ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 27, nº 2 (1998), p. 289-307

<http://www.numdam.org/item?id=ASNSP_1998_4_27_2_289_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVII (1998), pp. 289-307

Realizing Vector Fields Without Loss of Derivatives

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Abstract. In this paper we consider the scalar parabolic equation

$$\begin{aligned} u_t &= Lu + f(x, u, \nabla u), && t > 0,\ x \in \Omega \\ u(x,t) &= 0, && t > 0,\ x \in \partial\Omega \end{aligned}$$

on an open bounded set $\Omega \subset \mathbb{R}^N$, as well as the delay equation

$$\dot{y}(t) = Ly_t + F(y_t)$$

in $\mathbb{R}^n$. Using an idea of Poláčik and Rybakowski [18], we give a new elementary proof of the vector field realization results of Rybakowski [24], [25], which avoids the use of Nash-Moser theorem and the consequent loss of derivatives.

Mathematics Subject Classification (1991): 34K15 (primary), 34C30, 35K60, 58F39 (secondary).

1. – Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let us consider the scalar parabolic equation

$$\begin{aligned} u_t &= Lu + f(x, u, \nabla u), && t > 0,\ x \in \Omega \\ u(x,t) &= 0, && t > 0,\ x \in \partial\Omega \end{aligned} \tag{1.1}$$

where L is a second order differential operator and f is some nonlinearity. Also, let $r > 0$ and let us consider the functional differential equation

$$\dot{y}(t) = Ly_t + F(y_t) \tag{1.2}$$

in $\mathbb{R}^n$, where $L : C^0([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a linear operator and $F : C^0([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a nonlinearity having the special form

$$F(\phi) = f\big(\phi(0), \phi(-r_1), \dots, \phi(-r_l)\big), \tag{1.3}$$

Pervenuto alla Redazione il 26 giugno 1998 e in forma definitiva il 25 settembre 1998.

for some $0 < r_1 < \ldots < r_l \leq r$. It is well known that both equation (1.1) and (1.2) define a local semiflow in an appropriate functional space (see Sections 2 and 3 below for a precise setting). It is then interesting to investigate the complexity of the dynamical systems generated by these classes of equations. In both cases, though the phase space is infinite dimensional, the special structure of the equation that generates the dynamical system could in principle impose some restrictions on the dynamics of the corresponding semiflow. For example, this is the case for equation (1.1) when $N = 1$: infact, when $N = 1$, it is known that all bounded solutions of (1.1) converge to an equilibrium (see e.g. [12]). In the last years various authors have enquired about the complexity of the dynamics on finite dimensional invariant manifolds of the dynamical systems generated by (1.1) and (1.2), comparing it to the complexity observed in the dynamics of finite dimensional ODEs. The first result in this direction is due to J. Hale: in [10] he proved that, if the linear operator L in (1.2) has N eigenvalues on the imaginary axis ($N \geq 1$), then the flow generated by equation (1.2), restricted to a local center manifold at 0, is equivalent to a higher order scalar ODE of the form

$$z^{(N)} + a_1 z^{(N-1)} + \cdots + a_{N-1} z^{(1)} + a_N z = v_F(z, z^{(1)}, \ldots, z^{(N-1)}). \tag{1.4}$$

Moreover, if $l = N - 1$ in (1.3), any finite jet of a map $v : \mathbb{R}^N \to \mathbb{R}$ can be realized in (1.4) by an appropriate nonlinearity F. In [3] Faria and Magalhães, using their normal form theory [4], [5], extend this result to delay equations in $\mathbb{R}^n$. The case of parabolic equations was first considered by Poláčik: in [13] he proved that, if the operator L in (1.1) has an n-dimensional kernel ($n = N + 1$), and if the corresponding eigenfunctions satisfy a certain nondegeneracy condition, then every finite jet of a vector field $v : \mathbb{R}^n \to \mathbb{R}^n$ can be realized on the center manifold of equation (1.1). In [16] the same author improved the previous result and showed that, for a fixed $n \in \mathbb{N}$, two space dimensions are enough to obtain realizability of all jets on $\mathbb{R}^n$. As a consequence, he showed that there exists a dense subset (in the C^1-topology) of vector fields on $\mathbb{R}^n$, which can be realized (up to flow equivalence) on center manifolds of equations of type (1.1) in a two-dimensional domain Ω. For more jet-realization results, see the Reference section. In particular, the recent paper [2] proves realization of jets (and consequently of a dense subset of vector fields) in the class of spatially homogeneous equations, i.e. equations of the form $u_t = \Delta u + f(u, \nabla u)$.

Though jet realizations are an important step for a good understanding of the dynamics of (1.1) and (1.2), they are not sufficient to capture certain degenerate (but even more interesting) phenomena occurring in ODEs. This motivates the efforts towards realizing vector fields in parabolic equations as well as in delay equations. The first result in this direction is due to Poláčik: in [14] he proved that every vector field in $\mathbb{R}^{N+1}$ can be realized on some invariant manifold of an appropriate Neumann problem on an open set $\Omega \subset \mathbb{R}^N$. In [24], [25], Rybakowski proved a general result on realization of vector fields on center manifolds of parabolic equations as well as of delay equations. His technique, based on the Nash-Moser theorem, leads to a loss of derivatives and imposes certain "non-natural" restrictions on the smoothness of the vector field v and of the nonlinearity f (resp. F). Namely, Rybakowski proved that, if $m \geq 17$, then every vector field $v \in C^{m+15}$ can be realized on the center manifold of (1.1) (resp. (1.2)) by a suitable nonlinearity f (resp. F) of class C^m. An improvement of this result for parabolic equations was given by Poláčik and Rybakowski in [18]: in that paper they proved that every vector field in $\mathbb{R}^{N+1}$ can be realized on some invariant manifold of an appropriate Dirichlet problem on an open subset $\Omega \subset \mathbb{R}^N$, without any loss of derivatives. Also, as it was pointed out in [21], this invariant manifold is actually a center manifold, though it is obtained via a noncanonical imbedding.

In this paper we use an idea of Poláčik and Rybakowski [18] to give a short and surprisingly simple proof of the realization results of Rybakowski [24], [25]. Our proof is elementary and avoids the use of Nash-Moser theorem with the consequent loss of derivatives. Thus we improve the realization result for delay equations [25], eliminating all "non-natural" smoothness assumptions, and we improve also the result for parabolic equations [18], since we do not need to introduce noncanonical imbeddings of the center manifold.

ACKNOWLEDGEMENTS. I would like to thank Prof. K. P. Rybakowski for suggesting the problem to me and for all our stimulating and fruitful discussions.

