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On One-Sided BMO and Lipschitz Functions

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Abstract. Two basic results concerning functions with conditions on the mean oscillation are extended to the one-sided setting: the estimate of its distance to L^{∞} and the pointwise one-sided regularity. In addition we investigate the structure of these one-sided regular functions and their basic properties.

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1. - Introduction

In this note we study two natural questions induced by the theory of one-sided one dimensional weights (see [8], [9] and [10]). First we obtain upper and lower estimates for the distance of a function in $BMO^{+}(\mathbb{R})$ to the space $L^{\infty}(\mathbb{R})$, or even to a one-sided extension of the space of essentially bounded functions, generalizing a well-known result of Garnet and Jones [7].

The second problem has as a starting point the following remark. The product $w \cdot g$ of an $A_p(\mathbb{R})$ $(1 \leq p < \infty)$ Muckenhoupt weight w, and an increasing function g belongs to the class $A_p^{+}(\mathbb{R})$ (for example $|x|^{-1/2}e^{[x]} \in A_2^{+}(\mathbb{R})$, with $[x]$ the smallest integer greater than or equal to x), on the other hand it is not difficult to construct an $A_p^{+}(\mathbb{R})$ weight that cannot be decomposed as such a product. Taking logarithms, this remark can be re-written as: the sum $f_1 + f_2$ of a $BMO(\mathbb{R})$ function f_1 plus an increasing function f_2 belongs to the class $BMO^{+}(\mathbb{R})$ (for example $\log|x| + [x] \in BMO^{+}(\mathbb{R})$), but not every function in $BMO^{+}(\mathbb{R})$ can be written as $f_1 + f_2$ with $f_1 \in BMO(\mathbb{R})$ and f_2 increasing. Now, the function $f(x) = \sin x + [x]$, being the sum of a smooth function plus an increasing function, belongs to $BMO_{\alpha}^{+}(\mathbb{R})$ (see (1.3) below, with $\varphi(t) = t^{\alpha}$) for every $0 < \alpha \leq 1$. More generally $f_1 + f_2 \in BMO_{\alpha}^{+}(\mathbb{R})$

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whenever $f_1 \in BMO_\alpha(\mathbb{R})$ and f_2 is an increasing function. Given a function $f \in BMO_\alpha^+(\mathbb{R})$, can f be written as $f_1 + f_2$ with f_1 a $BMO_\alpha(\mathbb{R})$ function and f_2 an increasing function?

In [3], Campanato proves that $BMO_\alpha(\mathbb{R}^n)$ coincides with the usual class $\Lambda_\alpha(\mathbb{R}^n)$ of Lipschitz α functions and in [2] his result is extended to show that $BMO_\alpha^+(\mathbb{R})$ has a pointwise version $\Lambda_\alpha^+(\mathbb{R})$. Then, our second question can be re-stated in terms of the relationship between the functional cones $\Lambda_\alpha(\mathbb{R}) + \mathcal{I}$ and $\Lambda_\alpha^+(\mathbb{R})$, with $\mathcal{I}$ the class of increasing functions.

Even when pointwise one-sided regularity of vanishing lateral mean oscillation functions can be obtained from [2], we give a direct proof for the one-dimensional case, following [15], which does not make explicit use of the Calderón decomposition. Once we have a one-sided Lipschitz type condition on a real function f, we study its basic properties, such as type of discontinuities and size of the set of points where f is not continuous. Then, we show that $\Lambda_\alpha^+(\mathbb{R}) \neq \Lambda_\alpha(\mathbb{R}) + \mathcal{I}$ for $0 < \alpha < 1$, that $\Lambda_1^+(\mathbb{R}) = \Lambda_1(\mathbb{R}) + \mathcal{I}$, and we obtain some extensions to more general functions φ. As another way of comparison, we prove that the Hausdorff dimension of the Graph of a one-sided Lipschitz function satisfies the classical estimate for the (bilateral) Lipschitz case, so that at least in this sense the new situation is not worse than the usual one.

Let us observe that these problems, specially the first one, rely strongly on the characterization of lateral weights through lateral maximal functions and that this characterization is available only in one dimension (see [10], section 16.5). Nevertheless vanishing lateral mean oscillation and lateral pointwise regularity cones of functions can be extended to higher dimensions.

Since the pioneering work of J. Moser [11], it is known that from a parabolic BMO condition involving a cylinder $R = B(x,r) \times [r_0, r_0 + r^2]$ in $\mathbb{R}^{n+1}$ and its time translation, $TR = B(x,r) \times [r_0 + 2r^2, r_0 + 3r^2]$, John-Nirenberg type lemmas can be proved in order to produce an A_2 class of weights with "time lag" (see also [4]). This result holds also in the general setting of spaces of homogeneous type (see [1]). The more recent work of Martín-Reyes and de la Torre [8], reproduces the class BMO^+ as the log of A_∞^+ weights.

In 1982 Garnet and Jones [7] gave a precise estimate for the distance of a BMO function f to the subspace L^∞ in terms of the exponential integrability of f over cubes as an application of the factorization of A_p weights. On the other hand, after the work of Campanato, Meyers and Spanne ([3], [11], [14]), it is well known that some vanishing mean oscillation conditions on a function give pointwise regularity. Precisely, given a positive function φ defined on $\mathbb{R}^+$, a real function f of real variable is said to satisfy the BMO_φ condition if there exists a constant C such that for every interval I, there is a real number C_I for which

$$\frac{1}{|I|} \int_I |f - C_I| \leq C\varphi(|I|). \tag{1.1}$$

The number $||f||_{BMO_\varphi} = \sup_I \frac{1}{|I|\varphi(|I|)} \int_I |f - C_I|$, is a quasi-norm for this space, which becomes a Banach space if we identify functions through $f \sim g$

whenever $f - g$ is a constant. If $\varphi = 1$ this space is the BMO space of John and Nirenberg. Notice that if $f \in BMO_\varphi$ then $f \in L^1_{\text{loc}}$ and in (1.1) it is possible to take $C_I = f_I = \frac{1}{|I|}\int_I f$. Let us also observe that the inequality (1.1) is equivalent to the following pair of inequalities

$$\text{(1.2)} \qquad \begin{aligned} \frac{1}{|I|}\int_I (f - C_I)^+ &\leq C\varphi(|I|), \\ \frac{1}{|I|}\int_I (C_I - f)^+ &\leq C\varphi(|I|), \end{aligned}$$

and that none of the inequalities in (1.2) by itself gives an extra condition for a locally integrable function f. Instead, some one-sided control on the integral smoothness of f is obtained if in the second integral in (1.2) we change $I = [a, b]$ by its right neighbor $I^+ = [b, 2b - a]$, maintaining the same constant C_I in both integrals. Let us formally introduce the condition BMO^+_φ.

DEFINITION 1.3. Let f be a measurable function. We say that $f \in BMO^+_\varphi$ if there exists a constant C such that for every interval I, there is a real number C_I for which

$$\begin{aligned} \frac{1}{|I|}\int_I (f - C_I)^+ &\leq C\varphi(|I|), \\ \frac{1}{|I|}\int_{I^+} (C_I - f)^+ &\leq C\varphi(|I|), \end{aligned}$$

where $I^+ = (x + h, x + 2h)$ if $I = (x, x + h)$.

