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LAURA DE CARLI
MAURO NACINOVICH

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Unique Continuation in Abstract Pseudoconcave *CR* Manifolds

LAURA DE CARLI – MAURO NACINOVICH

Abstract. We prove a Carleman-type estimate for the $\bar{\partial}_M$-operator on functions on an abstract pseudoconcave CR manifold of arbitrary CR codimension. From this we deduce a weak unique continuation theorem for functions satisfying a differential inequality related to the tangential Cauchy-Riemann system. This implies the weak unique continuation for CR functions, as well as for CR sections of a CR line bundle.

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Introduction

Unique continuation for CR functions defined on a locally embeddable CR manifold is quite well understood from an extrinsic point of view. For a CR manifold which is minimal in the sense of Tumanov [Tu], the possibility of uniquely extending a CR function to a wedge in the ambient complex manifold implies the weak continuation property: namely, a CR function defined on a connected open subset Ω of a minimal embedded CR manifold M and vanishing on a nonempty open $\omega \subset \Omega$ is identically zero in Ω.

If M is a locally embedded strictly pseudoconcave CR manifold, then every CR function f defined on an open subset Ω of M uniquely extends to a holomorphic function $\tilde{f}$ on an open complex neighborhood $\tilde{\Omega}$ of Ω and hence we have the strong unique continuation principle: if f vanishes of infinite order at a point $x \in \Omega$, it also vanishes on the connected component of x in Ω. The assumption of strict pseudoconcavity can also be weakened by the use of the type function (cf. [DH]).

The situation is less clear for *abstract* CR manifolds, i.e. for those which are not known to be locally embeddable. Note that in particular there are several examples of strictly pseudoconcave CR manifolds which are not locally embeddable (cf. [H], [JT1], [JT2], [JT3] for the hypersurface type case and [HN2] for the higher codimensional case). The non embeddability being due to a lack of

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independent CR functions, we expect that the uniqueness results remain valid in the general abstract case. However, in this setting the technology of analytic disks is not available, making the extension of the results above nontrivial.

In this paper we take up the problem of unique continuation for *abstract* CR manifolds. We prove that the weak unique continuation property holds for CR-functions defined on *abstract* strictly pseudoconcave CR manifolds M. We note that the weak unique continuation principle is not valid, in general, for *abstract pseudoconvex* CR manifolds of the hypersurface type, as shown by a recent beautiful example of J.P. Rosay [R]. Besides the fact that we are dealing with possibly nonembedded CR manifolds, our results are new also because they apply more generally to solutions of a differential inequality related to the tangential Cauchy-Riemann system and moreover we only need a finite amount of regularity (related to the CR codimension k of M) on the CR structure.

In this way our work relates to the classical paper [AKS] and to various uniqueness results for solutions of differential inequalities, for which we refer to [Hö2] and to [Hö1] for a general survey. In our case we are far from an elliptic or hyperbolic setting, although the pseudoconcavity assumption implies that the Cauchy-Riemann equations define a $(1/2)$-subelliptic system (see [HN1]).

Our method is very classically based upon the proof of a Carleman type estimate. This can be obtained for weight functions whose differential is noncharacteristic for the tangential Cauchy-Riemann system: some geometrical considerations (see Section 2) allow to reduce the proof to this simpler case.

Some difficulty stems also from the need to understand the geometrical meaning of the analog of the complex Hessian that appears in the Carleman estimates. From [MN] it follows that this object has no intrinsic meaning for real functions on the manifold and maybe this also explains why the method of weighted estimates has not been up to now completely satisfactory in its applications to nonelliptic overdetermined complexes of partial differential operators (see [N]). Here we solved this question by an appropriate choice of local coordinates, so that the terms coming out in the estimates assume a geometrical meaning.

We feel that the study of the unique continuation property for abstract CR manifolds will provide an interesting example to pursue the same question in the case of general overdetermined system of linear partial differential operators.

1. – Preliminaries

Let M be a smooth real manifold of dimension $m = 2n+k$. A CR-structure of type (n,k) and class $\mathcal{C}^\mu$, with $2 \le \mu \le \infty$, on M can be defined by the datum of an n-dimensional distribution $\mathfrak{T}^{0,1}(M)$ of complex valued vector fields of class $\mathcal{C}^\mu$ on M with the properties:

$$\mathfrak{T}^{0,1}(M) \cap \overline{\mathfrak{T}^{0,1}(M)} = \{0\}\,; \tag{1.1}$$

$$[\mathfrak{T}^{0,1}(M), \mathfrak{T}^{0,1}(M)] \subset \mathcal{C}^{\mu-1}(M) \otimes \mathfrak{T}^{0,1}(M)\,. \tag{1.2}$$

A smooth differentiable manifold M with a CR-structure of type (n, k) and of class $\mathcal{C}^\mu$ is called a CR-manifold of type (n, k), of class $\mathcal{C}^\mu$.

We denote by HM the subbundle of the real tangent bundle TM of M defined by

$$H_xM = \left\{\operatorname{Re} X_x \mid X \in \mathfrak{T}^{0,1}(M)\right\}.$$

It is called the *analytic tangent bundle* of M and its annihilator bundle $H^0(M) \subset T^*M$ is called the *characteristic bundle* of M.

We note that condition (1.1) implies that for every $x \in M$ and every $X \in H_xM$ there is a unique $Y \in H_xM$ such that $X + \sqrt{-1}Y = L_x$ for some $L \in \mathfrak{T}^{0,1}(M)$. In this way we obtain a vector bundle automorphism $J : HM \to HM$ by associating to the tangent vector X the corresponding tangent vector Y. It satisfies $J^2 = -\mathrm{Id}_{HM}$ and therefore defines a complex structure on each fiber H_xM. This is called the *partial complex structure* of M. An equivalent definition of CR manifolds can be given in terms of the analytic tangent space and the partial complex structure (cf. [HN1]).

We use the characteristic bundle H^0M to parametrize the Levi form of M: if $x \in M$, $\omega \in H^0_xM$, $X \in H_xM$, we choose $L \in \mathfrak{T}^{0,1}(M)$ with $\Re L_x = X$ and an H^0M-valued 1-form $\tilde{\omega}$ of class $\mathcal{C}^1$, with $\tilde{\omega}_x = \omega$, to set

$$\mathcal{L}(\omega, X) = \mathrm{d}\tilde{\omega}(X, JX) = \left(1/2\sqrt{-1}\right)\omega\left([\bar{L}, L]\right). \tag{1.3}$$

The equality of the last two terms shows that the Levi form $\mathcal{L}(\omega, X)$ actually depends on ω and X and not on the choice of L and $\tilde{\omega}$.

For each $\omega \in H^0_xM$, $\mathcal{L}(\omega, \cdot)$ is a Hermitian quadratic form on H_xM for the complex structure defined by J_x.

We say that M is *strictly pseudoconcave* at $x \in M$ if the Levi form $\mathcal{L}(\omega, \cdot)$ has at least one negative eigenvalue for every choice of $\omega \in H^0_xM \setminus \{0\}$.

The CR manifold M is strictly pseudoconcave if it is such at every point.

Fix $p \in M$ and let U be an open neighborhood of p in M. We denote by $F(p, U)$ the set of points q of U such that there exist finitely many $\mathcal{C}^2$ integral curves $s_j : [0, 1] \to U$, $1 \le j \le N$, such that

(1) $\dot{s}_j(t) \in H_{s_j(t)}M$ for $0 \le t \le 1$ and $1 \le j \le N$;
(2) $s_1(0) = p$, $s_j(0) = s_{j-1}(1)$ for $1 < j \le N$, $s_N(1) = q$.

When $\mu = \infty$, the set $F(p, U)$ is the *Sussmann leaf* through p of HM in U and is a smooth submanifold of U (cf. [Su]).

