma} + A^\gamma \left[A^{-\gamma} S_\lambda - S_\lambda A^{-\gamma}\right], \lambda \in \Sigma_\eta, 0 < \eta < \pi - (\varphi_A + \varphi_B),$$

where γ is an arbitrary, but fixed real number with $0 < \gamma < 1$.

a) First we consider $A^\gamma S_\lambda A^{-\gamma}$. Since A belongs to the class $\mathcal{BIP}(X)$, the Banach space X_{A^γ} (i.e., the set $\mathcal{D}(A^\gamma)$ endowed with the graph norm) can be characterized as $X_{A^\gamma} \doteq [X, X_A]_\gamma$ (cf. [Tri78], 1.15.3 Theorem). Moreover, the assumption that X is a Hilbert space implies $[X, X_A]_\gamma \doteq (X, X_A)_{\gamma,2}$ (cf. [Pee69], Theorem 3.1 and Example 2.2). Consequently, we obtain $X_{A^\gamma} \doteq (X, X_A)_{\gamma,2}$.

On the other hand, S_λ defines a bounded linear map on $(X, X_A)_{\gamma,2}$ and satisfies

$$\|S_\lambda\|_{\mathcal{L}((X,X_A)_{\gamma,2})} \leq \frac{c}{|\lambda|}\left(1 + \frac{c_{AB}}{|\lambda|^\beta}\right), \forall \lambda \in \Sigma_\eta \qquad \text{(cf. Proposition 1.1)}.$$

Thus, we have the estimate

$$\left\|A^\gamma S_\lambda A^{-\gamma} x\right\|_X \leq \frac{c}{|\lambda|}\left(1 + \frac{c_{AB}}{|\lambda|^\beta}\right)\|x\|_X, \forall x \in X, \forall \lambda \in \Sigma_\eta,$$

which, in particular, states $A^\gamma S_\lambda A^{-\gamma} \in \mathcal{L}(X)$.

b) Therefore, the perturbation $A^\gamma[A^{-\gamma} S_\lambda - S_\lambda A^{-\gamma}]$ remains to be considered. Using the Dunford representation

$$A^{-\gamma} = \frac{1}{2\pi i}\int_{\Gamma_A^*} \xi^{-\gamma}(\xi - A)^{-1} \mathrm{d}\xi \text{ with } \Gamma_A^* := \Gamma_{\omega_A'}, \omega_A' \in (\omega_A, \pi),$$

the resolvent equation, and the identity $A(\zeta - A)^{-1} = -I + \zeta(\zeta - A)^{-1}$, $\zeta \in \rho(A)$, we derive

$$\begin{aligned}
&\left[A^{-\gamma} S_\lambda - S_\lambda A^{-\gamma}\right] x \\
&= \frac{1}{(2\pi i)^2}\int_{\Gamma_A}\int_{\Gamma_A^*} \frac{\xi^{-\gamma}}{\zeta}\left[(\xi - A)^{-1}(\lambda + \zeta B)^{-1} - (\lambda + \zeta B)^{-1}(\xi - A)^{-1}\right](\zeta - A)^{-1} A x \,\mathrm{d}\xi\,\mathrm{d}\zeta \\
&= \frac{1}{(2\pi i)^2}\int_{\Gamma_A}\int_{\Gamma_A^*} \frac{\xi^{-\gamma}}{\zeta(\zeta - \xi)}(\xi - A)^{-1}\left[A(\lambda + \zeta B)^{-1} - (\lambda + \zeta B)^{-1} A\right](\xi - A)^{-1} A x \,\mathrm{d}\xi\,\mathrm{d}\zeta \\
&\quad + \frac{1}{(2\pi i)^2}\int_{\Gamma_A}\int_{\Gamma_A^*} \frac{\xi^{-\gamma}}{\zeta(\zeta - \xi)}(\xi - A)^{-1}\left[A(\lambda + \zeta B)^{-1} - (\lambda + \zeta B)^{-1} A\right] x \,\mathrm{d}\xi\,\mathrm{d}\zeta \\
&\quad - \frac{1}{(2\pi i)^2}\int_{\Gamma_A}\int_{\Gamma_A^*} \frac{\xi^{-\gamma}}{\zeta - \xi}(\xi - A)^{-1}\left[A(\lambda + \zeta B)^{-1} - (\lambda + \zeta B)^{-1} A\right](\zeta - A)^{-1} x \,\mathrm{d}\xi\,\mathrm{d}\zeta, \\
&\forall x \in \mathcal{D}(A), \forall \lambda \in \Sigma_\eta.
\end{aligned}$$