2. – Semilinear parabolic equations

Throughout this section let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\rho}$ with $0 < \rho < 1$. Let L be a differential operator of the form

$$Lu = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j u) + au \,.$$

We assume throughout that L is uniformly elliptic and its coefficient functions satisfy $a_{ij} \in C^{1,\rho}(\overline{\Omega})$, $a_{ij} = a_{ji}$, $i, j = 1, \ldots, N$, and $a \in C^{\rho}(\overline{\Omega})$. Consider the semilinear parabolic equation (1.1). In order study this equation, we shall rewrite this problem in a more abstract way. Set

$$X := L^p(\Omega), \quad \text{for some } p > N \,.$$

The operator L with Dirichlet boundary conditions on $\partial\Omega$ defines a sectorial operator A on X with domain $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$; it is well known that A is the generator of an analytic semigroup e^{At}, $t \geq 0$, of linear operators. Moreover, the operator $-A$ generates the corresponding family X^α of fractional power spaces and fixing α with

$$(N+p)/(2p) < \alpha < 1$$

we have that

$$X^\alpha \subset C^1(\overline{\Omega})$$

with continuous inclusion. Let Y_m be the set of all functions

$$f\colon (x,s,w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto f(x,s,w) \in \mathbb{R}$$

such that for all $0 \le k \le m$ the Fréchet derivative $\mathrm{D}^k_{(s,w)} f$ exists and is continuous and bounded on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$.

Y_m is a linear space which becomes a Banach space when endowed with the norm

$$|f|_m := \sup_{(x,s,w)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N} \sup_{0\le k\le m} |\mathrm{D}^k_{(s,w)} f(x,s,w)|_{\mathcal{L}^k((\mathbb{R}\times\mathbb{R}^N)^k,\mathbb{R})}\,.$$

For $f \in Y_m$ the formula

$$\hat{f}(y)(x) := f\big(x, y(x), \nabla y(x)\big), \quad y \in X^\alpha, x \in \overline{\Omega} \tag{2.1}$$

defines the Nemitski operator

$$\hat{f}\colon X^\alpha \to X$$

of class C^m_b. We can rewrite problem (1.1) in the form

$$\dot{y} = Ay + \hat{f}(y)\,. \tag{2.2}$$

It is well known that equation (2.2) defines a local semiflow in the space X^α (see e.g. [11]), so concepts like those of local and global center manifolds are well defined for equation (2.2).

Define

$$X_0 := \ker A$$

and suppose $n := \dim X_0 \ge 1$. Since L is formally selfadjoint, and Ω is bounded, the spectrum $\sigma(A)$ of A consists of a sequence $(\lambda_i)_{i\in\mathbb{N}}$, $\lambda_i \to -\infty$ as $i \to \infty$, of real eigenvalues with the same (finite) geometric and algebraic multiplicity. So X_0 is the invariant subspace corresponding to the spectral set $\{0\}$ and, if $\phi_1, \ldots, \phi_n$ is an $L^2(\Omega)$-orthonormal basis of $\ker A$, then the spectral projection P_0 on X_0 is given by the formula

$$P_0 u := \sum_{j=1}^n \phi_j \int_\Omega u(x)\phi_j(x)dx\,. \tag{2.3}$$

Write

$$\phi(x) := \big(\phi_1(x), \ldots, \phi_n(x)\big)\,.$$

Note the assignement

$$Q\colon \mathbb{R}^n \to X_0, \qquad Q\xi := \xi \cdot \phi = \sum_{i=1}^{n} \xi_i \phi_i$$

is a linear isomorphism.

Now, let X_- (resp. X_+) be the eigenspace of all negative (resp. positive) eigenvalues of A. The subspaces X_+ and X_- are A-invariant, and $X = X_- \oplus X_0 \oplus X_+$. Let P_- (resp. P_+) be the spectral projection onto X_- (resp. X_+). Set

$$A_- := A|X_- \qquad A_+ := A|X_+\,.$$

Note X_+ is finite dimensional, so A_+ is bounded on X_+ and hence it generates a C^0-group e^{A_+t}, $t \in \mathbb{R}$, of linear operators. Moreover, A_- is sectorial on X_- and so it generates an analytic semigroup e^{A_-t}, $t \geq 0$, of linear operators. Let $\mu > 0$ be such that

$$0 < \mu < \min\big\{|\mathrm{Re}\sigma(A_-)|, \mathrm{Re}\sigma(A_+)\big\}\,;$$

then the following estimates hold:

$$\begin{aligned}
&\left\|e^{A_+t}\right\|_{\mathcal{L}(X_+,X_+)} \leq Ee^{\mu t}, && t \leq 0\,,\\
(2.4)\qquad &\left\|e^{A_-t}\right\|_{\mathcal{L}(X_-,X_-)} \leq Ee^{-\mu t}, && t \geq 0\,,\\
&\left\|e^{A_-t}\right\|_{\mathcal{L}(X_-,X_-^{\alpha})} \leq Et^{-\alpha}e^{-\mu t}, && t > 0.
\end{aligned}$$

For $\delta > 0$ and $m \geq 1$ define

$$\mathcal{V}(\delta) := \left\{ f \in Y_1 \mid \sup_{(x,s,w)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N} \left|D_{(s,w)} f(x,s,w)\right|_{\mathcal{L}((\mathbb{R}\times\mathbb{R}^N),\mathbb{R})} < \delta \right\}$$

and

$$\mathcal{V}_m(\delta) := \mathcal{V}(\delta) \cap Y_m\,.$$

The following result is well known (see e.g. [1] and [23]):

THEOREM 2.1. *For every $m \in \mathbb{N}$, there exist a positive constant δ_m and a map*

$$\Gamma\colon (f, \xi) \in \mathcal{V}_m(\delta_m) \times \mathbb{R}^n \mapsto \Gamma_f(\xi) \in X_-^{\alpha} \oplus X_+$$

satisfying the following properties:

(1) *Γ is of class C^m;*

(2) *For every* $f \in \mathcal{V}_m(\delta_m)$ *the set*

$$\mathcal{M}_f := \{Q\xi + \Gamma_f(\xi) \mid \xi \in \mathbb{R}^n\}$$

is the global center manifold of (2.2).

The map Γ_f *has the following characterization:* Γ_f *is the only bounded Lipschitz continuous map* $\Gamma : \mathbb{R}^n \to X_-^\alpha \oplus X_+$ *such that:*

$$\Gamma(\xi_0) = \int_{-\infty}^{0} e^{A_-(-s)} P_- \hat{f}\big(Q\xi(s) + \Gamma(\xi(s))\big) ds - \int_0^{+\infty} e^{A_+(-s)} P_+ \hat{f}\big(Q\xi(s) + \Gamma(\xi(s))\big) ds\,,$$

where $\xi(s)$ *is the solution of the ODE in* $\mathbb{R}^n$

$$\begin{aligned} \dot{\xi} &= Q^{-1} P_0 \hat{f}\big(Q\xi + \Gamma(\xi)\big) \\ \xi(0) &= \xi_0\,. \end{aligned}$$

Moreover, if $v_f\colon \mathbb{R}^n \to \mathbb{R}^n$ *is defined by*

$$v_f(\xi) := Q^{-1} P_0 \hat{f}\big(Q\xi + \Gamma_f(\xi)\big), \quad \xi \in \mathbb{R}^n\,,$$

then the flow on $\mathcal{M}_f$ *is governed by the ODE*

$$\dot{\xi} = v_f(\xi)\,, \tag{2.5}$$

in the sense that if $s \mapsto \xi(s)$ *is a solution of* (2.5), *then* $s \mapsto Q\xi(s) + \Gamma_f(\xi(s))$ *is a solution of* (2.2).

