ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

IRENA LASIECKA
ROBERTO TRIGGIANI

**Analyticity of thermo-elastic semigroups with free boundary conditions**

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 27, nº 3-4 (1998), p. 457-482

<http://www.numdam.org/item?id=ASNSP_1998_4_27_3-4_457_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVII (1998), pp. 457-482

# Analyticity of Thermo-Elastic Semigroups with Free Boundary Conditions

IRENA LASIECKA – ROBERTO TRIGGIANI

**Abstract.** We consider a thermo-elastic plate system where the elastic equation does not account for rotational forces. Of all canonical boundary conditions (B.C.), we focus on the most challenging case unsolved in the literature: that of free B.C., which are coupled. As in other simpler B.C.-cases, we show that the corresponding s.c. contraction semigroup (on a natural energy space) is *analytic*, and, hence, uniformly stable. The proof employs P.D.E. methods and estimates. Thus, this paper completes the authors' analysis [L-T.1], [L-T.2], spurred by the original important contribution [L-R.1], on analyticity of thermo-elastic semigroups with no rotational forces: under all canonical B.C., they are analytic, hence uniformly stable.

**Mathematics Subject Classification (1991):** 47F, 35K.

## 1. – Introduction. Problem statement. Main result

Dynamics. Let $\Omega$ be a two-dimensional domain with smooth boundary $\Gamma$. On $\Omega$ we consider a thermo-elastic plate problem in the displacement $w$ and in the temperature $\theta$, where the elastic equation does not account for rotational forces. Moreover, in this paper, we focus on the case of free boundary conditions (B.C.), which are *coupled* on the boundary (see literature below). The model, once stripped from lower-order terms and with inessential constants normalized to 1, is as follows [Lag.1]:

$$
\begin{cases}
w_{tt} + \Delta^2 w + \Delta\theta = 0 & \text{in } (0,T]\times\Omega = Q; & (1.1a)\\
\theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } Q; & (1.1b)\\
w(0,\cdot) = w_0;\ w_t(0,\cdot) = w_1;\ \theta(0,\cdot) = \theta_0 & \text{in } \Omega; & (1.1c)\\
\Delta w + (1-\mu)B_1 w + \theta = 0 & \text{in } (0,T]\times\Gamma \equiv \Sigma; & (1.1d)\\
\dfrac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w - w + \dfrac{\partial\theta}{\partial\nu} = 0 & \text{on } \Sigma,\ 0<\mu<1; & (1.1e)\\
\dfrac{\partial\theta}{\partial\nu} + b\theta = 0, \quad b>0 & \text{on } \Sigma; & (1.1f)
\end{cases}
$$

Pervenuto alla Redazione il 29 dicembre 1997 e in forma definitiva il 16 luglio 1998.

(1.1g) $\quad$ on $\Sigma : B_1 w = 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx},\ \nu = [\nu_1, \nu_2]$;

(1.1h) $\quad$ on $\Sigma : B_2 w = \dfrac{\partial}{\partial \tau}\left[(\nu_1^2 - \nu_2^2) w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right]$,

where $0 < \mu < 1$ is the Poisson modulus (physically, $0 < \mu < \frac{1}{2}$); $\nu$ is the unit outward normal to $\Gamma$; $\tau$ is the unit tangential vector along $\Gamma$, oriented counterclockwise. Thus $\frac{\partial}{\partial \nu}$ and $\frac{\partial}{\partial \tau}$ are the corresponding normal and tangential derivatives.

Abstract setting. We introduce several operators: (i) First, we let $\mathcal{A}$ be the following positive, self-adjoint operator on $L_2(\Omega)$ [Lag.1-2], [L-T.5, Chapter 3, Section 13],

(1.2a) $$\mathcal{A}h = \Delta^2 h,$$

(1.2b) $$\mathcal{D}(\mathcal{A}) = \left\{ h \in H^4(\Omega) : [\Delta h + (1-\mu) B_1 h]_\Gamma = 0; \left[\frac{\partial \Delta h}{\partial \nu} + (1-\mu) B_2 h - h\right]_\Gamma = 0 \right\},$$

whereby [Lag.1, p. 68], [Lag.2], [L-T.5, Chapter 3, Appendix C, Proposition C.5],

(1.3a) $$\left\| \mathcal{A}^{\frac{1}{2}} w(t) \right\|^2_{L_2(\Omega)} = \int_\Omega |\Delta w(t)|^2 d\Omega + 2(1-\mu) \int_\Omega [w_{xy}^2(t) - w_{xx}(t) w_{yy}(t)] d\Omega + \int_\Gamma |w(t)|^2 d\Gamma$$

(1.3b) $$= \int_\Omega \{\mu |\Delta w|^2 + (1-\mu)(w_{xx}^2 + w_{yy}^2) + 2(1-\mu) w_{xy}^2\} d\Omega + \int_\Gamma w^2 d\Gamma .$$

(ii) Next, let $\mathcal{A}_N$ be the positive self-adjoint operator

(1.4) $$\mathcal{A}_N h = -\Delta h;\ \mathcal{D}(\mathcal{A}_N) = \left\{ h \in H^2(\Omega) : \left[\frac{\partial h}{\partial \nu} + bh\right]_\Gamma = 0 \right\} \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}); \quad \mathcal{A}^{\frac{1}{2}} \mathcal{A}_N^{-1} \in \mathcal{L}(L_2(\Omega)).$$

(iii) Next, let $G_1$ be the Green operator corresponding to the first mechanical B.C. (1.1d):

(1.5a)–(1.5c) $$h \equiv G_1 g \iff \begin{cases} \Delta^2 h = 0 & \text{in } \Omega; \\ [\Delta h + (1-\mu) B_1 h]_\Gamma = g; & \\ \left[\dfrac{\partial \Delta h}{\partial \nu} + (1-\mu) B_2 h - h\right]_\Gamma = 0. & \end{cases}$$

which is a regular elliptic problem for $0 < \mu < 1$ (the Lopatinski-Shapiro condition is satisfied for $\mu \neq 1$). Elliptic regularity [L-M.1, p. 188-189] and [G.1] gives:

$$\text{(1.6a)} \quad G_1 : \text{ continuous } L_2(\Gamma) \to H^{\frac{5}{2}}(\Omega) \subset H^{\frac{5}{2}-4\epsilon}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{5}{8}-\epsilon}), \epsilon > 0,$$

$$\text{(1.6b)} \quad \mathcal{A}^{\frac{5}{8}-\epsilon} G_1 : \text{ continuous } L_2(\Gamma) \to L_2(\Omega);$$

(iv) Finally, let $G_2$ be the Green operator corresponding to the second mechanical B.C. (1.1e):

$$h \equiv G_2 g \iff \begin{cases} \Delta^2 h = 0 \quad \text{in } \Omega; & \text{(1.7a)} \\ [\Delta h + (1-\mu) B_1 h]_\Gamma = 0; & \text{(1.7b)} \\ \left[\dfrac{\partial \Delta h}{\partial \nu} + (1-\mu) B_2 h - h\right]_\Gamma = g, & \text{(1.7c)} \end{cases}$$

which is likewise a regular elliptic problem for $0 < \mu < 1$. Elliptic regularity [L-M.1; p. 188-189] and [G.1] give

$$\begin{cases} G_2 : \text{ continuous } L_2(\Gamma) \to H^{\frac{7}{2}}(\Omega) \subset H^{\frac{7}{2}-4\epsilon}(\Omega) = \mathcal{D}(\mathcal{A}^{\frac{7}{8}-\epsilon}) & \text{(1.8a)} \\ \quad = \{h \in H^{\frac{7}{2}-4\epsilon}(\Omega) : [\Delta h + (1-\mu) B_1 h]_\Gamma = 0\} & \text{(1.8b)} \\ \mathcal{A}^{\frac{7}{8}-\epsilon} G_2 : \text{ continuous } L_2(\Gamma) \to L_2(\Omega), & \text{(1.8c)} \end{cases}$$

Accordingly, we introduce the following space (equivalent norms):

$$\text{(1.9)} \quad Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega) \equiv H^2(\Omega) \times L_2(\Omega) \times L_2(\Omega).$$

Next, using the definitions of $G_1$ and $G_2$ in (1.5) and (1.7), we may rewrite equations (1.1a), (1.1d-e) for $w$, as usual, as:

$$\begin{cases} w_{tt} + \Delta^2 \left[ w + G_1(\theta|_\Gamma) + G_2 \left( \dfrac{\partial \theta}{\partial \nu} \right) \right] + \Delta \theta = 0 \text{ in } Q; & \text{(1.10)} \\ [\Delta + (1-\mu) B_1] \left[ w + G_1(\theta|_\Gamma) + G_2 \left( \dfrac{\partial \theta}{\partial \nu} \right) \right] \equiv 0 \text{ on } \Sigma; & \text{(1.11)} \\ \left[ \dfrac{\partial \Delta}{\partial \nu} + (1-\mu) B_1 - 1 \right] \left[ w + G_1(\theta|_\Gamma) + G_2 \left( \dfrac{\partial \theta}{\partial \nu} \right) \right] \equiv 0 \text{ on } \Sigma; & \text{(1.12)} \end{cases}$$

Hence, using the definition (1.2) of $\mathcal{A}$ on problem (1.10)-(1.12) and the definition (1.4) of $\mathcal{A}_N$ on the $\theta$-component of equation (1.10), we may rewrite problem (1.10)-(1.12) in the following abstract form

$$\text{(1.13)} \quad \begin{aligned} & w_{tt} + \mathcal{A} \left[ w + G_1(\theta|_\Gamma) + G_2 \left( \frac{\partial \theta}{\partial \nu} \right) \right] - \mathcal{A}_N \theta = 0, \\ & \left[ w + G_1(\theta|_\Gamma) + G_2 \left( \frac{\partial \theta}{\partial \nu} \right) \right] \in \mathcal{D}(\mathcal{A}). \end{aligned}$$