The choice of $\Gamma_A^* = \Gamma_{\omega_A'}$ and $\Gamma_A = \Gamma_{\omega_A}^{r_A}$ with $\omega_A' > \omega_A$ implies

$$\int_{\Gamma_A} \frac{1}{\zeta(\zeta-\xi)} (\lambda + \zeta B)^{-1} \mathrm{d}\zeta = 0, \forall \xi \in \Gamma_A^*, \quad \forall \lambda \in \Sigma_\eta,$$

by holomorphy of the integrand. Consequently, the first two summands on the right hand side of the previous equation vanish and we obtain

$$\begin{aligned} &A^\gamma \left[A^{-\gamma} S_\lambda - S_\lambda A^{-\gamma}\right] x \\ &= -\frac{1}{2\pi i} \int_{\Gamma_A} (\zeta - A)^{-1} \left[A(\lambda + \zeta B)^{-1} - (\lambda + \zeta B)^{-1} A\right] (\zeta - A)^{-1} x \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_{\Gamma_A} \frac{1}{\zeta} (\zeta - A)^{-1} Z_{A,B}(-\zeta, \lambda/\zeta) x \mathrm{d}\zeta, \forall x \in \mathcal{D}(A), \forall \lambda \in \Sigma_\eta. \end{aligned}$$

Now the application of the commutator condition (5) leads to the estimate

$$\begin{aligned} &\left\|A^\gamma \left[A^{-\gamma} S_\lambda - S_\lambda A^{-\gamma}\right] x\right\|_X \\ &\le \frac{c \cdot c_{AB}}{|\lambda|^{1+\beta}} \int_{\Gamma_A} \frac{|\mathrm{d}\zeta|}{|\zeta|^{2-(\alpha+\beta)}} \|x\|_X \le \frac{c \cdot c_{AB}}{|\lambda|^{1+\beta}} \|x\|_X, \forall x \in \mathcal{D}(A), \forall \lambda \in \Sigma_\eta, \end{aligned}$$

so that

$$\|S_\lambda x\|_X \le \left\|A^\gamma S_\lambda A^{-\gamma} x\right\|_X + \left\|A^\gamma \left[A^{-\gamma} S_\lambda - S_\lambda A^{-\gamma}\right] x\right\|_X \le \frac{c}{|\lambda|} \left(1 + \frac{c_{AB}}{|\lambda|^\beta}\right) \|x\|_X$$

holds for all $x \in \mathcal{D}(A)$ and $\lambda \in \Sigma_\eta$. Therefore, the densely defined operator S_λ has a unique extension $\mathrm{S}_\lambda \in \mathcal{L}(X)$, which satisfies

$$\|\mathrm{S}_\lambda\|_{\mathcal{L}(X)} \le \frac{c}{|\lambda|} \left(1 + \frac{c_{AB}}{|\lambda|^\beta}\right), \quad \forall \lambda \in \Sigma_\eta.$$

By the same arguments as in the proof of Lemma 1.10 we see that S_λ even defines a bounded linear operator from X into $X_B = (\mathcal{D}(B), \|\cdot\|_X + \|B \cdot\|_X)$.

c) The construction of $(\lambda + AB)^{-1}$ on the basis of the contour integral S_λ and its extension $\mathrm{S}_\lambda \in \mathcal{L}(X)$ can be adopted literally from the proof of Theorem 1.1. □

As a consequence we obtain the following modification of the Dore-Venni type Theorem 1.2.

THEOREM 1.4. *The operators A and B in a Hilbert space X are supposed to fulfil the conditions $A \in \mathcal{BIP}(X, \theta_A)$, $0 \in \rho(A)$, and $B \in \mathcal{S}(X, \phi_B)$ with $\theta_A + \phi_B < \pi$. We fix angles $\varphi_A > \theta_A$ and $\varphi_B > \phi_B$, such that $\varphi_A + \varphi_B < \pi$ holds. Let the commutator estimate* (6) *be satisfied for all $(\lambda, \mu) \in \Sigma_{\pi-\varphi_A} \times \Sigma_{\pi-\varphi_B}$.*

Then, $A + B$ is of class $\mathcal{L}\mathrm{is}(X_{A+B}, X)$, provided the constant $c_{A+B} \ge 0$ in (6) *is sufficiently small.*

We refer to the proof of Theorem 1.2 which can be adopted.

REMARK 1.5. The similarity argument used in the proof of Theorem 1.3 was originally employed to derive maximal regularity results for sums of operators (with commutative resolvents) in Hilbert spaces. For details we refer to Remark 2.11 in the paper [DV87] of G. Dore and A. Venni. Note that their result coincides with the resolvent commuting case of Theorem 1.4.

2. – Applications

2.1. – Degenerations of the time derivative

In this paragraph the abstract Cauchy problem (1) shall be considered. First we impose some conditions on the function $b : J_T \longrightarrow \mathbb{C}$, $J_T = [0, T)$, $T \le +\infty$, which gives rise to a singular perturbation of the time derivative.

ASSUMPTION 2.1. (A_1) $b \in C^1(\bar{J}_T, \mathbb{C})$.

(A_2) $b(t) \in \Sigma_\theta$, $\theta \in (0, \pi/2)$, for almost all $t \in J_T$.

(A_3) There are $q \in (1, \infty]$ and $\beta \in [0, 1 - 1/q)$, such that

$$b'(\cdot)b^{-(1-\beta)}(\cdot) \in L_q(J_T)\,.$$

Then the following statement concerning $\mathrm{d}(b \cdot u)/\mathrm{d}t$ can be shown.