If $n = N$ or $n = N+1$, it turns out that the vector field v_f is arbitrary in the following sense: given any sufficiently small map $v : \mathbb{R}^n \to \mathbb{R}^n$ of class C_b^m, there exists an appropriate nonlinearity $f \in Y_m$ such that $v_f = v$.

Let us recall the following fundamental concept:

DEFINITION 2.2. We say that the operator L satisfies the *Poláčik condition on* Ω if $\dim\ker A = N+1$ and for some (hence every) basis $\phi_1, \ldots, \phi_{N+1}$ of $\ker A$, $R(x) \neq 0$ for some $x \in \Omega$, where

$$R(\phi_1, \ldots, \phi_{N+1})(x) := \det \begin{pmatrix} \phi_1(x) & \nabla\phi_1(x) \\ \vdots & \vdots \\ \phi_{N+1}(x) & \nabla\phi_{N+1}(x) \end{pmatrix}, \quad x \in \Omega\,.$$

REMARK. We have $n = N+1$ in case the Poláčik condition holds. One can also define a (weaker and less interesting) version of the Poláčik condition with $n = N$ (cf. [24]).

REMARK. The existence of a potential $a(x)$ such that the operator $\Delta + a$ satisfies the Poláčik condition on the open set Ω was first proved by Poláčik in [13] and by Poláčik and Rybakowski in [18], when Ω is a ball in $\mathbb{R}^N$. In [21] and [22] the existence of such potentials was proved for arbitrary open sets in $\mathbb{R}^N$ and for second order differential operators with arbitrary principal part.

For $m \in \mathbb{N}_0$, let $C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ be the set of all maps

$$v: \mathbb{R}^n \to \mathbb{R}^n$$

such that, for all $0 \le k \le m$, the Fréchet derivative $\mathrm{D}^k v$ exists and is continuous and bounded on $\mathbb{R}^n$.

$C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ is a linear space which becomes a Banach space when endowed with the norm

$$|v|_m := \sup_{y \in \mathbb{R}^n} \sup_{0 \le k \le m} \left|\mathrm{D}^k v(y)\right|_{\mathcal{L}^k((\mathbb{R}^n)^k, \mathbb{R}^n)}.$$

Now we can state and prove the main theorem of this section.

THEOREM 2.3. *Assume L satisfies the Poláčik condition on Ω. Then there exists $\eta > 0$ such that for every $v \in C_b^1(\mathbb{R}^n, \mathbb{R}^n)$, with $|v|_1 \le \eta$, there exists $f \in \mathcal{V}_1(\delta_1)$ such that*

$$Q^{-1} P_0 \hat{f}\big(Q\xi + \Gamma_f(\xi)\big) = v(\xi), \quad \xi \in \mathbb{R}^n.$$

Moreover, for every $m \ge 1$, there exists $\eta_m > 0$ such that, if $v \in C_b^m(\mathbb{R}^n, \mathbb{R}^n)$, with $|v|_m \le \eta_m$, then f can be chosen in $\mathcal{V}_m(\delta_m)$.

Before proving the theorem, we cite two lemmas from [18]. Let B be a $n \times n$ matrix, such that $|e^{tB}| \le Ce^{\gamma|t|}$ for $t \in \mathbb{R}$, where γ and C are positive constants. Let $v : \mathbb{R}^n \to \mathbb{R}^n$ be a globally Lipschitz continuous vector field. We indicate by $\pi_v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ the global flow generated by the equation $\dot{\xi} = B\xi + v(\xi)$. By differentiating this equation along solutions and by applying Gronwall's Lemma, we get:

LEMMA 2.4. *For every $m \in \mathbb{N}$, there is a constant $\tilde{c}_m$ such that, for every vector field $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$, the flow π_v defined above is of class C^m and, for every $(t, \xi_0) \in \mathbb{R} \times \mathbb{R}^{N+1}$,*

$$\left|D_\xi^m \pi_v(t, \xi_0)\right|_{\mathcal{L}^m((\mathbb{R}^{N+1})^m, \mathbb{R}^{N+1})} \le \tilde{c}_m \exp\big((\gamma + mL_m)|t|\big),$$

where $L_m := |v|_{C_b^m}$.

Applying the higher-order chain rule to the composite map $v \circ \pi_v$ and using Lemma 2.4, we obtain:

LEMMA 2.5. *For every $m \in \mathbb{N}$, there is a constant c_m such that, for every vector field $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ and for every $(t, \xi_0) \in \mathbb{R} \times \mathbb{R}^{N+1}$,*

$$\left|D_\xi^m (v \circ \pi_v)(t, \xi_0)\right|_{\mathcal{L}^m((\mathbb{R}^{N+1})^m, \mathbb{R}^{N+1})} \le c_m L_m \exp\big((\gamma + mL_m)|t|\big),$$

where $L_m := |v|_{C_b^m}$.

PROOF OF THEOREM 2.3. Given $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$, $m \geq 1$, we want to find a couple (f, Γ), $f \in \mathcal{V}_m(\delta_m)$, $\Gamma : \mathbb{R}^{N+1} \to X_-^\alpha \oplus X_+$ of class C_b^m, such that:

(1) $Q^{-1}P_0\hat{f}(Q\xi + \Gamma(\xi)) = v(\xi)$, $\xi \in \mathbb{R}^{N+1}$;

(2) $\Gamma(\xi_0) = \int_{-\infty}^0 e^{A_-(-s)}P_-\hat{f}(Q\xi(s) + \Gamma(\xi(s)))ds - \int_0^{+\infty} e^{A_+(-s)}P_+\hat{f}(Q\xi(s) + \Gamma(\xi(s)))ds$, where $\xi(s)$ is the solution of the ODE in $\mathbb{R}^{N+1}$

$$\begin{aligned} \dot{\xi} &= Q^{-1}P_0\hat{f}(Q\xi + \Gamma(\xi)) = v(\xi) \\ \xi(0) &= \xi_0 \,. \end{aligned}$$

