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# A Floquet-Liapunov Theorem in Fréchet Spaces

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**Abstract.** Based on [4], we prove a variation of the theorem in title, for equations with periodic coefficients, in Fréchet spaces. The main result gives equivalent conditions ensuring the reduction of such an equation to one with constant coefficient. In the particular case of $\mathbb{C}^\infty$, we obtain the exact analogue of the classical theorem. Our approach essentially uses the fact that a Fréchet space is the limit of a projective sequence of Banach spaces. This method can also be applied for a geometric interpretation of the same theorem within the context of total differential equations in Fréchet fiber bundles.

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## Introduction

In spite of the great progress in the study of topological vector spaces and their rich structure, a general solvability theory of differential equations in the non Banach case is still missing. This led many authors (see e.g. R.S. Hamilton [5], R. Lemmert [7] and the survey article [2]) to study special classes of differential equations where some types of solutions can be found. Also in this context, G. Galanis in [4], published in this journal, has studied a class of linear differential equations in a Fréchet space $\mathbb{F}$, which can be always uniquely solved under given initial conditions. The key to this approach is the fact that $\mathbb{F}$ can be though of as the limit of a projective system of Banach spaces: $\mathbb{F} \equiv \varprojlim \mathbb{E}_i$. As a matter of fact, this class consists of all equations $\dot{x} = A(t) \cdot x$ whose coefficient $A$ can be factored in the form

$$A = \varepsilon \circ A^*,$$

where $A^* : [0, 1] \longrightarrow H(\mathbb{F})$ is a continuous map, $H(\mathbb{F})$ is the Fréchet subspace of $\prod_{i \in \mathbb{N}} (\mathcal{L}(\mathbb{E}_i))$ containing all the sequences $(f_i)_{i \in \mathbb{N}}$ which form projective systems and $\varepsilon : H(\mathbb{F}) \longrightarrow \mathcal{L}(\mathbb{F}) : (f_i)_{i \in \mathbb{N}} \mapsto \varprojlim f_i$.

Though the Cauchy problem for linear equations, as above, could be studied yet by more advanced methods (referring to equations in locally convex spaces, cf. e.g. [2], [7]), [4] expounds an elementary approach leading to an explicit description of the solutions (apart from their theoretical existence) as projective limits of solutions in Banach spaces, where the corresponding theory is fairly complete.

A natural question that arises now is whether we can exploit the same approach to obtain a kind of a Floquet-Liapunov theorem in the context of Fréchet spaces.

As it is well known, the classical theorem asserts that a differential equation $\dot{x} = A(t) \cdot x$, with $A(t)$ a periodic complex matrix, can always be reduced to an equation with constant coefficient (cf. e.g. [1], [10], [16]). The key to this reduction is the existence of the logarithm of $\Phi(\tau)$, if $\Phi$ is the fundamental solution of the equation and $\tau$ the period.

However, this result is not true, in general, even in the case of Banach spaces, since the existence of such a logarithm is not always ensured. In this respect we refer to [9] and [12], where the reader can also find some sufficient conditions for the validity of the result in this context.

In the present paper, which is a natural outgrowth of [4], we present a variation of the theorem in title. More precisely, in the first main result (Theorem 2.3) we obtain equivalent conditions implying the reduction of a periodic equation to one with constant coefficient. The basic idea, in this direction, is to consider a generalized monodromy homomorphism of the equation at hand, in which the pathological group $GL(\mathbb{F})$ (fundamentally involved in the classical or Banach case) is replaced by an appropriate subgroup $H^o(\mathbb{F})$ of $H(\mathbb{F})$ fully described in Section 1, since the classical homomorphism is useless in our context.

For the sake of completeness, we apply the main Theorem 2.3 also to the case of $\mathbb{C}^\infty$. It is worthy to note that, although we still work in a Fréchet space, now the reduction under discussion always holds true. Therefore (see Theorem 3.2), we obtain the exact analogue of the classical theory.