Let us first notice that the set BMO^+_φ is not a vector space, instead it is a functional cone in the sense that is only closed under multiplication by positive numbers and addition. The infimum of those constants C shall be considered as a "norm" on this cone of functions and will be denoted by $\|\cdot\|_{BMO^+_\varphi}$. The increasing functions have zero norm, also $BMO_\varphi \subset BMO^+_\varphi$. In a similar way we define BMO^-_φ changing the roles of I^+ and I in the integrals of Definition 1.3. It is clear that $BMO^+_\varphi \cap BMO^-_\varphi = BMO_\varphi$. When $\varphi = 1$ we get the one-sided versions BMO^+ and BMO^- of the John-Nirenberg BMO space. In Section 2 we use the factorization of one-sided weights proved by Martín-Reyes, de la Torre and Ortega-Salvador [9] in order to get an upper bound of the distance from one-sided BMO to the space L^∞, while, working the lower bound we introduce a natural substitute L^∞_+ of L^∞. In Section 3 we prove one-sided pointwise regularity from integral one-sided contidions for an even more general "lag mapping" than the "right neighbor" of the usual one-sided setting by and extension of the basic method in [15]. Finally, the last section is devoted to the analysis of the basic properties of one-sided Lipschitz functions.

2. – The distance from one-sided BMO to One-Sided L^∞

For BMO^+ functions, John-Nirenberg type inequalities can be found in different settings ([12], [4], [1], [8]). In fact a function f belongs to BMO^+ if and only if the following estimates for the distribution functions of $(f - C_I)^+$ and $(C_I - f)^+$ hold: there exist positive constants α and K such that

$$|\{x \in I : (f(x) - C_I)^+ > \lambda\}| \le K \exp\left(-\frac{\alpha\lambda}{\|f\|_{BMO^+}}\right),$$

$$|\{x \in I : (C_I - f(x))^+ > \lambda\}| \le K \exp\left(-\frac{\alpha\lambda}{\|f\|_{BMO^+}}\right),$$

for every interval I and every $\lambda > 0$. It is easy to see that for a BMO^+ function f we have the inequality

$$\frac{1}{|I|}\int_I (f - f_{I^+})^+ \le 2\|f\|_{BMO^+}, \tag{2.1}$$

where f_{I^+} is the average of f over I^+. From Theorem 3 of [8], (2.1) implies

$$|\{x \in I : (f(x) - f_{I^+})^+ > \lambda\}| \le K \exp\left(-\frac{\alpha\lambda}{\|f\|_{BMO^+}}\right) \tag{2.2}$$

with $\alpha = \frac{1}{16}\log\frac{4}{3}$ and $K = \frac{2\sqrt{3}}{9}$.

These results hold true even in the general setting of spaces of homogeneous type. In the one dimensional case, however, the theory of one sided maximal functions and weights is deeply connected with the spaces BMO^+ and BMO^- and gives the extensions of the tools used by Garnet and Jones in the classical context in order to study the distance from a fixed BMO function to the subspace L^∞.

The one sided maximal functions M^+ and M^- are given by

$$M^+f(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(y)|dy, \quad M^-f(x) = \sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f(y)|dy.$$

A weight function w satisfies A_p^+ $(1 < p < \infty)$ if

$$\frac{1}{|I|}\int_I w\left(\frac{1}{|I|}\int_{I^+} w^{-\frac{1}{p-1}}\right)^{p-1} \le C,$$

holds for every interval I, and w satisfies A_1^+ if $M^-w(x) \le Cw(x)$ for almost every $x \in \mathbb{R}$.

The results of this theory mainly developed by Sawyer, Martín-Reyes, de la Torre and Ortega-Salvador in [13], [8], [9] contain the fact that, in one-dimension, A_p^+ is the right condition on w for the $L^p(w)$ boundedness of M^+ and, the factorization theorem: $w \in A_p^+$ if and only if $w = w_1 {w_2}^{1-p}$ for some $w_1 \in A_1^+$ and $w_2 \in A_1^-$. The key argument in our proof of the upper bound will be based on this factorization theorem and on the next lemma which is itself a one sided extension of a well known result of Coifman and Rochberg [6].

LEMMA 2.3. *Let μ a positive Borel measure such that $M^-\mu(x) < \infty$ a.e. $x \in \mathbb{R}$. For each $0 < \gamma < 1$ the weight $w(x) = (M^-\mu)^\gamma(x) \in A_1^+$ and moreover, γ can be chosen in such a way that $M^-w(x) \leq 4w(x)$.*

PROOF. It is enough to prove that $\frac{1}{r}\int_{x-r}^x w \leq 4w(x)$ for every $r > 0$ and almost every $x \in \mathbb{R}$. Let us write $\mu = \mu_1 + \mu_2$ with $\mu_1 = \mu\chi_{(x-2r,x)}$ the restriction of μ to the interval $(x-2r, x)$. For $0 < \gamma < 1$, we have

$$(M^-\mu)^\gamma \leq (M^-\mu_1)^\gamma + (M^-\mu_2)^\gamma = (I) + (II).$$

Let us show that, both (I) and (II) satisfy the desired estimate. In order to estimate (I), let us first write the integral $\int_{x-r}^x (M^-\mu_1)^\gamma$ in terms of the distribution function, then let us divide the domain of integration at the level $R = \frac{\mu_1(\mathbb{R})}{r}$ and finally let us use the Hardy-Littlewood weak type inequality for M^-, to get

$$\begin{aligned}
\frac{1}{r}\int_{x-r}^x (M^-\mu_1)^\gamma &= \frac{\gamma}{r}\int_0^\infty t^{\gamma-1}|\{[x-r,x] : M^-\mu_1 > t\}|dt = \frac{\gamma}{r}\left(\int_0^R + \int_R^\infty\right) \\
&\leq \frac{\gamma}{r}\left[\frac{rR^\gamma}{\gamma} + 2\int_R^\infty t^{\gamma-2}\mu_1(\mathbb{R})dt\right] = R^\gamma\left[1 + \frac{2\gamma}{(1-\gamma)}\frac{\mu_1(\mathbb{R})}{rR}\right] \\
&\leq 2^\gamma\left(1 + \frac{2\gamma}{(1-\gamma)}\right)(M^-\mu(x))^\gamma \\
&\leq 2w(x),
\end{aligned}$$

for an appropriate choice of $\gamma \in (0,1)$. Notice that the estimate for (II) is a consequence of the fact that for every $y \in [x-r, x]$, $M^-\mu_2(y) \leq 2M^-\mu_2(x)$. To prove this inequality, pick $y \in [x-r, x]$ and h in such a way that $[y-h, y] \setminus [x-2r, x] \neq \emptyset$ and observe that $[y-h, y] \subset [x-2h, x]$, so that for $y \in [x-r, x]$ we have

$$\frac{1}{h}\int_{y-h}^y \mu_2 \leq \frac{1}{h}\int_{x-2h}^x \mu_2 \qquad \square$$

Let us now observe that Lemma 2.3 has an obvious symmetric version when M^- changes M^+ and A_1^+ to A_1^-. From the previous lemma and the fact that $\log A_p^+ \subset BMO^+$ with BMO^+ norm depending only on the A_p^+ constant C of w, in fact $||\log w||_{BMO^+} \leq \log(1+C)$ (see Theorem 1 of [8]), we have that $\log(M^-\mu)$ belongs to a fixed ball of BMO^+ for every positive Borel measure μ for which $M^-\mu(x) < \infty$ a.e. $x \in \mathbb{R}$. Symmetrically $\log(M^+\mu)$ belongs to a fixed ball of BMO^- for every positive Borel measure μ with $M^+\mu(x) < \infty$ a.e. Actually from the factorization theorem follows that $BMO^+ = \log A_p^+$ and $BMO^- = \log A_p^-$.

For the classical BMO space, Garnet and Jones realized that the inverse of the best constant A for which

$$\sup_I \frac{1}{|I|}\int_I e^{A|f-f_I|} < \infty$$

is a precise measure of the distance from f to L^∞. An immediate consequence of (2.2) is the following exponential integrability of a function f in BMO^+: there exists a constant A such that

$$\sup_I \frac{1}{|I|} \int_I e^{A(f-f_{I^+})^+} < \infty . \tag{2.4}$$

Notice that (2.4) holds for every $A < \alpha ||f||^{-1}_{BMO^+}$. Define $A^*(f) = \sup\{A : (2.4) \text{ is true}\}$, the purpose of this note is to prove that $(A^*(f))^{-1}$ is a good estimate for the distance from f to L^∞ or even to a larger class of functions L^∞_+ to be defined.