Finally, returning to equation (1.1b), (1.1f) for $\theta$, we rewrite problem (1.1) in abstract form via (1.13) as

$$\begin{cases} w_{tt} + \mathcal{A}w + \mathcal{A}G_1(\theta|_\Gamma) + \mathcal{A}G_2\left(\dfrac{\partial\theta}{\partial\nu}\right) - \mathcal{A}_N\theta = 0 & \text{in } [\mathcal{D}(\mathcal{A})]', \quad (1.14) \\ \theta_t + \mathcal{A}_N\theta - \Delta w_t = 0, & \quad (1.15) \end{cases}$$

after the usual extension of $\mathcal{A}$ in (1.2) to $\mathcal{A}: \ L_2(\Omega) \to [\mathcal{D}(\mathcal{A}^*)]' = [\mathcal{D}(\mathcal{A})]'$, by isomorphism, where the duality is with respect to $L_2(\Omega)$, as a pivot space. Setting $y = [w, w_t, \theta]$, we then rewrite the second-order system in (1.14), (1.15) as

$$\dot{y} = Ay, \ A = \begin{bmatrix} 0 & I & \\ -\mathcal{A} & 0 & -\mathcal{A}G_1(\cdot|_\Gamma) - \mathcal{A}G_2\dfrac{\partial\cdot}{\partial\nu} + \mathcal{A}_N \\ 0 & \Delta & -\mathcal{A}_N \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y, \tag{1.16}$$

to be interpreted in the sense that

$$A\begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} = \begin{bmatrix} w_2 \\ -\mathcal{A}\left[w_1 + G_1(\theta|_\Gamma) + G_2\dfrac{\partial\theta}{\partial\nu}\right] + \mathcal{A}_N\theta \\ \Delta w_2 - \mathcal{A}_N\theta \end{bmatrix}; \ \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \in \mathcal{D}(A), \tag{1.17}$$

where, recalling $Y$ in (1.9) and $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$, we define in connection with (1.17)

$$\begin{aligned} \mathcal{D}(A) = \Big\{ & w_1 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}); \ w_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}); \\ & \theta \in \mathcal{D}(\mathcal{A}_N): \left[w_1 + G_1(\theta|_\Gamma) + G_2\frac{\partial\theta}{\partial\nu}\right] \in \mathcal{D}(\mathcal{A}) \Big\}; \end{aligned} \tag{1.18}$$

The following lemma is readily proved by Green's second theorem (see [L-T.5, Chapter 3, Section 13] for details), where $(G_i u, y)_{L_2(\Omega)} = (u, G_i^* y)_{L_2(\Gamma)}$, $\forall u \in L_2(\Gamma), y \in L_2(\Omega)$.

LEMMA 1.1. *With reference to* (1.2), (1.5), *and* (1.7), *we have*

$$\begin{aligned} G_1^*\mathcal{A}f &= \frac{\partial f}{\partial\nu}, \ f \in \mathcal{D}(\mathcal{A}^{\frac{3}{8}+\epsilon}) = H^{\frac{3}{2}+4\epsilon}(\Omega); \\ G_2^*\mathcal{A}f &= -f|_\Gamma, \ f \in \mathcal{D}(\mathcal{A}^{\frac{1}{8}+\epsilon}) = H^{\frac{1}{2}+4\epsilon}(\Omega). \end{aligned} \tag{1.19}$$

SEMIGROUP GENERATION. The following result can be proved by standard methods: part (i) via the Lumer-Phillips Theorem [P.1]; part (ii) by direct computation; see [L-T.5, Chapter 3, Section 13] for details.

PROPOSITION 1.2. (i) *The operator* $A$ *in* (1.17), (1.18) *is densely defined, maximal dissipative, and thus generates a s.c. contraction semigroup:* $[w_1, w_2, \theta_0] \in Y \to e^{At}[w_1, w_2, \theta_0] = [w(t), w_t(t), \theta(t)]$ *on* $Y$.

(ii) *The operator* $A$ *has compact resolvent on* $Y$, *and there is no spectrum* (*i.e., no point spectrum*) *of* $A$ *on the closed half-plane* $\{\lambda\colon Re\ \lambda \geq 0\}$. □

ANALYTICITY OF $e^{At}$. The goal of this paper is to prove the following

THEOREM 1.3. *The s.c. contraction semigroup* $e^{At}$ *of Proposition* 1.2 *is, moreover, analytic on* $Y$, $t > 0$. □

UNIFORM STABILITY OF $e^{At}$. By Theorem 1.3 and Proposition 1.2 (ii), we have

COROLLARY 1.4. *The s.c. contraction analytic semigroup* $e^{At}$ *is also uniformly stable in* $\mathcal{L}(Y)$: *there exist constants* $M \geq 1$ *and* $\sigma > 0$ *such that* $\|e^{At}\|_{\mathcal{L}(Y)} \leq Me^{-\sigma t}, t \geq 0$. □

REMARK 1.1. We remark explicitly that the term $-w|_\Gamma$ in the B.C. (1.1e), while innocuous for the analyticity of $e^{At}$, is however critical for its stability. In fact, it is the presence of this term $-w|_\Gamma$ that makes $\mathcal{A}$ (strictly) positive (see (1.3b)); and then, it is the strict positivity of $\mathcal{A}$ that removes the eigenvalue $\lambda = 0$ from the spectrum of $A$. □

LITERATURE. Here, for brevity, we shall concentrate only on the case — which is pertinent to analyticity — where the elastic equation is of Euler-Bernoulli type, and thus does not account for rotational forces. A broader review of the literature is given in [Lag.1], [Las.1], [L-R.1], [L-L.1], [L-T.1-4]. The first result on the analyticity of a thermo-elastic system was given in [L-R.1] for equations (1.1a-b) with clamped/Dirichlet B.C. Later, [L-L.1] and [L-T.1] (see also [L-T.5, Chapter 3, Appendices E and F), showed, by very different techniques, analyticity of abstract thermo-elastic models, which include the clamped/Dirichlet B.C. case of [L-R.1], and other B.C. as well (see the numerous examples in [L-T.1]). However, the more demanding cases of *coupled* B.C. were excluded from the models (and the proofs) of [L-L.1], [L-T.2]. A first challenging case of analyticity for *coupled* B.C. (hinged/Neumann) was settled in [L-T.2], by means of P.D.E. methods and trace estimates. The proof of [L-T.2] serves as a guide for the present paper, where the most challenging case of free coupled B.C. (1.1d-e-f) is treated: to this end, we have to overcome additional serious difficulties over [L-T.2], as the proof below testifies.

The present paper completes the cycle: thermo-elastic semigroups generated by (1.1a-b) under *all* canonical B.C. are analytic on a natural energy space.

Research partially supported by the National Science Foundation under Grant DMS-9504822 and by the Army Research Office under Grant DAAH04-96-1-0059. Presented at Workshop on "Deterministic and stochastic evolution equations", Scuola Normale Superiore, Pisa, Italy, July 1997; IFIP Conference, Detroit, July 1997; MMAR'97, Miedzyzdroje, Poland, August 1997.

## 2. – Proof of Theorem 1.3

### 2.1. – General strategy and preliminaries

GENERAL STRATEGY. With reference to the space $Y$ in (1.9), let $f_0 \in Y$ be arbitrary

$$\text{(2.1.1)} \qquad \begin{cases} f_0 = [u_0, v_0, \theta_0] \in Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega), \\ \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \text{ (equivalent norms)}. \end{cases}$$

With reference to the operator $A$ in (1.17), let $\omega$ be real, $\omega \in \mathbb{R}$, and define

$$\text{(2.1.2)} \quad y(\omega) = [u(\omega), v(\omega), \theta(\omega)] = [i\omega I - A]^{-1} f_0 = R(i\omega, A) f_0 \in \mathcal{D}(A),$$

where the resolvent of $A$ is well-defined on the imaginary axis, see Proposition 1.2(ii).

Our *goal* is to show that the following uniform estimate holds true: there exists a constant $C > 0$ such that for all $\omega \in \mathbb{R}$, with $|\omega| > \omega_0 > 0$ for some suitable $\omega_0$,

$$\text{(2.1.3)} \qquad \left\| \begin{bmatrix} u(\omega) \\ v(\omega) \\ \theta(\omega) \end{bmatrix} \right\|_Y = \|y(\omega)\|_Y = \|R(i\omega, A) f_0\|_Y \leq \frac{C}{|\omega|} \|f_0\|_Y .$$

Once estimate (2.1.3) has been established for the generator $A$ of the s.c. contraction semigroup $e^{At}$ asserted by Proposition 1.2(i), we can invoke a known result [L-T.5, Chapter 3, Appendix E, Theorem E.3] and obtain that the s.c. semigroup $e^{At}$ is, in fact, analytic on $Y$, $t > 0$. Thus, in order to prove (2.1.3), we then seek to establish the following three simultaneous estimates for the components of $y(\omega)$ in (2.1.2): there exists a suitable $\omega_0 > 0$ such that, for all $\epsilon > 0$ there exists a constant $C_\epsilon > 0$, such that for all $\omega \in \mathbb{R}$, with $|\omega| > \omega_0 > 0$, the vector $y(\omega) = [u(\omega), v(\omega), \theta(\omega)]$ in (2.1.2) satisfies

$$\begin{cases} \text{(2.1.4)} \quad \|u(\omega)\|^2_{H^2(\Omega)} \leq \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|^2_Y ; \\ \text{(2.1.5)} \quad \|v(\omega)\|^2_{L_2(\Omega)} \leq \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|^2_Y ; \\ \text{(2.1.6)} \quad \|\theta(\omega)\|^2_{L_2(\Omega)} \leq \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|^2_Y . \end{cases}$$

Hereafter, we drop noting the explicit dependence on $\omega$ from $y(\omega) = [u(\omega), v(\omega), \theta(\omega)]$. Estimates (2.1.4)-(2.1.6) are proved below, in Proposition 2.2.6, equation (2.2.6) for $\theta$, and Corollary 4.4, equation (4.28) for $u$ and $v$.

