ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 27, nº 3-4 (1998), p. 483-497
<http://www.numdam.org/item?id=ASNSP_1998_4_27_3-4_483_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVII (1998), pp. 483-497

On the Scattering in Gevrey Classes for the Subcritical Hartree and Schrödinger Equations

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Abstract. We study the scattering problem and asymptotics for large time of solutions to the Cauchy problem for the subcritical cubic nonlinear Schrödinger and Hartree type equations

$$(A) \qquad iu_t + \frac{1}{2}u_{xx} = \mathcal{N}(|u|^2)u, \quad (t,x) \in \mathbb{R}^2; \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}$$

where the nonlinear interaction term is $\mathcal{N}(|u|^2) = \lambda|x|^{-\delta} * |u|^2 + \mu|t|^{1-\delta}|u|^2$. We suppose that the initial data u_0 are such that $e^{\beta|x|^\sigma}u_0 \in L^2$, $\beta > 0$, $1 - \frac{\delta}{2-\delta} < \sigma < 1$ and the norm $\epsilon = \|e^{\beta|x|^\sigma}u_0\|_{L^2}$ is sufficiently small. Then we prove the sharp decay estimate for the solution of the Cauchy problem (A) $\|u(t)\|_{L^p} \le C\epsilon t^{\frac{1}{p}-\frac{1}{2}}$, for all $t \ge 1$ and for every $2 \le p \le \infty$. Furthermore we show that for $\frac{1}{2} < \delta < 1$ there exists a unique final state $\hat{u}_+ \in L^2$ such that as $t \to \infty$

$$\left\| u(t) - \exp\left(-\frac{it^{1-\delta}}{1-\delta}\mathcal{N}(|\hat{u}_+|^2)\left(\frac{x}{t}\right)\right) U(t)u_+ \right\|_{L^2} = O(t^{1-2\delta})$$

and uniformly with respect to x

$$u(t,x) = \frac{1}{\sqrt{t}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}\mathcal{N}(|\hat{u}_+|^2)\left(\frac{x}{t}\right)\right) + O(t^{1/2-2\delta}),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ.

Mathematics Subject Classification (1991): 35Q55.

1. – Introduction

We study the asymptotic behavior for large time of solutions to the Cauchy problem for the subcritical cubic nonlinear Schrödinger with a growing in time

Pervenuto alla Redazione l'8 aprile 1998.

coefficient and Hartree type equation

$$(1.1)\qquad \begin{cases} iu_t + \frac{1}{2}u_{xx} = \mathcal{N}(|u|^2)u, & (t,x) \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where

$$\mathcal{N}(|u|^2) = \lambda \int |x-y|^{-\delta}|u(y)|^2 dy + \mu |t|^{1-\delta}|u|^2,$$

$0 < \delta < 1$, $\lambda, \mu \in \mathbb{R}$. Local and global existence in time of solutions to (1.1) were studied by many authors (see, e.g., [1], [2], [5], [11] and references therein). However there are few results about time decay estimates of solutions and scattering problems to (1.1).

In the previous paper [6] we proved that if the initial function u_0 decay exponentially rapidly at infinity, then the solution of (1.1) exists and satisfies the sharp decay estimate $\|u(t)\|_{L^p} \le C\epsilon t^{\frac{1}{p}-\frac{1}{2}}$, for all $t \ge 1$ and for every $2 \le p \le \infty$. Furthermore we showed that for $\frac{1}{2} < \delta < 1$ there exists a unique final state $u_+ \in L^2$ such that

$$\left\| u(t) - \exp\left(-\frac{it^{1-\delta}}{1-\delta} \mathcal{N}\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right)\right) U(t)u_+ \right\|_{L^2} = O\left(t^{1-2\delta}\right)$$

as $t \to \infty$, where $\hat{\phi}$ denotes the Fourier transform of ϕ.

Our purpose in this paper is to derive more exact exponential decay conditions on the initial data which lead to Gevrey classes in the investigation of the scattering theory.

Modified wave operators for the Hartree equation (considered as the critical case $\delta = 1$ in equation (1.1)) were constructed in [4] and the existence of modified scattering states for the Hartree equation was shown in [8] in higher space dimensions which are greater than or equal to 2. For the cubic nonlinear Schrödinger equation in the critical case ($\delta = 1$) in [4, 14] the modified wave operators were constructed and in [13] the existence of the modified scattering states was proved. However there are a few works on the scattering problem for subcritical case $0 < \delta < 1$. The asymptotics for large time of solutions to the Cauchy problem (1.1) with $\delta \in (0, \frac{1}{16})$ and $\lambda = 0$ was obtained in paper [13] by using the Gevrey classes. In the present paper we apply much more general and simple approach proposed in the paper [9]. Two types of the subcritical nonlinearities will be considered. If $\delta \in (0,1)$ then we prove in Theorem 1.1 below the sharp time decay estimates of the solutions and large time asymptotics (1.2). And for the case $\delta \in (1/2, 1)$ we will construct in Theorem 1.2 the modified scattering states (see inequality (1.4) below) and write the large time asymptotics of solutions more precisely compared to the asymptotic formula (1.2).

We state our strategy of the proofs of results to explain the reason why we need the condition that the data u_0 decay exponentially when $|x| \to \infty$.

As in [9] we define $w = (\mathcal{F}MU(-t)u(t))e^{ig}$, $t > 0$, where u is a solution of (1.1), and g is a solution of the heat equation $g_t = t^{-\delta}\mathcal{N}(|\mathcal{F}MU(-t)u(t)|^2) + \frac{1}{2t^2}(g_\chi)^2 + \frac{1}{2t^2}g_{\chi\chi}$. It is clear that g and w are defined by u and a simple computation shows that w and $h = t^{\delta/2-1}g_\chi$ satisfy the system (3.4). From (3.4) we see that we encounter a derivative loss in the first equation of (3.4) though it has nonlinearities with coefficients decaying in time sufficiently fast. By the previous works (see, e.g., [3], [7], [12]), it is known that suitable analytic function spaces are useful to overcome a derivative loss. Indeed in [6] some analytic function space were used to study the same problem as in this paper. Our proof of Theorem 1.1 shows that Gevrey function spaces of order $1/\sigma$, where $1 - \frac{\delta}{2-\delta} < \sigma < 1$, are sufficient to treat the derivative of unknown function w in (3.4) if h is a real valued function. More precisely, we can make use of integration by parts when h is a real valued function and we see from the important estimate (3.8) the remainder term which comes from integration by parts is controlled by Gevrey function spaces of order $1/\sigma$. Our proof is also useful to the local existence in time of solutions to nonlinear Schrödinger equations in Gevrey classes

$$iu_t + \frac{1}{2}u_{xx} = \mathcal{N}(u, u_x, \bar{u}, \bar{u}_x),$$

where $\mathcal{N}$ is a polynomial with respect to each arguments and $\partial_{u_x}\mathcal{N}$ is pure imaginary.