For a BMO^+ function f, pick A such that $\frac{A^*(f)}{2} < A < A^*(f)$, then $w = e^{Af} \in A_2^+$. From the one sided factorization of A_p^+, we obtain $w_1 \in A_1^+$ and $w_2 \in A_1^-$ such that $w = w_1 w_2^{-1}$. So that we have $Af = \log w = \log w_1 - \log w_2 = F_1 - F_2$ with $w_i = e^{F_i} < \infty$ a.e. for $i = 1, 2$. From the very definition of A_1^+ and A_1^- we see that the following inequalities

$$e^{F_1} \leq M^- e^{F_1} \leq Ce^{F_1} \text{ and } e^{F_2} \leq M^+ e^{F_2} \leq Ce^{F_2} ,$$

hold almost everywhere. Thus F_i $i = 1, 2$ can be written as

$$F_1 = \log\left(M^- e^{F_1}\right) + \log\left(e^{F_1}/M^- e^{F_1}\right) = \psi_1 + g_1 ,$$
$$F_2 = \log\left(M^+ e^{F_2}\right) + \log\left(e^{F_2}/M^+ e^{F_2}\right) = \psi_2 + g_2 ,$$

with g_i bounded functions for $i = 1, 2$ and

$$f = \frac{F_1 - F_2}{A} = \frac{\psi_1 - \psi_2}{A} + \frac{g_1 - g_2}{A}$$

On the other hand, from Lemma 2.3 and the remark following it we have that

$$\psi_1 = \log\left(M^- e^{F_1}\right) \in BMO^+ \text{ and } \psi_2 = \log\left(M^+ e^{F_2}\right) \in BMO^- ,$$

with norm independent of f. Then $f = \psi + h$ with $\psi = \frac{\psi_1 - \psi_2}{A}$, $||\psi||_{BMO^+} \leq \frac{\log 5}{A^*(f)}$ and $h \in L^\infty$, so that

$$\inf_{g \in L^\infty} ||f - g||_{BMO^+} \leq ||f - h||_{BMO^+} = ||\psi||_{BMO^+} \leq \frac{\log 5}{A^*(f)} .$$

We have just obtained the following result

Proposition 2.5. *For $f \in BMO^+$*

$$\inf_{g \in L^\infty} ||f - g||_{BMO^+} \leq \frac{\log 5}{A^*(f)} .$$

The opposite inequality is easier, in fact, we have

PROPOSITION 2.6. *For $f \in BMO^+$*

$$\frac{\alpha}{A^*(f)} \le \inf_{g\in L^\infty} \|f-g\|_{BMO^+},$$

with α as in (2.2).

PROOF. Let $f \in BMO^+$ and $g \in L^\infty$. We only need to prove that

$$\{A : (2.4) \text{ is true for } f-g\} \subset \{A : (2.4) \text{ is true for } f\}. \tag{2.7}$$

In fact, since every A such that $A < \frac{\alpha}{\|f-g\|_{BMO^+}}$ belongs also to the second set, then the result follows from the inequality $\frac{\alpha}{\|f-g\|_{BMO^+}} \le A^*(f)$. In order to proof (2.7), let A be such that (2.4) holds true for $f-g$, then

$$\begin{aligned}\frac{1}{|I|}\int_I e^{A(f-f_{I^+})^+} &\le \frac{1}{|I|}\int_I e^{A(f-g-(f-g)_{I^+})^+} e^{A(g-g_{I^+})^+}\\ &\le Ce^{2A\|g\|_\infty},\end{aligned} \tag{2.8}$$

which shows that (2.4) holds for f □

Notice that for the boundedness of the mean values on the left hand side of (2.8) we only need

$$g_+^*(x) = \sup_{I\in\mathcal{J}_x} (g(x)-g_{I^+})^+ \in L^\infty, \tag{2.9}$$

where $\mathcal{J}_x$ is the family of compact intervals containing x. This condition is weaker than L^∞, in fact any nondecreasing function satisfies (2.9). This observation suggests the definition of a new space between L^∞ and BMO^+. The set

$$L^\infty_{+,*} = \{f \in L^1_{\text{loc}} : f_+^* \in L^\infty\},$$

is a cone of real functions contained in BMO^+ and containing L^∞. This remark together with Propositions (2.5) and (2.6) give the following

THEOREM 2.10. *For $f \in BMO^+$ we have the inequalities*

$$\frac{\log\frac{4}{3}}{16A^*(f)} \le \inf_{g\in L^\infty_{+,*}} \|f-g\|_{BMO^+} \le \inf_{g\in L^\infty} \|f-g\|_{BMO^+} \le \frac{\log 5}{A^*(f)}.$$

Let us finally observe that if we define $f_+^\dagger(x) = \sup_{I\in\mathcal{I}_x}(f_{I^-}-f(x))^+$ and $L^\infty_{+,\dagger} = \{f \in L^1_{\text{loc}} : f_+^\dagger \in L^\infty\}$, we also get a cone in BNO^+. The function $f(x)$ equal to m on the interval $[m-1, m)$ and equal to $\log[(1-e^m)x+e^m-(1-e^m)m]$ on $[m, m+1)$ for m odd belongs to $L^\infty_{+,*}$ but not to $L^\infty_{+,\dagger}$. Another important fact is that an L^∞ function plus a nondecreasing function belongs to both $L^\infty_{+,*}$ and $L^\infty_{+,\dagger}$, but none of them reduces to $L^\infty+\mathcal{I}$ with $\mathcal{I}$ the class of nondecreasing functions. We define $L_+^\infty = L_{+,*}^\infty \cap L_{+,\dagger}^\infty$ and $L_-^\infty = L_{-,*}^\infty \cap L_{-,\dagger}^\infty$ associated to the maximal functions

$$f_-^*(x) = \sup_{I\in\mathcal{J}_x} (f(x) - f_{I^-})^+ \text{ and } f_-^\dagger(x) = \sup_{I\in\mathcal{J}_x} (f_{I^+} - f)^+ .$$

From the fact that the boundedness of a locally integrable function f is equivalent to the boundedness of the "maximal function" $\sup_{I\in\mathcal{J}_x} |f(x) - f_I|$, it is easy to see that $L^\infty = L_+^\infty \cap L_-^\infty$. So that from Theorem 2.10 and its obvious analogues for $L_{+,\dagger}^\infty$, $L_{-,*}^\infty$ and $L_{-,\dagger}^\infty$ we get as byproduct the classical result of Garnet and Jones for dimension one.

3. – Regularity of one-sided BMO functions

The purpose of this section is to give a direct method for the study of the pointwise one-sided regularity from integral one-sided mean oscillation conditions, for related results on spaces of homogeneous type using rearrangements see [2], based on the Calderón decomposition.

Let $\mathcal{J}$ be the class of the compact intervals of $\mathbb{R}$, we say that a one to one and onto function $T : \mathcal{J} \to \mathcal{J}$ is a "right lag mapping" of the intervals of $\mathbb{R}$ if

(3.1) there exist positive and finite constants A_1 and A_2 such that $A_1 \leq \frac{|TI|}{|I|} \leq A_2$ for every $I \in \mathcal{J}$,

(3.2) $\inf I \leq \inf\ TI$, for every $I \in \mathcal{J}$,

(3.3) dist $(I, TI) \leq c|I|$, for some non negative constant c,

(3.4) for every $x, y \in \mathbb{R}$ with $x < y$, there exists an interval I_1 such that $\frac{x+y}{2}$ is the center of $I_1 \cup J_1$ where $J_1 = TI_1$, and $x = \inf T^{-1} I_1$.