Let us fix $\Gamma : \mathbb{R}^{N+1} \to X_-^\alpha \oplus X_+$ of class C_b^m, $m \geq 1$, and look at (1). Using (2.1) and (2.3), (1) reads:

$$\int_\Omega \phi_i(x) f\left(x, \sum_{j=1}^{N+1} \xi_j\phi_j(x) + \Gamma(\xi)(x), \sum_{j=1}^{N+1} \xi_j\nabla\phi_j(x) + \nabla_x\Gamma(\xi)(x)\right) dx = v_i(\xi) \tag{2.6}$$

for $i = 1, \ldots, N+1$. Define the matrix

$$M_x := \begin{pmatrix} \phi_1(x) & \nabla\phi_1(x) \\ \vdots & \vdots \\ \phi_{N+1}(x) & \nabla\phi_{N+1}(x) \end{pmatrix}, \quad x \in \Omega \,.$$

Then (2.6) becomes:

$$\int_\Omega \phi_i(x) f\left(x, \xi^T \cdot M_x + (\Gamma(\xi)(x), \nabla_x\Gamma(\xi)(x))\right) dx = v_i(\xi) \,. \tag{2.7}$$

Since we have assumed that L satisfies the Poláčik condition on Ω, there exists an open subset $\Omega_0 \subset \Omega$ such that M_x^{-1} exists on $\overline{\Omega}_0$ and $|M_x^{-1}| \leq M$ for every $x \in \overline{\Omega}_0$. If

$$\left|D_\xi\Gamma(\xi)\right|_{\mathcal{L}(\mathbb{R}^{N+1}, X^\alpha)} < \frac{1}{M} \,,$$

then

$$\sup_{x\in\overline{\Omega}} \sup_{\xi\in\mathbb{R}^{N+1}} \sup_{h\in\mathbb{R}^{N+1}, |h|=1} \left(\left|D_\xi\Gamma(\xi)[h](x)\right| + \left|\nabla_x D_\xi\Gamma(\xi)[h](x)\right|\right) < \frac{1}{M} \,.$$

This implies that, for all $x \in \overline{\Omega}_0$, the map

$$\xi \to \xi^T \cdot M_x + \left(\Gamma(\xi)(x), \nabla_x\Gamma(\xi)(x)\right)$$

is a C^m-diffeomorphism of $\mathbb{R}^{N+1}$ onto $\mathbb{R}^{N+1}$. (We used the fact that $\Gamma \in C_b^m(\mathbb{R}^{N+1}, X^\alpha)$ and $X^\alpha \subset C^1(\overline{\Omega})$.) This means that there exists a map $a_x : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ such that

$$a_x\left(\xi^T \cdot M_x + (\Gamma(\xi)(x), \nabla_x\Gamma(\xi)(x))\right) \equiv \xi, \quad \xi \in \mathbb{R}^{N+1}, \quad x \in \overline{\Omega}_0 \,.$$

Moreover, for all $j = 0, \dots, m$, $D^j_{(s,w)} a_x(s, w)$ exists and is continuous on $\overline{\Omega}_0 \times \mathbb{R}^{N+1}$.

Now choose functions $b_1, \dots, b_{N+1}$ such that:

(1) $b_i \in C_0^\infty(\Omega_0)$, for $i = 1, \dots, N+1$;

(2) $\int_\Omega b_i(x)\phi_j(x)dx = \delta_{ij}$, for $i, j = 1, \dots, N+1$.

The existence of such functions was proved in [13]. Finally, set

$$f_\Gamma(x, s, w) := \begin{cases} 0, & \text{if } x \notin \Omega_0\,; \\ \sum_{i=1}^{N+1} b_i(x) v_i(a_x(s, w)), & \text{if } x \in \Omega_0\,. \end{cases} \tag{2.8}$$

Notice that $f_\Gamma \in Y_m$ if $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ and $\Gamma \in C_b^m(\mathbb{R}^{N+1}, X^\alpha)$. Notice also that there exist $\alpha_m > 0$, $\beta_m > 0$ such that, if $|v|_m \le \alpha_m$, $|\Gamma|_m \le \beta_m$, then $f_\Gamma \in \mathcal{V}_m(\delta_m)$. The function f_Γ has the following properties:

(1) $\hat{f}_\Gamma(Q\xi + \Gamma(\xi)) = \sum_{i=1}^{N+1} v_i(\xi) b_i$;

(2) $Q^{-1} P_0 \hat{f}_\Gamma(Q\xi + \Gamma(\xi)) = v(\xi)$.

In order to conclude, we have to choose Γ in such a way that the set $\{Q\xi + \Gamma(\xi),\ \xi \in \mathbb{R}^{N+1}\}$ is the center manifold of the equation $\dot{y} = Ay + \hat{f}_\Gamma(y)$, that is $\Gamma = \Gamma_{f_\Gamma}$. This means that Γ must satisfy:

$$\begin{aligned} \Gamma(\xi_0) = & \int_{-\infty}^{0} e^{A_-(-s)} P_- \hat{f}_\Gamma(Q\xi(s) + \Gamma(\xi(s)))ds \\ & - \int_0^{+\infty} e^{A_+(-s)} P_+ \hat{f}_\Gamma(Q\xi(s) + \Gamma(\xi(s)))ds\,, \end{aligned}$$

where $\xi(s)$ is the solution of the ODE in $\mathbb{R}^{N+1}$

$$\begin{aligned} \dot{\xi} &= Q^{-1} P_0 \hat{f}_\Gamma\big(Q\xi + \Gamma(\xi)\big) = v(\xi) \\ \xi(0) &= \xi_0. \end{aligned} \tag{2.9}$$

We have chosen f_Γ in such a way that

$$\hat{f}_\Gamma\big(Q\xi + \Gamma(\xi)\big) = \sum_{i=1}^{N+1} v_i(\xi) b_i\,.$$

So now Γ has to satisfy

$$\begin{aligned} \Gamma(\xi_0) = & \int_{-\infty}^{0} e^{A_-(-s)} P_- \left(\sum_{i=1}^{N+1} v_i(\xi(s)) b_i\right) ds \\ & - \int_0^{+\infty} e^{A_+(-s)} P_+ \left(\sum_{i=1}^{N+1} v_i(\xi(s)) b_i\right) ds\,, \end{aligned} \tag{2.10}$$

where $\xi(s)$ is the solution of (2.9). In the right hand side of (2.10) Γ does not appear in any way. So we can take (2.10) as a definition and (using the notation introduced in Lemmas 2.4. and 2.5) set:

$$\begin{aligned}\Gamma(\xi_0) := \sum_{i=1}^{N+1} \bigg(& \int_{-\infty}^{0} v_i\big(\pi_v(s,\xi_0)\big) e^{A_-(-s)} P_- b_i ds \\ & - \int_0^{+\infty} v_i\big(\pi_i(s,\xi_0)\big) e^{A_+(-s)} P_+ b_i ds \bigg) .\end{aligned} \tag{2.11}$$

Lemmas 2.4 and 2.5, together with the exponential estimates (2.4) for the semigroup e^{At}, easily imply that there exists η_m, $0 < \eta_m < \alpha_m$, such that, if $|v|_m < \eta_m$, then Γ defined by (2.11) is of class C_b^m and $|\Gamma|_m < \min(\beta_m, 1/M)$. So f_Γ defined by (2.8) is in $\mathcal{V}_m(\delta_m)$ and $\Gamma = \Gamma_{f_\Gamma}$. The theorem is proved. □

As a consequence of Theorem 2.3, we finally obtain the following realization result: fixed any sufficiently small vector field $v \in C_b^1(\mathbb{R}^n, \mathbb{R}^n)$, we can find a nonlinearity $f \in Y_1$ such that the flow of equation (2.2), reduced to its (n-dimensional) center manifold, is equivalent to the flow of the ODE

$$\dot{\xi} = v(\xi) \tag{2.12}$$

in $\mathbb{R}^n$.