In what follows we consider the positive time t only since for the negative one the results are analogous. Before stating our results we give some notations and function spaces. We let $\partial_x = \partial/\partial x$ and $\mathcal{F}\phi$ or $\hat{\phi}$ be the Fourier transform of ϕ defined by $\hat{\phi}(\chi) = \frac{1}{\sqrt{2\pi}}\int e^{-ix\chi}\phi(x)dx$ and $\mathcal{F}^{-1}\phi(x)$ or $\check{\phi}(x)$ be the inverse Fourier transform of ϕ, i.e. $\check{\phi}(x) = \frac{1}{\sqrt{2\pi}}\int e^{ix\chi}\phi(\chi)d\chi$. We introduce some function spaces. The usual Lebesgue space is $L^p = \{\phi \in \mathcal{S}'; \|\phi\|_p < \infty\}$, where $\|\phi\|_p = \left(\int |\phi(x)|^p dx\right)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_\infty = \text{ess.sup}_{x\in\mathbb{R}}|\phi(x)|$ if $p = \infty$. For simplicity we let $\|\phi\| = \|\phi\|_2$. Weighted Sobolev space we define as follows $H^{m,s} = \{\phi \in \mathcal{S}'; \|\phi\|_{m,s} = \|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\phi\| < \infty\}$, $m, s \in \mathbb{R}$. The homogeneous Sobolev space is $\dot{H}^{m,s} = \{\phi \in \mathcal{S}'; \||x|^s(-\partial_x^2)^{m/2}\phi\| < \infty\}$, $m, s \in \mathbb{R}$. Also we define the Gevrey function spaces of order $1/\sigma$ $\mathcal{G}_\alpha^\sigma = \{\phi \in L^2; \|\hat{\phi}\|_{\mathcal{Y}_\alpha^\sigma} < \infty\}$, $\sigma \in (0, 1]$, $\alpha > 0$ with the norm $\|\phi\|_{\mathcal{G}_\alpha^\sigma} = \|\hat{\phi}\|_{\mathcal{Y}_\alpha^\sigma}$, where $\|\phi\|_{\mathcal{Y}_\alpha^\sigma} = \|(1+\chi^2)e^{\alpha|\chi|^\sigma}\phi(\chi)\|$ and $\mathcal{Y}_\alpha^\sigma = \{\phi \in L^2; \|\phi\|_{\mathcal{Y}_\alpha^\sigma} < \infty\}$. The Gevrey functional spaces $\mathcal{G}_\alpha^\sigma$ can be equivalently defined in the x-representation by the norm $(\sum_{n=1}^\infty (\tilde{\alpha}n)^{-2n/\sigma}\|\partial^n\phi\|^2)^{1/2}$, $\tilde{\alpha} > 0$. Indeed we have the inequalities $\sum_{n=1}^\infty(\alpha_1 n)^{-2n/\sigma}\|\partial^n\phi\|^2 \le \|\phi\|^2_{\mathcal{G}_\alpha^\sigma} = \sum_{n=1}^\infty \int_{n-1\le|\chi|\le n} e^{2\alpha|\chi|^\sigma}|\check{\phi}(\chi)|^2 d\chi \le \sum_{n=1}^\infty(\alpha_2 n)^{-2n/\sigma}\|\partial^n\phi\|^2$ with some constants $\alpha_1, \alpha_2 > 0$. We let $(\psi, \varphi) = \int \overline{\psi}(x)\cdot\varphi(x)dx$. By $C(I; E)$ we denote the space of continuous functions from an interval I to a Banach space E.

The free Schrödinger evolution group $U(t) = e^{it\partial_x^2/2}$ gives us the solution of the Cauchy problem for the linear Schrödinger equation ((1.1) with $\mathcal{N} = 0$).

It can be represented explicitly in the following manner

$$U(t)\phi = \frac{1}{\sqrt{2\pi i t}} \int e^{i(x-y)^2/2t} \phi(y)dy = \mathcal{F}^{-1} e^{-it\chi^2/2} \mathcal{F}\phi .$$

Note that $U(t) = M(t)D(t)\mathcal{F}M(t)$, where $M = M(t) = \exp(\frac{ix^2}{2t})$ and $D(t)$ is the dilation operator defined by $(D(t)\psi)(x) = \frac{1}{\sqrt{it}}\psi(\frac{x}{t})$. Then since $D(t)^{-1} = iD(\frac{1}{t})$ we have $U(-t) = \overline{M}\mathcal{F}^{-1}D(t)^{-1}\overline{M} = \overline{M}i\mathcal{F}^{-1}D(\frac{1}{t}\overline{M}$, where $\overline{M} = M(-t) = \exp(-\frac{ix^2}{2t})$. Different positive constants might be denoted by the same letter C.

We now state our results in this paper.

THEOREM 1.1. *Let $\delta \in (0,1)$. We assume that $u_0 \in \mathcal{Y}^\sigma_{2\beta}$, $\beta > 0$, $1 - \frac{\delta}{2-\delta} < \sigma < 1$ and the norm $\epsilon = \|u_0\|_{\mathcal{Y}^\sigma_{2\beta}}$ is sufficiently small. Then*

(1) *there exists a unique global solution $u \in C(\mathbb{R}; L^2)$ of the Cauchy problem* (1.1) *such that the following decay estimate*

$$\|u(t)\|_p \le C\epsilon t^{\frac{1}{p}-\frac{1}{2}}$$

is valid for all $t \ge 1$, where $2 \le p \le \infty$;

(2) *there exists a unique final state u_+ such that $u_+ \in \mathcal{Y}^\sigma_\beta$ and the following asymptotics*

$$u(t,x) = \frac{1}{\sqrt{t}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}\mathcal{N}\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right) + O\left(1+t^{1-2\delta}\right)\right) + O\left(t^{-\frac{1}{2}-\delta}\right) \tag{1.2}$$

is true for $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$.

For the values $\delta \in (\frac{1}{2}, 1)$ we are able to construct the modified scattering states.

THEOREM 1.2. *Let $\delta \in (\frac{1}{2}, 1)$ and u be the solution of* (1.1) *obtained in Theorem* 1.1. *Then there exists a unique final state $u_+ \in \mathcal{Y}^\sigma_\beta$ such that the following asymptotics*

$$u(t,x) = \frac{1}{\sqrt{t}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}\mathcal{N}\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right)\right) + O\left(t^{\frac{1}{2}-2\delta}\right) \tag{1.3}$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$ and the estimate

$$\left\| u(t) - \exp\left(-\frac{it^{1-\delta}}{1-\delta}\mathcal{N}\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right)\right)U(t)u_+ \right\| \le Ct^{1-2\delta} \tag{1.4}$$

is true for all $t \ge 1$.