The CR manifold M is said to be *minimal* at p if for every open neighborhood U of p the set $F(p, U)$ is still a neighborhood of p in M. We say that M is *minimal* if it is minimal at every point.

Assume that M is a CR manifold of class $\mathcal{C}^\mu$. Let $\mathcal{D}_{-1}$ denote the distribution of $\mathcal{C}^\mu$ sections of HM. We define by recurrence $\mathcal{D}_p$ for $-1 \ge p \ge -\mu$ to be the $\mathcal{C}^{\mu+1+p}$ distribution of real vector fields generated by $\mathcal{D}_{1+p}+[\mathcal{D}_{1+p}, \mathcal{D}_{-1}]$, for $-2 \ge p \ge -\mu$. We say that M is of finite type μ (or kind μ) if $\mathcal{D}_{-\mu}$ is the distribution of all $\mathcal{C}^1$ real vector fields on M.

We note that strictly pseudoconcave CR manifolds are of the second kind.

Every CR manifold of finite type is minimal.

In order to clarify the definitions above, we give a short description of *embedded* CR manifolds, although in the paper we will be especially concerned with the *abstract* case.

Let X be a complex manifold of complex dimension $N > n$ with complex structure J. Given a real m-dimensional differentiable submanifold M of class $\mathcal{C}^{\mu+1}$ of X such that its analytic tangent space

$$H_x M = T_x M \cap J(T_x M)$$

has real dimension $2n$ at each point $x \in M$, we set

$$T_x^{0,1}(M) = \left\{X + \sqrt{-1}JX \mid X \in H_x M\right\}.$$

Then M becomes a CR-manifold of type $(n, m-2n)$ by setting

$$\mathfrak{T}^{0,1}(M) = \mathcal{C}^{\mu}\left(M, T^{0,1}(M)\right).$$

In this case we say that M is an embedded CR-manifold of class $\mathcal{C}^{\mu}$.

For a CR manifold M of type (n, k), embedded in a complex manifold X of complex dimension $n+k$ (generic embedding), the characteristic bundle H^0M can be canonically identified to the conormal bundle N^*M of M. Via this identification, the Levi form is described in local coordinates $z_1, \dots, z_{n+k}$ at $d\rho(x)$, for a real valued $\mathcal{C}^2$ function ρ vanishing on M, by the restriction to $H_x M$ of the Hermitian quadratic form defined by the matrix

$$\left(\frac{\partial^2 \rho(x)}{\partial z_\alpha \partial \bar{z}_\beta}\right)_{\alpha,\beta=1,\dots,n+k}.$$

In this way we recover the classical definition in the case M is the boundary of a domain in X.

Let us go back to the general case.

A complex valued $\mathcal{C}^1$ function f defined on an open subset Ω of a CR manifold M is called a CR-function if it satisfies the Cauchy-Riemann equations:

$$L(f) = 0 \quad \text{in} \quad \Omega \quad \forall L \in \mathfrak{T}^{0,1}(M). \tag{1.4}$$

It is convenient to give a more invariant formulation of (1.4), inserting it into a complex of linear partial differential operators on M. To avoid the use of densely defined operators and restrictions, we assume below that M is $\mathcal{C}^\infty$ smooth.

Let $\mathcal{J}(M)$ denote the ideal in the algebra $\mathcal{E}^*(M)$ of complex valued exterior forms on M generated by the differentials which annihilate the vector fields in $\mathfrak{T}^{0,1}M$. The integrability condition (1.2) can be expressed by

$$d\mathcal{J}(M) \subset \mathcal{J}(M). \tag{1.5}$$

Thus we obtain from the de Rham complex a quotient complex

$$\bar{\partial}_M : \frac{\mathcal{E}^*(M)}{\mathcal{J}(M)} \longrightarrow \frac{\mathcal{E}^*(M)}{\mathcal{J}(M)} \tag{1.6}$$

defined by the commutative diagram:

$$\begin{array}{ccc} \mathcal{E}^*(M) & \xrightarrow{d} & \mathcal{E}^*(M) \\ \downarrow & & \downarrow \\ \dfrac{\mathcal{E}^*(M)}{\mathcal{J}(M)} & \xrightarrow{\bar{\partial}_M} & \dfrac{\mathcal{E}^*(M)}{\mathcal{J}(M)} \end{array} \tag{1.7}$$

that is called the Cauchy-Riemann complex on M (see [HN1]). The graduation of $\mathcal{E}^*(M)$ induces a natural graduation on the quotient $\mathcal{Q}(M) = \mathcal{E}^*(M)/\mathcal{J}(M)$:

$$\mathcal{Q}(M) = \bigoplus_{j=0}^{n} \mathcal{Q}^{0,j}(M)\,; \tag{1.8}$$

we have $\mathcal{Q}^{0,0}(M) = \mathcal{E}^0(M) = \mathcal{C}^\infty(M, \mathbb{C})$ and smooth CR functions are just the kernel of $\bar{\partial}_M : \mathcal{Q}^{0,0}(M) \to \mathcal{Q}^{0,1}(M)$.

The elements of $\mathcal{Q}^{0,j}(M)$ are for each $0 \le j \le n$ the smooth sections of a smooth complex vector bundle $Q^{0,j}(M)$ of rank $\binom{n}{j}$.

Note that, in case we repeat this construction for a CR manifold of class $\mathcal{C}^\mu$, with $2 \le \mu < \infty$, $Q^{0,1}(M)$ turns out to be a complex vector bundle of class $\mathcal{C}^\mu$.

2. – A remark on minimal CR manifolds

Let M be a differentiable manifold of class $\mathcal{C}^2$. Denote by $\pi : T^*M \to M$ the projection onto M of the cotangent bundle T^*M of M. Let F be a closed subset of M. The set $N_e(F)$ of the exterior normals to F consists of all $\omega \in T^*M$ such that $\omega \ne 0$, $\pi(\omega) \in F$, and there exists a $\mathcal{C}^2$ real valued function $\chi : M \to \mathbb{R}$ satisfying:

$$\begin{cases} d\chi(\pi(\omega)) = \omega\,, \\ \chi(p) \le \chi(\pi(\omega)) \qquad \forall\, p \in F\,. \end{cases} \tag{2.1}$$

The main properties of the exterior normal set are collected in the following:

THEOREM 2.1. *Let F be any closed subset of M. Then*

(i) $\pi(N_e(F)) \subset \partial F$ *and is dense in* ∂F*;*

(ii) If X is a Lipschitz continuous real vector field on M such that

$$\langle X, \omega\rangle \le 0 \quad \forall\, \omega \in N_e(F)\,, \tag{2.2}$$

then F contains all integral curves of X that contain a point of F.

For the proof, see Proposition 8.5.8 and Theorem 8.5.11 in [Höl].

From this theorem and the discussion of minimality in Section 1, we obtain:

THEOREM 2.2. *Let M be a connected CR manifold of type (n, k). If M is minimal and F is a nonempty closed subset of M with $N_e(F) \subset H^0M$, then $F = M$.*

PROOF. Indeed, by (ii) of the previous theorem, F contains the Sussmann leaf $\mathcal{F}(p, M)$ of each of its points $p \in F$. By the minimality assumption, $\mathcal{F}(p, M)$ is a neighborhood of p. Hence F is a neighborhood of each of its points and thus open. Being open, closed and nonempty, it coincides with M because M is connected.