The results of this note can be also profitably used for a generalization of the Floquet-Liapunov theory within the geometric context of Fréchet fiber bundles. In the latter, ordinary linear equations correspond to equations with total differentials which, in turn, are interpreted as flat connections. Hence, the generalized analogue of the main result leads to the reduction of a flat connection to one with "constant" coefficients on a trivial bundle, as we briefly discuss in Section 4.

## 1. – Preliminaries

Let $\mathbb{F}$ be a Fréchet space. It is well known (see for instance [11]) that $\mathbb{F}$ can be identified with the limit of a projective system $\{\mathbb{E}_i; \rho_{ji}\}_{i,j \in \mathbb{N}}$ of Banach

spaces, i.e. $\mathbb{F} \equiv \varprojlim \mathbb{E}_i$. More precisely, each $\mathbb{E}_i$ is the completion of the normed space $\mathbb{F}/\operatorname{Ker}(p_i)$, where $\{p_i\}_{i \in \mathbb{N}}$ are the seminorms of $\mathbb{F}$ and $\rho_{ji}$ are the extensions of the natural projections

$$\mathbb{F}/\operatorname{Ker}(p_j) \longrightarrow \mathbb{F}/\operatorname{Ker}(p_i) : x + \operatorname{Ker}(p_j) \mapsto x + \operatorname{Ker}(p_i), \quad j \geq i.$$

As explained in the Introduction, in order to study linear differential equations in a Fréchet space $\mathbb{F}$ as above, we consider the following space:

$$H(\mathbb{F}) := \{(f_i)_{i \in \mathbb{N}} : f_i \in \mathcal{L}(\mathbb{E}_i) \text{ and } \varprojlim f_i \text{ exists}\}.$$

It can be proved that $H(\mathbb{F}) \equiv \varprojlim(H_k(\mathbb{F}))$, where each

$$H_k(\mathbb{F}) := \{(f_1, ..., f_k) : f_i \in \mathcal{L}(\mathbb{E}_i) \text{ and } \rho_{ji} \circ f_j = f_i \circ \rho_{ji} \ (j \geq i)\}; \ k \in \mathbb{N},$$

is a Banach space. Thus $H(\mathbb{F})$ is a Fréchet space.

Similarly, we set

$$H^o(\mathbb{F}) := H(\mathbb{F}) \cap \prod_{i \in \mathbb{N}} GL(\mathbb{E}_i).$$

This is is a topological group, as the limit (within an isomorphism) of the projective system of the Banach-Lie groups

$$H_k^o(\mathbb{F}) := H_k(\mathbb{F}) \cap \prod_{i=1}^{k} GL(\mathbb{E}_i), \quad k \in \mathbb{N}.$$

In the study of differential equations the exponential mapping plays a fundamental role (cf. e.g. [3]). However, in the case of a Fréchet space $\mathbb{F}$, an exponential map of the form $\exp : \mathcal{L}(\mathbb{F}) \longrightarrow GL(\mathbb{F})$ (in analogy to the Banach case) cannot be defined, as a consequence of the incompleteness of $\mathcal{L}(\mathbb{F})$ and the pathological structure of $GL(\mathbb{F})$. This difficulty is overcome by inducing a *generalized exponential* $\operatorname{Exp} : H(\mathbb{F}) \longrightarrow H^o(\mathbb{F})$ defined in the following way: If $\mathbb{F} \equiv \varprojlim \mathbb{E}_i$, we denote by $\exp_i : \mathcal{L}(\mathbb{E}_i) \longrightarrow GL(\mathbb{E}_i)$ the ordinary exponential mapping of the Banach-Lie group $GL(\mathbb{E}_i)$. We check that the finite products

$$\exp_1 \times \cdots \times \exp_k : H_k(\mathbb{F}) \longrightarrow H_k^o(\mathbb{F}); \quad k \in \mathbb{N},$$

form a projective system, thus we may define

$$\operatorname{Exp} := \varprojlim(\exp_1 \times \cdots \times \exp_k).$$

Using now the differentiability of mappings between Fréchet spaces in the sense of J. Leslie ([8]), the previous considerations lead to the following main result.