The inverse of T, T^{-1} is a "left lag mapping" satisfying (3.1) and (3.3), while (3.2) and (3.4) changes to

(3.2') $\sup T^{-1} I \leq \sup I$, for every $I \in \mathcal{J}$,

(3.4') for every $x, y \in \mathbb{R}$ with $x < y$, there exists an interval I_1 such that $\frac{x+y}{2}$ is the center of $I_1 \cup J_1$ where $J_1 = TI_1$, and $y = \sup TJ_1$.

Given such a T we define a one-sided type space of functions with mean oscillation controlled by φ.

Definition 3.5. We say that a function $f \in L^1_{\text{loc}}(\mathbb{R})$ belongs to $BMO_\varphi(T)$ if there exists a constant C such that for every interval I there exists C_I in such a way that

$$\frac{1}{|I|} \int_I (f - C_I)^+ \leq D\varphi(|I|) ,$$

$$\frac{1}{|TI|} \int_{TI} (C_I - f)^+ \leq D\varphi(|I|) .$$

The infimum of those constants D will be denoted by $||\cdot||_{BMO_\varphi(T)}$.

The main result of this section is the following pointwise one-sided regularity result for functions in $BMO_\varphi(T)$. We say that a function φ satisfies a Δ_2-condition if and only if $\varphi(2t) \leq C\varphi(t)$ for some positive constant C and every $t \in \mathbb{R}^+$.

THEOREM 3.6. *Let $f \in BMO_\varphi(T)$ with φ a Δ_2-function such that $\int_0^1 \frac{\varphi(s)}{s} ds < \infty$, then there exists a constants C depending only on the lag mapping constants such that*

$$(3.7) \qquad (f(x) - f(y))^+ \leq C||f||_{BMO_\varphi(T)} \int_0^{C(y-x)} \frac{\varphi(s)}{s} ds \text{ for } x < y.$$

In other words, the function satisfies a one-sided Lipschitz condition with a modulus of continuity $\tilde{\varphi}$ given by $\tilde{\varphi}(t) = \int_0^t \frac{\varphi(s)}{s} ds$. So that when φ is of positive lower type, i.e. $\varphi(st) \leq Cs^\alpha\varphi(t)$, $0 < s < 1$, $t > 0$, for some $\alpha > 0$, we have the one-sided Lipschitz-φ condition for f

$$(f(x) - f(y))^+ \leq C\varphi(y - x) \text{ for } x < y .$$

The constant C depends only on the norm $BMO_\varphi(T)$ of f and D on the lag mapping constants.

Let us start by proving two basic lemmas.

LEMMA 3.8. *Let $f \in BMO_\varphi(T)$ then there exists a constant C such that for every interval I we have that*

$$\frac{1}{|I|}\int_I (f - f_{TI})^+ \leq D\varphi(|I|),$$
$$\frac{1}{|TI|}\int_{TI} (f_I - f)^+ \leq D\varphi(|I|).$$

PROOF.

$$\frac{1}{|I|}\int_I (f - f_{TI})^+ \leq \frac{1}{|I|}\int_I (f - C_I)^+ + \frac{1}{|TI|}\int_{TI} (C_I - f)^+$$
$$\leq \widetilde{C}\varphi(|I|).$$

In a similar way we obtain the second inequality □

For each choice of $x < y$, property (3.4) gives two intervals I_1 and J_1. Through an iterative process we define two sequences of intervals starting from I_1 and J_1 in the following way, for $k = 2, 3, \dots$

$$(3.9) \qquad I_k = T\delta^-_{\frac{1}{2+c}} T^{-1}(I_{k-1}) \text{ and } J_k = T^{-1}\delta^+_{\frac{1}{2+c}} T(J_{k-1}),$$

where $\delta^-_{\frac{1}{2+c}}(I)$ s the left interval in the subdivision of I into $2+c$ equal parts, while $\delta^+_{\frac{1}{2+c}}(I)$ is its right portion, where c is the first integer for which (3.3) holds.

LEMMA 3.10. *For $i = 2, 3, \ldots,$ we have*

(3.10.1) $I_{i+1} \subset T^{-1}I_i = \delta^-_{\frac{1}{(2+c)}} T^{-1}I_{i-1}$.

(3.10.2) $\frac{A_1}{(2+c)}|\delta^-_{\frac{1}{(2+c)}} T^{-1}I_{i-1}| \leq |I_{i+1}| \leq \frac{A_2}{(2+c)}|\delta^-_{\frac{1}{(2+c)}} T^{-1}I_{i-1}|$.

(3.10.3) $|I_i| \leq \left(\frac{A_2}{A_1(2+c)}\right)^{i-1}|I_1|$.

PROOF. The inclusion and the equality in (3.10.1) follow both from the definition of the intervals I_i. To proof (3.10.2) we only give an upper estimate of the measure of I_{i+1} since the lower estimate is similar

$$\begin{aligned}|I_{i+1}| &= |T\delta^-_{\frac{1}{(2+c)}} T^{-1}I_i| \leq A_2|\delta^-_{\frac{1}{(2+c)}} T^{-1}I_i| = \frac{A_2}{(2+c)}|T^{-1}I_i| \\ &= \frac{A_2}{(2+c)}|\delta^-_{\frac{1}{(2+c)}} T^{-1}I_{i-1}|.\end{aligned}$$

The inequality (3.10.3) follow by iteration from

$$|I_k| = |T\delta^-_{\frac{1}{(2+c)}} T^{-1}I_{k-1}| \leq \frac{A_2}{(2+c)}|T^{-1}I_{k-1}| \leq \frac{A_2}{(2+c)}\frac{|I_{k-1}|}{A_1} \qquad \square$$

PROOF OF THEOREM 3.6. Given x and y with $x < y$, let $\{I_k\}$ and $\{J_k\}$ be the two sequences of intervals given by (3.9). Then

$$\begin{aligned}(f(x) - f(y))^+ &\leq (f(x) - f_{I_k})^+ + \sum_{i=1}^{k-1}(f_{I_{i+1}} - f_{I_i})^+ + (f_{I_1} - f_{J_1})^+ \\ &\quad + \sum_{i=1}^{k-1}(f_{J_i} - f_{J_{i+1}})^+ + (f_{J_k} - f(y))^+ \\ &= (1) + (2) + (3) + (4) + (5)\,.\end{aligned}$$

We only need to get the desired estimate (2), (3) and (4), since for every k

$$(f(x) - f(y))^+ \leq (f(x) - f_{I_k})^+ + C\int_0^{D(y-x)} \frac{\varphi(s)}{s} ds + (f_{J_k} - f(y))^+,$$

so that Lebesgue differentiation theorem applied to the function $f \in L^1_{\text{loc}}(\mathbb{R})$ and the sequences of intervals $\{I_k\}$ and $\{J_k\}$ converging to x and y, shows that (1) and (5) tends to zero for almost every x and y. We shall only estimate (2) and (3), the bound for (4) can be obtained in a similar way noticing that $-f \in BMO_\varphi(T^{-1})$ and applying Lemma 3.10. The estimate for (3) follows from the definition of $BMO_\varphi(T)$, in fact

$$(f_{I_1} - f_{J_1})^+ \leq \frac{1}{|I_1|}\int_{I_1}(f - f_{TI_1})^+ \leq C\varphi(y - x) \leq C\int_0^{y-x} \frac{\varphi(s)}{s} ds\,.$$