3. – Local description of the Cauchy-Riemann distribution

Let M be an abstract Cauchy-Riemann manifold of type (n, k). Its partial complex structure J defines, for each point $p \in M$, a complex structure on the analytic tangent space H_pM. Hence we can choose coordinates $(y^1, \dots, y^{2n}, t^1 \dots, t^k)$ at p in such a way that $(\frac{\partial}{\partial y^1})_p, \dots, (\frac{\partial}{\partial y^{2n}})_p$ are a basis of H_pM and

$$J\left(\frac{\partial}{\partial y^j}\right)_p = \left(\frac{\partial}{\partial y^{n+j}}\right)_p \qquad \text{for} \quad 1 \le j \le n .$$

Introducing complex coordinates

$$z^j = y^j + \sqrt{-1}y^{n+j} \qquad \text{for} \quad 1 \le j \le n ,$$

a basis for the Cauchy-Riemann distribution $\mathfrak{T}^{0,1}M$ on a neighborhood of p will be given by homogeneous partial differential operators of the form:

$$(3.1) \quad \begin{cases} L_j = \dfrac{\partial}{\partial \bar{z}^j} + \displaystyle\sum_{h=1}^{n} c_j^h \frac{\partial}{\partial z^h} + \sum_{\alpha=1}^{k} a_j^\alpha \frac{\partial}{\partial t^\alpha} , & 1 \le j \le n \\ \text{with } c_j^h(p) = a_j^\alpha(p) = 0 & \text{for } 1 \le j, h \le n, 1 \le \alpha \le k . \end{cases}$$

Such a system of coordinates will be called *CR-adapted at p*. Note that for every coordinate patch at p we can obtain one which is CR-adapted at p by composition with a linear change of coordinates in $\mathbb{R}^m$.

The integrability conditions (1.2) give for the complex vector fields of the basis (3.1):

$$(3.2) \qquad [L_j, L_h] = 0 \qquad \text{for} \quad 1 \le j, h \le n .$$

The complex vector fields $L_1, \dots, L_n, \overline{L}_1, \dots, \overline{L}_n, \frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^k}$ are a basis of the complex vector fields on a neighborhood U of p. Hence we have in U:

$$[L_j, \overline{L}_h] = \sqrt{-1}\sum_{\alpha=1}^{k} \ell^{\alpha}_{j,h}\frac{\partial}{\partial t^{\alpha}} + \sqrt{-1}\sum_{s=1}^{n}\left(\beta^s_{j,h}L_s + \overline{\beta}^s_{h,j}\overline{L}_s\right); \tag{3.3}$$

we have:

$$\ell^{\alpha}_{j,h}(p) = \left(1/\sqrt{-1}\right)\left(\frac{\partial \overline{a}^{\alpha}_h(p)}{\partial \overline{z}^j} - \frac{\partial a^{\alpha}_j(p)}{\partial z^h}\right), \tag{3.4}$$

$$\beta^s_{j,h}(p) = \left(1/\sqrt{-1}\right)\frac{\partial \overline{c}^s_h(p)}{\partial \overline{z}^j}. \tag{3.5}$$

The *Leviform* of M at p is given, in the coordinates $\xi \in \mathbb{C}^n$ of $T^{0,1}_p M$ associated to the basis given by (3.1) and the coordinates $\eta \in \mathbb{R}^k$ of $H^0_p M$ associated to the basis $(dt^1)_p, \dots, (dt^k)_p$, by

$$\mathcal{L}(\eta, \xi) = \sum_{\alpha=1}^{k}\sum_{j,h=1}^{n} \ell^{\alpha}_{j,h}(p)\eta_{\alpha}\xi^j\overline{\xi}^h. \tag{3.6}$$

In [MN] the complex Hessian was defined on abstract CR manifolds for *real transversal* 1*-jets*. These objects, after introducing a CR-gauge on M, are described by a pair $(\phi, \sqrt{-1}\omega)$ where ϕ is a smooth real valued function and ω a smooth section of H^0M. We give here an alternate and nonintrinsic discussion, substituting to the real transversal 1-jet a smooth complex valued function ψ and choosing a CR-adapted coordinate system in such a way that, at p, ϕ corresponds to the real part and ω is related to the imaginary part of ψ.

In order to compute the complex Hessian at p of a smooth function defined on a neighborhood of p, starting from a CR-adapted coordinate system (y, t), we modify it by a suitable change of coordinates in $\mathbb{C}^n \times \mathbb{R}^k$ that involves second degree polynomials. From (2.2) we obtain:

$$\frac{\partial c^h_i(p)}{\partial \overline{z}^j} = \frac{\partial c^h_j(p)}{\partial \overline{z}^i} \quad \text{and} \quad \frac{\partial a^{\alpha}_i(p)}{\partial \overline{z}^j} = \frac{\partial a^{\alpha}_j(p)}{\partial \overline{z}^i} \qquad \forall\, 1 \le i, j, h \le n,\ 1 \le \alpha \le k.$$

Denote by $(c^s_j)_{(1)}$ and $(a^{\alpha}_j)_{(1)}$ the sum of the homogeneous terms of the first order in the Taylor expansions with respect to the (y, t) variables of c^s_j and a^{α}_j respectively. Then by the integrability conditions above we can find homogeneous polynomials of the second degree $\tilde{c}^s$ and $\tilde{a}^{\alpha}$ such that

$$\frac{\partial \tilde{c}^s}{\partial \overline{z}^j} = \left(c^s_j\right)_{(1)} \quad \text{and} \quad \frac{\partial \tilde{a}^{\alpha}}{\partial \overline{z}^j} = \left(a^{\alpha}_j\right)_{(1)}$$

for $1 \leq s, j \leq n$ and $1 \leq \alpha \leq k$. The new coordinates:

$$\text{(3.7)} \qquad \begin{cases} \mathfrak{z}^j = z^j - \tilde{c}^j & \text{for} \quad 1 \leq j \leq n \\ \mathfrak{t}^\alpha = t^\alpha - \operatorname{Re} \tilde{a}^\alpha & \text{for} \quad 1 \leq \alpha \leq k \end{cases}$$

are still CR-adapted, but now $L_j(\mathfrak{z}^h)$ vanishes to the first order at p and $\mathfrak{t}^\alpha$ is the real part of the function $t^\alpha - \tilde{a}^\alpha$ which also has the property that $L_j(t^\alpha - \tilde{a}^\alpha)$ vanishes to the first order at p. This means that we have a basis for $\mathfrak{T}^{0,1}M$ in a neighborhood of p of the form:

$$\text{(3.8)} \qquad \begin{cases} L_j = \dfrac{\partial}{\partial \bar{\mathfrak{z}}^j} + \displaystyle\sum_{s=1}^n \mathfrak{c}^s \frac{\partial}{\partial \mathfrak{z}^s} + \sum_{\alpha=1}^k \mathfrak{a}_j^\alpha \frac{\partial}{\partial \mathfrak{t}^\alpha}, & 1 \leq j \leq n, \\ & \text{with } \mathfrak{c}_j^s(p) = 0,\ d\mathfrak{c}_j^s(p) = 0, \\ \mathfrak{a}_j^\alpha(p) = 0, \quad \dfrac{\partial \mathfrak{a}_j^\alpha(p)}{\partial \bar{\mathfrak{z}}^h} + \dfrac{\partial \bar{\mathfrak{a}}_j^\alpha(p)}{\partial \mathfrak{z}^h} = 0 & \\ & \text{for } 1 \leq s, j, h \leq n,\ 1 \leq \alpha \leq k. \end{cases}$$

We note that if M is $\mathcal{C}^\infty$, we can find actually smooth functions ζ^j and θ^α such that $L_h(\zeta^j)$ and $L_h(\theta^\alpha)$ vanish of infinite order at p and $d\zeta^j(p) = dz^j(p)$, $d\theta^\alpha(p) = dt^\alpha(p)$ for $1 \leq j, h \leq n$ and $1 \leq \alpha \leq k$. However, our computation is still valid if we only assume that M is of class $\mathcal{C}^2$ and suffices for our purposes.