REMARK. The smallness assumption on the vector field v is not a real restriction. Infact, we can always rescale the time in (2.12) by a small parameter ϵ, without changing the qualitative structure of the flow.

REMARK. An analogue result can be proved without any further effort for the case of Neumann boundary conditions.

3. – Delay equations

In this section we consider the functional differential equation (1.2) in $\mathbb{R}^n$. The linear operator $L : C^0([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is defined by

$$Lu := \int_{-r}^{0} u(\theta) d\eta(\theta) , \tag{3.1}$$

where r is a positive number and η is a Borel measure on $[-r, 0]$. The nonlinearity $F : C^0([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is taken of class C^m.

It is known (see e.g. [9]) that the linear equation

$$\dot{y}(t) = Ly_t \tag{3.2}$$

defines a strongly continuous semigroup $T(s)$ of linear operators in the functional space $C^0([-r,0],\mathbb{R}^n)$, whose infinitesimal generator A is given by

$$\begin{cases} D(A) = \left\{\phi \in C^0([-r,0],\mathbb{R}^n) \ : \ L\phi = \frac{d\phi}{d\theta}(0)\right\} \\ A\phi = \frac{d\phi}{d\theta} \end{cases}$$

If $f : [0,T) \to \mathbb{R}^n$ is in $L^1_{loc}([0,T),\mathbb{R}^n)$, the solution of the non-homogeneus linear equation

$$\begin{aligned} \dot{y}(t) &= Ly_t + f(t) \quad && t \geq 0 \\ y(\theta) &= \phi(\theta) && \theta \in [-r,0] \end{aligned}$$

is given by the abstract variation of constants formula

$$y_t = T(t)\phi + \int_0^t d[K(t,s)]f(s)\,, \tag{3.3}$$

where the kernel $K(t,\cdot) : [0,T) \to C^0([-r,0],\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))$ is given by

$$K(t,s)(\theta) = \int_0^s X(t+\theta-\alpha)d\alpha$$

and $X(t)$ is the fundamental matrix solution of equation (3.2). By using formula (3.3), it is easy to prove that equation (1.2) generates a local semiflow in the space $X = C^0([-r,0],\mathbb{R}^n)$, so that concepts like those of local and global center manifolds are well defined for equation (1.2).

We recall some spectral properties of the operator A (for the details see [9]). The spectrum $\sigma(A)$ of A is pure point and $\lambda \in \sigma(A)$ if and only if

$$\det \Delta(\lambda) = 0, \quad \text{where} \quad \Delta(\lambda) = \lambda I - \int_{-r}^0 d\eta(\theta)e^{\lambda\theta}\,.$$

Let σ_0, σ_-, σ_+ denote the parts of the spectrum of A with respectively null, negative and positive real part. Then $X = C^0([-r,0],\mathbb{R}^n)$ is decomposed into A-invariant subspaces:

$$X = X_0 \oplus X_- \oplus X_+\,.$$

We indicate by P_0, P_-, P_+ the corresponding spectral projections and by A_0, A_-, A_+ the restrictions of A to X_0, X_- and X_+ respectively. We have that $\sigma(A_0) = \sigma_0$, $\sigma(A_-) = \sigma_-$, $\sigma(A+) = \sigma_+$. Since X_0 and X_+ are finite dimensional, A_0 and A_+ generate C^0 groups of linear operators $T_0(s)$ and $T_+(s)$ respectively, $t \in \mathbb{R}$. Moreover, an explicit representation of P_0 and P_+ can be obtained in terms of the eigenvectors of the transpose problem. Set

$C^* := C^0([0,r],\mathbb{R}^{n*})$ and define the transpose operator $A^T : D(A^T) \subset C^* \to C^*$ by

$$\begin{cases} D(A^T) := \left\{ \psi \in C^* \ : \ \frac{d\psi}{d\theta}(0) = -\int_{-r}^{0} \psi(-\theta) d\eta(\theta) \right\} \\ A^T\psi = \frac{d\psi}{d\theta} \end{cases}$$

The spectrum of A^T is pure point and $\lambda \in \sigma(A^T)$ if and only if $\det \Delta(\lambda) = 0$, so that $\sigma(A) = \sigma(A^T)$. Moreover, if $\lambda \in \sigma(A) = \sigma(A^T)$, then λ has the same Jordan structure as an eigenvalue of A and A^T. We indicate by C_0^* and C_+^* the generalised eigenspaces corresponding to the eigenvalues of A^T with null (resp. positive) real part. Let $\phi_1, \ldots, \phi_N$ be a basis of X_0, $\phi_1^+, \ldots, \phi_H^+$ be a basis of X_+, and choose a basis $\psi_1, \ldots, \psi_N$ of C_0^* and a basis $\psi_1^+, \ldots, \psi_H^+$ of C_+^* such that $(\psi_i, \phi_j) = \delta_{ij}$, $i, j = 1, \ldots, N$, $(\psi_i^+, \phi_j^+) = \delta_{ij}$, $i, j = 1, \ldots, H$, where the bilinear form $(\cdot,\cdot) : C^* \times X \to \mathbb{R}$ is defined by

$$(\psi, \phi) := \psi(0)\phi(0) - \int_{-r}^{0}\int_{0}^{\tau} \psi(\tau-\theta) d\eta(\theta)\phi(\tau) d\tau .$$

We set

$$\Phi := (\phi_1, \ldots, \phi_N) \in \mathcal{M}^{n\times N}, \qquad \Psi := \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \in \mathcal{M}^{N\times n} .$$

Moreover, we set

$$\Phi^+ := (\phi_1^+, \ldots, \phi_H^+) \in \mathcal{M}^{n\times H}, \qquad \Psi^+ := \begin{pmatrix} \psi_1^+ \\ \vdots \\ \psi_H^+ \end{pmatrix} \in \mathcal{M}^{H\times n} .$$