THEOREM 1.1. *Let $\mathbb{F}$ be a Fréchet space and the (homogeneous) linear equation*

$$(1) \qquad \dot{x} = A(t) \cdot x,$$

*where the coefficient $A : I = [0, 1] \longrightarrow \mathcal{L}(\mathbb{F})$ is continuous and factors in the form $A = \varepsilon \circ A^*$, with $A^* : I \longrightarrow H(\mathbb{F})$ being a continuous mapping and $\varepsilon : H(\mathbb{F}) \longrightarrow \mathcal{L}(\mathbb{F})$ given by $\varepsilon((f_i)) = \varprojlim f_i$. Then (1) admits a unique solution for any given initial condition.*

The complete study of equations of type (1) has been already given in [4]. However, since we need the exact form of the solutions, we outline the basic steps of the proof.

PROOF. Let $(t_0, x_0) \in \mathbb{R} \times \mathbb{F}$ be an initial condition. Considering a projective system $\{\mathbb{E}_i; \rho_{ji}\}_{i,j \in \mathbb{N}}$ of Banach spaces with $\mathbb{F} \equiv \varprojlim \mathbb{E}_i$, we obtain the equation (on $\mathbb{E}_i$):

$$(1.i) \qquad \dot{x}_i = A_i(t) \cdot x_i; \quad i \in \mathbb{N},$$

where $A_i$ is the projection of $A$ to the $i$-factor. Since $\mathbb{E}_i$ is Banach, there exists a unique solution $f_i$ of (1.i) through $(t_0, \rho_i(x_0))$, if $\rho_i : \mathbb{F} \longrightarrow \mathbb{E}_i$ denotes the canonical projection. For any $j \geq i$, we check that $\rho_{ji} \circ f_j$ is also a solution of (1.i) with the same initial data, hence $\rho_{ji} \circ f_j = f_i$. As a result, the $C^1$-mapping $f := \varprojlim f_i$ can be defined and yields

$$\dot{f}(t) = (\dot{f}_i(t))_{i \in \mathbb{N}} = (A_i(t)(f_i(t)))_{i \in \mathbb{N}} = A(t)(f(t)),$$

$$f(t_0) = (\rho_i(x_0))_{i \in \mathbb{N}} = x_0.$$

Therefore, $f$ is the desired solution of (1). Moreover, $f$ is unique with respect to $(t_0, x_0)$, since for any other solution $g$ we may check, using analogous techniques as before, that $\rho_i \circ g = \rho_i \circ f$, $i \in \mathbb{N}$.          $\square$

REMARKS. From the preceding proof it is clear that the method applied therein provides explicit solutions of equation (1). Moreover, the present approach, apart from being elementary (in contrast to more general methods known for equations in arbitrary locally convex spaces; cf. [2], [7]), is very convenient for the geometric generalization discussed in Section 4.

## 2. – Floquet-Liapunov theory

In this section we consider again a differential equation of the form (1) with a *periodic coefficient* $A$, satisfying $A = \varepsilon \circ A^*$ as in Theorem 1.1. Without loss of generality, we may assume that *the period is* 1.

LEMMA 2.1. *The coefficient A is periodic if and only if $A^*$ is.*

PROOF. The periodicity of $A^*$ clearly implies that of $A$. Conversely, equality $A(t+1) = A(t)$ implies that $\varprojlim(A_i(t+1)) = \varprojlim(A_i(t))$, thus

$$(2) \qquad A_i(t+1) \circ \rho_i = A_i(t) \circ \rho_i; \quad i \in \mathbb{N}, \ t \in I,$$

where $\rho_i : \mathbb{F} \longrightarrow \mathbb{E}_i$ are the canonical projections. Taking into account that any element of $\mathbb{E}_i$ is the limit of a sequence of elements of $\mathbb{F}/\operatorname{Ker}(p_i)$ and $\mathbb{F}$ is projected onto the previous quotient (for each $i \in \mathbb{N}$), (2) results in $A_i(t+1) = A_i(t)$, thus ensuring the periodicity of $A^*$. $\qquad\square$

In the proof of the main Theorem we shall also need the following auxiliary result