PROPOSITION 2.1. *Let E be a Banach space of class $\mathcal{HT}$. The function b is supposed to satisfy Assumption 2.1. Moreover, let κ be a nonnegative real number which is strictly positive if $T = +\infty$. By X we denote the space $L_p(J_T, E)$ with an arbitrary, but fixed $p \in (1, \infty)$, $p \le q$.*

Then, $\nu + AB$ with $\nu \ge 0$, defined by

$$(ABu)(t) := \left(\kappa + \frac{\mathrm{d}}{\mathrm{d}t}\right)[b(t) \cdot u(t)] \text{ on}$$

$$\mathcal{D}(AB) = \left\{u \in X : b \cdot u \in W_p^1(J_T, E), b(0)u(0) = 0\right\}\,,$$

is of class $\mathcal{S}(L_p(J_T, E), \varphi_A + \varphi_B)$, $\varphi_A > \pi/2$, $\varphi_B > \theta$, $\varphi_A + \varphi_B < \pi$, provided

$$\sup_{\delta > \nu}\left\{\left\|\frac{b'}{b^{1-\beta}}\right\|_{L_q(J_T)} \cdot \delta^{-\beta}\right\}$$

is sufficiently small.

PROOF. In order to apply the abstract Theorem 1.1, we interpret AB as a product of two sectorial operators A and B in the evolution space $X := L_p(J_T, E)$, which (by virtue of the assumption $E \in \mathcal{HT}$ and the restriction $p \in (1, \infty)$) belongs to the class $\mathcal{HT}$ (cf. Lemma 1.1 (S_4)).

First let A be defined by

$$(Au)(t) := \kappa \cdot u(t) + \frac{\mathrm{d}u}{\mathrm{d}t}(t) \text{ on } \mathcal{D}(A) := \left\{u \in W_p^1(J_T, E) : u(0) = 0\right\}.$$

It is well-known that the operator A has bounded imaginary powers on X with the power angle $\theta_A = \pi/2$ (cf. Example 1.1). Its resolvents are given by

$$\left[(\lambda + A)^{-1}u\right](t) = (k_\lambda * u)(t) = \int_0^t k_\lambda(t - s)u(s)\mathrm{d}s$$

$$\text{with } k_\lambda(t) = \mathrm{e}^{-(\lambda+\kappa)t}, t \in J_T, |\arg\lambda| < \pi/2\,,$$

so that the positivity of κ in the case $T = +\infty$ guarantees $0 \in \rho(A)$.

Assumption 2.1 ensures that the operator B, defined by $(Bu)(t) := b(t)u(t)$ on $\mathcal{D}(B) := \{u \in X : Bu \in X\}$, belongs to the class $\mathcal{BIP}(X, \theta)$. Moreover, $b \in C^1(\bar{J}_T)$ implies $(\mu + B)^{-1}\mathcal{D}(A) \subseteq \mathcal{D}(A)$ for $\mu \in \rho(-B)$.

Therefore, the commutator estimate (5) remains to be verified. To this end we fix two arbitrary angles $\varphi_A > \pi/2$, $\varphi_B > \theta$ which satisfy the condition $\varphi_A + \varphi_B < \pi$. It is easily seen that $Z_{A,B}(\lambda, \mu)$ is represented by

$$\left[Z_{A,B}(\lambda, \mu)u\right](t) = g'_\mu(t) \cdot (k_\lambda * u)(t), u \in X, (\lambda, \mu) \in \Sigma_{\pi-\varphi_A} \times \Sigma_{\pi-\varphi_B},$$

where $g_\mu(t) := (\mu + b(t))^{-1}$. From Assumption 2.1 ($A_3$) we deduce

$$\left\|g'_\mu\right\|_{L_q(J_T)} = \left\|g_\mu^2 b'\right\|_{L_q(J_T)} \leq \frac{c_B}{|\mu|^{1+\beta}}, \forall \mu \in \Sigma_{\pi-\varphi_B},$$
$$\text{with } c_B = \frac{1}{\sin^2\left[(\varphi_B - \theta)/2\right]} \left\|\frac{b'}{b^{1-\beta}}\right\|_{L_q(J_T)}.$$

Moreover, the kernel $k_\lambda(t) = \mathrm{e}^{-(\lambda+\kappa)t}$ satisfies the estimate

$$\|k_\lambda\|_{L_{q'}(J_T)} \leq \frac{c_A}{(1 + |\lambda|)^{1-1/q}}, \forall \lambda \in \Sigma_{\pi-\varphi_A} \text{ with } c_A = \mathcal{O}\left\{[\sin(\varphi_A - \pi/2)]^{-1}\right\},$$

where $q' = q/(q-1)$ denotes the conjugate exponent of q. Using Hölder's and Young's inequality we consequently obtain

$$\begin{aligned}
&\left\|Z_{A,B}(\lambda, \mu)u\right\|_{L_p(J_T,E)} \\
&\leq \left\|g'_\mu\right\|_{L_q(J_T)} \|k_\lambda * u\|_{L_{pq/(q-p)}(J_T,E)} \leq \left\|g'_\mu\right\|_{L_q(J_T)} \|k_\lambda\|_{L_{q'}(J_T)} \|u\|_{L_p(J_T,E)} \\
&\leq \frac{c_A \cdot c_B}{(1 + |\lambda|)^{1-1/q}|\mu|^{1+\beta}} \|u\|_{L_p(J_T,E)}, \forall u \in X, \forall (\lambda, \mu) \in \Sigma_{\pi-\varphi_A} \times \Sigma_{\pi-\varphi_B}.
\end{aligned}$$