Each term in the sum of (2) with index i greater than or equal to 3 can be bounded applying (3.10.1) and (3.10.2),

$$\begin{aligned}(f_{I_{i+1}} - f_{I_i})^+ &\le \frac{1}{|I_{i+1}|}\int_{I_{i+1}} (f - f_{I_i})^+ \le \frac{|T^{-1}I_i|}{|I_{i+1}|}\frac{1}{|T^{-1}I_i|}\int_{T^{-1}I_i} (f - f_{I_i})^+ \\ &\le C\frac{(2+c)}{A_1}\varphi\left(|\delta^-_{1/(2+c)}T^{-1}I_{i-1}|\right) = C\frac{(2+c)}{A_1}\varphi\left(\frac{|T^{-1}I_{i-1}|}{(2+c)}\right) \\ &\le C\frac{(2+c)}{A_1}\varphi\left(\frac{|I_{i-1}|}{A_1(2+c)}\right).\end{aligned}$$

For the first two terms in (2), we have

$$\begin{aligned}(f_{I_2} - f_{I_1})^+ &\le \frac{1}{|I_2|}\int_{I_2} (f - f_{I_1})^+ = \frac{1}{|I_2|}\int_{I_2} (f - f_{TT^{-1}I_1})^+ \\ &\le \frac{(2+c)}{A_1}\frac{1}{|T^{-1}I_1|}\int_{T^{-1}I_1} (f - f_{TT^{-1}I_1})^+ \le \frac{(2+c)}{A_1}\varphi\left(|T^{-1}I_1|\right),\end{aligned}$$

and

$$(f_{I_3} - f_{I_2})^+ \le \frac{1}{|I_3|}\int_{I_3} (f - f_{I_2})^+ \le \frac{(2+c)}{A_1}\varphi\left(\frac{|T^{-1}I_1|}{2+c}\right).$$

Adding term by term this estimates, applying (3.10.3) and noticing that $|T^{-1}I_1| \le \frac{y-x}{A_1}$, we obtain the desired inequality

$$\begin{aligned}(2) &\le \frac{(2+c)}{A_1}\left[\sum_{i=3}^{k-1}\varphi\left(\left(\frac{A_2}{A_1}\right)^{i-2}\frac{(y-x)^{i-2}}{A_1(2+c)^{i-1}}\right) + \varphi\left(\frac{y-x}{A_1(2+c)}\right) + \varphi\left(\frac{y-x}{A_1}\right)\right] \\ &\le \frac{(2+c)}{A_1}\left[\int_0^{\frac{A_2}{4A_1}\frac{(y-x)}{A_1}} \frac{\varphi(s)}{s}ds + \int_0^{\frac{y-x}{2A_1}} \frac{\varphi(s)}{s}ds + \int_0^{\frac{y-x}{A_1}} \frac{\varphi(s)}{s}ds\right] \\ &\le C\int_0^{D(y-x)} \frac{\varphi(s)}{s}ds\end{aligned}$$

□

Theorem 3.6 has a "left" analogue, which actually follows as a corollary when we notice that T is a left lag mapping if and only if T^{-1} is a right lag mapping and $f \in BMO_\varphi(T)$ if and only if $-f \in BMO_\varphi(T^{-1})$.

COROLLARY 3.11. *Let T be a "left lag mapping" and φ a Δ_2-function such that $\int_0^1 \frac{\varphi(s)}{s}ds < \infty$, then for $f \in BMO_\varphi(T)$ there exist constants C and D such that*

$$(f(y) - f(x))^+ \le C\int_0^{D(y-x)} \frac{\varphi(s)}{s}ds,$$

for every $x < y$.

The standard result on regularity of functions satisfying usual bilateral conditions of BMO_φ type is now an inmediate consequence of (3.6) and (3.11). In fact, more generally we have:

COROLLARY 3.12. *Let φ a Δ_2-function such that $\int_0^1 \frac{\varphi(s)}{s}ds < \infty$ and assume that T is a right and left lag mapping, then every function f in $BMO_\varphi(T)$ coincides almost everywhere with a continuos function for which*

$$|f(x) - f(y)| \leq C \int_0^{D|x-y|} \frac{\varphi(s)}{s} ds \,.$$

4. – One-sided Lipschitz functions

The results in Section 3. leads us to the study of the properties of functions with some one-sided pointwise regularity condition like (3.7). These ideas have also a geometric starting point. Given a function f on $\mathbb{R}$ we say that f belongs to the class Lipschitz 1 if there exists a positive constant C such that for every x, y in $\mathbb{R}$ we have $|f(x) - f(y)| \leq C|x - y|$. The geometrical meaning of this condition is the following: for every x the Graph of f remains outside of the cone defined by $\{(y, z) \in \mathbb{R}^2 : |z - f(x)| > C|x - y|\}$. In a similar way, the classical Lipschitz or Hölder α-spaces, $0 < \alpha < 1$, of functions satisfying $|f(x) - f(y)| \leq C|x - y|^\alpha$, admit a geometric formulation changing standard cones by the more general cones defined by $\{(y, z) \in \mathbb{R}^2 : |z - f(x)| > c|x - y|^\alpha\}$. It is also usual to study function spaces with a prescribed modulus of continuity. For $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, non decreasing with $\varphi(0^+) = 0$ we say that f is a Lipschitz φ function, and write $f \in \Lambda_\varphi$, if and only if there exists a positive constant C such that $|f(x) - f(y)| \leq C\varphi(|x - y|)$ for every x, $y \in \mathbb{R}$. The infimum of those constants C is a seminorm that we will denote by $[f]_{\Lambda_\varphi}$. Now the cones are given by $\{(y, z) : |f(x) - z| > C\varphi(|x - y|)\}$.

With $\bar{\Lambda}_\varphi$ we denote the functions in Λ_φ that belong to L^∞, which become a Banach space if we add to $[f]_{\Lambda_\varphi}$ the L^∞ norm of f.

We are interested in the one-sided versions of the Lipschitz spaces. To introduce this spaces from a geometric starting point, let us first observe some elementary properties of the Lipschitz condition. Suposse that g is a real function such that $g(0) = 0$, satisfying the Lipschitz 1 condition then

$$\text{(A)} \qquad |g(x)| \leq C|x|, \quad x \in \mathbb{R}\,,$$

in other words the Graph of g does not intersect the cone $\Gamma_A = \{(x, y) : |y| > c|x|\}$; $\Gamma_A \cap$ Graph $(f) = \emptyset$. Condition (A), implies in particular the continuity at the origin of the function g. Moreover, notice that (A) is invariant under the action the two operators of symmetry

$$\tilde{g}(x) = g(-x)\,; \quad \bar{g}(x) = -g(x)\,.$$

The conditions

$$\text{(B)} \qquad |g(x)| \le -Cx\,, \quad x \le 0\,,$$

$$\text{(C)} \qquad g^+(x) \le C|x|\,, \quad x \in \mathbb{R}\,,$$

are strictly weaker than (A) and different from each other. In fact (B) and (C) admit discontinuos functions at the origin and while (B) is invariant for the operator $\bar{g}$, (C) is invariant for the operator $\tilde{g}$. Let us observe that (B) is equivalent to $\Gamma_B \cap$ Graph $(g) = \emptyset$ with $\Gamma_B = \{(x, y) : |y| > -cx;\ x \le 0\}$ and (C) is equivalent to $\Gamma_C \cap$ Graph $(g) = \emptyset$ for $\Gamma_C = \{(x, y) : y^+ > |x|;\ x \in \mathbb{R}\}$. Notice that Lipschitz 1 condition, can be written in terms of (A) in the following way

$$\Lambda_1 = \{f : g_y(x) = f(y - x) - f(y) \text{ satisfies (A) for each } y \in \mathbb{R}\}\,.$$

But, moreover, the sets of functions

$$\Lambda_1^B = \{f : g_y(x) = f(y) - f(y - x) \text{ satisfies (B) for each } y \in \mathbb{R}\} \text{ and}$$
$$\Lambda_1^C = \{f : g_y(x) = f(y - x) - f(y) \text{ satisfies (C) for each } y \in \mathbb{R}\}\,,$$

with properties (B) or (C) at each point their of Graph, both give the Λ_1. Therefore, conditions (A), (B) and (C) at each point of the Graph of a function f are equivalent and define the Λ_1 condition.