LEMMA 2.2 ([4, Proposition 3.1]). *Let $f \in \mathcal{L}(\mathbb{F})$. Then $f = \varprojlim f_i$, $f_i \in \mathcal{L}(\mathbb{E}_i)$, if and only if, for each $i \in \mathbb{N}$, there exists $M_i > 0$ such that*

$$p_i(f(u)) \leq M_i \cdot p_i(u), \quad u \in \mathbb{F}.$$

From Lemma 2.1, it follows that each differential equation of type $(1.i)$ has periodic coefficient. Therefore, the corresponding monodromy homomorphisms ([1,6]) have the form

$$\alpha_i^{\#} : \mathbb{Z} \longrightarrow GL(\mathbb{E}_i) : n \mapsto \Phi_i(n),$$

where $\Phi_i$ is the fundamental solution (resolvent) of $(1.i)$.

Since the solutions of (1) are limits of solutions of the projective system of equations $(1.i)$, the family $(\alpha_i^{\#}(n))_{i \in \mathbb{N}}$, for each $n \in \mathbb{Z}$, forms a projective system. Hence we can define the homomorphism

$$(3) \qquad \alpha^{\#} : \mathbb{Z} \longrightarrow H^o(\mathbb{F}) : n \mapsto (\alpha_i^{\#}(n))_{i \in \mathbb{N}}.$$

The previous homomorphism is the key to our approach as replacing the classical monodromy homomorphism with values in $GL(\mathbb{F})$. The latter, as is well known, is useless in the Fréchet framework since it does not admit even a topological group structure.

We are now in a position to prove the first main result of this note.

THEOREM 2.3 (Floquet-Liapunov). *Assume that the coefficient A of (1) is periodic. Then the following conditions are equivalent:*
(i) *Equation (1) reduces to an (equivalent) equation*

$$\dot{y} = B \cdot y, \quad B \ constant$$

*by means of a periodic transformation $y = (\varepsilon \circ Q^*)(t) \cdot x$, for some continuous $Q^* : \mathbb{R} \longrightarrow H^o(\mathbb{F})$.*
(ii) *There exists $\overline{B} \in H(\mathbb{F})$ such that $\operatorname{Exp}(\overline{B}) = \alpha^{\#}(1)$.*
(iii) *The monodromy homomorphism $\alpha^{\#}$ can be extended to a homomorphism $F : \mathbb{R} \longrightarrow H^o(\mathbb{F})$.*

PROOF. Assume that (i) holds true. Then, setting $Q := \varepsilon \circ Q^*$, we see that $Q(t) = \varprojlim(Q_i(t))$, for every $t \in \mathbb{R}$, where $Q_i : \mathbb{R} \longrightarrow GL(\mathbb{E}_i)$. Since $B \cdot Q = Q \cdot \dot{x} + \dot{Q} \cdot x$ (as a result of the transformation $Q$) we check that

$$(4) \qquad\qquad B(u) = Q(0)(A(0) \cdot u) + \dot{Q}(0) \cdot u, \quad u \in \mathbb{F}.$$

By Lemma 2.2, along with $A(0) = \varprojlim(A_i(0))$ and $Q(0) = \varprojlim(Q_i(0))$, (4) implies that $B = \varprojlim B_i$, where $B_i \in \mathcal{L}(\mathbb{E}_i)$. Transforming now each (1.$i$) via $y_i = Q_i(t) \cdot x_i$, we observe that

$$\dot{y}_i = \rho_i(\dot{y}) = \rho_i(B \cdot y) = B_i \cdot \rho_i(y) = B_i \cdot y_i$$

with $B$ constant. Therefore, applying the Theorem of Floquet in Banach spaces, $\exp_i(B_i) = \alpha_i^\#(1)$, $i \in \mathbb{N}$. Hence, $\mathrm{Exp}((B_i)_{i \in \mathbb{N}}) = \alpha^\#(1)$, which proves (ii) for $\overline{B} := (B_i)_{i \in \mathbb{N}}$.