Thus, the abstract Theorem 1.1 shows that $\nu + AB$ is sectorial (with spectral angle $\phi_{AB} \leq \varphi_A + \varphi_B$), provided $\sup_{\delta>\nu}\{c_A \cdot c_B \cdot \delta^{-\beta}\}$ is sufficiently small. □

REMARK 2.1. Proposition 2.1 remains true for $\kappa = 0$ (and $T = +\infty$) if the function b is bounded. Since $b \in L_\infty(J_\infty, \mathbb{C})$ defines a bounded linear operator $(Bu)(t) = b(t)u(t)$ on $X = L_p(J_\infty, E)$, this is an immediate consequence of a well-known perturbation result for the class $\mathcal{S}(X)$, stated e.g. in [HP].

The following examples are chosen to illustrate Proposition 2.1.

EXAMPLE 2.1. Consider $b(t) = t^\gamma$ with $\gamma > 1$ on the interval $J_T = [0, T)$, $T < +\infty$. For an arbitrary $\beta \in (1/\gamma, 1)$ this function satisfies the estimate

$$\left\|\frac{b'}{b^{1-\beta}}\right\|_{L_\infty(J_T)} \leq \gamma \cdot T^{\gamma\beta - 1}.$$

Consequently, the following statement holds:

Let E be a Banach space of class $\mathcal{HT}$, $p \in (1, +\infty)$, and $\varepsilon > 0$. Then, there is some $\delta = \delta(\varepsilon)$, such that the operator $\nu + AB$, $\nu > 0$, defined by

$$(ABu)(t) := \frac{\mathrm{d}}{\mathrm{d}t}\left[t^{\gamma}u(t)\right]$$

$$\text{on } \mathcal{D}(AB) = \left\{u \in L_p(J_T, E) : b \cdot u \in W_p^1(J_T, E)\right\},$$

belongs to $\mathcal{S}(L_p(J_T, E), \pi/2 + \varepsilon)$, provided $T/\nu \le \delta(\varepsilon)$.

EXAMPLE 2.2. Now we consider $b(t) = \mathrm{e}^{at}$ with $a \in \mathbb{R}$ on the nonnegative real axis $J_\infty = [0, +\infty)$. Since b satisfies the identity

$$\left\|\frac{b'}{b^{1-\beta}}\right\|_{L_\infty(J_\infty)} = |a| \text{ for } \beta = \beta(a) \in \begin{cases} [0, \infty) & \text{if } a \le 0 \\ \{0\} & \text{if } a > 0 \end{cases},$$

Proposition 2.1 and Remark 2.1 lead to the following result:

Let E be a Banach space of class $\mathcal{HT}$, $p \in (1, +\infty)$, and $\varepsilon > 0$. The real number $\kappa \ge 0$ is supposed to be strictly positive in the case $a > 0$. Then, there is some $\delta = \delta(\varepsilon)$, such that the operator $\nu + AB$, $\nu \ge 0$, given by

$$(ABu)(t) := \left(\kappa + \frac{\mathrm{d}}{\mathrm{d}t}\right)\left[\mathrm{e}^{at}u(t)\right] \text{ on}$$

$$\mathcal{D}(AB) = \left\{u \in L_p(J_\infty, E) : \mathrm{e}^{a\cdot}u(\cdot) \in W_p^1(J_\infty, E), u(0) = 0\right\},$$

belongs to $\mathcal{S}(L_p(J_\infty, E), \pi/2 + \varepsilon)$, provided $\sup_{\delta > \nu}\{|a| \cdot \delta^{-\beta(a)}\} \le \delta(\varepsilon)$.

Finally we combine Proposition 2.1, concerning the perturbed time derivative $\mathrm{d}(b \cdot u)/\mathrm{d}t$, with the Dore-Venni type Theorem 1.4 (for Hilbert spaces). This yields the following maximal regularity result for Cauchy problem (1).

THEOREM 2.1. *Let H be a Hilbert space, and $\mathfrak{L}$ an invertible operator of class $\mathcal{BIP}(H, \theta_L)$, $\theta_L < \pi/2$. Recall that the subspace $(\mathcal{D}(\mathfrak{L}), \|\cdot\|_H + \|\mathfrak{L}\cdot\|_H)$ is denoted by $H_{\mathfrak{L}}$. The bounded function $b : J_T \longrightarrow \mathbb{C}$ on $J_T = [0, T)$, $T \le +\infty$, is supposed to satisfy Assumption 2.1 with $\theta < \pi/2 - \theta_L$ and $q \ge 2$.*

Then, the following statement holds: For each $f \in L_2(J_T, H)$ Cauchy problem (1) has a unique solution $u \in L_2(J_T, H_{\mathfrak{L}})$ with $b \cdot u \in W_2^1(J_T, H)$ and

$$\|u\|_{L_2(J_T,H)} + \|b \cdot u\|_{W_2^1(J_T,H)} + \|\mathfrak{L}u\|_{L_2(J_T,H)} \le c\|f\|_{L_2(J_T,H)},$$

$$\text{provided } c_B(\nu) := \sup_{\delta > \nu}\left\{\left\|\frac{b'}{b^{1-\beta}}\right\|_{L_q(J_T)} \cdot \delta^{-\beta}\right\} \text{ is sufficiently small.}$$

PROOF. Let ϕ_{AB} be an angle with $\pi/2 + \theta < \phi_{AB} < \pi - \theta_L$. Then, $\nu + AB$, given in Proposition 2.1 with $\kappa = 0$, is an operator of class $\mathcal{S}(L_2(J_T, H), \phi_{AB})$, provided $c_B(\nu)$ is sufficiently small.