PRELIMINARIES. By (1.17), we obtain explicitly from (2.1.2),

$$(2.1.7)\qquad (i\omega - A)\begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} i\omega u - v \\ i\omega v + \mathcal{A}\left[u + G_1(\theta|_\Gamma) + G_2 \dfrac{\partial\theta}{\partial\nu}\right] - \mathcal{A}_N\theta \\ i\omega\theta - \Delta v + \mathcal{A}_N\theta \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \\ \theta_0 \end{bmatrix} = f_0 \in Y,$$

or upon dividing by $\omega \neq 0$,

$$\begin{cases} \text{I: } iu - \dfrac{v}{\omega} = \dfrac{u_0}{\omega}; & (2.1.8) \\ \text{II: } iv + \dfrac{1}{\omega}\mathcal{A}\left[u + G_1(\theta|_\Gamma) + G_2 \dfrac{\partial\theta}{\partial\nu}\right] - \dfrac{1}{\omega}\mathcal{A}_N\theta = \dfrac{v_0}{\omega}; & (2.1.9) \\ \text{III: } i\theta - \dfrac{1}{\omega}\Delta v + \dfrac{1}{\omega}\mathcal{A}_N\theta = \dfrac{\theta_0}{\omega}, & (2.1.10) \end{cases}$$

where, recalling (1.18), we have *a-fortiori* the following regularity properties:

$$(2.1.11)\qquad y = [u, v, \theta] \in \mathcal{D}(A) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}_N), \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega).$$

ORIENTATION. The basic "driving" term in the present proof is the thermal estimate (2.2.2) below for $\theta$, which follows at once from the basic *a-priori* dissipativity condition (2.2.1). To achieve the desired estimates (2.1.4) through (2.1.6), we shall employ the driving estimate (2.2.2) repeatedly, along with *a-priori* bounds in the right norms, to dominate each norm quantity $\|q\|$ of interest as follows

$$(2.1.12)\qquad \|q\| \le [a + b][\epsilon a + k_\epsilon b] \le 2\epsilon a^2 + C_\epsilon b^2, \quad a, b \ge 0,$$

to be specialized with $a = \|y\|_Y$ and $b = \|\frac{f_0}{\omega}\|_Y$. We shall divide the proof of the present free B.C. case into three parts. Part I, dealt with in Section 2, follows closely the proof given in [L-T.2] of the case of coupled hinged mechanical B.C. and Neumann thermal B.C., up to the breaking point of that proof, which will be duly noted: see Remark 3.1 below. It collects the "driving" estimate (2.2.2), as well as the *a-priori* bounds on $u, v, \theta$. With Part II, expounded in Section 3, we begin a radical departure from the proof of [L-T.2], to compensate for the lack, at this stage, of the "good" $\epsilon$-estimate for $\|v\|_{H^1(\Omega)}$, such as in [L-T.2, equation (2.5.5)]. More precisely, Part II collects all those new results which can be obtained, without making explicit use of the *structure* of the boundary operators $B_1$ and $B_2$ in (1.1g-h). This includes the required estimate (2.1.6) for $\theta$ (see (2.2.19) of Proposition 2.2.6 below), as well as the "right", desired $\epsilon$-estimate for the *difference* $[\|v\|^2_{L_2(\Omega)} - \|u\|^2_{H^2(\Omega)}]$ of the first two variables, see (3.58) of Proposition 3.6 below. Finally, we complete the proof in Part III (Section 4), by showing simultaneously the required estimates (2.1.5) for $v$ and (2.1.6) for $u$. To this end, we shall exploit the special *structure* of the boundary operator $B_1$: see equation (4.6) in terms of tangential and normal derivatives, rather than in terms of the original $x$ and $y$ variables.

## 2.2. – The "driving" estimate for $\theta$, and *a-priori* bounds for $u, v, \theta$

In this section we collect results on equation I = (2.1.8), II = (2.1.9), III = (2.1.10), which can be proved exactly as in the case of hinged/Neumann B.C. in [L-T.2]. Accordingly, they will only be listed, with a proof after [L-T.2] being relegated to the Appendix for completeness.

Part (i) of the following lemma is obtained by integration by parts, and is in fact behind the property of dissipativity of $A$ noted in Proposition 1.2(i). See [L-T.5, Chapter 3, Section 13] for details. Throughout, equivalence in norm is denoted by $\doteq$.

LEMMA 2.2.1 (Preliminary *a-priori* bounds for $\theta$). *Recalling* (2.1.1), (2.1.2), *we have*

(i)

$$(2.2.1)\qquad (\mathcal{A}_N\theta, \theta)_{L_2(\Omega)} = Re\left([i\omega I - A]\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}, \begin{bmatrix} u \\ v \\ \theta \end{bmatrix}\right)_Y = Re\,(f_0, y)_Y\,;$$

(ii) *for any* $\epsilon > 0$ *and* $\omega \in \mathbb{R}$, $\omega \neq 0$:

$$(2.2.2)\qquad \frac{1}{|\omega|}\,\|\theta\|^2_{H^1(\Omega)} \doteq \frac{1}{|\omega|}\left\|\mathcal{A}_N^{\frac{1}{2}}\theta\right\|^2_{L_2(\Omega)} \leq \frac{\epsilon}{2}\,\|y\|^2_Y + \frac{1}{2\epsilon}\left\|\frac{f_0}{\omega}\right\|^2_Y. \qquad \Box$$

LEMMA 2.2.2 (*A-priori* bounds for $v$). *For* $f_0$ *and* $y$ *as in* (2.1.1), (2.1.2), *we have*

(i)

$$(2.2.3)\qquad \frac{1}{|\omega|}\,\|v\|_{H^2(\Omega)} \leq \|u\|_{H^2(\Omega)} + \left\|\frac{u_0}{\omega}\right\|_{H^2(\Omega)} \leq \|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\,;$$

(ii)

$$(2.2.4)\qquad \frac{1}{\sqrt{|\omega|}}\,\|v\|_{H^1(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right].$$

PROOF. See proof of [L-T.2, Lemma 2.2.2], given in the Appendix. $\Box$

LEMMA 2.2.3 (Further *a-priori* bound for $\theta$). *For* $f_0$ *and* $y$ *as in* (2.1.1), (2.1.2), *we have*

$$(2.2.5)\qquad \frac{1}{|\omega|}\|\theta\|^2_{H^2(\Omega)} \doteq \frac{1}{|\omega|}\|\mathcal{A}_N\theta\|_{L_2(\Omega)} \leq 2\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right].$$

PROOF. See proof of [L-T.2, Lemma 2.2.3] given in the Appendix. $\Box$

LEMMA 2.2.4 (*A-priori* bounds for $u$). *Recalling* (2.1.1), (2.1.2), *we have for* $\omega \in \mathbb{R}$,

(i)

$$\frac{1}{|\omega|}\|u\|_{H^4(\Omega)} \le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]; \tag{2.2.6}$$

(ii)

$$\frac{1}{\sqrt{|\omega|}}\|u\|_{H^3(\Omega)} \le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{2.2.7}$$

PROOF. (i) As in the proof of [L-T.2, Lemma 2.2.4], we shall obtain (2.2.6) by elliptic regularity, except that now the elliptic problem has different B.C. Referring to (2.1.9) and to the definition of $\mathcal{A}$ in (1.2), we have that

$$\frac{1}{\omega}\mathcal{A}\left[u + G_1(\theta|_\Gamma) + G_2\left(\frac{\partial\theta}{\partial\nu}\right)\right] = -iv + \frac{1}{\omega}\mathcal{A}_N\theta + \frac{v_0}{\omega} \tag{2.2.8}$$

is equivalent, via the definitions of the Green operators $G_1$ and $G_2$, given in (1.5) and (1.7), to the following elliptic boundary value problem (i.e., the original elliptic problem (1.5), of which (2.2.8) is the abstract version):

$$\begin{cases} \Delta^2\left(\dfrac{u}{\omega}\right) = -iv + \dfrac{1}{\omega}\mathcal{A}_N\theta + \dfrac{v_0}{\omega} & \text{in } \Omega; \quad (2.2.9)\\[2mm] \Delta\left(\dfrac{u}{\omega}\right) + (1-\mu)B_1\left(\dfrac{u}{\omega}\right) = -\dfrac{1}{\omega}\theta & \text{on } \Gamma; \quad (2.2.10)\\[2mm] \dfrac{\partial\Delta}{\partial\nu}\left(\dfrac{u}{\omega}\right) + (1-\mu)B_2\left(\dfrac{u}{\omega}\right) - \left(\dfrac{u}{\omega}\right) = -\dfrac{1}{\omega}\dfrac{\partial\theta}{\partial\nu} & \text{on } \Gamma. \quad (2.2.11)\end{cases}$$

From the right-hand side of (2.2.9), we readily estimate by virtue of (2.2.5) for $\mathcal{A}_N\theta/\omega$, majorizing $v$ and $\theta_0$ by $y$ and $f_0$, via (2.1.1), (2.1.2),

$$\left\|\Delta^2\left(\frac{u}{\omega}\right)\right\|_{L_2(\Omega)} \le 3\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{2.2.12}$$

(This step is the same as in [L-T.2, equation (2.2.19)].) Moreover, from the first B.C. in (2.2.10) we estimate by trace theory on $\theta$, followed by estimate (2.2.5),

$$\left\|\Delta\left(\frac{u}{\omega}\right) + (1-\mu)B_1\left(\frac{u}{\omega}\right)\right\|_{H^{\frac{3}{2}}(\Gamma)} = \frac{1}{|\omega|}\|\theta|_\Gamma\|_{H^{\frac{3}{2}}(\Gamma)} \tag{2.2.13}$$

$$\text{(by (2.2.5))} \quad \le \frac{C}{|\omega|}\|\theta\|_{H^2(\Omega)} \le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{2.2.14}$$

Finally, we likewise estimate the second B.C. (2.2.11), via trace theory on $\theta$, and (2.2.5),

$$\left\| \frac{\partial \Delta}{\partial \nu}\left(\frac{u}{\omega}\right) + (1-\mu) B_2 \left(\frac{u}{\omega}\right) - \left(\frac{u}{\omega}\right) \right\|_{H^{\frac{1}{2}}(\Gamma)} = \frac{1}{|\omega|} \left\| \left.\frac{\partial \theta}{\partial \nu}\right|_{\Gamma} \right\|_{H^{\frac{1}{2}}(\Gamma)} \tag{2.2.15}$$

$$\text{(by (2.2.5))} \quad \le \frac{C}{|\omega|} \|\theta\|_{H^2(\Omega)} \le C \left[ \|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right]. \tag{2.2.16}$$

We can then apply elliptic regularity theory on problem (2.2.9), (2.2.10), (2.2.11), satisfying estimates (2.2.12), (2.2.14), (2.2.16), thus obtaining

$$\left\| \frac{u}{\omega} \right\|_{H^4(\Omega)} \le C \left[ \|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right], \tag{2.2.17}$$

and (2.2.6) is proved. [We note that the right-hand side estimate (2.2.12) and the boundary estimates (2.2.14), (2.2.15) produce, *independently*, the same regularity of $\frac{u}{\omega}$ in $H^4(\Omega)$ for the corresponding elliptic problem in $(\frac{u}{\omega})$.] Part (ii), equation (2.2.7) then follows from (2.2.6) by interpolation (moment inequality):

$$\|u\|_{H^3(\Omega)} \le C \|u\|^{\frac{1}{2}}_{H^4(\Omega)} \|u\|^{\frac{1}{2}}_{H^2(\Omega)} \le C|\omega|^{\frac{1}{2}} \left[ \|y\|^{\frac{1}{2}}_Y + \left\| \frac{f_0}{\omega} \right\|^{\frac{1}{2}}_Y \right] \|y\|^{\frac{1}{2}}_Y$$
$$\le C|\omega|^{\frac{1}{2}} \left[ \|y\|_Y + \frac{1}{2} \left\| \frac{f_0}{\omega} \right\|_Y + \frac{1}{2}\|y\|_Y \right],$$

and (2.2.7) is proved. □

LEMMA 2.2.5. *For $f_0$ and $y$ as in* (2.1.1), (2.1.2), *given $\epsilon > 0$ there exists $C_\epsilon > 0$, such that for all $\omega \in \mathbb{R}$, $|\omega| \ge 1$, we have*

$$\left| \frac{1}{\omega} (\Delta v, \theta)_{L_2(\Omega)} \right| \le \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y. \tag{2.2.18}$$

PROOF. Same proof of [L-T.2, Proposition 2.3.1; equation (2.3.1)], given in the Appendix. □

We can then obtain the desired estimate (2.1.6) for $\theta$.