Then we have:

$$P_0\phi = \sum_{i=1}^{N} \phi_i(\psi_i, \phi) = \Phi(\Psi, \phi)$$

$$P_+\phi = \sum_{i=1}^{H} \phi_i^+(\psi_i^+, \phi) = \Phi^+(\Psi^+, \phi)$$

The variation of constant formula (3.3) is then decomposed in the following way:

$$\begin{aligned} P_0 y_t &= T_0(t)\Phi(\Psi,\phi) + \int_0^t T_0(t-s)\Phi\Psi(0) f(s) ds \\ P_+ y_t &= T_+(t)\Phi^+(\Psi^+,\phi) + \int_0^t T_+(t-s)\Phi^+\Psi^+(0) f(s) ds \qquad (3.4) \\ P_- y_t &= T(t)P_-\phi + \int_0^t d[K^-(t,s)] f(s) \end{aligned}$$

where

$$K^-(t,s) = K(t,s) - \Phi\big(\Psi, K(t,s)\big) - \Phi^+\big(\Psi^+, K(t,s)\big)\,.$$

Moreover, if B is the matrix that represents A_0 with respect to the basis ϕ_1, ..., ϕ_N, we have

$$(\Psi, y_t) = e^{tB}(\Psi, \phi) + \int_0^t e^{(t-s)B}\Psi(0)f(s)ds\,.$$

This formula gives the coordinates of $P_0 y_t$ with respect to $\phi_1, \dots, \phi_N$.

Let μ be such that

$$0 < \mu < \min\left\{|\mathrm{Re}\sigma_-|, \mathrm{Re}\sigma_+\right\},$$

and let $0 < \gamma < \mu$. Then there exists $E > 0$ such that the following estimates hold:

$$\begin{aligned} |T_0(t)| &\le Ee^{\gamma|t|}, && t \in \mathbb{R} \\ |T_+(t)| &\le Ee^{\mu t}, && t \le 0 \\ |T(t)P_-| &\le Ee^{-\mu t}, && t \ge 0 \\ \mathrm{Var}_{[0,t)}[K^-(t,\cdot)] &\le Ee^{-\mu t}, && t \ge 0 \end{aligned} \tag{3.5}$$

Let us turn back to the nonlinear equation (1.2). We consider the special case of a delay equation. Fix positive numbers $r_1, \dots, r_{l-1}$ such that $0 > -r_1 > \dots > -r_{l-1} \ge -r$, and let $f : \mathbb{R}^{ln} \to \mathbb{R}^n$ be of class C_b^m. We define $F := \hat{f}$, where $\hat{f}$ is the Nemitski operator associated with f:

$$\hat{f}(\phi) := f(\phi(0), \phi(-r_1), \dots, \phi\big(-r_{l-1}\big)), \quad \phi \in C^0([-r,0], \mathbb{R}^n)\,. \tag{3.6}$$

We set

$$\begin{aligned} Y_m &:= C_b^m(\mathbb{R}^{nl}, \mathbb{R}^n), \\ \mathcal{V}(\delta) &:= \{f \in Y_1 \;:\; |Df| \le \delta\} \\ \mathcal{V}_m(\delta) &:= Y_m \cap \mathcal{V}(\delta) \end{aligned}$$

Then, using the variation of constants formula (3.3) and its spectral decomposition (3.4), one can prove the following (see [9]):

THEOREM 3.1. *For every $m \in \mathbb{N}$, there exist a positive constant δ_m and a map*

$$\Gamma\colon (f,\xi) \in \mathcal{V}_m(\delta_m) \times \mathbb{R}^N \mapsto \Gamma_f(\xi) \in X_- \oplus X_+$$

satisfying the following properties:

(1) *Γ is of class C^m;*
(2) *For every $f \in \mathcal{V}_m(\delta_m)$ the set*

$$\mathcal{M}_f := \left\{\Phi\xi + \Gamma_f(\xi) \mid \xi \in \mathbb{R}^N\right\}$$

is the global center manifold of the semiflow generated by (1.2) *in the space* X. *The map* Γ_f *has the following characterization:* Γ_f *is the only bounded Lipschitz continuous map* $\Gamma : \mathbb{R}^N \to X_- \oplus X_+$ *such that:*

$$\Gamma(\xi_0) = \int_{-\infty}^{0} d[K^-(0,s)]\hat{f}\big(\Phi\xi(s) + \Gamma(\xi(s))\big) - \int_{0}^{+\infty} T_+(-s)\Phi^+\Psi^+(0)\hat{f}\big(\Phi\xi(s) + \Gamma(\xi(s))\big)ds\,,$$

where $\xi(s)$ *is the solution of the ODE in* $\mathbb{R}^N$

$$\dot{\xi} = B\xi + \Psi(0)\hat{f}\big(\Phi\xi + \Gamma(\xi)\big)$$
$$\xi(0) = \xi_0$$

Moreover, if $v_f : \mathbb{R}^N \to \mathbb{R}^N$ *is defined by*

$$v_f(\xi) := \Psi(0)\hat{f}\big(\Phi\xi + \Gamma_f(\xi)\big), \quad \xi \in \mathbb{R}^N\,,$$

then the flow on $\mathcal{M}_f$ *is governed by the ODE*

$$\dot{\xi} = B\xi + v_f(\xi)\,, \tag{3.7}$$

in the sense that if $s \mapsto \xi(s)$ *is a solution of* (3.7) *then* $s \mapsto \Phi\xi(s) + \Gamma_f(\xi(s))$ *is a solution of* (1.2).

We briefly recall the analysis of the structure of the vector field

$$B\xi + \Psi(0)\hat{f}\big(\Phi\xi + \Gamma_f(\xi)\big)$$

carried on in [3]. Let $p := \operatorname{rank}\Psi(0)$ and let $\Psi_{j_1}(0), \ldots, \Psi_{j_p}(0)$, $j_1 < \ldots < j_p$, be a set of linearly independent columns of $\Psi(0)$. Then a "good" basis can be found in X_0 in such a way that, with respect to this basis,

$$B = \begin{pmatrix} B_{11} & \ldots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{p1} & \ldots & B_{pp} \end{pmatrix}, \qquad \Psi(0) = \begin{pmatrix} \overline{M}_1 \\ \vdots \\ \overline{M}_p \end{pmatrix},$$

where B_{ii} is a $d_i \times d_i$ matrix in companion form, B_{ij} $(i \neq j)$ has all elements zero, except possibly the ones in the last row and $\overline{M}_i$ is a $d_i \times n$ matrix with all elements zero, except in the last row, where the element in the j_i-th column is 1 and the other elements can be possibly different from zero. More precisely,

$$B_{ii} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ b_1^i & b_2^i & \cdots & b_{d_i}^i \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ b_1^{ij} & \cdots & b_{d_j}^{ij} \end{pmatrix}$$

and

$$\overline{M}_i = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \overline{m}_1^i & \cdots & \overline{m}_{j_i-1}^i & 1 & \overline{m}_{j_i+1}^i & \cdots & \overline{m}_n^i \end{pmatrix}.$$