We organize our paper as follows. In section 2 we prove the local in time existence of the solutions to the Cauchy problem (1.1) in the functional space

$$\mathcal{X} = \left\{ \varphi \in C([0,T]; L^2); |||\varphi|||_{\mathcal{X}}^2 \equiv \sup_{0\le t\le T} \|U(-t)\varphi(t)\|_{\mathcal{Y}_{\alpha(t)}^\sigma}^2 \right.$$
$$\left. + \int_0^T |\alpha'(t)| \left\| |x|^{\sigma/2} U(-t)\varphi(t) \right\|_{\mathcal{Y}_{\alpha(t)}^\sigma}^2 dt < \infty \right\},$$

where $\|\varphi(t)\|_{\mathcal{Y}_{\alpha(t)}^\sigma} = \|E(t)\varphi(t)\|$, $E(t,\chi) = (1+\chi^2)e^{\alpha(t)|\chi|^\sigma}$ and $\alpha(t) = \beta + \beta(1+t)^{-\gamma}$, $1 - \frac{\delta}{2-\delta} < \sigma < 1$ $\gamma > 0$ is sufficiently small satisfying $0 < \gamma < \frac{1}{2}(\sigma - \frac{2-2\delta}{2-\delta})$. We let $\mathcal{X}_\rho$ be the closed ball in'$\mathcal{X}$ with a center at the origin and a radius ρ. Then in Section 3 we transform equation (1.1) to a new system of equations (3.2) which describes explicitly the time decay velocity of each term. As stated before the new system (3.2) has a derivative loss, therefore we use Gevrey classes to prove the existence of global solutions and to obtain in Lemma 3.1 time decay estimates of solutions of system (3.2). The rest of Section 3 is devoted to the proof of Theorems 1.1-1.2.

2. – Local existence

In this section we prove the local existence of solutions of the Cauchy problem (1.1) in Gevrey classes $\mathcal{G}_{\alpha(t)}^\sigma$.

Lemma 2.1. *Suppose that the initial data $u_0 \in \mathcal{Y}_{2\beta}^\sigma$. Then there exists a time $T > 0$ and a unique solution $u \in C([0,T]; L^2)$ of the Cauchy problem* (1.1). *Moreover if the norm of the initial data $\|u_0\|_{\mathcal{Y}_{2\beta}^\sigma} = \epsilon$ is sufficiently small, then there exists a time $T > 1$ and a unique solution $u \in C([0,T]; L^2)$ of the Cauchy problem* (1.1) *with the estimate $\|u\|_{\mathcal{X}} \le 2\epsilon$.*

Proof. The proof is established by the standard contraction mapping principle. We consider the linearized Cauchy problem (1.1)

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = \mathcal{N}\left(|v|^2\right) v, & (t,x) \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $v \in \mathcal{X}_\epsilon$. Multiplying both sides of (2.1) by $E(t)U(-t)$ with $E(t,x) = (1+x^2)e^{\alpha(t)|x|^\sigma}$ we obtain

$$i\partial_t E(t)U(-t)u - i\alpha'(t)|x|^\sigma E(t)U(-t)u = E(t)U(-t)\mathcal{N}\big(|v|^2\big)v\,,$$

whence multiplying both sides by $E(t)\overline{U(-t)u(t)}$ integrating with respect to x and taking the imaginary part we get

$$\begin{aligned}\frac{d}{dt}\|U(-t)u(t)\|^2_{\mathcal{Y}^\sigma_\alpha} + 2|\alpha'(t)|\big\||\chi|^{\sigma/2}U(-t)u(t)\big\|^2_{\mathcal{Y}^\sigma_\alpha} \\ \le 2\big\|U(-t)\mathcal{N}\big(|v|^2\big)v\big\|_{\mathcal{Y}^\sigma_\alpha}\|U(-t)u(t)\|_{\mathcal{Y}^\sigma_\alpha}.\end{aligned} \tag{2.2}$$

Since

$$\begin{aligned}\|\phi\psi\|_{\mathcal{G}^\sigma_\alpha} &= \left\|(1+\chi^2)e^{\alpha|\chi|^\sigma}\int\hat{\phi}(\chi-y)\hat{\psi}(y)dy\right\| \\ &\le \left\|(1+\chi^2)\int e^{\alpha|\chi-y|^\sigma+\alpha|y|^\sigma}|\hat{\phi}(\chi-y)||\hat{\psi}(y)|dy\right\| \le C\|\phi\|_{\mathcal{G}^\sigma_\alpha}\|\psi\|_{\mathcal{G}^\sigma_\alpha}\end{aligned}$$

by Hölder's and Sobolev's inequalities we obtain

$$\begin{aligned}\big\|U(-t)\mathcal{N}\big(|v|^2\big)v\big\|_{\mathcal{Y}^\sigma_\alpha} &= \big\|E\overline{M}\mathcal{F}^{-1}D^{-1}\overline{M}\mathcal{N}\big(|v|^2\big)v\big\| = \big\|D^{-1}\mathcal{N}\big(|\overline{M}v|^2\big)\overline{M}v\big\|_{\mathcal{G}^\sigma_\alpha} \\ &\le Ct^{-\delta}\big\|D^{-1}\overline{M}v\big\|^3_{\mathcal{G}^\sigma_\alpha} = Ct^{-\delta}\|U(-t)v\|^3_{\mathcal{Y}^\sigma_\alpha} \le C\epsilon^3t^{-\delta}.\end{aligned}$$

Hence we have by (2.2)

$$\|U(-t)u(t)\|_{\mathcal{Y}^\sigma_\alpha} \le \epsilon + C\epsilon^3T^{1-\delta}. \tag{2.3}$$

Substituting estimate (2.3) to the right hand side of inequality (2.2) we get $\|u\|^2_{\mathcal{X}} \le \epsilon^2 + C\epsilon^3T^{1-\delta}(\epsilon + C\epsilon^3T^{1-\delta})$ which implies

$$\|u\|_{\mathcal{X}} \le 2\epsilon, \tag{2.4}$$

if we take ϵ or $T > 0$ to be sufficiently small: $C\epsilon^3T^{1-\delta}(\epsilon + C\epsilon^3T^{1-\delta}) \le \epsilon^2$. In the same way we prove the estimate

$$\|u_1 - u_2\|_{\mathcal{X}} \le \frac{1}{2}\|v_1 - v_2\|_{\mathcal{X}}, \tag{2.5}$$

where u_j, $j = 1, 2$ are the corresponding solutions of the Cauchy problems

$$\begin{cases} i\partial_t u_j + \frac{1}{2}\partial_x^2 u_j = \mathcal{N}\big(|v_j|^2\big)v_j, & (t,x)\in\mathbb{R}^2, \\ u_j(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases}$$