Coordinates $(\mathfrak{z}, \mathfrak{t})$ for which (3.8) is valid will be said to be *CR-adapted of the second order* at p. A basis of $\mathfrak{T}^{0,1}M$ near p satisfying (3.8) will be called $\mathcal{C}^2$ canonical at p.

Assume that (3.8) is a $\mathcal{C}^2$ canonical basis for $\mathfrak{T}^{0,1}M$ at p. For $\xi \in \mathbb{C}^n$ we denote by L_ξ the partial differential operator

$$L_\xi = \sum_{j=1}^n \xi^j L_j\,.$$

Then we have for every complex valued smooth function ψ, defined on a neighborhood of p:

$$\text{(3.9)} \qquad \begin{aligned} \left(L_\xi \overline{L_\xi(\psi)} + \overline{L_\xi} L_\xi(\psi)\right)(p) &= 2 \sum_{j,h=1}^n \frac{\partial^2 \operatorname{Re} \psi(p)}{\partial \mathfrak{z}^h \partial \bar{\mathfrak{z}}^j} \xi^j \bar{\xi}^h \\ &+ \sqrt{-1} \sum_{j,h=1}^n \sum_{\alpha=1}^k \left(\frac{\partial a_j^\alpha(p)}{\partial \mathfrak{z}^h} - \frac{\partial \bar{a}_h^\alpha(p)}{\partial \bar{\mathfrak{z}}^j} \right) \frac{\partial \operatorname{Im} \psi(p)}{\partial \mathfrak{t}^\alpha} \xi^j \bar{\xi}^h\,. \end{aligned}$$

We call this expression the *complex Hessian* at p of the smooth function ψ.

4. – Uniqueness for solutions of a system of differential inequalities

We formulate the weak uniqueness result for the Cauchy-Riemann system on functions:

THEOREM 4.1. *Let M be a connected abstract strictly pseudoconcave CR manifold of type (n, k) and of class $\mathcal{C}^\mu$ with $\frac{k}{2} + 2 \leq \mu \leq \infty$. Let $u \in \mathrm{L}^2_{\mathrm{loc}}(M)$ satisfy the following:*

$$\text{(4.1)}\qquad \begin{aligned} &\textit{for every } L \in \mathfrak{T}^{0,1}(M),\ L(u) \in L^2_{loc}(M)\\ &\textit{and there exists } \kappa_L \in L^\infty_{loc}(M) \textit{ such that}\\ &\big|L(u)(x)\big| \leq \kappa_L(x)|u(x)| \textit{ a.e. in } M\,. \end{aligned}$$

If u vanishes on a nonempty open subset of M, then u is identically zero on M.

Clearly this theorem implies the weak uniqueness property for the solutions of the Cauchy-Riemann equations (1.2).

The proof of this theorem involves several steps and will be described in the remaining sections of the paper.

Using Theorem 2.2 we reduce the proof to the uniqueness in the noncharacteristic Cauchy problem for solutions of (4.1). Let indeed u be a solution of (4.1) and consider the support F of u. If $\partial F \neq \emptyset$, then $N_e(F)$ is not contained in H^0M. Assuming that $u = 0$ in an open subset Ω of M, in case F was nonempty, we could find a real valued $\mathcal{C}^2$ function χ on M such that, for a point $p_0 \in \partial F$, $d\chi(p_0) \notin H^0M$ and $F \subset \{\chi(p) \leq \chi(p_0)\}$. Thus Theorem 4.1 follows from:

THEOREM 4.2. *Let U be an open domain in a strictly pseudoconcave CR manifold M of type (n, k) and of class $\mathcal{C}^\mu$ with $\frac{k}{2} + 2 \leq \mu \leq \infty$. If U^- is an open subset of U such that $\partial U^- \cap U$ is smooth and $N_e(\overline{U^-}) \cap H^0M|_U = \emptyset$, then every $u \in \mathrm{L}^2_{\mathrm{loc}}(U)$ which satisfies* (4.1) *and vanishes on U^- is zero a.e. on a neighborhood of $\overline{U^-}$ in U.*

5. – A Carleman type estimate

We introduce a smooth Riemannian metric on the CR manifold M and a smooth Hermitian metric on the fibers of $Q^{0,1}(M)$; in this way the L^2 norm $\|\cdot\|$ is well defined for functions and sections of $Q^{0,1}M$ on M. We shall deduce Theorem 4.2 and hence Theorem 4.1 from the following Carleman type estimate:

THEOREM 5.1. *Let M be a strictly pseudoconcave CR manifold of type (n, k), with $k \geq 1$, and of class $\mathcal{C}^\mu$ with $\frac{k}{2} + 2 \leq \mu \leq \infty$. Let ϕ be a real valued smooth* *function on M and $p \in M$ a point where $d\phi(p) \notin H^0M$. Then we can find $r > 0$, $A > 0$, $c > 0$, $\tau_0 > 0$ such that*

$$\text{(5.1)} \qquad \begin{cases} \sqrt{\tau}\|\exp(\tau(\phi + A\phi^2)) \cdot f\| \leq c\,\|\exp(\tau(\phi + A\phi^2)) \cdot \bar{\partial}_M(f)\| \\ \qquad \forall f \in \mathcal{C}_0^\infty(B_r(p)),\, \tau \geq \tau_0 , \end{cases}$$

where $B_r(p)$ denotes the ball of radius r centered in p.

Note that the statement of this theorem is invariant with respect to the choice of the Riemannian metric on M and the Hermitian metric on the fiber.

6. – Localization of the estimate

We fix a point $p \in M$ and coordinates $(\mathfrak{z}, \mathfrak{t})$ in an open neighborhood U which are CR-adapted of the second order at p, so that (3.8) gives a basis for $\mathfrak{T}^{0,1}M$ on U. We can assume that the coordinate neighborhood is defined on $B_1 = \{|\mathfrak{z}|^2 + |\mathfrak{t}|^2 < 1\}$ and that the metric on M coincides with the Euclidean metric in the coordinates $(\mathfrak{z}, \mathfrak{t})$. We denote by HB_1, $T^{0,1}B_1$, H^0B_1, $Q^{0,1}B_1$ the bundles on B_1 corresponding to those on the open U in M. We can assume (by taking a smaller U) that they are all trivial. The projection of the trivial bundle generated by $d\bar{\mathfrak{z}}^1, \ldots, d\bar{\mathfrak{z}}^n$ over B_1 is an isomorphism with $Q^{0,1}B_1$; we consider on the fibers the pullback of the Hermitian metric for which these differentials form an orthonormal basis.

We denote by $\mathfrak{f}$ the principal bundle on B_1 whose fiber at each point x_0 consists of the orthogonal frames $g_{x_0} : C^n \times R^k \to \mathbb{R}^m$ such that $g_{x_0}(\mathbb{C}^n \times \{0\}) = H_{x_0}B_1$. We note that $\mathfrak{f}$ is smooth and locally trivial. Hence we can fix a smooth section g on a neighborhood B_r of 0 in B_1 such that g_0 is the identity. We can assume that $r = 1$. We shall use the section g to construct slowly varying metrics in $B(0, 1)$ (cf. [Höl]).

Given a point $x_0 = (\mathfrak{z}_0, \mathfrak{t}_0) \in B_1$, the coordinates $(z, t) = g_{x_0}^{-1}(x - x_0)$ are adapted CR-coordinates at $(\mathfrak{z}_0, \mathfrak{t}_0)$. We call them *$g$ coordinates* at $x_0 = (\mathfrak{z}_0, \mathfrak{t}_0)$. Note that for $x_0 = 0$ they coincide with our $(\mathfrak{z}, \mathfrak{t})$ at p and are CR-adapted of the second order. We will write $z_{x_0}(x)$ and $t_{x_0}(x)$ to indicate the dependence of the g coordinates (z, t) of x at $x_0 = (\mathfrak{z}_0, \mathfrak{t}_0)$.