Let us now consider the class of function g, whose Graph does not intersect $\Gamma_B \cap \Gamma_C = \Gamma_D$, more precisely

$$\text{(D)} \qquad g^+(x) \le -x\,, \quad x \le 0\,.$$

Notice that (D) is neither invariant for the operator $\bar{g}$ nor for $\tilde{g}$. Consider the class of functions for which (D) holds at each point, in other words

$$\Lambda_1^+ = \{f : g_y(x) = f(y) - f(y - x) \text{ satisfies (D) for each } y \in \mathbb{R}\}$$

Such a function f admits at each point of its Graph an antisymmetric cone which is a translation of $\Gamma_D \cup (-\Gamma_D)$, notice that for both, $\tilde{f}$ and $\bar{f}$, a complementary condition is valid? $g_y(x) = f(y - x) - f(y)$ satisfies (D) for each $y \in \mathbb{R}$, and that f belong to Λ_1^+ if and only if $\tilde{\bar{f}}$ does. Moreover $f \in \Lambda_1^+$ if and only if $(f(x) - f(y))^+ \le C(y - x)$ for $x \le y$.

Let us now precise the definition and basic notation.

DEFINITION 4.1. Assume that $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a non decreasing function with $\varphi(0^+) = 0$, define the following set of functions with real values

$$\Lambda_\varphi^+ = \{f : (f(x) - f(y))^+ \le C\varphi(y - x) \text{ for every } x \le y \text{ and some } C > 0\}\,.$$

In other words, a function f belong to Λ_φ^+ if its Graph does not intersect the cone $\{(x, y) \in \mathbb{R}^2 : (y - f(x_0))^+ > C\varphi(|x_0 - x|)$ for all $x \leq x_0\}$.

Definition 4.1 does not imply the regularity Λ_φ of f, because this condition allows jumps when the function is growing, in fact, any non-decreasing functions satisfies Λ_φ^+ for every φ.

The set Λ_φ^+ is a cone within the vector space of functions on $\mathbb{R}$. Given f in this cone the infimun of those constants C, for which (4.1) holds, denoted by $[\ ]_{\Lambda_\varphi^+}$, satisfy the following properties: 1) $[f]_{\Lambda_\varphi^+} = 0$ if and only if f is an increasing function, 2) if $\alpha > 0$ then $[\alpha f]_{\Lambda_\varphi^+} = \alpha[f]_{\Lambda_\varphi^+}$, 3) $[f + g]_{\Lambda_\varphi^+} \leq [f]_{\Lambda_\varphi^+} + [g]_{\Lambda_\varphi^+}$. The number $[\ ]_{\Lambda_\varphi^+}$ shall be called the "norm" of f.

In a similar way we define $\Lambda_\varphi^- = \{f : (f(x) - f(y))^+ \leq C\varphi(|y - x|)$ for every $y \leq x\}$. Now, these functions have a control on their regularity when growing, but for example arbitrary non-increasing functions belong to Λ_φ^-.

The following proposition collects some elementary properties of these sets of functions. Here $\widetilde{\Lambda}_\varphi^+$ denotes the class $\{\tilde{f} : f \in \Lambda_\varphi^+\}$ and $\bar{\Lambda}_\varphi^+ = \{\bar{f} : f \in \Lambda_\varphi^+\}$.

Proposition 4.2.
(4.2.1) $\Lambda_\varphi^- \cap \Lambda_\varphi^+ = \Lambda_\varphi$.
(4.2.2) $\widetilde{\Lambda}_\varphi^+ = \bar{\Lambda}_\varphi^+ = \Lambda_\varphi^-$.
(4.2.3) $\widetilde{\Lambda}_\varphi^- = \bar{\Lambda}_\varphi^- = \Lambda_\varphi^+$.
(4.2.4) $\bar{\widetilde{\Lambda}}_\varphi^+ = \Lambda_\varphi^+$ *and* $\bar{\widetilde{\Lambda}}_\varphi^- = \Lambda_\varphi^-$.

If $\mathcal{I}$ is the class of all non-decreasing functions then, we have the trivial inclusion $\mathcal{I} + \Lambda_\varphi \subset \Lambda_\varphi^+$.

The following propositions give us elementary properties of boundedness, existence of lateral limits and examples that show that this spaces are generally non-trivial.

Proposition 4.3. *The functions Λ_φ^+ are locally bounded, in other words $\Lambda_\varphi^+ \subset L^\infty_{\text{loc}}$.*

Proof. Let $f \in \Lambda_\varphi^+$ and $[a, b]$ be a compact interval of $\mathbb{R}$. We will prove that f is bounded in $[a, b]$. with

$$l(t) = \begin{cases} \varphi(-t) & t < 0 \\ -\varphi(t) & t \geq 0 \end{cases},$$

let us define the following functions in $[a, b]$: $g(t) = l(t - a) + f(a) = -\varphi(t - a) + f(a)$ and $h(t) = l(t - b) + f(b) = \varphi(b - t) + f(b)$. Since f is a one-sided Lipschitz φ functions, we have that $g(x) \leq f(x) \leq h(x)$, on $a \leq x \leq b$. On the other hand, since φ takes finite values on $\mathbb{R}$ and is non-decreasing, the functions g and h have upper and lower bounds on every compact interval and therefore f is locally bounded □

Proposition 4.4. *Every non-increasing function f in Λ_φ^+ to Λ_φ.*

PROOF. For $x < y$, $|f(x) - f(y)| = f(x) - f(y) \leq C\varphi(y - x)$. □

Even when condition Λ_φ^+ allows discontinuities, the next proposition shows that these can only be of jump type. Given a point $x_0 \in \mathbb{R}$ and a function f defined on $\mathbb{R}$ we denote with $\alpha^+(x_0)$ the lim inf from the right of f at the point x_0, in other words

$$\alpha^+(x_0) = \lim_{\delta \to 0} \inf_{x_0 < x < x_0 + \delta} f(x),$$

for the lim sup from the right of f al x_0 we write

$$\beta^+(x_0) = \lim_{\delta \to 0} \sup_{x_0 < x < x_0 + \delta} f(x).$$

The existence of limit from the right of f at x_0, $f(x_0^+)$, is equivalent to the equality $\alpha^+(x_0) = \beta^+(x_0)$ and the existence of limit from the left of the function f at x_0, $f(x_0^-)$, is equivalent to $\alpha^-(x_0) = \beta^-(x_0)$, with the obvious definitions of α^- and β^-.