PROPOSITION 2.2.6. *For $f_0$ and $y$ as in* (2.1.1), (2.1.2), *given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \ge 1$, we have*

$$\|\theta\|^2_{L_2(\Omega)} \le \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y. \tag{2.2.19}$$

PROOF. As in [L-T.2, Proposition 2.4.1], we return to equation III = (2.1.10), take here the $L_2(\Omega)$-inner product with $\theta$, use estimate (2.2.18) and (2.2.2) and obtain (2.2.19). □

LEMMA 2.2.7. *For $f_0$ and $y$ as in* (2.1.1), (2.1.2), *given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \ge 1$, we have*

$$\left| \frac{1}{\omega} (\mathcal{A}_N \theta, v)_{L_2(\Omega)} \right| \le \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y. \tag{2.2.20}$$

PROOF. See the proof of [L-T.2, Lemma 2.5.1] given in the Appendix. □

## 3. – Desired $\epsilon$-estimates for $\theta$, $\Delta u$, and $[\|v\|^2 - \|\mathcal{A}^{\frac{1}{2}} u\|^2]$

In the case of hinged mechanical/Neumann thermal B.C. of [L-T.2], we had $v \in \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega)$. Instead, in the present case, we only have $v \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$. The consequence is that, while in the case of hinged/Neumann B.C. where $v|_\Gamma = 0$, we could get at this stage the good $\epsilon$-estimate for $\frac{1}{|\omega|}\|v\|^2_{H^1(\Omega)}$ as in [L-T.2, equation (2.5.5)], instead, in the present development, we obtain at this stage only a *weaker* result, as follows.

LEMMA 3.1. *With reference to* (2.1.1), (2.1.2), *given* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that for all* $\omega \in \mathbb{R}$, $|\omega| \geq 1$, *we have*

$$\left|\frac{1}{\omega}(\Delta v, v)_{L_2(\Omega)}\right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3.1}$$

PROOF. Same as the one in [L-T.2, Lemma 2.5.2]. We return to equation III = (2.1.10), take here the $L_2(\Omega)$-inner product with $v$, and obtain

$$\begin{aligned}
\left|\frac{1}{\omega}(\Delta v, v)_{L_2(\Omega)}\right| &= \left|\left(\frac{\theta_0}{\omega} - i\theta - \frac{1}{\omega}\mathcal{A}_N\theta, v\right)_{L_2(\Omega)}\right| \\
&\leq \left\|\frac{\theta_0}{\omega}\right\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} + \|\theta\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} + \left|\frac{1}{\omega}(\mathcal{A}_N\theta, v)_{L_2(\Omega)}\right| \\
&\text{(by (2.2.19)) and (2.2.20))} \leq \left[\frac{\epsilon_1}{2}\|v\|^2_{L_2(\Omega)} + \frac{1}{2\epsilon_1}\left\|\frac{\theta_0}{\omega}\right\|^2_{L_2(\Omega)}\right] \\
&+ \left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right]\|y\|_Y \\
&+ \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2,
\end{aligned} \tag{3.2}$$

majorizing $v$ by $y$ twice via (2.1.1), (2.1.2). equation (3.2) proves (3.1). □

REMARK 3.1. In [L-T.2, Lemma 2.5.2], for the left-hand side of (3.1), we obtained, instead:

$$\begin{aligned}
\frac{1}{|\omega|}\|v\|^2_{H^1(\Omega)} &\doteq \frac{1}{|\omega|}\|\mathcal{A}_D^{\frac{1}{2}} v\|^2_{L_2(\Omega)} = \left|\frac{1}{\omega}(\mathcal{A}_D v, v)_{L_2(\Omega)}\right. \\
&\leq \epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y,
\end{aligned} \tag{3.3}$$

which is a stronger result than (3.1). Equation (3.3) was then used in the next step of the proof in [L-T.2, Proposition 2.6.1] after the $L_2(\Omega)$-inner product of equation II with $v$, in combination with the *a-priori* bound (2.2.7) for $u$. In the present development, where (3.1) represents a *loss* over (3.3), the variable $v$

is still not good enough. Thus a *major departure from the proof of [L-T.2] takes place here: we must carry out still with the "good" variable $\theta$* (satisfying the "driving" estimate (2.2.2)). Accordingly, in our next step, we take the $L_2(\Omega)$-inner product of equation II with $\theta$, not with $v$ as in [L-T.2]. In the present case, the proof of the required estimates (2.1.5) and (2.1.6) for $u$ and $v$ is much more complicated.

LEMMA 3.2. *With reference to* (2.1.1), (2.1.2), *given* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that for all* $\omega \in \mathbb{R}$, $|\omega| \geq 1$, *we have:*

(i)

$$\left|\frac{1}{\omega}(\Delta^2 u, \theta)_{L_2(\Omega)}\right| = \left|\frac{1}{\omega}(\Delta u, \Delta\theta)_{L_2(\Omega)} + \frac{1}{\omega}\int_\Gamma \frac{\partial \Delta u}{\partial \nu}\bar{\theta}d\Gamma - \frac{1}{\omega}\int_\Gamma \Delta u \frac{\partial \bar{\theta}}{\partial \nu}d\Gamma\right| \tag{3.4}$$

$$\leq \epsilon\|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2 . \tag{3.5}$$

(ii) *Similarly,*

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial \Delta u}{\partial \nu}\bar{\theta}d\Gamma\right| \leq \epsilon\|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2 ; \tag{3.6}$$

(iii)

$$\left|\frac{1}{\omega}\int_\Gamma \Delta u \frac{\partial \bar{\theta}}{\partial \nu}d\Gamma\right| \leq \epsilon\|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2 . \tag{3.7}$$

(iv) *Finally,*

$$\left|\frac{1}{\omega}(\Delta u, \Delta\theta)_{L_2(\Omega)}\right| \leq \epsilon\|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2 . \tag{3.8}$$

PROOF. (i) We return to equation II = (2.1.9), take here the $L_2(\Omega)$-inner product with $\theta$ and obtain, recalling (1.2) for $\mathcal{A}$; (1.5) for $G_1$; (1.7) for $G_2$, and using the estimates (2.2.19) on $\theta$, and (2.2.2) on $\mathcal{A}_N^{\frac{1}{2}}\theta$:

$$\left|\frac{1}{\omega}(\Delta^2 u, \theta)_{L_2(\Omega)}\right| \leq \left|\left(\frac{v_0}{\omega}, \theta\right)_{L_2(\Omega)}\right| + \frac{1}{|\omega|}\|\mathcal{A}_N^{\frac{1}{2}}\theta\|_{L_2(\Omega)}^2 + \left|i(v,\theta)_{L_2(\Omega)}\right| \tag{3.9}$$

$$\text{(by (2.2.19), (2.2.2))} \leq \left\|\frac{v_0}{\omega}\right\|_{L_2(\Omega)}\left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right] + \left[\frac{\epsilon_1}{2}\|y\|_Y^2 + \frac{1}{2\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] + \|v\|_{L_2(\Omega)}\left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right] \tag{3.10}$$

$$\leq 3\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2 , \tag{3.11}$$

majorizing, in the last step, $v_0$ by $f_0$ and $v$ by $y$, via (2.1.1), (2.1.2). Then, (3.11) proves estimate (3.5), while (3.4) is just an application of Green's second theorem.

(ii) We estimate by [Th.1, p. 26], [B-S.1, p. 39],

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial \Delta u}{\partial \nu}\bar{\theta}\,d\Gamma\right| \le \frac{1}{|\omega|}\left\|\frac{\partial \Delta u}{\partial \nu}\Big|_\Gamma\right\|_{L_2(\Gamma)} \|\theta|_\Gamma\|_{L_2(\Gamma)} \tag{3.12}$$

$$\le \frac{1}{|\omega|}\left[\|\Delta u\|^{\frac{1}{2}}_{H^2(\Omega)}\|\Delta u\|^{\frac{1}{2}}_{H^1(\Omega)}\right]\left[\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}\|\theta\|^{\frac{1}{2}}_{L_2(\Omega)}\right] \tag{3.13}$$

$$\le C\left(\frac{\|u\|^{\frac{1}{2}}_{H^4(\Omega)}}{|\omega|^{\frac{1}{2}}}\right)\left(\frac{\|u\|^{\frac{1}{2}}_{H^3(\Omega)}}{|\omega|^{\frac{1}{4}}}\right)\left(\frac{\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}}\right)\|\theta\|^{\frac{1}{2}}_{L_2(\Omega)}. \tag{3.14}$$

But, invoking inequalities (2.2.6), (2.2.7), we estimate

$$\frac{\|u\|^{\frac{1}{2}}_{H^4(\Omega)}}{|\omega|^{\frac{1}{2}}}\;\frac{\|u\|^{\frac{1}{2}}_{H^3(\Omega)}}{|\omega|^{\frac{1}{4}}} \le C\left[\|y\|^{\frac{1}{2}}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right]\left[\|y\|^{\frac{1}{2}}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right] \tag{3.15}$$

$$\le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3.16}$$

Next, invoking the fourth root estimate of (2.2.2) and majorizing $\theta$ by $y$ via (2.1.1), (2.1.2), we obtain

$$\frac{\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}}\|\theta\|^{\frac{1}{2}}_{L_2(\Omega)} \le \left[\left(\frac{\epsilon_1}{2}\right)^{\frac{1}{4}}\|y\|^{\frac{1}{2}}_Y + \left(\frac{1}{2\epsilon_1}\right)^{\frac{1}{4}}\left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right]\|y\|^{\frac{1}{2}}_Y \tag{3.17}$$

$$\le \epsilon_2\|y\|_Y + C_{\epsilon_2}\left\|\frac{f_0}{\omega}\right\|_Y. \tag{3.18}$$

Using (3.16) and (3.18) in (3.14), we obtain via inequality (2.1.12),

$$C\left(\frac{\|u\|^{\frac{1}{2}}_{H^4(\Omega)}}{|\omega|^{\frac{1}{2}}}\right)\left(\frac{\|u\|^{\frac{1}{2}}_{H^3(\Omega)}}{|\omega|^{\frac{1}{4}}}\right)\left(\frac{\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}}\right)\|\theta\|^{\frac{1}{2}}_{L_2(\Omega)} \tag{3.19}$$

$$\le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\epsilon_2\|y\|_Y + C_{\epsilon_2}\left\|\frac{f_0}{\omega}\right\|_Y\right]$$

$$(\text{by (2.1.12)}) \le \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3.20}$$

Then, (3.20) used in (3.14) yields (3.6), as desired.