We have the desired result from estimates (2.4), (2.5) and the contraction mapping principle. Lemma 2.1 is proved. □

3. – Proof of Theorems

We define a new function $u(t,x) = MDv = \frac{1}{\sqrt{it}} e^{\frac{ix^2}{2t}} v(t, \frac{x}{t})$ as in [10], whence we see that the function $v(t) = \mathcal{F}M(t)U(-t)u(t)$ satisfies the equation

$$iv_t + \frac{1}{2t^2} v_{\chi\chi} = t^{-\delta} \mathcal{N}\big(|v|^2\big) v \,, \tag{3.1}$$

where $\chi = \frac{x}{t}$. The function v is well defined by Lemma 2.1 and $v \in \mathcal{G}^{\sigma}_{\alpha(t)}$ since $\|v(t)\|_{\mathcal{G}^{\sigma}_{\alpha(t)}} = \|M(t)U(-t)u(t)\|_{\mathcal{Y}^{\sigma}_{\alpha(t)}} = \|U(-t)u(t)\|_{\mathcal{Y}^{\sigma}_{\alpha(t)}} \le \|u\|_{\mathcal{X}} \le 2\epsilon$ for all $0 \le t \le T$, where $T > 1$. In order to remove the nonlinear term from the right hand side of equation (3.1) as in paper [9] we put $w = ve^{ig}$, where the phase function g obeys the following equation $g_t = t^{-\delta}\mathcal{N}(|v|^2) + \frac{1}{2t^2}(g_\chi)^2 + \frac{1}{2t^2} g_{\chi\chi}$ for all $\chi \in \mathbb{R}, t > 1$, with the initial condition $g(1) = 0$. Then the function w satisfies the following Cauchy problem

$$\begin{cases} w_t = \dfrac{1}{t^2} w_\chi g_\chi + \dfrac{1+i}{2t^2} w g_{\chi\chi} + \dfrac{i}{2t^2} w_{\chi\chi}, & \chi \in \mathbb{R}, \quad t > 1\,, \\ w(1) = v(1) = \mathcal{F}M(1)U(-1)u(1), & \chi \in \mathbb{R}\,. \end{cases}$$

Therefore we consider the system of equations

$$\begin{cases} w_t = \dfrac{1}{t^2} w_\chi g_\chi + \dfrac{1+i}{2t^2} w g_{\chi\chi} + \dfrac{i}{2t^2} w_{\chi\chi}, & \chi \in \mathbb{R}, \quad t > 1\,, \\ g_t = t^{-\delta}\mathcal{N}\big(|w|^2\big) + \dfrac{1}{2t^2}(g_\chi)^2 + \dfrac{1}{2t^2} g_{\chi\chi}, & \chi \in \mathbb{R}, \quad t > 1\,, \\ g(1) = 0, \quad w(1) = v(1) = \mathcal{F}M(1)U(-1)u(1), & \chi \in \mathbb{R}\,. \end{cases} \tag{3.2}$$

Thus we have removed the nonlinear term with insufficient time decay from equation (3.1) but instead we now encounter in system (3.2) the derivative loss. This is the reason why we need the Gevrey function space $\mathcal{G}^{\sigma}_{\alpha}$. Note that the analytic function spaces $\mathcal{G}^{1}_{\alpha}$ were used to solve some nonlinear evolution equations with nonlinearities involving the derivatives of unknown function (see, e.g., [3], [7], [12]). First we prove the global existence in time of solutions to (3.2) under the condition that the norm of the initial data $\|v(1)\|_{\mathcal{G}^{\sigma}_{\alpha(1)}}$ is sufficiently small. As we mentioned above the value $\|v(1)\|_{\mathcal{G}^{\sigma}_{\alpha(1)}}$ is sufficiently small provided that the initial data u_0 of the Cauchy problem (1.1) are sufficiently small.

LEMMA 3.1. *Let $\delta \in (0,1)$. Suppose that the initial data $v(1) \in \mathcal{G}^{\sigma}_{\alpha(1)}$ have sufficiently small norm $\|v(1)\|_{\mathcal{G}^{\sigma}_{\alpha(1)}} = \epsilon$, where $\alpha(t) = \beta + \beta(1+t)^{-\gamma}$, $1 - \frac{\delta}{2-\delta} < \sigma < 1$, $\gamma > 0$ is sufficiently small $0 < \gamma < \frac{1}{2}(\sigma - \frac{2-2\delta}{2-\delta})$. Then there exist unique global solutions $w \in C([1,\infty), \mathcal{G}^{\sigma}_{\beta})$, $g \in C([1,\infty), L^{\infty})$, $g_\chi \in C([1,\infty), \mathcal{G}^{\sigma}_{\beta})$ of the Cauchy problem* (3.2) *satisfying the following estimates*

$$\begin{gathered} \big\|\big(1-\partial_\chi^2\big) w\big\|_{\mathcal{G}^{\sigma}_{\alpha}} < 2\epsilon, \quad t^{\delta-1}\big(\|g\|_{\infty} + \|g_\chi\|_{\mathcal{G}^{\sigma}_{\beta}}\big) < \sqrt{\epsilon}, \\ t^{\delta/2-1}\big\|\big(1-\partial_\chi^2\big) g_\chi\big\|_{\mathcal{G}^{\sigma}_{\alpha}} < 2\epsilon\,. \end{gathered} \tag{3.3}$$

PROOF. By Lemma 2.1 the solution of the Cauchy problem for the system (3.2) exists locally in some time interval $[0, T]$. If we prove estimates (3.3) on $[0, T]$, then the global existence follows by a standard continuation argument. Let us prove estimates (3.3) by contradiction. Suppose that at least one of estimates (3.3) is violated at some moment of time. By Lemma 2.1 and the continuity of the left hand sides of (3.3) we can find a maximal time $T > 1$ such that the nonstrict inequalities (3.3) are valid for all $t \in [0, T]$. To estimate the value $J^2 = \|(1-\partial_\chi^2)w\|^2_{\mathcal{G}^\sigma_\alpha} + \|(1-\partial_\chi^2)h\|^2_{\mathcal{G}^\sigma_\alpha}$ we differentiate the second equation of the system (3.2) to get with $h = t^{\delta/2-1}g_\chi$

$$(3.4)\qquad \begin{cases} w_t = \dfrac{1}{t^{1+\delta/2}} w_\chi h + \dfrac{1+i}{2t^{1+\delta/2}} w h_\chi + \dfrac{i}{2t^2} w_{\chi\chi}\,, \\ h_t = \dfrac{1}{t^{1+\delta/2}} \partial_\chi \mathcal{N}\big(|w|^2\big) + \dfrac{1}{t^{1+\delta/2}} h h_\chi - \dfrac{1-\delta/2}{t} h + \dfrac{1}{2t^2} h_{\chi\chi}\,, \\ g(1) = 0, \quad w(1) = v(1)\,. \end{cases}$$