Moreover, let L be defined by $(Lu)(t) := \mathfrak{L}u(t)$ on $\mathcal{D}(L) = \{u \in L_2(J_T, H_{\mathfrak{L}})\}$. From $\mathfrak{L} \in \mathcal{BIP}(H, \theta_L)$ and $0 \in \rho(\mathfrak{L})$ follows that L is an invertible operator of class $\mathcal{BIP}(L_2(J_T, H), \theta_L)$.

Since the resolvents of AB and L commute, and $\phi_{AB} + \theta_L < \pi$ is satisfied, Theorem 1.4 leads to the asserted maximal regularity

$$\begin{aligned} &\nu + AB + L \in \mathcal{L}\text{is}(X_{AB+L}, X) \\ &\text{on } X_{AB+L} = (\mathcal{D}(AB) \cap \mathcal{D}(L) : \|\cdot\|_X + \|(AB+L)\cdot\|_X) \end{aligned}$$

in the Hilbert space $X := L_2(J_T, H)$. □

Theorem 2.1 applies, in particular, to the $L_2(\Omega)$-realization of $\mathfrak{L}$, defined in Example 1.2. For further examples of operators $\mathfrak{L} \in \mathcal{BIP}(H, \theta_L)$ confer the references, given in Paragraph 1.1.

REMARK 2.2. For an extension of Theorem 2.1 to Banach spaces $X = L_p(J_T, E)$ (with $E \in \mathcal{HT}$ and $p \in (1, \infty)$) by means of Dore-Venni type theorems it remains to be proven that $\nu + AB$ (defined in Proposition 2.1) has bounded imaginary powers. We shall address this topic in a forthcoming paper.

2.2. – Degenerations of the Laplacian

In this paragraph we deal with the initial value problem

$$\begin{aligned} \partial_t u(t,x) - \Delta_x[b(x)u(t,x)] &= f(t,x), \quad (t,x) \in J_T \times \mathbb{R}^n \\ u(0,x) &= 0, \qquad\qquad x \in \mathbb{R}^n \end{aligned} \tag{16}$$

where J_T denotes the time interval $[0, T)$, $T \leq +\infty$, and the complex-valued function b gives rise to a singular perturbation of $-\Delta_x = -\Delta = -\sum_{i=1}^n \partial_i^2$. On the basis of our abstract results it shall be investigated, under which conditions maximal regularity in $L_p(J_T, L_2(\mathbb{R}^n))$, $1 < p < +\infty$, can be obtained.

First we formulate our assumptions on b.

ASSUMPTION 2.2. (A_1) $b \in C^2(\mathbb{R}^n, \mathbb{C}) \cap L_\infty(\mathbb{R}^n, \mathbb{C})$.

(A_2) $b(x) \in \Sigma_\theta$, $\theta \in (0, \pi)$, for almost all $x \in \mathbb{R}^n$.

(A_3) There are $q \in (\max\{1, n/2\}, \infty]$ and $\beta \in [0, 1 - n/(2q))$, such that

$$\frac{\partial_i b(\cdot)}{b^{1-\beta/2}(\cdot)} \in L_{2q}(\mathbb{R}^n) \text{ and } \frac{\partial_i^2 b(\cdot)}{b^{1-\beta}(\cdot)} \in L_q(\mathbb{R}^n), i \in \{1, \ldots, n\}.$$

Then the following result can be shown.

PROPOSITION 2.2. *Let the bounded function $b : \mathbb{R}^n \longrightarrow \mathbb{C}$ satisfy Assumption 2.2. We fix $p \in (1, \infty)$ with $p \leq q$. Then, $\nu + \mathfrak{L}$ with $\nu \geq 0$, defined by*

$$(\mathfrak{L}u)(x) := -\Delta\,[b(x)u(x)] \text{ on } \mathcal{D}(\mathfrak{L}) := \left\{u \in L_p(\mathbb{R}^n) : b(\cdot)u(\cdot) \in W_p^2(\mathbb{R}^n)\right\},$$

is of class $\mathcal{S}(L_p(\mathbb{R}^n), \varphi_A + \varphi_B)$, $\varphi_A > 0$, $\varphi_B > \theta$, $\varphi_A + \varphi_B < \pi$, provided

$$\sup_{\delta>\nu}\left\{c_B \cdot \delta^{-\beta}\right\} \text{ with } c_B = \max_{i=1,\dots,n}\left\|\frac{\partial_i b}{b^{1-\beta/2}}\right\|_{L_{2q}(\mathbb{R}^n)} + \left\|\frac{\Delta b}{b^{1-\beta}}\right\|_{L_q(\mathbb{R}^n)} \tag{17}$$

is sufficiently small.