(iii) This is similar to the proof of part (ii). We likewise estimate by [Th.1, p. 26], [B-S.1, p. 39],

$$\left|\frac{1}{\omega}\int_\Gamma \Delta u \frac{\partial\bar\theta}{\partial\nu} d\Gamma\right| \le \frac{1}{|\omega|}\|\Delta u|_\Gamma\|_{L_2(\Gamma)} \left\|\frac{\partial\theta}{\partial\nu}\Big|_\Gamma\right\|_{L_2(\Gamma)} \tag{3.21}$$

$$\le \frac{C}{|\omega|}\left[\|\Delta u\|^{\frac12}_{H^1(\Omega)}\|\Delta u\|^{\frac12}_{L_2(\Omega)}\right]\left[\|\theta\|^{\frac12}_{H^2(\Omega)}\|\theta\|^{\frac12}_{H^1(\Omega)}\right] \tag{3.22}$$

$$\le C\left(\frac{\|u\|^{\frac12}_{H^3(\Omega)}}{|\omega|^{\frac14}}\right)\left(\|u\|^{\frac12}_{H^2(\Omega)}\right)\left(\frac{\|\theta\|^{\frac12}_{H^2(\Omega)}}{\sqrt{|\omega|}}\right)\left(\frac{\|\theta\|^{\frac12}_{H^1(\Omega)}}{|\omega|^{\frac14}}\right). \tag{3.23}$$

Recalling again estimate (2.2.7) and majorizing $u$ by $y$ via (2.1.1), (2.1.2), we obtain

$$\frac{\|u\|^{\frac12}_{H^3(\Omega)}}{|\omega|^{\frac14}}\|u\|^{\frac12}_{H^2(\Omega)} \le C\left[\|y\|^{\frac12}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac12}_Y\right]\|y\|^{\frac12}_Y \tag{3.24}$$

$$\le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3.25}$$

Moreover, recalling estimate (2.2.5) and (2.2.2) on $\theta$, we obtain via inequality (2.1.12),

$$\frac{\|\theta\|^{\frac12}_{H^2(\Omega)}}{\sqrt{|\omega|}}\frac{\|\theta\|^{\frac12}_{H^1(\Omega)}}{|\omega|^{\frac14}} \le C\sqrt{2}\left[\|y\|^{\frac12}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac12}_Y\right] \times\left[\left(\frac{\epsilon_1}{2}\right)^{\frac14}\|y\|^{\frac12}_Y + \left(\frac{1}{2\epsilon_1}\right)^{\frac14}\left\|\frac{f_0}{\omega}\right\|^{\frac12}_Y\right] \tag{3.26}$$

$$\text{(by (2.1.12))} \le \epsilon_2\|y\|_Y + C_{\epsilon_2}\left\|\frac{f_0}{\omega}\right\|_Y. \tag{3.27}$$

Using both (3.25) and (3.27) in (3.23), we obtain

$$C\left(\frac{\|u\|^{\frac12}_{H^3(\Omega)}}{|\omega|^{\frac14}}\right)\left(\|u\|^{\frac12}_{H^2(\Omega)}\right)\left(\frac{\|\theta\|^{\frac12}_{H^2(\Omega)}}{\sqrt{|\omega|}}\right)\left(\frac{\|\theta\|^{\frac12}_{H^1(\Omega)}}{|\omega|^{\frac14}}\right) \le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\epsilon_2\|y\|_Y + C_{\epsilon_2}\left\|\frac{f_0}{\omega}\right\|_Y\right] \tag{3.28}$$

$$\text{(by (2.1.12))} \le \epsilon\|y\|^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y, \tag{3.29}$$

invoking once more inequality (2.1.12). Finally, (3.29) used in (3.23) yields (3.7), as desired.

(iv) equation (3.8) is an immediate consequence of estimates (3.6) and (3.7), once used in (3.5). □

The next result is a first serious step in achieving the desired estimates (2.1.5) and (2.1.6) for $u$ and $v$. Its part (ii) improves upon estimate (3.1) of Lemma 3.1.

LEMMA 3.3. *Recalling* (2.1.1), (2.1.2), *given* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

(i)

$$\int_\Omega |\Delta u|^2 d\Omega \le \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2 . \tag{3.30}$$

[*See* (1.3) *for the difference between* $\|\mathcal{A}^{\frac{1}{2}} u\|_{L_2(\Omega)}^2$ *and* $\int_\Omega |\Delta u|^2 d\Omega$.]

(ii)

$$\frac{1}{|\omega|^2} \int_\Omega |\Delta v|^2 d\Omega \le \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2 . \tag{3.31}$$

PROOF. (i) We return to equation III = (2.1.10), take the $L_2(\Omega)$-inner product with $\Delta u$ and obtain by use of (2.2.19), (3.8), after majorizing $\Delta u$ in $L_2(\Omega)$ by $y$ in $Y$ via (2.1.1), (2.1.2):

$$\left| \frac{1}{\omega} \int_\Omega \Delta v \Delta \bar{u} d\Omega \right| \tag{3.32}$$

$$= \left| (i\theta, \Delta u)_{L_2(\Omega)} - \frac{1}{\omega} (\Delta\theta, \Delta u)_{L_2(\Omega)} - \left( \frac{\theta_0}{\omega}, \Delta u \right)_{L_2(\Omega)} \right|$$

$$\le \|\theta\|_{L_2(\Omega)} \|\Delta u\|_{L_2(\Omega)} + \frac{1}{|\omega|} |(\Delta\theta, \Delta u)_{L_2(\Omega)}| + \left\| \frac{\theta_0}{\omega} \right\|_{L_2(\Omega)} \|\Delta u\|_{L_2(\Omega)} \tag{3.33}$$

$$(\text{by } (2.2.19), (3.8)) \le \left[ \epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] C \|y\|_Y + \left[ \epsilon_2 \|y\|_Y^2 + C_{\epsilon_2} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right] + C \left\| \frac{f_0}{\omega} \right\|_Y \|y\|_Y . \tag{3.34}$$

Hence, (3.34) yields

$$\left| \frac{1}{\omega} \int_\Omega \Delta v \Delta \bar{u}\, d\Omega \right| \le \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2 . \tag{3.35}$$

Next, we recall equation I = (2.1.8), apply $\Delta$ throughout, and take the $L_2(\Omega)$-inner product with $\Delta u$, to obtain the identity

$$i\int_\Omega \Delta u \Delta \bar{u} d\Omega = \frac{1}{\omega}\int_\Omega \Delta v \Delta \bar{u} d\Omega + \int_\Omega \Delta\left(\frac{u_0}{\omega}\right)\Delta \bar{u} d\Omega, \tag{3.36}$$

from which we estimate by use of (3.35) and majorizing $\Delta u$ in $L_2(\Omega)$ by $y$ in $Y$ via (2.1.1), (2.1.2):

$$\int_\Omega |\Delta u|^2 d\Omega \leq \left|\frac{1}{\omega}\int_\Omega \Delta v \Delta \bar{u} d\Omega\right| + \left\|\Delta\left(\frac{u_0}{\omega}\right)\right\|_{L_2(\Omega)} \|\Delta u\|_{L_2(\Omega)} \tag{3.37}$$

$$(\text{by } (3.35)) \leq \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] + C\left\|\frac{f_0}{\omega}\right\|_Y \|y\|_Y \tag{3.38}$$

$$\leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3.39}$$

Thus (3.39) proves (3.30), as desired.

(ii) A further use of equation I = (2.1.8) gives via (3.30), majorizing $\Delta u_0$ in $L_2(\Omega)$ by $f_0$ in $Y$ via (2.1.1), (2.1.2):

$$\int_\Omega \left|\frac{\Delta v}{\omega}\right|^2 d\Omega \leq \int_\Omega |\Delta u|^2 d\Omega + \int_\Omega \left|\frac{\Delta u_0}{\omega}\right|^2 d\Omega \tag{3.40}$$

$$(\text{by } (3.30)) \leq \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] + C\left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3.41}$$

and (3.41) proves (3.31), as desired. □

As a corollary we obtain

LEMMA 3.4. *With reference to* (2.1.1), (2.1.2), *given* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

$$\frac{1}{|\omega|}\|\mathcal{A}_N\theta\|_{L_2(\Omega)} \leq \epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y. \tag{3.42}$$

PROOF. We return to equation III = (2.1.10) and estimate by use of (3.31) and (2.2.19) and majorizing $\theta_0$ by $f_0$ via (2.1.1), (2.1.2):

$$\frac{1}{|\omega|}\|\mathcal{A}_N\theta\|_{L_2(\Omega)} \leq \frac{1}{|\omega|}\|\Delta v\|_{L_2(\Omega)} + \|i\theta\|_{L_2(\Omega)} + \left\|\frac{\theta_0}{\omega}\right\|_{L_2(\Omega)} \tag{3.43}$$

$$(\text{by } (3.31), (2.2.19)) \leq \left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right] + \left[\epsilon\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right] + \left\|\frac{f_0}{\omega}\right\|_Y \tag{3.44}$$

$$\leq \epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y, \tag{3.45}$$

and (3.45) proves (3.42), as desired. □

LEMMA 3.5. *With reference to* (2.1.1), (2.1.2), *we have*
(i)

$$\begin{aligned} &i\|v\|^2_{L_2(\Omega)} - i\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)} + \frac{1}{\omega}\left(\theta|_\Gamma, \frac{\partial v}{\partial \nu}\Big|_\Gamma\right)_{L_2(\Gamma)} - \frac{1}{\omega}\left(\frac{\partial \theta}{\partial \nu}\Big|_\Gamma, v|_\Gamma\right)_{L_2(\Gamma)} \\ &= \left(\mathcal{A}^{\frac{1}{2}}u, \frac{\mathcal{A}^{\frac{1}{2}}u_0}{\omega}\right)_{L_2(\Omega)} + \left(\frac{1}{\omega}\mathcal{A}_N\theta, v\right)_{L_2(\Omega)} + \left(\frac{v_0}{\omega}, v\right)_{L_2(\Omega)}. \end{aligned} \tag{3.46}$$