PROPOSITION 4.5. *For f in Λ_φ^+ both $f(x^+)$ and $f(x^-)$ exist at each point x and $f(x^-) \leq f(x) \leq f(x^+)$, moreover, the set of discontinuities is countable.*

PROOF. We shall prove that for each point x in $\mathbb{R}$ we have that $\beta^+(x) = \alpha^+(x)$. We only need to see that $\beta^+(x) \leq \alpha^+(x)$. Given $\epsilon > 0$ small, there exists $\delta_0 > 0$ such that for every $\delta < \delta_0$ we can pick x_1 and x_2 such that $x_0 < x_2 < x_1 < x_0 + \delta$, $f(x_1) - \epsilon < \alpha^+(x) + \epsilon$ and $f(x_2) + \epsilon > \beta^+(x) - \epsilon$, these inequalities lead us to the following

$$\begin{aligned}(\beta^+(x) - 2\epsilon - \alpha^+(x) - 2\epsilon)^+ &< (f(x_2) - \alpha^+(x) - 2\epsilon)^+ \\ &< (f(x_2) - f(x_1))^+ \leq C\varphi(|x_2 - x_1|),\end{aligned}$$

now, since $\varphi(|x_2 - x_1|) \leq \varphi(\delta) \to 0$, $\delta \to 0$ we have that $(\beta^+(x) - \alpha^+(x) - 4\epsilon)^+ = 0$ for every $\epsilon > 0$, so that $\beta^+(x) = \alpha^+(x)$. In a similar way we can prove that $\alpha^-(x) = \beta^-(x)$. The inequality $f(x^-) \leq f(x) \leq f(x^+)$ follows immediately from the definition of Λ_φ^+. To prove that the set of points of which f is no continuous, we only need to show that the set $A - \{x \in [-N, N] : f(x^+) - f(x^-) > \delta\}$ is finite. In fact, assume that A is not finite and take $x_0 \in [-N, N]$ a limit point of A. Let $\{x_k\} \subset A$ such that $x_k \to x_0$, $k \to \infty$. The given sequence has a monotone subsequence, let us assume that it is non-increasing, and call it again $\{x_k\}$. We know that $f(x_k^+) > \delta + f(x_k^-)$ for each k. Let us now pick a sequence ϵ_k of positive numbers such that $x_{k+1} + \epsilon_{k+1} < x_k + \epsilon_k$ and $f(x_k + \epsilon_k) > \frac{\delta}{2} + f(x_k - \epsilon_k)$. Since both $\{x_k + \epsilon_k\}$ and $\{x_k - \epsilon_k\}$ tend to x_0 from the right we have that $f(x_0^+) \geq \frac{\delta}{2} + f(x_0^+)$ which is impossible □

PROPOSITION 4.6. *Let $\mathcal{I}$ be the class of non-decreasing functions on $\mathbb{R}$ and $0 < \alpha < 1$, then $\Lambda_\alpha - \mathcal{I} \subset \Lambda_\alpha^+$, and there are functions f in Λ_α^+ that can not be decomposed as a sum of a Λ_α-function plus a non-decreasing function.*

PROOF. The first part of the proposition is obvious. The prove the second, let $\gamma = \frac{1}{\alpha} > 1$ and $A = \sum_{n=2}^{\infty} \frac{1}{n^\gamma} < \infty$. Define $f(x) = (\sum_{i=2}^{n} \frac{1}{i^\gamma} - x)$ for $x \in |\sum_{i=2}^{n-1} \frac{1}{i^\gamma}, \sum_{i=2}^{n} \frac{1}{i^\gamma}]$, $n = 3, 4, \ldots$, and $f(x) = 0$ otherwise. There is no way to write $f = g + h$ with $g \in \mathcal{I}$ and $h \in \Lambda_\alpha$. In fact if we suppose the opposite and consider the sequence $x_n = \sum_{i=2}^{n} \frac{1}{i^\gamma}$ for $n \geq 2$, then since h is a continuos function, we have that $f(x_n^+) = g(x_n^+) + h(x_n)$ and $f(x_n^-) = g(x_n^-) + h(x_n)$, therefore $f(x_n^+) - f(x_n^-) = \frac{1}{n} = g(x_n^+) - g(x_n^-)$, with g a non-decreasing function. Since $g(A) \geq g(x_n^+)$ for every n, we have that $g(A) = +\infty$, in fact

$$\begin{aligned} g(x_n^+) &= g(x_n^+) - g(x_n^-) + g(x_n^-) \\ &\geq g(x_n^+) - g(x_n^-) + g(x_{n-1}^-) \\ &\geq \big(g(x_n^+) - g(x_n^-)\big) + (g(x_{n-1}^+) - g(x_{n-1}^-) + \ldots + \big(g(x_1^+) - g(x_1^-)\big) \\ &\geq \frac{1}{n} + \frac{1}{n-1} + \ldots + 1 \\ &\geq \log n \end{aligned}$$

□

Proposition 4.6 can be extended to the general case when the function φ satisfies the Δ_2-condition.

PROPOSITION 4.7. *Given a Δ_2-function φ satisfying $\int_0^1 \frac{\varphi^{-1}(t)}{t^2} dt < \infty$ there is a function f in Λ_φ^+ that can not be decomposed as a sum of a Λ_φ-function plus a nondecreasing function. On the other hand $\Lambda_\varphi + \mathcal{I} \subset \Lambda_\varphi^+$.*

PROOF. If we define

$$f(x) = \varphi\left(\sum_{i=2}^{n} \varphi^{-1}\left(\frac{1}{i}\right) - x\right) \quad \text{for } x \in \left[\sum_{i=2}^{n-1} \varphi^{-1}\left(\frac{1}{i}\right), \sum_{i=2}^{n} \varphi^{-1}\left(\frac{1}{i}\right)\right],$$

with $n = 3, 4, \ldots$ and $f = 0$ otherwise, to obtain the result we only need to observe that $A = \sum_{n=2}^{\infty} \varphi^{-1}(\frac{1}{n}) < \infty$. But the finiteness of A is equivalent to the integrability of the function $\frac{\varphi^{-1}(t)}{t^2}$ □

Let us notice that the case $\varphi(t) = t$ is not contained in the previous propositions. Let us next show that in fact Λ_1^+ is the "trivial" space $\Lambda_1 + \mathcal{I}$.

PROPOSITION 4.8.

$$\Lambda_1^+ = \Lambda_1 + \mathcal{I}.$$

PROOF. Let $x < y$ then if $f(y) > f(x)$ we have that $f(x) - f(y) < 0 \leq C(y - x)$, and if $f(y) \leq f(x)$ we have that $f(x) - f(y) = (f(x) - f(y))^+ \leq C(y - x)$. In both cases we obtain that $f(x) - f(y) \leq C(y - x)$, in other words $f(x) + Cx \leq f(y) + Cy$, so that the function $g(x) = f(x) + Cx$ is increasing, and we have the desired decomposition of f as g plus $(-C)x$ □

As a consequence we see that Λ_1^+-functions are locally of bounded variation, this fact follows also from the obvious observation that the negative variation of f over each interval in finite.

We have the following one-sided version of the triviality of Λ_α when $\alpha > 1$.

PROPOSITION 4.9. *If $\alpha > 1$ then $\Lambda_\alpha^+ = \mathcal{I}$. Moreover when φ is strictly convex then $\Lambda_\varphi^+ = \mathcal{I}$.*

PROOF. Let us suppose that there exists $f \in \Lambda_\alpha^+$ but $f \notin \mathcal{I}$. It is clear that if γ and β are positive real numbers then $\gamma f(\beta x)$ also belongs to the class Λ_α^+. This allow us, after scaling, tu assume that $f(0) = 1$ and $f(1) = 0$. We divide the interval $[0, 1] = [0, f(0)]$ in n intervals of lenght $\frac{1}{n}$, at the point $\frac{1}{n^{\frac{1}{\alpha}}}$ the function can take only values larger than $f(0) - \frac{1}{n}$. Moreover $f(\frac{1}{n^{\frac{1}{\alpha}}})$ can only be bigger than $1 - \frac{1}{2^\alpha n}$, since in the opposite situation the Graph of f would necessarily touch the forbiden region for Λ_α^+. Then in the image interval, the function can descend at most $\frac{1}{2^\alpha n}$, then in n intervals of lenght $\frac{1}{n^{\frac{1}{\alpha}}}$, the values of f, can descend at most $\frac{1}{2^\alpha} < 1$. Since the function f is not allowed to have a decreasing jump, this fact contradicts that $f(1) = 0$. For the general case the extension of the geometric approach used for the case of $\varphi(t) = t^\alpha$ leads us to ask ourselves for the point x at which the slopes of the cones at zero and at $\varphi^{-1}(\frac{1}{i})$ coincide. This is equivalent to solve the equation $\varphi(x) + \varphi(\varphi^{-1}(\frac{1}{i}) - x) = C$. A solution is given by $x_i = \frac{\varphi^{-1}(\frac{1}{i})}{2}$, now since φ has lower type bigger than one, we obtain that $n\varphi(\frac{\varphi^{-1}(\frac{1}{i})}{2}) < 1$, which is again impossible being $f(1) = 0$ □

It is a well-known fact the finiteness of the length of the Graph of a function of bounded variation on an interval. So that the Graph of a Lipschitz 1 function has finite lenght. This fact is no longer true for Lipschitz α functions with $0 < \alpha < 1$. In fact for each $\alpha \in (0, 1)$ there exists a Lipschitz α function whose Graph has Hausdorff dimension equal to $2 - \alpha > 1$. Moreover it is known that $2 - \alpha$ is an upper bound for the Hausdorff dimension of the Graph of a Lipschitz α function. Both facts can be foun in [5]. After Proposition 4.6 one could expect that also in this sense of Hausdorff, one-sided Lipschitz are worse than usual Lipschitz. This is not the case, in the next proposition we prove that for $f \in \Lambda_\alpha^+$, dim (Graph $(f)) \leq 2 - \alpha$.