This means that in $\Psi(0)\hat{f}(\Phi\xi + \Gamma_f(\xi))$ all elements are zero, except the ones in the k_i-th position, $i = 1, \ldots p$, where $k_1 := d_1$, $k_2 := d_1 + d_2, \ldots$, $k_p := d_1 + \cdots + d_p$; these elements are linear combinations of $\hat{f}_1(\Phi\xi + \Gamma_f(\xi))$, $\ldots$, $\hat{f}_n(\Phi\xi + \Gamma_f(\xi))$, with at least one coefficient different from zero. Then, as it was pointed out in [3] and in [10], the reduced equation

$$\dot{\xi} = B\xi + \Psi(0)\hat{f}\big(\Phi\xi + \Gamma_f(\xi)\big)$$

can be interpreted as the normal form of an ODE in $\mathbb{R}^N$ or alternatively as a system of p higher order scalar differential equations, coupled by mean of nonlinear terms, namely:

$$z_i^{(d_i)} - \sum_{\nu=1}^{d_i} b_\nu^i z_i^{(d_i-\nu)} = \sum_{j\neq i}\sum_{\nu=1}^{d_j} b_\nu^{ij} z_j^{(d_j-\nu)} + \sum_{\nu=1}^{n} \overline{m}_\nu^i \hat{f}_\nu\left(\Phi\zeta + \Gamma_f(\zeta)\right), \qquad i = 1, \ldots, p$$

where

$$\zeta := \left(z_1^{(d_1-1)}, \ldots, z_1, \ldots, z_p^{(d_p-1)}, \ldots, z_p\right).$$

The numbers p and d_i and the coefficients b_ν^i, b_ν^{ij}, $\overline{m}_\nu^i$ are independent of the nonlinearity f. In the rest of this section we fix a basis $\phi_1, \ldots, \psi_N$ of X_0 with the above properties. If the number of delays l is sufficiently large, then the nonlinear part

$$\Psi(0)\hat{f}\big(\phi\xi + \Gamma_f(\xi)\big)$$

is arbitrary in the following sense: given any sufficiently small map $v : \mathbb{R}^N \to \mathbb{R}^p$ of class C_b^m, there exists an appropriate nonlinearity $f : \mathbb{R}^{nl} \to \mathbb{R}^n$ of class C_b^m such that

$$\Big[\Psi(0)\hat{f}(\Phi\xi + \Gamma_f(\xi))\Big]_{k_i} = v_i(\xi)$$

for $i = 1, \ldots, p$. The following condition is an analogue of Poláčik condition for parabolic PDEs:

DEFINITION 3.2. We say that the operator L satisfies *condition* ($\star$) with the $l-1$ delays $r_1, \ldots, r_{l-1}$, if $\dim X_0 = N < nl$ and for some (hence every) basis $\phi_1, \ldots, \phi_N$ of X_0, the map

$$R : \xi \in \mathbb{R}^N \mapsto (\Phi(0)\xi, \Phi(-r_1)\xi, \ldots, \Phi(-r_{l-1})\xi) \in \mathbb{R}^{nl}$$

is a linear injection.

In [3], Faria and Magalhães showed that if $q := \operatorname{rank}\Phi(0)$, then, with $l = N - q + 1$, it is possible to find $l-1$ delays $r_1, \ldots, r_{l-1}$ so that that *condition* ($\star$) is satisfied.

Now we can state and prove the main theorem of this section.

THEOREM 3.3. *Assume L satisfies condition* ($\star$) *with the* $l-1$ *delays* $r_1, \dots, r_{l-1}$. *Then there exists* $\eta > 0$ *such that for every* $v \in C_b^1(\mathbb{R}^N, \mathbb{R}^p)$, *with* $|v|_1 \leq \eta$, *there exists* $f \in \mathcal{V}_1(\delta_1)$ *such that*

$$\left[\Psi(0)\hat{f}(\Phi\xi + \Gamma_f(\xi))\right]_{k_i} = v_i(\xi)$$

for $i = 1, \dots, p$. *Moreover, for every* $m \geq 1$, *there exists* $\eta_m > 0$ *such that, if* $v \in C_b^m(\mathbb{R}^N, \mathbb{R}^p)$, *with* $|v|_m \leq \eta_m$, *then* f *can be chosen in* $\mathcal{V}_m(\delta_m)$.

PROOF. Given $v \in C_b^m(\mathbb{R}^N, \mathbb{R}^p)$, $m \geq 1$, we want to find $f \in \mathcal{V}_m(\delta_m)$ such that

$$\left[\Psi(0)\hat{f}(\Phi\xi + \Gamma_f(\xi))\right]_{k_i} \equiv v_i(\xi), \quad \xi \in \mathbb{R}^N,$$

for $i = 1, \dots, p$. We take an $n \times p$ matrix U such that $\Psi(0)U = M$,

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_p \end{pmatrix},$$

where M_i is a $d_i \times p$ matrix whose elements are all zero, except in the last row, where the element in the i-th column is 1. The existence of such a matrix U is obvious. For $\mathrm{f} : \mathbb{R}^{nl} \to \mathbb{R}^p$, define $f : \mathbb{R}^{nl} \to \mathbb{R}^n$ by $f := U\mathrm{f}$; note $\hat{f} := U\hat{\mathrm{f}}$, where

$$\hat{\mathrm{f}}(\phi) := \mathrm{f}(\phi(0), \phi(-r_1), \dots, \phi(-r_{l-1})), \quad \phi \in C^0([-r, 0], \mathbb{R}^n). \tag{3.8}$$

If $\mathrm{f} \in C_b^m(\mathbb{R}^{nl} \to \mathbb{R}^p)$ is chosen in such a way that $f \in \mathcal{V}_m(\delta_m)$ and $\hat{\mathrm{f}}(\Phi\xi + \Gamma_f(\xi)) = v(\xi)$, $\xi \in \mathbb{R}^N$, we are done.