If (ii) holds, $\overline{B}$ has necessarily the form $\overline{B} := (B_i)_{i \in \mathbb{N}}$, with $B_i \in \mathcal{L}(\mathbb{E}_i)$ and $\exp_i(B_i) = \alpha_i^\#(1)$. Therefore, once again by Floquet's Theorem, there exists a homomorphism $F_i : \mathbb{R} \longrightarrow GL(\mathbb{E}_i)$ extending $\alpha_i^\#$. Since $F_i(t) = \exp_i(t \cdot B_i)$, we check that $(F_i(t))_{i \in \mathbb{N}}$ is a projective system, for any $t \in \mathbb{R}$, thus we obtain condition (iii) by defining

$$F : \mathbb{R} \longrightarrow H^o(\mathbb{F}) : t \mapsto (F_i(t))_{i \in \mathbb{N}}.$$

Finally, (iii) implies (i) as follows: For an $F$ as in the statement, we have that $F = (F_i)_{i \in \mathbb{N}}$ with $F_i | \mathbb{Z} = \alpha_i^\#$. Thus, each equation (1.$i$) (with periodic coefficient) reduces to the equation with constant coefficient

$$(5) \qquad\qquad \dot{y}_i = B_i \cdot y_i; \quad i \in \mathbb{N},$$

by means of the transformation $y_i = Q_i(t) \cdot x_i$, where

$$Q_i(t) = \exp_i(t B_i) \cdot \Phi_i^{-1}(t),$$

and $B_i = \log_i(\alpha_i^\#(1))$. We check that $(Q_i(t))_{i \in \mathbb{N}}$ is a projective system, for each $t \in \mathbb{R}$, and we set $Q^*(t) := (Q_i(t))_{i \in \mathbb{N}}$. On the other hand, for any $j \geq i$, $B_j(u) = \dot{y}_j(0)$, where $y_j$ is the solution of (5) with initial condition $(0, u)$. Since $x_j = Q_j(t)^{-1} \cdot y_j$ is a solution of (1.$i$) with coefficient $A_j(t)$, clearly $\rho_{ji} \circ x_j$ is a solution of (1.$i$) with coefficient $A_i(t)$. Therefore,

$$y_i = Q_i(t)(\rho_{ji} \circ x_j)$$

is the solution of (5) with coefficient $B_i$ and initial condition $(0, \rho_{ji}(u))$. Moreover, $\rho_{ji} \circ y_j = y_i$. Hence,

$$(B_i \circ \rho_{ji})(u) = \dot{y}_i(0) = \rho_{ji}(\dot{y}_j(0)) = (\rho_{ji} \circ B_j)(u), \quad u \in \mathbb{E}_j.$$

As a result, $B := \varprojlim B_i \in \mathcal{L}(\mathbb{F})$ exists and is constant. Applying now the transformation $y = (\varepsilon \circ Q^*)(t) \cdot x$, we see that

$$\dot{y}(t) = ((\varprojlim Q_i(t)) \cdot (x_i(t))_{i \in \mathbb{N}})^{\cdot} = ((Q_i(t)(x_i(t)))^{\cdot})_{i \in \mathbb{N}} =$$
$$= (\dot{y}_i(t))_{i \in \mathbb{N}} = (B_i \cdot y_i(t))_{i \in \mathbb{N}} = B \cdot y(t),$$

for every $t \in \mathbb{R}$, thus completing the proof. $\qquad\qquad\qquad\qquad\square$

REMARK. We note that in the previous theorem the quantities $B$, $\overline{B}$ do not coincide, as in the Banach case, but they are related via $\varepsilon(\overline{B}) = B$.

## 3. – An application: equations in $\mathbb{C}^\infty$

In this section we shall prove that the conditions of Theorem 2.3 are always satisfied in the case of a periodic equation of type (1) with $\mathbb{F} = \mathbb{C}^\infty$. This illustrates the elementary methods of Section 2 in the concrete case of $\mathbb{C}^\infty$. Here $\mathbb{C}^\infty$ is viewed as the limit $\varprojlim \mathbb{C}^n$, $n \in \mathbb{N}$, with corresponding connecting morphisms $\rho_{ji} : \mathbb{C}^j \longrightarrow \mathbb{C}^i$ $(i, j \in \mathbb{N};\ j \geq i)$ the natural projections.