(ii) *Given* $\epsilon > 0$, *there exists* $C_\epsilon > 0$ *such that*

$$\left|\frac{1}{\omega}(\mathcal{A}_N\theta, v)_{L_2(\Omega)}\right| = \left|\frac{1}{\omega}\int_\Omega \Delta\theta\bar{v}\,d\Omega\right| \le \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3.47}$$

(iii) *Given* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial \nu}\bar{\theta}\,d\Gamma - \frac{1}{\omega}\int_\Gamma v\frac{\partial\bar{\theta}}{\partial \nu}\,d\Gamma\right| \le \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3.48}$$

PROOF. (i) We return to equation II = (2.1.9), and take here the $L_2(\Omega)$-inner product with $v$, thereby obtaining

$$\begin{aligned} &i\|v\|^2_{L_2(\Omega)} + \left(\mathcal{A}u, \frac{v}{\omega}\right)_{L_2(\Omega)} + \frac{1}{\omega}(\theta|_\Gamma, G_1^*\mathcal{A}v)_{L_2(\Gamma)} \\ &+ \frac{1}{\omega}\left(\frac{\partial\theta}{\partial\nu}\Big|_\Gamma, G_2^*\mathcal{A}v\right)_{L_2(\Gamma)} = \left(\frac{1}{\omega}\mathcal{A}_N\theta, v\right)_{L_2(\Omega)} + \left(\frac{v_0}{\omega}, v\right)_{L_2(\Omega)}. \end{aligned} \tag{3.49}$$

Next, we substitute $\frac{v}{\omega} = iu - \frac{u_0}{\omega}$ from equation I = (2.1.8) into the second term on the left-hand side of (3.49), and we recall that

$$G_1^*\mathcal{A}v = \frac{\partial v}{\partial \nu}; \quad G_2^*\mathcal{A}v = -v|_\Gamma, \quad v \in H^2(\Omega), \tag{3.50}$$

from Lemma 1.1, equation (1.19) to obtain (3.46), as desired, from (3.49).
(ii) By (3.42) we estimate, majorizing also $v$ by $y$ via (2.1.1), (2.1.2),

$$\left|\left(\frac{1}{\omega}\mathcal{A}_N\theta, v\right)_{L_2(\Omega)}\right| \le \frac{1}{|\omega|}\|\mathcal{A}_N\theta\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} \tag{3.51}$$

$$(\text{by } (3.42)) \le \left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right]\|y\|_Y, \tag{3.52}$$

$$\le \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y, \tag{3.53}$$

and (3.53) proves (3.47), as desired.

(iii) By Green's second theorem we compute

$$\frac{1}{\omega}\int_\Omega \Delta v\bar\theta\, d\Omega = \frac{1}{\omega}\int_\Omega v\Delta\bar\theta\, d\Omega + \frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial\nu}\bar\theta\, d\Gamma - \frac{1}{\omega}\int_\Gamma v\frac{\partial\bar\theta}{\partial\nu}\, d\Gamma . \tag{3.54}$$

Thus, by (3.54), recalling (3.31) and (3.47), we estimate

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial\nu}\bar\theta d\Gamma - \frac{1}{\omega}\int_\Gamma v\frac{\partial\bar\theta}{\partial\nu}d\Gamma\right| \le \left\|\frac{1}{\omega}\Delta v\right\|_{L_2(\Omega)} \|\theta\|_{L_2(\Omega)} + \left|\frac{1}{\omega}(v,\Delta\theta)_{L_2(\Omega)}\right| \tag{3.55}$$

$$(\text{by } (3.31), (3.47)) \le \left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right]\|y\|_Y + \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] \tag{3.56}$$

$$\le \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2 , \tag{3.57}$$

and (3.57) proves (3.48), as desired. □

As a corollary to Lemma 3.5, we obtain the desired good estimate for the difference $[\|v\|^2_{L_2(\Omega)} - \|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)}]$. This is a second serious step (the first was Lemma 3.3) in achieving the final desired estimates (2.1.5) and (2.1.6) for $u$ and $v$.

PROPOSITION 3.6. *Recalling* (2.1.1), (2.1.2), *given* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

$$\left|\|v\|^2_{L_2(\Omega)} - \|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)}\right| \le \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2 . \tag{3.58}$$

PROOF. We return to identity (3.46), and use here estimates (3.47), (3.48), obtaining

$$\left|\|v\|^2_{L_2(\Omega)} - \|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)}\right| \le \left|\frac{1}{\omega}\left(\theta|_\Gamma, \frac{\partial v}{\partial\nu}\Big|_\Gamma\right)_{L_2(\Gamma)} - \frac{1}{\omega}\left(\frac{\partial\theta}{\partial\nu}\Big|_\Gamma, v|_\Gamma\right)_{L_2(\Gamma)}\right| + \|\mathcal{A}^{\frac{1}{2}}u\|_{L_2(\Omega)}\left\|\mathcal{A}^{\frac{1}{2}}\left(\frac{u_0}{\omega}\right)\right\|_{L_2(\Omega)} + \left|\frac{1}{\omega}(\mathcal{A}_N\theta, v)_{L_2(\Omega)}\right| + \left\|\frac{v_0}{\omega}\right\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} \tag{3.59}$$

$$(\text{by } (3.47), (3.48)) \le \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] + \left[\frac{\epsilon_1}{2}\|y\|_Y^2 + \frac{1}{2\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right], \tag{3.60}$$

majorizing $\mathcal{A}^{\frac{1}{2}}u$, $v$, and $\mathcal{A}^{\frac{1}{2}}u_0$, $v_0$ in $L_2(\Omega)$ by $y$ and $f_0$ in $Y$, respectively, via (2.1.1), (2.1.2). Then, (3.60) yields (3.58), as desired.

## 4. – Proof of estimates (2.1.5) and (2.1.6) for $u$ and $v$

ORIENTATION. So far, throughout the arguments of Sections 2 and 3, we have made *no use* of the *special structure* of the boundary operators $B_1$ and $B_2$, see (1.1g), (1.1h).

This way, we have achieved only the "right" $\epsilon$-estimates for the following quantities: for $\theta$ in (2.2.19); for $\Delta u$, or $\frac{1}{\omega}\Delta v$, in (3.30), (3.31); finally, for the difference $[\|v\|^2_{L_2(\Omega)} - \|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)}]$ in (3.58). On the other hand, formula (1.3) shows the relationship between $\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)} = \|u\|^2_{H^2(\Omega)}$ and $\|\Delta u\|^2_{L_2(\Omega)}$. In the present section, we shall finally complete the proof, by achieving the desired estimates (2.1.5) and (2.1.6) for $\|v\|^2_{L_2(\Omega)}$ and $\|u\|^2_{H^2(\Omega)}$, in fact simultaneously. To this end, we need to work with a corresponding elliptic problem: we already know by (3.30), (3.31) that

$$\|\Delta u\|^2_{L_2(\Omega)} \le \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y ; \tag{4.1}$$
$$\left\|\Delta\left(\frac{v}{\omega}\right)\right\|^2_{L_2(\Omega)} \le \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y .$$

Therefore, *if we manage to show* that

$$\textit{either } \|u|_\Gamma\|^2_{H^{\frac{3}{2}}(\Gamma)} \le \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y , \tag{4.2}$$

$$\text{or } \textit{else } \left\|\left(\frac{v}{\omega}\right)\right|_\Gamma \Big\|^2_{H^{\frac{3}{2}}(\Gamma)} \le \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y , \tag{4.3}$$

then we can appeal to elliptic theory *either* for the $u$-problem (4.1)(left), (4.2); or *else* for the $\left(\frac{v}{\omega}\right)$-problem (4.1)(right), (4.3), and obtain, respectively,

$$\textit{either } \|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)} = \|u\|^2_{H^2(\Omega)} \le \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y , \tag{4.4}$$

$$\text{or } \textit{else } \left\|\frac{v}{\omega}\right\|^2_{H^2(\Omega)} \le \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y . \tag{4.5}$$

Once either one of the estimates (4.4), or (4.5), has been established, the other readily follows via I = (2.1.8). Then, (4.4) *proves* (2.1.5), *as desired.* Moreover, (4.4) used in (3.58), *proves* (2.1.6), as well, and the proof of Theorem 1.3

is complete. Thus, *the remaining key estimate to prove* is either estimate (4.2) for $u$, or else estimate (4.3) for $\left(\frac{v}{\omega}\right)$. To this end, we shall take advantage, *for the first time*, of the special structure of the boundary operator $B_1$, rewritten as [L-T.5, Chapter 3, Proposition C.1, equation (C.2)],

$$B_1 = -\left[D_\tau^2 + k\,\frac{\partial}{\partial\nu}\right], \tag{4.6}$$

where $D_\tau^2$ denotes the second tangential derivative, and $-k(x) = \text{ div } \nu(x)$ is the mean curvature at the point $x \in \Gamma$. Due to the required smoothness of $\Gamma$, we may assume that $k \in L_\infty(\Gamma)$. A first step is the following result on $u|_\Gamma$ in $H^2(\Gamma)$:

LEMMA 4.1. *With reference to* (2.1.1) *and* (2.1.2), *given* $\epsilon > 0$, *there exists* $C_\epsilon > 0$, *such that for all* $\omega \in \mathbb{R}$, *say* $|\omega| \geq 1$, *we have*

(i)

$$\|\Delta u|_\Gamma\|_{L_2(\Gamma)} \leq |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y\right]; \tag{4.7}$$

(ii)

$$\|\theta|_\Gamma\|_{L_2(\Gamma)} \leq |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y\right]; \tag{4.8}$$

(iii)

$$\|u|_\Gamma\|_{H^2(\Gamma)} \leq |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y\right] + C_\mu\left\|k\,\frac{\partial u}{\partial\nu}\right\|_{L_2(\Gamma)}; \tag{4.9}$$

(iv) *moreover, given* $\epsilon > 0$, *there exists* $C_\epsilon > 0$ *such that for all* $|\omega| \geq \omega_0 = C\,(\max_{x\in\Gamma}|k|)\,/[\epsilon(1-\mu)]$, *we have*

$$\|u|_\Gamma\|_{H^2(\Gamma)} \leq |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{4.10}$$

PROOF. (i) By use of the usual trace estimates [Th.1, p. 26], [B-S.1, p. 39], of estimate (2.2.7) and of estimate (3.36), we obtain, say, for $|\omega| \geq 1$:

$$\|\Delta u|_\Gamma\|_{L_2(\Gamma)} \leq C\|\Delta u\|^{\frac{1}{2}}_{H^1(\Gamma)}\|\Delta u\|^{\frac{1}{2}}_{L_2(\Omega)} \leq C\|u\|^{\frac{1}{2}}_{H^3(\Omega)}\|\Delta u\|^{\frac{1}{2}}_{L_2(\Omega)} \tag{4.11}$$

$$\text{(by (2.2.7), (3.36))} \leq C|\omega|^{\frac{1}{4}}\left[\|y\|_Y^{\frac{1}{2}} + \left\|\frac{f_0}{\omega}\right\|_Y^{\frac{1}{2}}\right] \tag{4.12}$$

$$\times\left[\epsilon_1\|y\|_Y^{\frac{1}{2}} + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^{\frac{1}{2}}\right]$$

$$\leq |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|\right], \tag{4.13}$$

and (4.13) proves (4.7), as desired.