Taking the inverse Fourier transformation of system (3.4), multiplying the resulting system by the factor $\tilde{E}(t,x) = (1+x^2)^2 e^{\alpha(t)|x|^\sigma}$ we get

$$(3.5)\qquad \begin{cases} (\tilde{E}\check{w})_t = \tilde{E}\check{G}_1 + \alpha'(t)|x|^\sigma \tilde{E}\check{w} - \dfrac{ix^2}{2t^2}\tilde{E}\check{w}\,, \\ (\tilde{E}\check{h})_t = \tilde{E}\check{G}_2 + \alpha'(t)|x|^\sigma \tilde{E}\check{h} - \dfrac{2-\delta}{2t}\tilde{E}\check{h} - \dfrac{x^2}{2t^2}\tilde{E}\check{h}\,, \\ \check{h}(1) = 0, \quad \check{w}(1) = \check{v}(1)\,, \end{cases}$$

where

$$\begin{cases} G_1 = \dfrac{1}{t^{1+\delta/2}} w_\chi h + \dfrac{1+i}{2t^{1+\delta/2}} w h_\chi \\ G_2 = \dfrac{1}{t^{1+\delta/2}} \partial_\chi \mathcal{N}(|w|^2) + \dfrac{1}{t^{1+\delta/2}} h h_\chi\,. \end{cases}$$

Multiplying equations of system (3.5) by $\tilde{E}(t,x)\overline{\check{w}(t,x)}$ and $\tilde{E}(t,x)\overline{\check{h}(t,x)}$ respectively, integrating with respect to the space variable and taking the real part of the result we obtain

$$(3.6)\qquad \begin{cases} \dfrac{d}{dt}\big\|(1-\partial_\chi^2)w\big\|^2_{\mathcal{G}^\sigma_\alpha} - 2\alpha'(t)\big\||x|^{\sigma/2}\tilde{E}\check{w}\big\|^2 = 2\mathrm{Re}\big(\tilde{E}\check{w}, \tilde{E}\check{G}_1\big), \\ \dfrac{d}{dt}\big\|(1-\partial_\chi^2)h\big\|^2_{\mathcal{G}^\sigma_\alpha} - 2\alpha'(t)\big\||x|^{\sigma/2}\tilde{E}\check{h}\big\|^2 = 2\mathrm{Re}\big(\tilde{E}\check{h}, \tilde{E}\check{G}_2\big) \\ \qquad -\dfrac{2-\delta}{t}\big\|(1-\partial_\chi^2)h\big\|^2_{\mathcal{G}^\sigma_\alpha} - \dfrac{1}{t^2}\big\|(1-\partial_\chi^2)h_\chi\big\|_{\mathcal{G}^\sigma_\alpha}. \end{cases}$$

Note that

$$\begin{cases} \check{G}_1 = \dfrac{1}{t^{1+\delta/2}} \displaystyle\int iy\check{w}(t,y)\check{h}(t,x-y)dy + \dfrac{1+i}{2t^{1+\delta/2}} \int \check{w}(t,x-y)iy\check{h}(t,y)dy\,, \\ \check{G}_2 = \dfrac{(\lambda(ix)^\delta + i\mu x)}{t^{1+\delta/2}} \displaystyle\int \check{w}(t,x-y)\check{w}(t,y)dy + \dfrac{1}{t^{1+\delta/2}} \int \check{h}(t,x-y)iy\check{h}(t,y)dy. \end{cases}$$

We write the following representation for nonlinearity $\check{G}_1$ which will allow us to make integration by parts in the first summand

$$
\begin{aligned}
&t^{1+\delta/2}\mathrm{Re}\big(\tilde{E}\check{w}, \tilde{E}\check{G}_1\big)\\
&\quad = \mathrm{Re}\int \tilde{E}(t,x)\overline{\check{w}(t,x)}\int \check{h}(t,x-y)\check{w}(t,y)iye^{\mathcal{K}(x,y)}\tilde{E}(t,y)dydx\\
&\quad + \mathrm{Re}\int \tilde{E}(t,x)\overline{\check{w}(t,x)}\int \check{h}(t,x-y)\check{w}(t,y)iy\big(\tilde{E}(t,x)\\
&\quad - e^{\mathcal{K}(x,y)}\tilde{E}(t,y)\big)dydx\\
&\quad + \frac{1}{2}\mathrm{Re}(1+i)\int \tilde{E}(t,x)\overline{\check{w}(t,x)}\tilde{E}(t,x)\int \check{w}(t,x-y)iy\check{h}(t,y)dydx\,,
\end{aligned}
\tag{3.7}
$$

where $\mathcal{K}(x,y) = \alpha|x-y|^{\sigma}\varphi(\frac{|x-y|}{|x|+|y|})$, here $\varphi(z) = z^{1-\sigma}$ for $0 < z < 1$ and $\varphi(z) = 1$ for $z > 1$. We see from this definition that the function $\mathcal{K}(x,y)$ is symmetric with respect to x and y. Therefore if we rename the variables of integration $x = y'$ and $y = x'$ in the first summand of representation (3.7) then by the property $\overline{\check{h}(t,x)} = \check{h}(t,-x)$ of the Fourier transformation of the real valued function $h(t,\chi)$ we get the analogy to the integration by parts

$$
\begin{aligned}
&\mathrm{Re}\int \tilde{E}(t,x)\overline{\check{w}(t,x)}\int e^{\mathcal{K}(x,y)}\check{h}(t,x-y)\tilde{E}(t,y)iy\check{w}(t,y)dydx\\
&\quad = \mathrm{Re}\int \tilde{E}(t,x)\overline{ix\check{w}(t,x)}\int e^{\mathcal{K}(x,y)}\check{h}(t,x-y)\tilde{E}(t,y)\check{w}(t,y)dydx\\
&\quad = -\frac{1}{2}\mathrm{Re}\int \tilde{E}(t,x)\overline{\check{w}(t,x)}\int e^{\mathcal{K}(x,y)}i(x-y)\check{h}(t,x-y)\tilde{E}(t,y)\check{w}(t,y)dydx\,.
\end{aligned}
$$