To this end we first need the following

LEMMA 3.1. *Let* $\{f_n : \mathbb{C}^n \longrightarrow \mathbb{C}^n\}_{n \in \mathbb{N}}$ *be a family of continuous linear mappings and let* $\{M_n \in \mathcal{M}_n(\mathbb{C}^n)\}_{n \in \mathbb{N}}$ *be the corresponding matrices with respect to the natural basis of* $\mathbb{C}^n$. *Then the following conditions are equivalent:*
(i) $\{f_n\}_{n \in \mathbb{N}}$ *is a projective system.*
(ii)

$$M_{n+1} = \begin{pmatrix} M_n & 0 \\ \mu_n & \lambda_n \end{pmatrix}; \quad n \in \mathbb{N},$$

*where* $\mu_n \in \mathbb{C}^n$, *and* $\lambda_n \in \mathbb{C}$.

PROOF. If (i) holds, then $\rho_{n+1,n} \circ f_{n+1} = f_n \circ \rho_{n+1,n}$ $(n \in \mathbb{N})$ implies that

$$f_{n+1}(e_k^{n+1}) = (f_n(e_k^n), \mu_k^n), \quad 1 \leq k \leq n,$$

$$f_{n+1}(e_{n+1}^{n+1}) = (0, \lambda_n),$$

where $\{e_k^n\}_{1 \leq k \leq n}$ is the natural basis of $\mathbb{C}^n$ and $\mu_k^n, \lambda_n \in \mathbb{C}$. Therefore,

$$M_{n+1} = \left( \begin{pmatrix} f_n(e_k^n) \\ \mu_k^n \end{pmatrix}_{1 \leq k \leq n} \quad \begin{matrix} 0 \\ \lambda_n \end{matrix} \right) = \begin{pmatrix} M_n & 0 \\ (\mu_k^n)_{1 \leq k \leq n} & \lambda_n \end{pmatrix}$$

which gives (ii) for $\mu_n := (\mu_k^n)_{1 \leq k \leq n}$.

Conversely, if we assume (ii), then for every $(x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1}$,

$$(\rho_{n+1,n} \circ f_{n+1})(x_1, \ldots, x_{n+1}) = \rho_{n+1,n} \left( M_{n+1} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \right) =$$

$$= M_n \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (f_n \circ \rho_{n+1,n})(x_1, \ldots, x_{n+1})$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now in a position to show that the exact analogue of the classical Floquet-Liapunov theorem is true in $\mathbb{C}^\infty$. More precisely, we have

THEOREM 3.2. *Consider the equation* (1) *with a periodic coefficient of the form* $A : I \longrightarrow \mathcal{L}(\mathbb{C}^\infty)$. *Then, the (equivalent) conditions of Theorem 2.3 are always satisfied.*

PROOF. Clearly, it suffices to prove the existence of a projective system $(B_i)_{i \in \mathbb{N}} \in H(\mathbb{C}^\infty)$ such that $\mathrm{Exp}((B_i)_{i \in \mathbb{N}}) = \alpha^{\#}(1)$. As we have seen in Section 2 (see also (3)), $\alpha^{\#}(1) = \varprojlim(\alpha_n^{\#}(1))$. For the sake of simplicity we set

$$M_n := \alpha_n^{\#}(1) \in GL(n, \mathbb{C}).$$

The previous matrices satisfy condition (ii) of Lemma 3.1. We define

$$B_1 := \log M_1 = \log \lambda_1 \in \mathbb{C},$$

where $\log \lambda_1$ is arbitrarily chosen but fixed. By induction, we set

$$B_{n+1} := \begin{pmatrix} B_n & 0 \\ y_n & \log \lambda_n \end{pmatrix}, \quad n \in \mathbb{N}.$$