Having fixed a real parameter $\tau > 1$, we set, for each point $x_0 \in B(0, 1)$,

$$\|x\|^2_{\tau, x_0} = \tau |z_{x_0}(x)|^2 + \tau^2 |t_{x_0}(x)|^2 .$$

By further restricting the open neighborhood of p under consideration, we obtain that

$$\|x - x_0\|_{\tau, x_0} < 1 \Rightarrow \|v\|_{\tau, x} \leq 2\|v\|_{\tau, x_0} \quad \forall\, v \in \mathbb{R}^m ,$$

so that $\| \quad \|_x$ is slowly varying in $B(0, 1)$, uniformly with respect to $\tau > 1$.

For each $\tau > 1$ we obtain therefore a partition of unity $\{\kappa_{\tau,\nu}\}$ in the neighborhood $B(0, 1 - 1/\sqrt{\tau})$ of 0 by smooth real valued functions with compact support contained in balls $\{x \in B(0,1) \,|\, \|x - x_\nu\|_{x_\nu} < 1\}$ for a finite subset $\{x_\nu\}$ of $B(0,1)$. We note that these balls correspond to the cubes $\{|z| < 1/\sqrt{\tau}, |t| < 1/\tau\}$ for the g-coordinates centered at x_ν. Moreover, for the functions of such a partition of the unity, there are uniform bounds, independent of $\tau > 1$, for the maximum number C of intersecting supports.

There are also τ-dependent bounds for their derivatives. In fact, if L is a smooth vector field in $\mathfrak{T}^{0,1}B_1$, then we obtain, for every τ and ν such that the support of $\kappa_{\tau,\nu}$ is contained in B_1,

$$\left|L(\kappa_{\tau,\nu})\right| < \text{constant } \cdot \sqrt{\tau}$$

for a constant which only depends on the supremum of the modulus of coefficients of L and of their first derivatives.

It is therefore apparent that the Carleman estimate (5.1) is valid if we are able to prove:

LEMMA 6.1. *There are $\epsilon_0 > 0$, $r > 0$ such that* (5.1) *is valid with the same constant $c > 0$ when $0 < \epsilon < \epsilon_0$, $\tau > \tau_0(\epsilon)$ and the function f has support contained in the intersection of $B_r(p_0)$ with a cube $\{|z| < 1/\sqrt{\tau\epsilon},\ |t| < 1/(\tau\epsilon)\}$, for g-coordinates CR-adapted at the center of the cube.*

Indeed, we shall have, with $\psi = \phi + A\phi^2$:

$$\begin{aligned}
\|\exp(\tau\psi)\bar{\partial}_M(f)\| &\simeq \sum_{j=1}^{n} \|\exp(\tau\psi)L_j(f)\| \\
&\geq C^{-1}\sum_{j=1}^{n}\sum_{\nu} \|\exp(\tau\psi)\,\kappa_{\epsilon\tau,\nu}L_j(f)\| \\
&\geq C^{-1}\sum_{j=1}^{n}\sum_{\nu} \|\exp(\tau\psi)L_j(\kappa_{\epsilon\tau,\nu}f)\| \\
&\quad - \sqrt{\epsilon\tau}\cdot \text{constant}\,\|\exp(\tau\psi)f\| \\
&\geq \sqrt{\tau}\,(c\cdot C^{-1} - \sqrt{\epsilon}\cdot\text{constant})\,\|\exp(\tau\psi)f\|\,,
\end{aligned}$$

(where C is a τ-independent upper bound for the number of intersecting supports of the functions of the partition of unity, and we assume that $\epsilon\cdot\tau > 1$) from which (5.1) follows if we take $\epsilon > 0$ sufficiently small.

Note that the restriction on the radius $r > 0$ of the ball $B_r(p_0)$ will be determined by the condition that the coefficients $\beta^s_{j,h}$ in (3.3) be small. Since the coordinates $(\mathfrak{z}, \mathfrak{t})$ were chosen adapted to the second order at p_0, they are actually bounded, in the g-coordinates, by a constant times the radius r of the ball for $0 < r < 1$.

7. – Microlocalization

We denote by Γ the positive cone of $\mathbb{R}^k$:

$$\Gamma = \left\{ (t^1, \dots, t^k) \in \mathbb{R}^k \,\middle|\, t^\alpha \geq 0 \text{ for } 1 \leq \alpha \leq k \right\} . \tag{7.1}$$

For $a \in \mathbf{GL}(k, \mathbb{R})$, we set $\Gamma_a = a(\Gamma)$: it is a closed proper convex cone with vertex at 0 and its intersection σ_a with the unit sphere $\mathbf{S}^{k-1} = \{|t| = 1\} \subset \mathbb{R}^k$ is called a geometrical $(k-1)$-simplex of $\mathbf{S}^{k-1}$.

Since we assumed that M is strictly pseudoconcave at p_0, for each $\eta \in \mathbf{S}^{k-1}$ the set of $\xi \in \mathbf{S}^{2n-1} \subset \mathbb{C}^n$ such that

$$\sum_{\alpha=1}^{k} \sum_{j,h=1}^{n} \ell^\alpha_{j,h}(p_0) \eta_\alpha \xi^j \bar{\xi}^h < 0 \tag{7.2}$$

is nonempty. Thus there exists a triangulation $\{\sigma_{a_1}, \dots, \sigma_{a_N}\}$ of $\mathbf{S}^{k-1}$ by geometrical $(k-1)$-simplices such that for each $1 \leq i \leq N$ the set of $\xi \in \mathbf{S}^{2n-1}$ that satisfy (7.2) for all $\eta \in \sigma_i$ has a nonempty interior in $\mathbf{S}^{n-1}$. By convexity (7.2) for all $\eta \in \sigma_{a_i}$ is equivalent to

$$\sum_{\alpha=1}^{k} \sum_{j,h=1}^{n} \ell^\alpha_{j,h}(p_0) (a_i^{-1})^\beta_\alpha \xi^j \bar{\xi}^h < 0 \quad \text{for} \quad \beta = 1, \dots, k . \tag{7.3}$$

Denote by $G_i \subset \mathbf{S}^{2n-1}$ the set of ξ for which (7.3) is valid.

Consider now the complex Hessian of the function $\psi = \phi + A\phi^2$ at p_0. By our choice of coordinates, it is given by:

$$\begin{aligned} &L_\xi \left(\overline{L_\xi(\psi)} \right)(p_0) + \overline{L_\xi} \left(L_\xi(\psi) \right)(p_0) \\ &\qquad = 2 \sum_{j,h=1}^{n} \frac{\partial^2 \phi(x_0)}{\partial \bar{z}^j \partial z^h} \xi^j \bar{\xi}^h + 4A \left| \sum_{j=1}^{n} \xi^j \frac{\partial \phi(p_0)}{\partial \bar{z}^j} \right|^2 . \end{aligned} \tag{7.4}$$

Since we assumed that $d\phi(p_0) \notin H^0 M$, the second summand is nonzero. Hence the complex Hessian is positive for large $A > 0$ in a cone V_A of the form $|L_\xi(\phi)(p_0)| > (C/A)|\xi|$ for some constant $C > 0$. For A sufficiently large, V_A intersects each G_i for $1 \leq i \leq N$. We pick for each $1 \leq i \leq N$ a point $\xi_i \in V_A \cap G_i$. By taking A very large we can also assume that

$$L_{\xi_i} \left(\overline{L_{\xi_i}(\psi)} \right)(p_0) + \overline{L_{\xi_i}} \left(L_{\xi_i}(\psi) \right)(p_0) > 1 \quad \text{for} \quad 1 \leq r \leq N . \tag{7.5}$$

By continuity we can find $r_0 > 0$, which will be taken also $< 1/2$, such that (7.3) and (7.5) are still valid for points of B_{r_0} in the g-coordinates, when we substitute $\psi(z, 0)$ to $\psi(z, t)$ for $|t| \leq r_0$.