(ii) Similarly, by the trace estimates [Th.1, p. 26], [B-S.1, p. 39], recalling the "driving" estimate (2.2.2), we obtain

$$(4.14)\qquad \|\theta|_\Gamma\|_{L_2(\Gamma)} \le C\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}\|\theta\|^{\frac{1}{2}}_{L_2(\Omega)}$$

$$(4.15)\qquad (\text{by } (2.2.2)) \le |\omega|^{\frac{1}{4}}\left[\left(\frac{\epsilon_1}{2}\right)^{\frac{1}{4}}\|y\|_Y^{\frac{1}{2}} + \left(\frac{1}{2\epsilon_1}\right)^{\frac{1}{4}}\left\|\frac{f_0}{\omega}\right\|_Y^{\frac{1}{2}}\right]\|y\|_Y^{\frac{1}{2}}$$

$$(4.16)\qquad \le |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|\right],$$

and (4.16) proves (4.8), as desired. [In going from (4.14) to (4.15), we have simply majorized $\theta$ by $y$, with no need of invoking the finer estimate (2.2.19).]

(iii) We use the first B.C. (1.1d) for $u$ and (for the first time) the structure (4.6) for the boundary operator $B_1$, thus obtaining

$$(4.17)\quad \text{on } \Gamma: \Delta u + (1-\mu)B_1 u + \theta = \Delta u - (1-\mu)\left[D_\tau^2 u + k\frac{\partial u}{\partial \nu}\right] + \theta = 0\,.$$

Thus (with $0 < \mu < 1$), by (4.17), recalling (4.7) and (4.8), we estimate

$$(4.18)\qquad \|D_\tau^2 u|_\Gamma\|_{L_2(\Gamma)} \le \frac{1}{1-\mu}\left[\|\Delta u|_\Gamma\|_{L_2(\Gamma)} + \|\theta|_\Gamma\|_{L_2(\Gamma)}\right] + \left\|k\frac{\partial u}{\partial \nu}\right\|_{L_2(\Gamma)}$$

$$(4.19)\ (\text{by } (4.7), (4.8)) \le \frac{1}{1-\mu}|\omega|^{\frac{1}{4}}\left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right] + \left\|k\frac{\partial u}{\partial \nu}\right\|_{L_2(\Gamma)},$$

and (4.19) yields (4.9), as desired.

(iv) To prove (4.10) from (4.9), we may use trace theory, with $k \in L_\infty(\Gamma)$, and majorize $u$ by $y$ via (2.1.1), (2.1.2):

$$(4.20)\qquad \left\|k\frac{\partial u}{\partial \nu}\right\|_{L_2(\Gamma)} \le \left(\max_{x\in\Gamma}|k|\right)C\|u\|_{H^2(\Omega)} \le C_k\|y\|_Y$$

$$(4.21)\qquad \le \frac{C_k}{|\omega|^{\frac{1}{4}}}|\omega|^{\frac{1}{4}}\|y\|_Y \le C_\mu\epsilon|\omega|^{\frac{1}{4}}\|y\|_Y\,,$$

for all $\omega \in \mathbb{R}$ with $|\omega|^{\frac{1}{4}} \ge C_k/[C_\mu\epsilon]$, and (4.10) follows from (4.9), by use of (4.21). □

It remains to establish the desired estimate (4.2) for $u$ [or (4.3) for $\left(\frac{v}{\omega}\right)$] in $H^{\frac{3}{2}}(\Gamma)$ from inequality (4.10) in $H^2(\Gamma)$: this requires getting rid of the factor $|\omega|^{\frac{1}{4}}$ while lowering the boundary norm of $u$ from $H^2(\Gamma)$ to $H^{\frac{3}{2}}(\Gamma)$. Below we shall prove (4.3).

LEMMA 4.2. *With reference to* (2.1.1), (2.1.2) *we have, say, for* $|\omega| \geq 1$*:*

$$\left\| \frac{v}{\omega}\Big|_\Gamma \right\|_{H^1(\Gamma)} \leq C \left\| \frac{v}{\omega} \right\|_{H^{\frac{3}{2}}(\Omega)} \leq \frac{C}{|\omega|^{\frac{1}{4}}} \left[ \|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right]. \tag{4.22}$$

PROOF. We interpolate (moment inequality) between estimate (2.2.3) and (2.2.4), rewritten here as

$$\left\| \frac{v}{\omega} \right\|_{H^2(\Omega)} \leq \|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \quad \text{and} \quad \left\| \frac{v}{\omega} \right\|_{H^1(\Omega)} \leq \frac{C}{|\omega|^{\frac{1}{2}}} \left[ \|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right], \tag{4.23}$$

to obtain (4.22). □

However, estimate (4.22) for $\left(\frac{v}{\omega}\right)$ does not yield the same estimate for $u$, because of the datum $\frac{u_0}{\omega}$, via equation I = (2.1.8). Accordingly, we shall proceed by taking appropriate initial conditions $u_0$ in a dense set of $H^2(\Omega)$, prove inequalities (2.1.5) and (2.1.6) in this case, and then extend them to all $u_0$ in $H^2(\Omega)$ by density.

PROPOSITION 4.3. *With reference to* (2.1.1), (2.1.2), *given* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that for all* $|\omega| > \omega_0$, $w_0 > 0$ *as in Lemma* 4.1(*iii*), *we have that inequality* (4.3) *for* $\left(\frac{v}{w}\right)$ *holds true.* □

PROOF. First we have

$$\left\| \frac{v}{w}\Big|_\Gamma \right\|_{H^{\frac{3}{2}}(\Gamma)} \leq \left( \left\| \frac{v}{w}\Big|_\Gamma \right\|^{\frac{1}{2}}_{H^2(\Gamma)} \right) \left( \left\| \frac{v}{w}\Big|_\Gamma \right\|^{\frac{1}{2}}_{H^1(\Gamma)} \right), \tag{4.24}$$

by [Th.1, p. 26], [B-S.1, p. 39]. Next, we define the subspace $S_0$ of $H^2(\Omega)$ of initial data

$$S_0 = \{u_0 \in H^3(\Omega) : D_\tau^2 u_0 = 0\}, \tag{4.25}$$

which is dense in $H^2(\Omega)$. Let $u_0 \in S_0$, then $i D_\tau^2 u = D_\tau^2 \left(\frac{v}{\omega}\right)$ via equation I = (2.1.8), and hence we obtain the equivalence $\|u|_\Gamma\|_{H^2(\Gamma)} \doteq \|\frac{v}{\omega}|_\Gamma\|_{H^2(\Gamma)}$, which used in (4.24) yields by virtue of (4.10), (4.22):

$$\left\| \frac{v}{\omega}\Big|_\Gamma \right\|_{H^{\frac{3}{2}}(\Gamma)} \leq C \|u|_\Gamma\|^{\frac{1}{2}}_{H^2(\Gamma)} \left\| \frac{v}{\omega}\Big|_\Gamma \right\|^{\frac{1}{2}}_{H^1(\Gamma)} \tag{4.26}$$

$$(\text{by } (4.10), (4.22)) \leq C|\omega|^{\frac{1}{8}} \left[ \epsilon_1 \|y\|^{\frac{1}{2}}_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|^{\frac{1}{2}}_Y \right] \frac{C}{|\omega|^{\frac{1}{8}}} \times \left[ \|y\|^{\frac{1}{2}}_Y + \left\| \frac{f_0}{\omega} \right\|^{\frac{1}{2}}_Y \right] \tag{4.27}$$

$$\leq \epsilon \|y\|_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y, \quad u_0 \in S_0. \tag{4.28}$$

Finally, we extend the validity of estimate (4.28) to all $u_0 \in H^2(\Omega)$, by density of $S_0$ in $H^2(\Omega)$ and thus obtain inequality (4.3), as desired. □

Inequality (4.3) was our targeted goal: as explained in the Orientation of Section 4, from (4.3) we then deduce (4.5) by appealing to the elliptic problem for $(\frac{v}{w})$ in (4.1); next, (4.5) yields (4.4) = (2.1.5) via I = (2.1.8). Finally, (4.4) used in (3.58) proves (2.1.6), as desired. We summarize all this in the next corollary.

COROLLARY 4.4. *With reference to* (2.1.1), (2.1.2), *given* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that for all* $|\omega| > \omega_0$, $\omega_0 > 0$ *as in Lemma* 4.1(iii), *we have*

$$\|u\|^2_{H^2(\Omega)} + \|v\|^2_{L_2(\Omega)} \le \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y . \tag{4.29}$$ □

Thus, via (2.2.19), (4.29), the desired estimates (2.1.4) through (2.1.6) are proved. Theorem 1.3 is established. □

## 5. – Appendix to Section 2.2

PROOF OF LEMMA 2.2.2. (i) The validity of estimate (2.2.3) stems at once from equation I = (2.1.8), the norm equivalence in (2.1.1), where one majorizes $u$ and $u_0/\omega$ in $H^2(\Omega)$ by $y$ and $f_0/\omega$ in $Y$ via (2.1.1).