To estimate the second summand in (3.7) we use the inequality

$$
\begin{aligned}
&|y|\big|\tilde{E}(t,x) - \tilde{E}(t,y)e^{\mathcal{K}(x,y)}\big| = |y|\Big|\Big(\big(1+x^2\big)^2 - \big(1+y^2\big)^2\Big)e^{\alpha|x|^{\sigma}}\\
&\quad + \big(1+y^2\big)^2\Big(e^{\alpha|x|^{\sigma}} - e^{\alpha|y|^{\sigma}+\mathcal{K}(x,y)}\Big)\Big|\\
&\quad \le C|y|\big((x-y)^4 + y^3|x-y|\big)e^{\alpha|x|^{\sigma}}\\
&\quad + C|y|\big(1+y^2\big)^2\big|\alpha|x|^{\sigma} - \alpha|y|^{\sigma} - \mathcal{K}(x,y)\big|e^{\max(\alpha|x|^{\sigma},\alpha|y|^{\sigma}+\mathcal{K}(x,y))}\\
&\quad \le C\tilde{E}(t,x-y)\tilde{E}(t,y)\left(\frac{1}{1+|y|^3} + \frac{1+|y|^{\sigma}}{1+|x-y|^3}\right),
\end{aligned}
\tag{3.8}
$$

where we have used the inequality $|x|^{\sigma} \le |y|^{\sigma} + |x-y|^{\sigma}$ and the following estimate

$$
\phi \equiv |y|\left||x|^{\sigma} - |y|^{\sigma} - |x-y|^{\sigma}\varphi\left(\frac{|x-y|}{|x|+|y|}\right)\right| \le C|x-y||y|^{\sigma}
$$

for all $x, y \in \mathbb{R}$. Indeed for the case $|y| \le 2|x-y|$ by virtue of the inequalities $|x| = |(x-y)+y| \le 3|x-y|$ and $0 \le \varphi \le 1$ we easily obtain $\phi \le |y|(|x|^\sigma + |y|^\sigma + |x-y|^\sigma) \le 6|y||x-y|^\sigma \le 6|x-y||y|^\sigma$. And if $|y| > 2|x-y|$ then denoting $z = \frac{x-y}{y}$ so that $|z| < \frac{1}{2}$x we get the inequality

$$\phi = |y|^{1+\sigma}\left|(1+z)^\sigma - 1 - \frac{|z|}{(2+z)^{1-\sigma}}\right| \le |y|^\sigma\left(\sigma\left|\int_0^z (1+\xi)^{\sigma-1}d\xi\right| + |z|\right)$$
$$\le (1+2^{1-\sigma}\sigma)|y|^{1+\sigma}|z| \le C|x-y||y|^\sigma .$$

Now by virtue of the Hölder inequality we obtain from (3.7) and (3.8)

$$\begin{aligned}\left|\mathrm{Re}(\tilde{E}\check{w}, \tilde{E}\check{G}_1)\right| &\le \frac{C}{t^{1+\delta/2}}\Big(\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\|^2\|\tilde{E}\check{h}\| + \|\tilde{E}\check{w}\|^2\|\tilde{E}\check{h}\| \\ &\quad + \|\tilde{E}\check{w}\|\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\|\left\||x|^{1-\sigma/2}\tilde{E}\check{h}\right\|\Big) \\ &\le \frac{C}{t^{1+\delta/2}}\Big(\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\|^2\|\tilde{E}\check{h}\| \\ &\quad + \|\tilde{E}\check{w}\|\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\|\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\|^{\frac{\sigma}{2-\sigma}}\left\||x|\tilde{E}\check{h}\right\|^{1-\frac{\sigma}{2-\sigma}}\Big).\end{aligned}$$

Then by Young's inequality $abc \le \frac{a^p}{p} + \frac{b^q}{q} + \frac{c^r}{r}$ with $a = t^{-\frac{1+\gamma}{2}}\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\|$, $b = t^{-\frac{(1+\gamma)\sigma}{2(2-\sigma)}}\left\||x|^{\sigma/2}\tilde{E}\check{h}\right\|^{\frac{\sigma}{2-\sigma}}$ and $c = t^{-1-\delta/2+\frac{1+\gamma}{2-\sigma}}\left\||x|\tilde{E}\check{h}\right\|^{\frac{2-2\sigma}{2-\sigma}}$, where $p = 2$, $q = \frac{4-2\sigma}{\sigma}$ and $r = \frac{2-\sigma}{1-\sigma}$ so that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ we obtain

$$\begin{aligned}\left|\mathrm{Re}(\tilde{E}\check{w}, \tilde{E}\check{G}_1)\right| &\le \frac{C\epsilon^3}{t^{1+\gamma}} + \frac{C\epsilon}{t^{1+\gamma}}\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\|^2 + \frac{C\epsilon}{t^{1+\gamma}}\left\||x|^{\sigma/2}\tilde{E}\check{h}\right\|^2 \\ &\quad + \frac{C\epsilon}{t^2}\left\||x|\tilde{E}\check{h}\right\|^2,\end{aligned} \tag{3.9}$$

since $\frac{1+\gamma-\delta}{1-\delta/2} \le \sigma < 1$. Analogously to (3.9) we get the estimate

$$\begin{aligned}\left|\mathrm{Re}(\tilde{E}\check{h}, \tilde{E}\check{G}_2)\right| &\le \frac{C}{t^{1+\delta/2}}\Big(\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\|\|\tilde{E}\check{w}\|\left(|\lambda|\left\||x|^{\delta-\sigma/2}\tilde{E}\check{h}\right\|\right. \\ &\quad \left. + |\mu|\left\||x|^{1-\sigma/2}\tilde{E}\check{h}\right\|\right) + \|\tilde{E}\check{h}\|\left\||x|^{\sigma/2}\tilde{E}\check{h}\right\|\left\||x|^{1-\sigma/2}\tilde{E}\check{h}\right\|\Big) \\ &\le \frac{C\epsilon}{t^{1+\delta/2}}\left(\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\| + \left\||x|^{\sigma/2}\tilde{E}\check{h}\right\|\right)\left(\left\||x|^{1-\sigma/2}\tilde{E}\check{h}\right\| + \left\|\tilde{E}\check{h}\right\|\right) \\ &\le \frac{C\epsilon^3}{t^{1+\gamma}} + \frac{C\epsilon}{t^{1+\gamma}}\left\||x|^{\sigma/2}\tilde{E}\check{w}\right\|^2 + \frac{C\epsilon}{t^{1+\gamma}}\left\||x|^{\sigma/2}\tilde{E}\check{h}\right\|^2 + \frac{C\epsilon}{t^2}\left\||x|\tilde{E}\check{h}\right\|^2.\end{aligned}$$