Finally, we introduce the pseudodifferential operators of order 0:

$$(7.6)\quad P_i(D)(v)(z,t) = (2\pi)^{-k}\int_{\Gamma_{a_i}} \tilde{v}(z,\eta)\exp\left(\sqrt{-1}\langle t,\eta\rangle\right)d\eta \qquad 1 \le i \le N\,,$$

where

$$\tilde{v}(z,\eta) = \int_{\mathbb{R}^k} v(z,t)\exp\left(\sqrt{-1}\langle t,\eta\rangle\right)dt$$

is the partial Fourier transform of a function $v \in \mathrm{L}^2(\mathbb{R}^m)$. We note that

$$(7.7)\qquad \left\|P_i(D)(v)\right\| \le \|v\| \qquad \forall\, v \in \mathrm{L}^2(\mathbb{R}^m)$$

and

$$(7.8)\qquad \sum_{i=1}^{N}\left\|P_i(D)(v)\right\|^2 = \|v\|^2 \qquad \forall\, v \in \mathrm{L}^2(\mathbb{R}^m)$$

by the Plancherel formula. This remains valid if we consider L^2 functions with support in $K_\tau = \{|z| \le \sqrt{\epsilon\tau},\ |t| \le \epsilon\tau\}$ and make $P_i(D)$ into a properly supported pseudodifferential operator $\tilde{P}_i(t,D)$ of order 0 by multiplying it by a real valued cutoff function of the t variables with values in $[0,1]$ and which is 1 on a neighborhood of K_τ.

Namely, we will set, for a smooth real valued function $\tilde{\kappa}$, defined for $t \in \mathbb{R}^k$ which is equal to 1 for $|t| < 1$ and is 0 for $|t| > 2$,

$$(7.9)\qquad \tilde{\kappa}_{\epsilon\tau}(t) = \tilde{\kappa}(\epsilon\tau t)$$

and define

$$(7.10)\qquad \tilde{P}_i(t,D)(v) = \kappa_{\epsilon\tau}P_i(v)\,.$$

We note that for each $L \in \mathfrak{T}^{0,1}B_1$, the commutator $[L, \tilde{P}_i(t,D)]$ is of order 0 and we have

$$(7.11)\quad \left\|[L,\tilde{P}_i(t,D)](v)\right\| \le \text{ constant } (L)\cdot(1+\sqrt{\epsilon\tau})\cdot\|v\| \qquad \forall\, v \in \mathrm{L}^2(K_\tau)\,.$$

8. – Proof of the Carleman estimate

After the preparation of the previous sections, we turn now to the proof of the Carleman estimate (5.1). We fix a point $(\mathfrak{z}_0, \mathfrak{t}_0) \in B_r$, for some $0 < r < r_0$ that will be made more precise later on and we assume that f, in the g-coordinates (z, t) at $(\mathfrak{z}_0, \mathfrak{t}_0)$, which are CR-adapted, has support contained in $K_\tau = \{|z| < 1/\sqrt{\epsilon\tau},\ |t| < 1/(\epsilon\tau)\}$.

With $v = f \cdot \exp(\tau\psi)$, we have

$$\left\|\exp(\tau\psi)L_\xi(f)\right\| = \left\|L_\xi(v) - \tau v L_\xi(\psi)\right\| \tag{8.1}$$

for every $\xi \in \mathbb{C}^n$.

Then (5.1) is equivalent to

$$c\sum_{i=1}^{N}\left\|L_{\xi_i}(v) - \tau\, v\, L_{\xi_i}(\psi)\right\|^2 \geq \tau\|v\|^2 \qquad \forall\, v \in C_0^\infty(K_\tau)\,. \tag{8.2}$$

In order to apply microlocalization, it is convenient to substitute the terms in the left hand side with analogous terms in which the coefficient involving the parameter τ is independent of t. To this aim we note that:

$$\left|L_\xi(\psi)(z,t) - L_\xi(\psi)(z,0)\right| \leq \frac{\text{constant}}{\epsilon\tau} \quad \text{in} \quad K_\tau\,. \tag{8.3}$$

Setting

$$|\!|\!| v |\!|\!|^2 = \sum_{i=1}^{N}\left\|L_{\xi_i}(v) - \tau\, v\, L_{\xi_i}(\psi)(z,0)\right\|^2, \tag{8.4}$$

we have

$$\sum_{i=1}^{N}\left\|L_{\xi_i}(v) - \tau\, v\, L_{\xi_i}(\psi)\right\|^2 \geq \frac{1}{2}\,|\!|\!| v |\!|\!|^2 - \frac{\text{constant}}{\epsilon^2}\|v\|^2\,. \tag{8.5}$$

By Lemma 6.1 it suffices to show that, for some positive constant c:

$$c\,|\!|\!| v |\!|\!|^2 \geq \tau\|v\|^2 \quad \forall\, v \in C_0^\infty(K_\tau)\,. \tag{8.6}$$

For every $\xi \in \mathbb{C}^n$, we obtain by integration by parts:

$$\begin{aligned} \left\|L_\xi(v) - \tau v L_\xi(\psi)(z,0)\right\|^2 &= \left\|L_\xi^*(v) - \tau\, v\, \overline{L_\xi(\psi)(z,0)}\right\|^2 \\ &+ \int [L_\xi^*, L_\xi](v)\,\overline{v}dx + 2\,\tau\,\mathrm{Re}\int L_\xi\left(\overline{L_\xi(\psi(z,0))}\right)|v|^2 dx\,, \end{aligned} \tag{8.7}$$

where L_ξ^* is the L^2 formal adjoint of L_ξ.

The first term on the right is nonnegative and will be discarded. The inequality looked for comes essentially from the third term. The second term is going to be an error term, after appropriate microlocalization.

We have $[L_\xi^*, L_\xi] = [L_\xi, \overline{L_\xi}] + G_\xi$ for a bounded function G_ξ. Thus we can use formula (3.3) to estimate the second summand in the right hand side of (8.7). We obtain, also using integration by parts:

$$\begin{aligned}
\frac{1}{\sqrt{-1}}\int [L_\xi, \overline{L_\xi}](v)\,\bar{v}\,dx &= \sum_{\alpha=1}^{k}\int\sum_{j,h=1}^{n} \ell_{j,h}^{\alpha}\xi^j\bar{\xi}^h\frac{\partial v}{\partial t^\alpha}\,\bar{v}\,dx \\
&\quad+\sum_{j,h,s=1}^{n}\int 2\,(\operatorname{Im}\beta_{j,h}^{s})\xi^j\bar{\xi}^h\,L_s(v)\,\bar{v}dx \\
&\quad+\sum_{j,h,s=1}^{n}\int (\overline{L_s^*\beta_{j,h}^{s}})\xi^j\bar{\xi}^h\,|v|^2\,dx \\
&=\sum_{\alpha=1}^{k}\int\sum_{j,h=1}^{n} \ell_{j,h}^{\alpha}\xi^j\bar{\xi}^h\frac{\partial v}{\partial t^\alpha}\,\bar{v}\,dx \\
&\quad+\sum_{j,h,s=1}^{n}\int 2\,(\operatorname{Im}\beta_{j,h}^{s})\xi^j\bar{\xi}^h\,(L_s(v) \\
&\quad-\tau\, v\, L_s(\psi)(z,0))\,\bar{v}dx \\
&\quad+\sum_{j,h,s=1}^{n}\int (\overline{L_s^*\beta_{j,h}^{s}})\xi^j\bar{\xi}^h\,|v|^2\,dx \\
&\quad+\tau\sum_{j,h,s=1}^{n}\int 2\,(\operatorname{Im}\beta_{j,h}^{s})\xi^j\bar{\xi}^h\,(v\,L_s(\psi)(z,0))\,\bar{v}dx\,.
\end{aligned}$$

Since we noted that the coefficients $\beta_{j,h}^{s}$ are bounded by a constant times r on the ball $B_r(p_0)$, we obtain:

$$\begin{aligned}
\operatorname{Re}\int [L_\xi, \overline{L_\xi}](v)\,\bar{v}\,dx &\geq \sqrt{-1}\sum_{\alpha=1}^{k}\int\sum_{j,h=1}^{n} \ell_{j,h}^{\alpha}\xi^j\bar{\xi}^h\frac{\partial v}{\partial t^\alpha}\,\bar{v}\,dx \\
&\quad-\text{constant}|\xi|^2\left(\|v\|^2 + r\,\tau\|v\|^2 + r\,\|v\|\,|\!|\!|v|\!|\!|\right),
\end{aligned} \tag{8.8}$$

provided the support of v is also contained in the ball $B_r(p_0)$.