Here $y_n$ is determined through the following equivalent conditions:

$$\mathrm{exp}_{n+1}(B_{n+1}) = M_{n+1} \quad \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} \mathrm{exp}_n(B_n) & 0 \\ y_n \cdot \left( \sum_{k \geq 1} \frac{1}{k!} \left( \sum_{j=1}^{k} B_n^{k-j} \cdot (\log \lambda_n)^{j-1} \right) \right) & \lambda_n \end{pmatrix} = M_{n+1} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad y_n \cdot \left( \sum_{k \geq 1} \frac{1}{k!} \left( \sum_{j=1}^{k} B_n^{k-j} \cdot (\log \lambda_n)^{j-1} \right) \right) = \mu_n.$$

Note that the last equation can be solved since the $(n \times n)$ matrix figuring in it is always non singular. Indeed, the above mentioned matrix is triangular, since $B_n$ is. Therefore, its determinant equals the product of

$$\gamma_i := \sum_{k \geq 1} \frac{1}{k!} \left( \sum_{j=1}^{k} b_i^{k-j} \cdot (\log \lambda_n)^{j-1} \right); \qquad i = 1, \ldots n,$$

where $b_i$ are the diagonals of $B_n$ satisfying $\mathrm{exp}(b_i) = \lambda_i$. If $\lambda_n \neq \lambda_i$ ($1 \leq i \leq n-1$), then $\gamma_i = (\lambda_i - \lambda_n)/(b_i - \log \lambda_n) \neq 0$, for every $i$. If $\lambda_n = \lambda_i$, for some $i$, then $\gamma_i = \lambda_n$ which is also non zero since $M_{n+1} \in GL(n+1, \mathbb{C})$. As a result, the determinant of

$$\sum_{k \geq 1} \frac{1}{k!} \cdot \left( \sum_{j=1}^{k} B_n^{k-j} \cdot (\log \lambda_n)^{j-1} \right)$$

is, in any case, non zero.

Using once again Lemma 3.1, we check that $(B_n)_{n \in \mathbb{N}} \in H(\mathbb{C}^\infty)$. Moreover,

$$\mathrm{Exp}((B_n)_{n \in \mathbb{N}}) = (\mathrm{exp}_n(B_n))_{n \in \mathbb{N}} = \alpha^{\#}(1)$$

and the proof is now complete. □

REMARK. Clearly, the proof of the above Theorem is essentially based on the appropriate choice of the logarithms $B_n$, $n \in \mathbb{N}$, satisfying condition (ii) of Theorem 2.3, instead of considering arbitrarily chosen logarithms of $\alpha_n^{\#}(1)$.

## 4. – A geometric generalization

In this section we briefly describe an application of the previous methods to the geometric framework of Fréchet fiber bundles. Since the complete details are beyond the scope of this paper, we restrict ourselves to a mere outline of these ideas, the Banach analogues of which can be found in [13], [14].

Assume that the coefficient $A$ of (1) is periodic. With the notations of Section 2, if we interpret $\mathbb{Z}$ as the fundamental group of the unit circle; that is, $\mathbb{Z} \cong \pi_1(S^1)$), and $\mathbb{R}$ as the universal covering space of $S^1$ (: $\mathbb{R} \cong \widetilde{S^1}$), then the corresponding fundamental solution of (1), given by

$$\Phi : \mathbb{R} \longrightarrow H^o(\mathbb{F}) : t \mapsto (\Phi_i(t))_{i \in \mathbb{N}},$$

determines the $H(\mathbb{F})$-valued smooth 1-form

$$\widetilde{\theta} := d\Phi \cdot \Phi^{-1} \in \Lambda^1(\widetilde{S^1}, H(\mathbb{F})).$$

We recall that $d\Phi \cdot \Phi^{-1}$ denotes the right total differential

$$(d\Phi \cdot \Phi^{-1})_x = (dR_{\Phi(x)^{-1}})_{\Phi(x)} \circ (d\Phi)_x; \quad x \in \mathbb{R} \cong \widetilde{S^1},$$

where $R_{\Phi(x)^{-1}}$ is the right translation of $H^o(\mathbb{F})$ by $\Phi(x)^{-1}$.