If, in addition, Assumption 2.2 is satisfied with $\theta < \pi/2$, then $-\mathfrak{L}$ generates a strongly continuous analytic semigroup of bounded linear operators on $L_p(\mathbb{R}^n)$.

PROOF. Analogously to the proof of Proposition 2.1 we consider $\mathfrak{L}$ as a product of two sectorial operators A and B in the Banach space $X := L_p(\mathbb{R}^n) \in \mathcal{HT}$.

Let A be defined by $(Au)(x) := -\Delta u(x)$ on $\mathcal{D}(A) = \{u \in W_p^2(\mathbb{R}^n)\}$. Then $\kappa + A$ with $\kappa > 0$ is an invertible operator of class $\mathcal{BIP}(X, 0)$ (cf. Example 1.2).

Moreover, we define $(Bu)(x) := b(x)u(x)$ on $\mathcal{D}(B) := \{u \in X : b \cdot u \in X\}$. From Assumption 2.2 ($A_1$), ($A_2$) follows that B belongs to $\mathcal{BIP}(X, \theta)$, and satisfies the compatibility condition $(\mu + B)^{-1}\mathcal{D}(A) \subseteq \mathcal{D}(A)$ for $\mu \in \rho(-B)$.

In order to apply Theorem 1.1 to the product $(\kappa + A)B$, the commutator estimate (5) remains to be verified. To this end we fix two arbitrary angles $\varphi_A > 0$ and $\varphi_B > \theta$, which satisfy $\varphi_A + \varphi_B < \pi$. It is easily seen that

$$\left\{\left[A(\mu + B)^{-1} - (\mu + B)^{-1}A\right] v\right\}(x) = -2\sum_{i=1}^{n} \partial_i g_\mu(x)\partial_i v(x) - \Delta g_\mu(x)v(x)$$

holds for $v \in \mathcal{D}(A)$ and $\mu \in \Sigma_{\pi-\varphi_B}$, where g_μ is given by $g_\mu(x) := (\mu+b(x))^{-1}$. Using Hölder's inequality we consequently obtain

$$\begin{aligned}&\left\|\left[A(\mu + B)^{-1} - (\mu + B)^{-1}A\right] v\right\|_{L_p(\mathbb{R}^n)}\\ &\leq 2\max_{i=1,\dots,n}\left\|\partial_i g_\mu\right\|_{L_{2q}(\mathbb{R}^n)}\sum_{i=1}^{n}\|\partial_i v\|_{L_{2pq/(2q-p)}(\mathbb{R}^n)} + \left\|\Delta g_\mu\right\|_{L_q(\mathbb{R}^n)}\|v\|_{L_{pq/(q-p)}(\mathbb{R}^n)},\\ &\quad \forall v \in \mathcal{D}(A), \forall \mu \in \Sigma_{\pi-\varphi_B}.\end{aligned}$$

By virtue of Assumption 2.2 (A_3), the function g_μ satisfies

$$\max_{i=1,\dots,n}\left\|\partial_i g_\mu\right\|_{L_{2q}(\mathbb{R}^n)} = \max_{i=1,\dots,n}\left\|g_\mu^2\partial_i b\right\|_{L_{2q}(\mathbb{R}^n)} \leq \frac{c_{B,1}}{|\mu|^{1+\beta/2}} \qquad \text{and}$$

$$\left\|\Delta g_\mu\right\|_{L_q(\mathbb{R}^n)} \leq \left\|g_\mu^2\Delta b\right\|_{L_q(\mathbb{R}^n)} + 2\sum_{i=1}^{n}\left\|g_\mu^{3/2}\partial_i b\right\|_{L_{2q}(\mathbb{R}^n)}^2 \leq \frac{c_{B,2}}{|\mu|^{1+\beta}}, \forall \mu \in \Sigma_{\pi-\varphi_B},$$

where the constants $c_{B,i}$, $i \in \{1,2\}$, are given by

$$c_{B,1} := c_{\varphi_B}^{-2} \cdot \max_{i=1,\dots,n} \left\| \frac{\partial_i b}{b^{1-\beta/2}} \right\|_{L_{2q}(\mathbb{R}^n)} \quad \text{with } c_{\varphi_B} := \sin\left[(\varphi_B - \theta)/2\right] \quad \text{or}$$

$$c_{B,2} := c_{\varphi_B}^{-2} \cdot \left\| \frac{\Delta b}{b^{1-\beta}} \right\|_{L_q(\mathbb{R}^n)} + 2n \cdot c_{\varphi_B}^{-3} \cdot \max_{i=1,\dots,n} \left\| \frac{\partial_i b}{b^{1-\beta/2}} \right\|_{L_{2q}(\mathbb{R}^n)}^2 ,$$

respectively. Thus, the estimate

$$\left\| \left[A(\mu + B)^{-1} - (\mu + B)^{-1} A \right] v \right\|_{L_p(\mathbb{R}^n)}$$
$$\leq \frac{c_{B,1}}{|\mu|^{1+\beta/2}} \sum_{i=1}^{n} \|\partial_i v\|_{L_{2pq/(2q-p)}(\mathbb{R}^n)} + \frac{c_{B,2}}{|\mu|^{1+\beta}} \|v\|_{L_{pq/(q-p)}(\mathbb{R}^n)}$$