To deal with the term:

$$I(\xi)(v) = \sqrt{-1}\sum_{\alpha=1}^{k}\int\sum_{j,h=1}^{n} \ell_{j,h}^{\alpha}\xi^j\bar{\xi}^h\frac{\partial v}{\partial t^\alpha}\,\bar{v}\,dx \tag{8.9}$$

we use microlocalization.

Let $v_i = \tilde{P}_i(t, D)(v)$. Then

$$\begin{aligned}\|L_{\xi_i}(v) - \tau\, v\, L_{\xi_i}(\psi)(z,0)\| &\geq \|\tilde{P}_i(t,D)\left(L_{\xi_i}(v) - \tau\, v\, L_{\xi_i}(\psi)(z,0)\right)\| \\ &\geq \|L_{\xi_i}(v_i) - \tau\, v_i\, L_{\xi_i}(\psi)(z,0)\| - \text{constant}\sqrt{\epsilon\tau}\|v\|\end{aligned}$$

where the constant is independent of τ because the term containing the parameter τ is independent of t.

Hence we obtain:

$$\begin{aligned}\|\!|v|\!\|^2 &\geq (1/2)\sum_{i=1}^{N}\|L_{\xi_i}(v_i) - \tau\, v_i\, L_{\xi_i}(\psi)(z,0)\|^2 \\ &\quad - \text{constant}\,\epsilon\,\tau\,\|v\|^2 \\ &\geq \sum_{i=1}^{N}\left(I(\xi_i)(v_i) + 2\tau\,\mathrm{Re}\int L_{\xi_i}(\overline{L_{\xi_i}\psi(z,0)})\,|v_i|^2\,dx\right) \\ &\quad - \text{constant}\left((1+(r+\epsilon)\tau)\|v\|^2 + (\sqrt{\epsilon\tau} + r\sqrt{\tau})\|v\|\,\|\!|v|\!\|\right).\end{aligned} \tag{8.10}$$

For this estimate we also used the fact that, for every index i,

$$\left\|L_{\xi_i}(v_i) - \tau\, v_i\, L_{\xi_i}(\psi)(z,0)\right\| \leq \left\|L_{\xi_i}(v) - \tau\, v\, L_{\xi_i}(\psi)(z,0)\right\| + \text{constant}\sqrt{\epsilon\tau}\|v\|.$$

Then the desired estimate follows because by (7.5) and (7.8)

$$2\tau\,\mathrm{Re}\sum_{i=1}^{N}\int L_{\xi_i}\left(\overline{L_{\xi_i}\psi(z,0)}\right)|v_i|^2\,dx \geq \tau\sum_{i=1}^{n}\|v_i\|^2 = \tau\|v\|^2, \tag{8.11}$$

while the terms $I(\xi_i)(v_i)$ are bounded from below by $-\text{constant}\sqrt{\epsilon\tau}\|v\|^2$.

To prove this last fact, we choose $\tilde{a}_i \in \mathbf{GL}(k,\mathbb{R})$ such that $\tilde{a}_i(\Gamma)$ properly contains the cone $a_i(\Gamma)$. Taking $\tilde{a}_i$ sufficiently close to a_i, the functions:

$$\left(b_i^{\beta}(z,t)\right)^2 = -\sum_{\alpha=1}^{k}\sum_{j,h=1}^{n}\ell_{j,h}^{\alpha}(z,t)\left(\tilde{a}_i^{-1}\right)_{\alpha}^{\beta}\xi_i^j\bar{\xi}_i^h \quad \text{for} \quad \beta = 1,\dots,k \tag{8.12}$$

are still positive. With new variables $\tilde{t}^1,\dots,\tilde{t}^k$ corresponding to the matrix $\tilde{a}_i$, we obtain:

$$I_{\xi_i}(v_i) \geq \frac{1}{\sqrt{-1}}\sum_{\beta=1}^{k}\int\frac{\partial}{\partial\tilde{t}^{\beta}}(b_i^{\beta}v_i)\cdot\overline{(b_i^{\beta}v_i)}\,dx - \text{constant}\,\|v\|^2. \tag{8.13}$$

We estimate the integral in the right hand side of (8.13) using the partial Fourier transform with respect to the $\tilde{t}$-variables. The cone $\Gamma' = \tilde{a}_i^{-1}(\Gamma_{a_i})$ is properly contained in Γ and therefore we have, with $\delta > 0$,

$$|\eta - \eta'| \geq \delta|\eta| \quad \forall\,\eta' \in \Gamma', \quad \forall\,\eta \in \mathbb{R}^k \setminus \Gamma. \tag{8.14}$$

Since $\eta^\beta \geq 0$ in Γ when $\beta = 1, \dots, k$, we obtain, using the Young's estimate for convolution (i.e. the continuous inclusion $L^1 * L^2 \subset L^2$):

$$\frac{1}{\sqrt{-1}} \sum_{\beta=1}^{k} \int \frac{\partial}{\partial \tilde{t}^\beta}(b_i^\beta v_i) \cdot \overline{(b_i^\beta v_i)}\, dx \geq -\sum_{\beta=1}^{k} \int_{\mathbb{C}^n} d\lambda_{2n}(z) \int_{\mathbb{R}^k \setminus \Gamma} |\eta| |\widetilde{b_i^\beta v_i}(z,\eta)|^2\, d\eta$$

$$\geq -C \sum_{\beta=1}^{k} \int_{\mathbb{C}^n} d\lambda_{2n}(z) \int_{\eta \in \mathbb{R}^k \setminus \Gamma} |\eta| \left| \int_{\eta' \in \Gamma'} \widetilde{b_i^\beta \kappa_{\epsilon\tau}}(z, \eta - \eta') \tilde{v}(z, \eta')\, d\eta' \right|^2 d\eta$$

$$\geq -(C/\delta) \sum_{\beta=1}^{k} \int_{\mathbb{C}^n} d\lambda_{2n}(z) \int_{\eta \in \mathbb{R}^k} \left| \int_{\eta' \in \mathbb{R}^k} |\eta - \eta'|^{1/2} |\widetilde{b_i^\beta \kappa_{\epsilon\tau}}(z, \eta - \eta')| |\tilde{v}(z, \eta')| d\eta' \right|^2 d\eta$$

$$\geq -(C/\delta) \sum_{\beta=1}^{k} \int_{\mathbb{C}^n} d\lambda_{2n}(z) \left(\int_{\eta' \in \mathbb{R}^k} |\eta'|^{1/2} |\widetilde{b_i^\beta \kappa_{\epsilon\tau}}(z, \eta')| d\eta' \right)^2 \int_{\eta \in \mathbb{R}^k} |\tilde{v}(z,\eta)|^2 d\eta$$

for a constant C depending on the determinant of the matrix $\tilde{a}_i$. Although the estimate of the last integral above is standard, we give the complete argument to show that it actually involves only the derivatives up to the order $[(k+1)/2]+2$ of the coefficients of the complex vector fields L_i.