Thus for the value J^2 we get from the system (3.6)

$$\frac{d}{dt}J^2 \le \left(\frac{C\epsilon}{t^{1+\gamma}} - 2|\alpha'(t)|\right)\left(\left\|\left(1-\partial_\chi^2\right)(-\partial_\chi^2)^{\sigma/4}w\right\|_{\mathcal{G}_\alpha^\sigma}^2 + \left\|\left(1-\partial_\chi^2\right)(-\partial_\chi^2)^{\sigma/4}h\right\|_{\mathcal{G}_\alpha^\sigma}^2\right)$$
$$+\left(\frac{C\epsilon}{t^2} - \frac{1}{2t^2}\right)\left\|\left(1-\partial_\chi^2\right)\partial_\chi h\right\|_{\mathcal{G}_\alpha^\sigma}^2 + \frac{C\epsilon^3}{t^{1+\gamma}} \le \frac{C\epsilon^3}{t^{1+\gamma}},$$

whence integrating with respect to t, we obtain

$$J^2(t) \le J^2(1) + C\epsilon^3 < 5\epsilon^2. \tag{3.10}$$

And for the L^∞ norm of g we get directly from equation (3.2)

$$\|g\|_\infty = \left\|\int_1^t g_t dt\right\|_\infty \le \int_1^t t^{-\delta}\left\|\mathcal{N}\left(|w|^2\right)\right\|_\infty dt + \int_1^t \left\|(g_\chi)^2\right\|_\infty \frac{dt}{2t^2}$$
$$+\int_1^t \|g_{\chi\chi}\|_\infty \frac{dt}{2t^2} \le Ct^{1-\delta}\sup_{t\ge 1}(J+J^2) \le C\epsilon t^{1-\delta}.$$

Similarly from equation (3.2) we estimate the norm $\|g_\chi\|_{\mathcal{G}_\beta^\sigma}$

$$\|g_\chi\|_{\mathcal{G}_\beta^\sigma} \le \int_1^t t^{-\delta}\|\partial_\chi\mathcal{N}(|w|^2)\|_{\mathcal{G}_\beta^\sigma}dt + \int_1^t \|g_\chi g_{\chi\chi}\|_{\mathcal{G}_\beta^\sigma}\frac{dt}{t^2}$$
$$\le Ct^{1-\delta}\sup_{t\ge 1} J^2 \le C\epsilon^2 t^{1-\delta}.$$

Hence

$$t^{\delta-1}\left(\|g\|_\infty + \|g_\chi\|_{\mathcal{G}_\beta^\sigma}\right) < \sqrt{\epsilon}. \tag{3.11}$$

Estimates (3.10) and (3.11) give us the desired contradiction. Thus estimates (3.3) are valid for all $t \ge 1$. This completes the proof of Lemma 3.1. □

We are now in a position to prove Theorems 1.1 - 1.2.

Proof of Theorem 1.1. By virtue of the local existence result of Lemma 2.1 along with the a-priori estimates of solutions obtained in Lemma 3.1, and the standard continuation argument we easily get the existence of global solutions of the Cauchy problem (1.1). Now in view of the definitions of the functions w and g we have

$$\begin{aligned} u(t) &= M(t)D(t)w(t)\exp(-ig) \\ &= \frac{1}{\sqrt{it}}M(t)w\left(t,\frac{x}{t}\right)\exp\left(-ig\left(t,\frac{x}{t}\right)\right). \end{aligned} \tag{3.12}$$

Whence we easily get the estimate

$$
\begin{aligned}
\|u(t)\|_p &\le Ct^{-1/2}\left\|w\left(t,\frac{\cdot}{t}\right)\right\|_p \le Ct^{-1/2}\left(\int\left|w\left(t,\frac{x}{t}\right)\right|^p dx\right)^{1/p} \\
&= Ct^{1/p-1/2}\left(\int|w(t,y)|^p dy\right)^{1/p} = Ct^{1/p-1/2}\|w\|_p \le C\epsilon t^{1/p-1/2}
\end{aligned}
\tag{3.13}
$$

for all $2 \le p \le \infty$. Inequality (3.13) yields the first part of Theorem 1.1.

Furthermore via estimates of Lemma 3.1 we have

$$
\begin{aligned}
\|w(t)-w(s)\|_{\mathcal{G}^\sigma_\beta} &\le \int_s^t \|w_\tau(\tau)\|_{\mathcal{G}^\sigma_\beta} d\tau \le C\int_s^t \big(\|g_\chi w_\chi\|_{\mathcal{G}^\sigma_\beta} \\
&+ \|w_{\chi\chi}\|_{\mathcal{G}^\sigma_\beta} + \|wg_{\chi\chi}\|_{\mathcal{G}^\sigma_\beta}\big)\frac{d\tau}{\tau^2} \le C\epsilon\int_s^t \frac{d\tau}{\tau^{1+\delta}} \le C\epsilon s^{-\delta}
\end{aligned}
\tag{3.14}
$$

for all $t > s > 1$. Therefore there exists a unique limit $W_+ \in \mathcal{G}^\sigma_\beta$ such that $\lim_{t\to\infty} w(t) = W_+$ in $\mathcal{G}^\sigma_\beta$ and thus we get

$$
u(t,x) = \frac{1}{\sqrt{it}}M(t)w\left(t,\frac{x}{t}\right)e^{-ig\left(t,\frac{x}{t}\right)} = \frac{1}{\sqrt{it}}M(t)W_+\left(\frac{x}{t}\right)e^{-ig\left(t,\frac{x}{t}\right)} + O\left(\epsilon t^{-\frac{1}{2}-\delta}\right)
$$

uniformly with respect to $x \in \mathbb{R}$ since for all $2 \le p \le \infty$ we have

$$
\begin{aligned}
&\left\|u(t) - \frac{1}{\sqrt{it}}M(t)W_+\left(\frac{\cdot}{t}\right)e^{-ig\left(t,\frac{\cdot}{t}\right)}\right\|_p \le Ct^{-1/2}\left\|w\left(t,\frac{\cdot}{t}\right) - W_+\left(\frac{\cdot}{t}\right)\right\|_p \\
&\le Ct^{1/p-1/2}\|w(t)-W_+\|_p \le Ct^{1/p-1/2}\|w(t)-W_+\|_{1/2-1/p,0} \\
&\le C\epsilon t^{1/p-1/2-\delta}.
\end{aligned}
$$