is valid for all $v \in \mathcal{D}(A)$ and $\mu \in \Sigma_{\pi-\varphi_B}$. We consider now the diagram

$$(X, X_A)_{\gamma,1} \hookrightarrow \left(L_p(\mathbb{R}^n), W_p^2(\mathbb{R}^n) \right)_{\gamma,1} \doteq B_{p,1}^{2\gamma}(\mathbb{R}^n) \hookrightarrow B_{s,1}^k(\mathbb{R}^n) \hookrightarrow W_s^k(\mathbb{R}^n) ,$$

which holds for $k \in \{0,1\}$, $s \in [p, \infty)$ and $\gamma = (k + n/p - n/s)/2$ (cf. [Tri78], 2.4.2 Remark 4, 2.8.1 Theorem (a), and 2.3.3 Remark 4). In particular, the continuous imbeddings

$$\begin{aligned} (X, X_A)_{\alpha_1,1} &\hookrightarrow W^1_{2pq/(2q-p)}(\mathbb{R}^n) \quad \text{with} \quad \alpha_1 = [1 + n/(2q)]/2 , \\ (X, X_A)_{\alpha_2,1} &\hookrightarrow L_{pq/(q-p)}(\mathbb{R}^n) \quad \text{with} \quad \alpha_2 = n/(2q) , \end{aligned}$$

are valid, so that

$$\left\| \left[A(\mu + B)^{-1} - (\mu + B)^{-1} A \right] v \right\|_X \leq \sum_{i=1}^{2} \frac{c \cdot c_{B,i}}{|\mu|^{1+\beta_i}} \|v\|_{(X,X_A)_{\alpha_i,1}}, \forall \mu \in \Sigma_{\pi-\varphi_B} ,$$

holds with $\beta_1 = \beta/2$ and $\beta_2 = \beta$ for $v \in \mathcal{D}(A)$. Since $v = (\lambda + \kappa + A)^{-1} u$ fulfills

$$\|v\|_{(X,X_A)_{\alpha_i,1}} \leq c(\alpha_i) \left\| (\lambda + \kappa + A)^{-1} u \right\|_X^{1-\alpha_i} \left\| (\lambda + \kappa + A)^{-1} u \right\|_{X_A}^{\alpha_i}$$
$$\leq \frac{c_{A,i}}{(1 + |\lambda|)^{1-\alpha_i}} \|u\|_X, \forall u \in X, \forall \lambda \in \Sigma_{\pi-\varphi_A} \text{ with } c_{A,i} = \mathcal{O}\left[(\sin \varphi_A)^{\alpha_i - 1} \right] ,$$

we obtain the desired commutator estimate

$$\left\| \left[(\kappa + A)(\mu + B)^{-1} - (\mu + B)^{-1}(\kappa + A) \right] (\lambda + \kappa + A)^{-1} \right\|_{\mathcal{L}(X)}$$
$$\leq \sum_{i=1}^{2} \frac{c \cdot c_{A,i} \cdot c_{B,i}}{(1 + |\lambda|)^{1-\alpha_i} |\mu|^{1+\beta_i}}, \forall (\lambda, \mu) \in \Sigma_{\pi-\varphi_A} \times \Sigma_{\pi-\varphi_B} ,$$

where $\alpha_i, \beta_i \geq 0, \alpha_i + \beta_i < 1, i \in \{1,2\}$.

Considering Remark 1.2, Theorem 1.1 shows that $\nu + (\kappa + A)B$ is sectorial in X (with spectral angle $\phi_{AB} \le \varphi_A + \varphi_B$), provided $\sum_{i=1}^{2} \sup_{\delta>\nu}\{c_B \cdot \delta^{-\beta_i}\}$ is sufficiently small. Since our assumption $b \in L_\infty(\mathbb{R}^n, \mathbb{C})$ implies $B \in \mathcal{L}(X)$, we consequently obtain the assertion $\nu + \mathfrak{L} = \nu + (\kappa + A)B - \kappa B \in \mathcal{S}(X, \varphi_A + \varphi_B)$ (cf. e.g. [HP]).

If Assumption 2.2 is satisfied with $\theta < \pi/2$, the angles $\varphi_A > 0$ and $\varphi_B > \theta$ can be chosen in such a manner that $\varphi_A + \varphi_B < \pi/2$. In this case $-\mathfrak{L}$ generates an analytic C_0-semigroup on X (cf. e.g. [Ama95], I.1.2.2 Theorem and I.1.2.1 Remark (a)), and our proof is complete. □

From Proposition 2.2 we deduce the following maximal regularity result.