For a fixed $\epsilon > 0$, we can restrict to $\tau > 1/\epsilon$, so that all $\kappa_{\epsilon\tau}$'s have support contained in a fixed compact set K. By multiplying the b_i^β's by a smooth function with compact support which is equal to 1 on a neighborhood of K, we can assume in the following argument that the b_i^β's are $\mathcal{C}^{\mu-1}$ functions with compact support. Then

$$\widetilde{b_i^\beta \kappa_{\epsilon\tau}}(z,\eta) = \left[\tilde{b}_i^\beta(z,\,\cdot\,) * \tilde{\kappa}_{\epsilon\tau}(z,\,\cdot\,) \right](\eta)\,.$$

We have:

$$\tilde{\kappa}_{\epsilon\tau}(z,\eta) = (2\epsilon\tau)^{-k} \tilde{\kappa}\left(\sqrt{2\epsilon\tau} z, \eta/(2\epsilon\tau) \right)\,.$$

We use the estimate:

$$(2\epsilon\tau)^{-k} \int_{\eta \in \mathbb{R}^k} \left| \int_{\eta' \in \mathbb{R}^k} |\eta|^{1/2} \tilde{b}_i^\beta(z, \eta - \eta') \cdot \tilde{\kappa}\left(\sqrt{2\epsilon\tau} z, \frac{\eta'}{2\epsilon\tau} \right) d\eta' \right| d\eta$$

$$\leq (2\epsilon\tau)^{-k} \int_{\eta \in \mathbb{R}^k} \int_{\eta' \in \mathbb{R}^k} |\eta - \eta'|^{1/2} |\tilde{b}_i^\beta(z, \eta - \eta')| \cdot \left| \tilde{\kappa}\left(\sqrt{2\epsilon\tau} z, \frac{\eta'}{2\epsilon\tau} \right) \right| d\eta' d\eta$$

$$+ (2\epsilon\tau)^{-k} \int_{\eta \in \mathbb{R}^k} \int_{\eta' \in \mathbb{R}^k} |\tilde{b}_i^\beta(z, \eta - \eta')| \cdot |\eta'|^{1/2} \cdot \left| \tilde{\kappa}\left(\sqrt{2\epsilon\tau} z, \frac{\eta'}{2\epsilon\tau} \right) \right| d\eta' d\eta$$

$$\leq \int_{\eta \in \mathbb{R}^k} |\eta|^{1/2} |\tilde{b}_i^\beta(z,\eta)| d\eta \int_{\eta \in \mathbb{R}^k} \left| \tilde{\kappa}\left(\sqrt{2\epsilon\tau} z, \eta \right) \right| d\eta$$

$$+ \sqrt{2\epsilon\tau} \int_{\eta \in \mathbb{R}^k} |\tilde{b}_i^\beta(z,\eta)| d\eta \int_{\eta \in \mathbb{R}^k} |\eta|^{1/2} \left| \tilde{\kappa}\left(\sqrt{2\epsilon\tau} z, \eta \right) \right| d\eta\,.$$

Note that the last integrals, by a classical theorem of Bernstein, can be estimated using the $\frac{k}{2}+1$ Sobolev norms of the b_i^β's and of κ. From this we obtain that

$$\int_{\eta\in\mathbb{R}^k} |\eta|^{1/2}\cdot|\widetilde{b_i^\beta\kappa_{\epsilon\tau}}(z,\eta)|\,d\eta \le \text{constant}(1+\epsilon\tau)^{1/2}$$

uniformly with respect to z.

Hence we obtain the estimate

$$I_{\xi_i}(v_i) \ge -\text{constant}(1+\epsilon\tau)\|v\|^2$$

from which the Carleman estimate (5.1) follows.

9. – Proof of Theorem 4.2

The proof of the uniqueness in the noncharacteristic Cauchy problem for $\overline{\partial}_M$ follows in a standard way from the Carleman type estimate (5.1).

Fix a point $p_0 \in \partial U^- \cap U$ and a defining function ρ for U^- in a neighborhood $V \subset U$ of p_0:

$$\begin{cases} \rho \in \mathcal{C}^2(V,\mathbb{R}), \\ U^-\cap V = \{p\in V \mid \rho(p)<0\}, \\ \rho(p_0)=0, \quad d\rho(p_0)\notin H^0M. \end{cases} \tag{9.1}$$

Assuming that V is a coordinate patch, with coordinates $x\in\mathbb{R}^m$ vanishing at p_0, we set

$$\phi(p) = \rho(p) - C\cdot|x(p)|^2 \quad \text{for} \quad p\in V, \tag{9.2}$$

with C sufficiently large, so that $\phi(p) < -1$ outside a compact neighborhood of p_0 in V. By Theorem 5.1, there are $r>0$, $A>0$, $c>0$ and $\tau_0>0$ such that (5.1) is valid for the weight function ϕ. Fix a function $\nu:\mathbb{R}\to\mathbb{R}$ with:

$$\begin{cases} 0\le\nu(\theta)\le 1 & \forall\theta\in\mathbb{R}, \\ \nu(\theta)=1 & \text{if} \quad \theta>1, \\ \nu(\theta)=0 & \text{if} \quad \theta<-1. \end{cases} \tag{9.3}$$

Given a solution $u\in L^2_{\text{loc}}(U)$ of (4.1) that vanishes in U^-, for real $\delta>0$ we consider the function

$$f_\delta = u\cdot\nu(\phi/\delta). \tag{9.4}$$

Its support is contained in

$$(U\setminus U^-)\cap\{\phi(p)\ge-\delta\}.$$

and therefore is compact and contained in $V \cap B_r(p_0)$ if $\delta > 0$ is sufficiently small, say $\delta < \delta_0$. The estimate (5.1) is valid for f_δ when $\delta < \delta_0$ by Friedrichs extension theorem (cf. [F]). For a fixed $0 < \delta < \delta_0$ and $\psi = \phi + A\phi^2$, we obtain

$$\tau \left\| f_\delta \cdot \exp(\tau\psi) \right\|^2 \leq c \left\| \exp(\tau\psi) \cdot \overline{\partial}_M(f_\delta) \right\|^2 . \tag{9.5}$$

For $\lambda = \delta + A\delta^2$, we obtain:

$$\begin{aligned} \tau \| \exp(\tau\psi - \lambda) \cdot u \|^2_{\phi \geq \delta} &\leq c(\| \exp(\tau\psi - \lambda)\overline{\partial}_M(u) \|_{\phi \geq \delta} \\ &\quad + c \| \exp(\tau\psi - \lambda)\overline{\partial}_M(f_\delta) \|^2_{\phi \leq \delta} \\ &\leq \text{constant} \| \exp(\tau\psi - \lambda) \cdot u \|^2_{\phi \geq \delta} + \\ &\quad + c \| \exp(\tau\psi - \lambda)\overline{\partial}_M(f_\delta) \|^2_{\phi \leq \delta} \end{aligned} \tag{9.6}$$

This gives:

$$(\tau - \text{constant}) \|u\|^2_{\phi \geq \delta} \leq c \left\| \overline{\partial}_M(f_\delta) \right\|^2_{\phi \leq \delta} \tag{9.7}$$

for all $\tau \geq \tau_0$ and hence $u = 0$ a.e. for $\phi \geq \delta$, showing that u vanishes on a neighborhood of p_0.

The proof is complete.

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Dipartimento di Matematica
Via Cintia
I-80126 Napoli
decarli@matna2.dma.unina.it

Dipartimento di Matematica
Via F. Buonarroti, 2
I-56127 Pisa
nacinovi@dm.unipi.it