For the phase g we obtain

$$
\begin{aligned}
g(t) &= \int_1^t \mathcal{N}(|w|^2)\frac{d\tau}{\tau^\delta} + \int_1^t (g_\chi)^2\frac{d\tau}{2\tau^2} + \int_1^t g_{\chi\chi}\frac{d\tau}{2\tau^2} \\
&= \int_1^t \mathcal{N}(|w|^2)\frac{d\tau}{\tau^\delta} + O(t^{1-2\delta})
\end{aligned}
$$

uniformly with respect to $x \in \mathbb{R}$. Then we write the identity

$$
\begin{aligned}
\int_1^t \mathcal{N}(|w|^2)\frac{d\tau}{\tau^\delta} &= \mathcal{N}(|W_+|^2)\frac{(t^{1-\delta}-1)}{1-\delta} + \int_1^t \left(\mathcal{N}(|w(\tau)|^2) - \mathcal{N}(|w(t)|^2)\right)\frac{d\tau}{\tau^\delta} \\
&+ \left(\mathcal{N}(|w|^2) - \mathcal{N}(|W_+|^2)\right)\frac{(t^{1-\delta}-1)}{1-\delta}.
\end{aligned}
$$

Since $\|\mathcal{N}(|w(t)|^2) - \mathcal{N}(|w(\tau)|^2)\|_\infty \le C\epsilon\|w(t)-w(\tau)\|_{1,0} \le C\epsilon^2\tau^{-\delta}$ we get $\int_1^t \mathcal{N}(|w|^2)\frac{d\tau}{\tau^\delta} = \frac{t^{1-\delta}}{1-\delta}\mathcal{N}(|W_+|^2) + O(1+t^{1-2\delta})$. From these estimates the second result of Theorem 1.1 follows with $\hat{u}_+ = \frac{W_+}{\sqrt{i}}$. □

PROOF OF THEOREM 1.2. Denote

$$\Phi(t) = \int_1^t \mathcal{N}(|w(\tau)|^2)\frac{d\tau}{\tau^\delta} - \mathcal{N}(|w(t)|^2)\frac{t^{1-\delta}-1}{1-\delta} + \int_1^t (g_\chi(\tau))^2\frac{d\tau}{2\tau^2} + \int_1^t g_{\chi\chi}(\tau)\frac{d\tau}{2\tau^2}.$$

Then we have

$$\begin{aligned}\Phi(t) - \Phi(s) = &\int_s^t (\mathcal{N}(|w(\tau)|^2) - \mathcal{N}(|w(t)|^2))\frac{d\tau}{\tau^\delta} \\ &- (\mathcal{N}(|w(t)|^2) - \mathcal{N}(|w(s)|^2))\frac{s^{1-\delta}-1}{1-\delta} \\ &+ \int_s^t (g_\chi(\tau))^2\frac{d\tau}{2\tau^2} + \int_1^t g_{\chi\chi}(\tau)\frac{d\tau}{2\tau^2},\end{aligned} \tag{3.15}$$

where $1 < s < t$. We apply Lemma 3.1 and (3.14) to (3.15) to get $\|\Phi(t) - \Phi(s)\|_{\mathcal{G}^\sigma_\beta} \le C\epsilon s^{1-2\delta}$ for $1 < s < t$. This implies that there exists a unique limit $\Phi_+ = \lim_{t\to\infty}\Phi(t) \in \mathcal{G}^\sigma_\beta$ such that

$$\|\Phi(t) - \Phi_+\|_{\mathcal{G}^\sigma_\beta} \le C\epsilon t^{1-2\delta} \tag{3.16}$$

since we now consider the case $\frac{1}{2} < \delta < 1$.

Furthermore $\Phi(t) = g(t) - \frac{t^{1-\delta}-1}{1-\delta}\mathcal{N}(|w(t)|^2)$ so we have by virtue of (3.14) and (3.16)

$$\left\|g(t) - \frac{t^{1-\delta}-1}{1-\delta}\mathcal{N}(|W_+|^2) - \Phi_+\right\|_\infty \le C\epsilon t^{1-2\delta}. \tag{3.17}$$

We now put $\hat{u}_+ = \frac{1}{\sqrt{i}}W_+\exp(-i\Phi_+ + \frac{i}{1-\delta}\mathcal{N}(|W_+|^2)) \in \mathcal{G}^\sigma_\beta$. Therefore we obtain the asymptotics (1.3) for $t\to\infty$ uniformly with respect to $x\in\mathbb{R}$. Via (3.17) and Lemma 3.1 we have

$$\begin{aligned}&\left\|\mathcal{F}MU(-t)u(t) - \hat{u}_+\exp\left(-i\frac{t^{1-\delta}}{1-\delta}\mathcal{N}(|\hat{u}_+|^2)\right)\right\| \\ &= \left\|w(t)\exp(-ig(t)) - W_+\exp\left(-i\frac{t^{1-\delta}-1}{1-\delta}\mathcal{N}(|\hat{W}_+|^2) - i\Phi_+\right)\right\| \\ &\le \|w(t) - W_+\| + \|W_+\|\left\|g(t) - \mathcal{N}(|W_+|^2)\frac{t^{1-\delta}-1}{1-\delta} - \Phi_+\right\|_\infty \le C\epsilon t^{1-2\delta},\end{aligned}$$

whence we get the estimate (1.4) in the following way

$$\begin{aligned}
&\left\| u(t) - \exp\left(-i \frac{t^{1-\delta}}{1-\delta} \mathcal{N}\left(|\hat{u}_+|^2\right) \left(\frac{x}{t}\right) \right) U(t) u_+ \right\| \\
&= \left\| u(t) - M(t) D(t) \exp\left(-i \frac{t^{1-\delta}}{1-\delta} \mathcal{N}\left(|\hat{u}_+|^2\right) \right) \mathcal{F} M(t) u_+ \right\| \\
&\leq \left\| M(t) D(t) \left(\mathcal{F} M(t) U(-t) u(t) - \hat{u}_+ \exp\left(-i \frac{t^{1-\delta}}{1-\delta} \mathcal{N}\left(|\hat{u}_+|^2\right) \right) \right) \right\| \\
&+ \left\| M(t) D(t) \exp\left(-i \frac{t^{1-\delta}}{1-\delta} \mathcal{N}\left(|\hat{u}_+|^2\right) \right) \mathcal{F}(M(t) - 1) u_+ \right\| \\
&\leq C t^{1-2\delta} + C \|\mathcal{F}(M(t) - 1) u_+\| \leq C t^{1-2\delta} + C t^{-1} \|x^2 u_+\| \leq C t^{1-2\delta}
\end{aligned}$$

since $\|x^2 u_+\| = \|\partial_\chi^2 \hat{u}_+\| = \|\partial_\chi^2 (W_+ e^{-i\Phi_+ + \frac{i}{1-\delta} \mathcal{N}(|W_+|^2)})\| \leq C\epsilon$. This completes the proof of Theorem 1.2. □

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