We procede as in the proof of Theorem 2.3. Given $v \in C_b^m(\mathbb{R}^N, \mathbb{R}^p)$, $m \geq 1$, we want to find a couple (f, Γ), $\mathrm{f} \in C_b^m(\mathbb{R}^{nl}, \mathbb{R}^p)$, $f = U\mathrm{f} \in \mathcal{V}_m(\delta_m)$, and $\Gamma : \mathbb{R}^N \to X_- \oplus X_+$ of class C_b^m, such that:

(1) $\hat{\mathrm{f}}(\Phi\xi + \Gamma(\xi)) = v(\xi)$, $\xi \in \mathbb{R}^N$;
(2) $\Gamma(\xi_0) = \int_{-\infty}^0 d[K^-(0,s)]\hat{f}(\Phi\xi(s) + \Gamma(\xi(s))) - \int_0^{+\infty} T_+(-s)\Phi_+\Psi_+(0)\hat{f}(\Phi\xi(s) + \Gamma(\xi(s)))ds$, where $\xi(s)$ is the solution of the ODE in $\mathbb{R}^N$

$$\begin{aligned} \dot{\xi} &= B\xi + \Psi(0)\hat{f}(\Phi\xi + \Gamma(\xi)) = B\xi + Mv(\xi) \\ \xi(0) &= \xi_0. \end{aligned}$$

Let us fix $\Gamma : \mathbb{R}^N \to X_- \oplus X_+$ of class C_b^m, $m \geq 1$, and look at (1). Using (3.8), (1) reads:

$$\begin{aligned} &\mathrm{f}\big(\Phi(0)\xi + \Gamma(\xi)(0), \Phi(-r_1)\xi + \Gamma(\xi)(-r_1), \dots, \Phi(-r_{l-1})\xi \\ &\qquad + \Gamma(\xi)(-r_{l-1})\big) \equiv v(\xi), \end{aligned} \tag{3.9}$$

that is

$$\mathrm{f}\big(R\xi + (\Gamma(\xi)(0), \Gamma(\xi)(-r_1), \dots, \Gamma(\xi)(-r_{l-1}))\big) \equiv v(\xi). \tag{3.10}$$

Since we have assumed that L satisfies *condition* $(\star)$, the map R has a left inverse S, $S : \mathbb{R}^{nl} \to \mathbb{R}^N$, $SR = I_N$. If

$$\left|D\Gamma(\xi)\right|_{\mathcal{L}(\mathbb{R}^N, X)} < \frac{1}{|S|},$$

by the contraction mapping theorem we can find a map $h_\Gamma : \mathbb{R}^N \to \mathbb{R}^N$ of class C_b^m such that

$$h_\Gamma\big(\xi + S(\Gamma(\xi)(0), \Gamma(\xi)(-r_1), \dots, \Gamma(\xi)(-r_{l-1}))\big) \equiv \xi\,.$$

Then

$$h_\Gamma\big(S(R\xi + (\Gamma(\xi)(0), \Gamma(\xi)(-r_1), \dots, \Gamma(\xi)(-r_{l-1})))\big) \equiv \xi\,.$$

We define

$$\mathrm{f}_\Gamma(w) := v\big(h_\Gamma(Sw)\big), \quad w \in \mathbb{R}^{nl}\,. \tag{3.11}$$

Notice that, if $v \in C_b^m(\mathbb{R}^N, \mathbb{R}^p)$ and $\Gamma \in C_b^m(\mathbb{R}^N, X)$, then $\mathrm{f}_\Gamma \in C_b^m(\mathbb{R}^{nl}, \mathbb{R}^p)$, so that $f_\Gamma = U\mathrm{f}_\Gamma \in Y_m$. Note also that there exist $\alpha_m > 0$, $\beta_m > 0$ such that, if $|v|_m \le \alpha_m$, $|\Gamma|_m \le \beta_m$, then $f_\Gamma \in \mathcal{V}_m(\delta_m)$. The map f_Γ has by construction the following property:

$$\hat{\mathrm{f}}_\Gamma(\Phi\xi + \Gamma(\xi)) \equiv v(\xi)\,.$$

In order to conclude, we have to choose Γ in such a way that the set $\{\Phi\xi + \Gamma(\xi),\ \xi \in \mathbb{R}^N\}$ is the global center manifold of the semiflow generated by (1.2) in the space X with $f = f_\Gamma$, that is $\Gamma = \Gamma_{f_\Gamma}$. This means that Γ must satisfy:

$$\begin{aligned}\Gamma(\xi_0) = &\int_{-\infty}^{0} d[K^-(0,s)]\hat{f}_\Gamma(\Phi\xi(s) + \Gamma(\xi(s)))\\ &- \int_0^{+\infty} T_+(-s)\Phi_+\Psi_+(0)\hat{f}_\Gamma(\Phi\xi(s) + \Gamma(\xi(s)))ds\,,\end{aligned}$$

where $\xi(s)$ is the solution of the ODE in $\mathbb{R}^N$

$$\begin{aligned}&\dot{\xi} = B\xi + \Psi(0)\hat{f}_\Gamma(\Phi\xi + \Gamma(\xi)) = B\xi + Mv(\xi)\\ &\xi(0) = \xi_0.\end{aligned} \tag{3.12}$$

We have chosen f_Γ in such a way that

$$\hat{f}_\Gamma(\Phi\xi + \Gamma(\xi)) = Uv(\xi)\,.$$

So now Γ has to satisfy

$$\begin{aligned}\Gamma(\xi_0) = &\int_{-\infty}^{0} d\big[K^-(0,s)\big]Uv\big(\xi(s)\big)\\ &- \int_0^{+\infty} T_+(-s)\Phi_+\Psi_+(0)Uv\big(\xi(s)\big)ds\,,\end{aligned} \tag{3.13}$$

where $\xi(s)$ is the solution of (3.12). In the right hand side of (3.13) Γ does not appear in any way. So we can take (3.13) as a definition and (using the notation introduced in Lemmas 2.4 and 2.5) set:

$$\tag{3.14} \begin{aligned} \Gamma(\xi_0) = & \int_{-\infty}^{0} d[K^-(0,s)]Uv(\pi_v(s,\xi_0)) \\ & - \int_0^{+\infty} T_+(-s)\Phi_+\Psi_+(0)Uv(\pi_v(s,\xi_0))ds\,. \end{aligned}$$

Lemmas 2.4 and 2.5, together with the exponential estimates (3.5) for $T_+(s)$ and $K^-(t,s)$, easily imply that there exists η_m, $0 < \eta_m < \alpha_m$, such that, if $|v|_m < \eta_m$, then Γ defined by (3.14) is of class C_b^m and $|\Gamma|_m < \min(\beta_m, 1/|S|)$. So $f_\Gamma = U\mathrm{f}_\Gamma$, with f_Γ defined by (3.11), is in $\mathcal{V}_m(\delta_m)$ and $\Gamma = \Gamma_{f_\Gamma}$. The theorem is proved. □

As a consequence of Theorem 3.3, we finally obtain the following realization result: fixed any sufficiently small map $v \in C_b^1(\mathbb{R}^N, \mathbb{R}^p)$, we can find a nonlinearity $f \in Y_1$ such that the semiflow generated by equation (1.2), reduced to its (N-dimensional) center manifold, is equivalent to the flow of the ODE

$$\tag{3.15} \dot{\xi} = B\xi + Mv(\xi)$$

in $\mathbb{R}^N$, with the matrices B and M defined above.

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