Let us review the basic facts on Hausdorff measure and dimension. For given subset E of $\mathbb{R}^n$, $s > o$ and $\delta > 0$ we define

$$\mathcal{H}_\delta^s(E) = \inf \sum_{i=1}^{\infty} |U_i|^s ,$$

where the infimum is taken over all covering $\{U_i\}$ of E with $|U_i| < \delta$, where $|U_i|$ denotes the diameter of the set U_i. For fixed E anf fixed $s > 0$, $\mathcal{H}_\delta^s(E)$ is an increasing function of δ then

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E) .$$

becomes an outer measure on $\mathbb{R}^n$. By the usual Caratheodory process, when restricted to the $\mathcal{H}^s$-measurable sets, $\mathcal{H}^s$ is a measure called the s-dimensional Hausdorff measure on $\mathbb{R}^n$. It easy to show that for fixed $E \subset \mathbb{R}^n$, there exists one and only one s_0 such that $\mathcal{H}^s(E) = +\infty$ if $0 \le s < s_0$ and $\mathcal{H}^s(E) = 0$ if $s_0 < s < \infty$. The number s_0 is colled the Hausdorff dimension of E and denoted by $\dim E$.

PROPOSITION 4.10. *For $f \in \Lambda^+_\alpha$ and $G =$ Graph (f), we have that* $\dim G \le 2 - \alpha$.

The following lemma is easy to prove and will be of some help in proving Proposition 4.10.

LEMMA 4.11. *Let f be a real function and let α and β be two positive real numbers, then the Graphs of $f(x)$ and $\alpha f(\beta x)$ have the same Hausdorff dimension.*

PROOF OF 4.10. It is enough to show that for $s_0 = 2 - \alpha$ we have that $\mathcal{H}^{s_0}(G) < \infty$. Since $G = \cup_{k \in \mathbb{Z}} \{(x, f(x)) : x \in [k, k+1)\}$ it is enough to prove that

$$\mathcal{H}^{s_0}(\{(x, f(x)) : x \in [k, k+1)\}) < \infty .$$

From the traslation invariance of condition Λ^+_α we only need to obtain the desired estimate for $k = 0$. Let $\Gamma = \{(x, f(x)) : x \in [0, 1)\}$. From Lemma 4.11 we may assume that the Graph Γ of f is contained in the unit square $[0, 1] \times [0, 1]$ and the Lipschitz constant of f is 1. We will construct a rectangular mesh covering the square $[0, 1] \times [0, 1]$. Let $h = \frac{1}{n}$, we divide the basis in n intervals of lenght h, and consider the mesh of height h^α. In this way the square $[0, 1] \times [0, 1]$ will be covered by a mesh of $n \times m$ rectangles R_{ij} of basis h and height h^α with m of the order $\frac{1}{h^\alpha}$. Observe now that $\Gamma \subset \bigcup_{i,j} R_{ij}$. In each R_{ij} we have at most $[h^{\alpha-1}] + 1$ squares $\{Q^k_{ij}\}_k$ of side h in such a way that $R_{ij} \subset \bigcup_{k=1}^{[h^{\alpha-1}]+1} Q^k_{ij}$ so that Γ is covered by $\bigcup_{i,j \in L} \bigcup_{k=1}^{[h^{\alpha-1}]+1} Q^k_{ij}$, where L is a subset of $\{1, \ldots, n\} \times \{1, \ldots, m\}$. The proposition will follow from the following estimate of the size of L,

$$\#L \le Cn . \tag{4.12}$$

In fact, let us assume (4.12) then

$$\begin{aligned} \mathcal{H}^{s_0}_h(\Gamma) &\le \sum_{i,j \in L} \sum_{k=1}^{[h^{\alpha-1}]+1} |Q^k_{ij(i)}|^{s_0} \le C h^{s_0} n([h^{\alpha-1}]+1) \\ &\le C h^{\alpha - 2 + s_0} \\ &\le C , \end{aligned}$$

for all $h \in (0, 1)$. In order to prove (4.12) observe that the hypothesis $f \in \Lambda_\alpha^+$, when translated to the discrete level induced by the mesh $\{R_{ij}\}$ means that

$$\text{if } (i_o, j_o) \in L \text{ then } (i_o + 1, j) \notin L \text{ for all } j < j_o - 1 .$$

To count the set L for $i \in \{1, \dots, n\}$ we define $g(i) = \max\{k : (i, k) \in L\}$. For the remainder points of the interval $[0, n]$ we defines $g(t)$ by using the following Λ_1^+ interpolation: if there exists i such that $g(i) \geq g(i+1)$ we define $g(t)$ by linear interpolation of the points $g(i)$ and $g(i+1)$ on the interval $(i, i+1)$. On the other hands if $g(i) < g(i+1)$ then we take $g(t)$ to be the constant $g(i)$ on $(i, i+1)$. By construction the function g belongs to the one sided Lipschitz 1 space, therefore it is of bounded variation on the interval $[1, n]$, moreover since from (4.8), $g(t) = -t + h(t)$ with h an increasing function, then the variation of g is bounded by a Cn with C a positive constant. Finally we observe that the total variation of g on $[1, n]$ is a bound for $\#L$ □

Let us finally observe that the $\Lambda_\varphi^+(\mathbb{R})$ can be extended to higher dimension by taking cones instead of half-lines. In facts, for a given open set Γ in $\mathbb{R}^n$ with $0 \in \overline{\Gamma}$, we denote by $\Gamma(x) = x + \Gamma$ and we define

$$\Lambda_{\varphi,+}^{\Gamma}(\mathbb{R}^n) = \{f : (f(x) - f(y))^+ \leq C\varphi(|x - y|) \text{ for every } y \in \Gamma(x)\} .$$

Of course if $0 \in \Gamma$, then every function in $\Lambda_{\varphi,+}^{\Gamma}(\mathbb{R}^n)$ is locally in $\Lambda_\varphi(\mathbb{R}^n)$. If Γ is a semispace with $0 \in \partial\Gamma$, then $\Lambda_{\varphi,+}^{\Gamma}(\mathbb{R}^n)$ contains discontinuos functions. But if for example, $\Gamma = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n : x < -|y|\}$, then $\Lambda_{\varphi,+}^{\Gamma}(\mathbb{R}^{n+1}) = \Lambda_\varphi(\mathbb{R}^{n+1})$. Moreover, with $\Gamma = \{(x, y) \in \mathbb{R}^2 : x < -y^2\}$, we have that if $f \in \Lambda_{\alpha,+}^{\Gamma}(\mathbb{R}^2)$ then f is locally in $\Lambda_{\alpha/2}(\mathbb{R}^2)$.

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