(ii) By interpolation (moment inequality), we compute via (2.2.3), and majorizing $v$ by $y$, by (2.1.1), (2.1.2),

$$\begin{aligned} \|v\|_{H^1(\Omega)} &\le C\|v\|^{\frac{1}{2}}_{H^2(\Omega)} \|v\|^{\frac{1}{2}}_{L_2(\Omega)} \le C|\omega|^{\frac{1}{2}} \left[ \|y\|^{\frac{1}{2}}_Y + \left\| \frac{f_0}{\omega} \right\|^{\frac{1}{2}} \right] \|y\|^{\frac{1}{2}}_Y \\ &\le C|\omega|^{\frac{1}{2}} \left[ \|y\|_Y + \frac{1}{2} \left\| \frac{f_0}{\omega} \right\|_Y + \|y\|_Y \right], \end{aligned} \tag{A.1}$$

and (A.1) proves estimate (2.2.4). □

PROOF OF LEMMA 2.2.3. We return to equation III = (2.1.10), where we use estimate (2.2.3) for $v$,

$$\begin{aligned} \left\| \frac{1}{\omega} \mathcal{A}_N \theta \right\|_{L_2(\Omega)} &= \left\| \frac{\theta_0}{\omega} - i\theta + \frac{1}{\omega} \Delta v \right\|_{L_2(\Omega)} \le \left\| \frac{\theta_0}{\omega} \right\|_{L_2(\Omega)} \\ &\quad + \|\theta\|_{L_2(\Omega)} + \frac{1}{|\omega|} \|\Delta v\|_{L_2(\Omega)} \\ (\text{by (2.2.3)}) &\le \left\| \frac{f_0}{\omega} \right\|_Y + \|y\|_Y + \left[ \|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right], \end{aligned} \tag{A.2}$$

majorizing, in the last step, $\theta_0$ and $\theta$ by $f_0$ and $y$ via (2.1.1), (2.1.2). Then, (A.2) proves (2.2.5). □

PROOF OF LEMMA 2.2.5. STEP 1. By Green's first theorem with $v \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$ and $\theta \in \mathcal{D}(\mathcal{A}_N)$, we have

$$\frac{1}{\omega}(\Delta v, \theta)_{L_2(\Omega)} = \frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial \nu}\bar{\theta} d\Gamma - \frac{1}{\omega}\int_\Omega \nabla v \cdot \nabla\bar{\theta} d\Omega . \tag{A.3}$$

But by the *a-priori* bound (2.2.4) and the "driving" bound (2.2.2), we estimate

$$\begin{aligned}\left|\frac{1}{\omega}\int_\Omega \nabla v \cdot \nabla \bar{\theta} d\Omega\right| &\leq \left(\frac{1}{\sqrt{|\omega|}}\|v\|_{H^1(\Omega)}\right)\left(\frac{1}{\sqrt{|\omega|}}\|\theta\|_{H^1(\Omega)}\right) \\ &\leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\sqrt{\epsilon_1}\|y\|_Y + \frac{1}{\sqrt{\epsilon_1}}\left\|\frac{f_0}{\omega}\right\|_Y\right] \\ (\text{by (2.1.12)}) &\leq C\sqrt{\epsilon_1}\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2 ,\end{aligned} \tag{A.4}$$

with $\epsilon_1 > 0$ arbitrary. Moreover, recalling the *a-priori* inequalities [Th.1, p. 26], [B-S.1, p. 39],

$$\left\|\frac{\partial v}{\partial \nu}\right\|_{L_2(\Gamma)} \leq C\|v\|_{H^2(\Omega)}^{\frac{1}{2}}\|v\|_{H^1(\Omega)}^{\frac{1}{2}}; \ \|\theta|_\Gamma\|_{L_2(\Gamma)} \leq C\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}}\|\theta\|_{L_2(\Omega)}^{\frac{1}{2}} . \tag{A.5}$$

Using (A.5) we estimate

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial \nu}\bar{\theta} d\Gamma\right| \leq C\frac{\|v\|_{H^2(\Omega)}^{\frac{1}{2}}}{\sqrt{|\omega|}}\frac{\|v\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}}\frac{\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}}\|\theta\|_{L_2(\Omega)}^{\frac{1}{2}} . \tag{A.6}$$

Taking the $\frac{1}{2}$-th power of the *a-priori* bounds (2.2.3) and (2.2.4), we obtain the following uniform bound for $|\omega| \geq 1$:

$$\begin{aligned}\frac{\|v\|_{H^2(\Omega)}^{\frac{1}{2}}}{\sqrt{|\omega|}}\frac{\|v\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}} &\leq C\left[\|y\|_Y^{\frac{1}{2}} + \left\|\frac{f_0}{\omega}\right\|_Y^{\frac{1}{2}}\right]\left[\|y\|_Y^{\frac{1}{2}} + \left\|\frac{f_0}{\omega}\right\|_Y^{\frac{1}{2}}\right] \\ &\leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right] .\end{aligned} \tag{A.7}$$

On the other hand, taking the $\frac{1}{4}$-power of the "driving" bound (2.2.2) and majorizing $\theta$ by $y$ by (2.1.1), (2.1.2), we obtain the following uniform bound for $|\omega| \geq 1$,

$$\begin{aligned}\frac{\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}}\|\theta\|_{L_2(\Omega)}^{\frac{1}{2}} &\leq 2^{-\frac{1}{4}}\left[\epsilon_1\|y\|_Y^{\frac{1}{2}} + \frac{1}{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^{\frac{1}{2}}\right]\|y\|_Y^{\frac{1}{2}} \\ &\leq 2^{-\frac{5}{4}}\left[3\epsilon_1\|y\|_Y + \frac{1}{\epsilon_1^3}\left\|\frac{f_0}{\omega}\right\|_Y\right] .\end{aligned} \tag{A.8}$$

Using estimates (A.7) and (A.8) on the right-hand side of (A.6), we obtain

$$\text{(A.9)} \qquad \left|\frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial \nu}\bar{\theta}\, d\Gamma\right| \le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\epsilon_1\|y\|_Y + \frac{1}{\epsilon_1^3}\left\|\frac{f_0}{\omega}\right\|_Y\right]$$

$$\text{(A.10)} \qquad \text{(by (2.1.12))} \le C\left[2\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right],$$

recalling (2.1.12) in the last step, $\epsilon_1 > 0$ being arbitrary. Finally, using (A.4) and (A.9) on the right-hand side of (A.3) yields, as desired, (2.2.18):

$$\text{(A.11)} \qquad \left|\frac{1}{\omega}(\Delta v, \theta)_{L_2(\Omega)}\right| \le \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2 .$$

Proof of Lemma 2.2.7. With $\theta \in \mathcal{D}(\mathcal{A}_N)$ and $v \in \mathcal{D}(\mathcal{A}_D) \subset H^1(\Omega) = \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$, we estimate by (2.2.2) and (2.2.4):

$$\text{(A.12)} \qquad \left|\frac{1}{\omega}(\mathcal{A}_N\theta, v)_{L_2(\Omega)}\right| = \left|\left(\frac{\mathcal{A}_N^{\frac{1}{2}}\theta}{\sqrt{|\omega|}}, \frac{\mathcal{A}_N^{\frac{1}{2}}v}{\sqrt{|\omega|}}\right)_{L_2(\Omega)}\right|$$

$$\le \frac{\|\mathcal{A}_N^{\frac{1}{2}}\theta\|_{L_2(\Omega)}}{\sqrt{|\omega|}}\ \frac{\|\mathcal{A}_N^{\frac{1}{2}}v\|_{L_2(\Omega)}}{\sqrt{|\omega|}}$$

$$\text{(A.13)} \quad \text{(by (2.2.2), (2.2.4))} \le C\left[\epsilon_1\|y\|_Y + \frac{1}{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]$$

$$\text{(A.14)} \qquad \text{(by (2.1.2))} \le C\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2,$$

after invoking (2.1.12) in the last step, for an arbitrary $\epsilon_1 > 0$. Then, (A.14) proves (2.2.20), as desired. □

## REFERENCES

[B-S.1] S. Brenner – L. R. Scott, "The Mathematical Theory of Finite Element Methods", TAM 15, Springer-Verlag, 1994.

[G.1] P. Grisvard, *Caracterization de quelques espaces d'interpolation*, Arch. Rational Mech. Anal. **25** (1967), 40-63.

[Lag.1] J. Lagnese, "Boundary Stabilization of Thin Plates", SIAM, Philadelphia, 1989.

[Lag.2] J. Lagnese, *Uniform boundary stabilization of homogeneous, isotropic plates*, Springer-Verlag LNCIS, **102** (1987), 204-215.

[Las.1] I. Lasiecka, *Control and stabilization of interactive structures*, Systems and Control in the 21st Century, Birkhäuser (1997), 245-262.

[L-T.1] I. Lasiecka – R. Triggiani, *Two direct proofs on the analyticity of the s.c. semigroup arising in abstract thermo-elastic equations*, Adv. Differential Equations **3** (1998), 387-416.

[L-T.2] I. Lasiecka – R. Triggiani, *Analyticity of thermo-elastic semigroups with coupled hinged/Neumann B.C.*, Abstr. Appl. Anal., to appear.

[L-T.3] I. Lasiecka – R. Triggiani, *Structural decomposition of thermo-elastic semigroups with rotational forces*, IFIP Workshop, University of Floriday, February 1997, Semigroup Forum, to appear.

[L-T.4] I. Lasiecka – R. Triggiani, *Analyticity and lack thereof*, of thermo-elastic semigroups, ESAIM to appear.

[L-T.5] I. Lasiecka – R. Triggiani, "Control Theory for Partial Differential Equations: continuous and Approximation Theories", Cambridge University Press, Encyclopedia of Mathematics and its Applications, in press to appear in 1999.

[L-M.1] J. L. Lions – E. Magenes, "Nonhomogeneous Boundary Value Problems and Applications I", 1972, Springer-Verlag.

[L-L.1] K. Liu – Z. Liu, *Exponential stability and analyticity of abstract linear thermo-elastic systems*, preprint, December 1996.

[L-R.1] Z. Liu – M. Renardy, *A note on the equations of thermoelastic plate*, Appl. Math. Let. **8** (1995), 1-6.

[P.1] A. Pazy, "Semigroups of Linear Operators and Application to Partial Differential Equations", Springer-Verlag, 1983.

[Th.1] V. Thomee, "Galerkin Finite Element Methods for Parabolic Problems", Lecture Notes in Math., Vol. 1054, Springer, Berlin, 1984. Expanded version reprinted as a Springer-Verlag's book, 1997.

Department of Mathematics
Kerchof Hall
University of Virginia
Charlottesville, VA 22903