Using projective limits, along with [13] and [14,] we check that there exists an *integrable* $\theta \in \Lambda^1(S^1, H(\mathbb{F}))$ (that is, $d\theta = 1/2 \cdot [\theta, \theta]$) such that

$$p^*\theta = \widetilde{\theta},$$

where $p : \widetilde{S^1} \cong \mathbb{R} \longrightarrow S^1 : t \mapsto \exp(2\pi i t)$. Then, we can find a unique connection $\omega_\Phi$ on the trivial bundle $\ell_o = (S^1 \times H^o(\mathbb{F}), H^o(\mathbb{F}), S^1, pr_1)$ with unique (local) connection form $-\theta$, i.e. $-\theta = \sigma^*\omega_\Phi$, if $\sigma : S^1 \longrightarrow S^1 \times H^o(\mathbb{F})$ is the natural section of the bundle. It turns out that $\omega_\Phi$ is a flat connection with corresponding holonomy homomorphism coinciding, up to conjugation, with $\alpha^\#$. Moreover, the coefficient of (1) is related with $\omega$ by

$$A(t) = ((\sigma \circ p)^*\omega_\Phi)(\partial_t),$$

if $\partial$ is the fundamental vector field of $\mathbb{R}$. The converse is also true. Thus, there is a bijection between flat connections on $\ell_o$ and equations of type (1) with periodic coefficients. Similarly, we can construct an $\omega_F$ for any morphism $F : \mathbb{R} \longrightarrow H^o(\mathbb{F})$. Now, the geometric analogue of Theorem 2.3 can be stated as follows:

> *If the monodromy homomorphism $\alpha^\#$ of (1) (with periodic coefficient) extends to a morphism $F : \mathbb{R} \longrightarrow H^o(\mathbb{F})$, then the connection $\omega_F$ corresponds to a constant coefficient $B$.*

We note that $\omega_F$, $\omega_\Phi$ are gauge-equivalent and that the coefficient $B$ is precisely the coefficient of Theorem 2.3.

The previous result could motivate an analogous study on arbitrary (non trivial) $H^o(\mathbb{F})$-bundles. This idea has been applied in [15]

# REFERENCES

[1]   H. AMANN, "Ordinary Differential Equations", W. de Gruyter, Berlin, 1990.

[2]   V. I. BOGACHEV, *Deterministic and stochastic differential equations in infinite-dimensional spaces*, Acta Appl. Math. **40** (1995), 25-93.

[3]   H. CARTAN, "Differential Calculus", Hermann, Paris, 1971.

[4]   G. GALANIS, *On a type of linear differential equations in Fréchet spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24** (1997), 501-510.

[5]   R. S. HAMILTON, *The Inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. **7** (1982), 65-222.

[6]   S. G. KREIN – N. I. JACKIN, "Linear Differential Equations on Manifolds", Veronez. Gos. Univ., 1980 (in Russian).

[7]   R. LEMMERT, *On ordinary differential equations in locally convex spaces*, Nonlinear Anal. **10** (1986), 1385-1390.

[8]   J. A. LESLIE, *On a differential structure for the group of diffeomorphisms*, Topology **6** (1967), 263-271.

[9]   J. L. MASSERA – J. J. SCHAEFFER, "Linear Differential Equations and Function Spaces", Academic Press, London, 1966.

[10]  L. S. PONTRYAGIN, "Ordinary Differential Equations", Addison-Wesley, Reading Mass., 1962.

[11]  H. H. SCHAEFFER, "Topological Vector Spaces", Springer-Verlag, 1980.

[12]  J. J. SCHAEFFER, *On Floquet's theorem in Hilbert spaces*, Bull. Amer. Math. Soc. **70** (1964), 243-245.

[13]  E. VASSILIOU, *Floquet-type connections on principal bundles*, Tensor (N.S.) **43** (1986), 189-195.

[14]  E. VASSILIOU, *Total differential equations and the structure of fibre bundles*, Bull. Greek Math. Soc. **27** (1986), 149-159.

[15]  E. VASSILIOU – G. GALANIS, *Flat Fréchet principal bundles and their holonomy homomorphisms*, (submitted).

[16]  H. K. WILSON, "Ordinary Differential Equations", Addison-Wesley, Reading Mass., 1971.

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