THEOREM 2.2. *The function $b : \mathbb{R}^n \longrightarrow \mathbb{C}$ is supposed to satisfy Assumption 2.2 with $\theta < \pi/2$ and $q \ge 2$. Moreover, let $f \in L_p(J_T, L_2(\mathbb{R}^n))$ with $p \in (1, \infty)$ and $J_T = [0, T)$, $T \in (0, +\infty)$ be given.*

Then, the initial value problem (16) has a unique solution $u \in L_p(J_T, L_2(\mathbb{R}^n))$ with $\partial_t u$, $\Delta_x(bu) \in L_p(J_T, L_2(\mathbb{R}^n))$, which satisfies the estimate

$$\|u\|_{W_p^1(J_T, L_2(\mathbb{R}^n))} + \|\Delta_x(bu)\|_{L_p(J_T, L_2(\mathbb{R}^n))} \le c\|f\|_{L_p(J_T, L_2(\mathbb{R}^n))} ,$$

provided that $\beta > 0$, or the constant c_B, defined in (17), is sufficiently small.

PROOF. Let ϕ_L be an angle with $\theta < \phi_L < \pi/2$. Then there is some $\nu > 0$, such that $\nu + \mathfrak{L}$, given in Proposition 2.2, belongs to $\mathcal{S}(H, \phi_L)$, where $H := L_2(\mathbb{R}^n)$. Consequently,

$$(Lu)(t) := \mathfrak{L}u(t), \mathcal{D}(L) := \{u \in L_p(J_T, H_\mathfrak{L})\} \text{ with}$$
$$H_\mathfrak{L} = \Big(\mathcal{D}(\mathfrak{L}) = \Big\{u \in H : b \cdot u \in W_2^2(\mathbb{R}^n)\Big\}, \|\cdot\|_{L_2(\mathbb{R}^n)} + \|\Delta(b\,\cdot)\|_{L_2(\mathbb{R}^n)}\Big)$$

defines an operator $\nu + L \in \mathcal{S}(L_p(J_T, H), \phi_L)$.

Now let $p = 2$. Since $\partial := \mathrm{d}/\mathrm{d}t$ on $\mathcal{D}(\partial) = \{u \in W_2^1(J_T, H) : u(0) = 0\}$ is an invertible operator of class $\mathcal{BIP}(L_2(J_T, H), \pi/2)$ (cf. Example 1.1), Theorem 1.4 applies to the sum $\nu + \partial + L$ and shows

$$\nu + \partial + L \in \mathcal{L}\mathrm{is}\left(W_2^1(J_T, H) \cap L_2(J_T, H_\mathfrak{L}), L_2(J_T, H)\right) .$$

This implies $\partial + L \in \mathcal{L}\mathrm{is}(\mathrm{e}^\nu[W_2^1(J_T, H) \cap L_2(J_T, H_\mathfrak{L})], \mathrm{e}^\nu L_2(J_T, H))$, where $\mathrm{e}^\nu X$ denotes the space $\{u \in L_{1,\mathrm{loc}}(J_T, H) : \mathrm{e}^{-\nu\cdot}u(\cdot) \in X\}$, equipped with the norm $\|\mathrm{e}^{-\nu\cdot}u(\cdot)\|_X$ (cf. [Ama95], III.1.5.3 Proposition). By virtue of the assumed boundedness of the time interval J_T, we consequently obtain

$$\partial + L \in \mathcal{L}\mathrm{is}\left(W_2^1(J_T, H) \cap L_2(J_T, H_\mathfrak{L}), L_2(J_T, H)\right) . \tag{18}$$

Finally, this result shall be extended to the case of an arbitrary $p \in (1, \infty)$. For that purpose we employ Theorem 4.2 of the survey article [Dor93] on maximal L_p-regularity. It states that if $-\mathfrak{L}$ is a closed linear operator which generates an analytic C_0-semigroup on H, then (18) implies

$$\partial + L \in \mathcal{L}\mathrm{is}\left(W_p^1(J_T, H) \cap L_p(J_T, H_\mathfrak{L}), L_p(J_T, H)\right)$$

for each $p \in (1, +\infty)$. Thus, our proof is complete. □

In order to extend Theorem 2.2 to spaces $X = L_p(J_T, L_q(\mathbb{R}^n))$, $p, q \in (1, +\infty)$, by means of Dore-Venni type theorems, it remains to be shown that L is of class $\mathcal{BIP}(X)$. We shall address this topic in a forthcoming paper.

REMARK 2.3. Combining Proposition 2.2 with [Ama95], II.1.2.1 Theorem, we obtain the following existence and regularity result.

The function $b : \mathbb{R}^n \longrightarrow \mathbb{C}$ *is supposed to satisfy Assumption* 2.2 *with* $\theta < \pi/2$. *Moreover, let* $f \in C^\gamma([0, T), L_p(\mathbb{R}^n))$, $\gamma \in (0, 1)$, $p \in (1, +\infty)$, $p \leq q$, $T \leq +\infty$, *be given. Then, problem* (16) *has a unique solution* u *in* $C^{1+\gamma}((0, T), L_p(\mathbb{R}^n)) \cap C^1([0, T), L_p(\mathbb{R}^n))$ *with* $b \cdot u \in C^\gamma((0, T), W_p^2(\mathbb{R}^n))$, *provided that* $\beta > 0$, *or* c_B, *defined in* (17), *is sufficiently small.*

Maximal regularity in (singular) Hölder spaces of $L_p(\mathbb{R}^n)$-valued functions can be shown by means of results, proven in [Ama95], Section III.2.5, or [Lun95], Chapter 4.

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