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## Gradient Flow for the One-Dimensional Mumford-Shah Functional

MASSIMO GOBBINO

**Abstract.** In order to introduce a notion of gradient flow for the one-dimensional Mumford-Shah functional  $MS(u)$ , we consider a family  $\{\hat{F}_\varepsilon\}$  of regular functionals, defined in spaces of piecewise constant functions, which converge in a variational sense to  $MS(u)$ .

Moreover, given an initial datum  $u_0$ , with  $MS(u_0) < +\infty$ , and a family  $\{u_{0\varepsilon}\}$  of piecewise constant approximations of  $u_0$ , we consider the evolution problems

$$u'_\varepsilon(t) = -[\nabla \hat{F}_\varepsilon](u_\varepsilon(t)), \quad u_\varepsilon(0) = u_{0\varepsilon}.$$

We show that for large classes of initial data, the family  $\{u_\varepsilon(t)\}$  converges, as  $\varepsilon \rightarrow 0^+$ , to a certain  $u(t)$ , which is the solution of the heat equation with homogeneous Neumann boundary conditions in a suitable variable domain. On the other hand, we show that, for some special  $u_0$ , the family  $\{u_\varepsilon(t)\}$  has infinitely many limit points as  $\varepsilon \rightarrow 0^+$ .

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### 1. – Introduction

In last years many variational problems with free discontinuities have been studied. The canonical examples are the minimum problems related to the so called Mumford-Shah functional, defined by

$$\mathcal{F}(u) = \int_{\Omega} |\nabla u(x)|^2 dx + \mathcal{H}^{n-1}(S_u),$$

where  $\Omega$  is an open subset of  $\mathbb{R}^n$ ,  $u$  belongs to the space  $SBV(\Omega)$  of special functions with bounded variation (see Section 2),  $\nabla u$  is the approximate gradient

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of  $u$ ,  $S_u$  is the set of essential discontinuity points of  $u$ , and  $\mathcal{H}^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure.

This functional is the weak formulation in the space  $SBV(\Omega)$  of the functional introduced by D. Mumford and J. Shah in [21] to approach image segmentation problems.

On the other hand, a competition between bulk and surface energies had already been considered seventy years before by Griffiths [19] to model fractures in materials.

By the semicontinuity and compactness theorem in  $SBV$  proved by Ambrosio in [1], variational problems involving  $\mathcal{F}$  or more general functionals defined in  $SBV$  can be solved using the direct methods of the calculus of variations. Moreover, also the regularity of minimizers has long been studied after the pioneering paper by E. De Giorgi, M. Carriero & A. Leaci [14]: the interested reader can find appropriate references in the survey [3].

On the contrary, evolution problems with free discontinuities seem to be a still unexplored research field, despite of the possible applications to fracture dynamic. The prototype of these evolution problems is the gradient flow for the Mumford-Shah functional.

A first difficulty is to establish what “gradient flow” means in this case, since  $\mathcal{F}(u)$  is neither regular nor convex, and therefore it is not possible to apply standard theories, such as maximal monotone operators (cf. [8]). A possible approach to a similar problem was considered by A. Chambolle and F. Doveri in [12]. They studied a model of fracture propagation introduced by Ambrosio and Braides in [4], based on the evolution by minimizing movements of the two-dimensional Mumford-Shah energy, with a few additional assumptions on the discontinuity set.

In this paper we pursue a different path. In the one-dimensional case, we approximate  $\mathcal{F}$  by means of regular functionals  $\hat{F}_\varepsilon$  defined in Hilbert spaces of piecewise constant functions, and then we define the gradient flow associated to  $\mathcal{F}$  as the limit of the gradient flows associated to  $\hat{F}_\varepsilon$ . We recall that approximations of the Mumford-Shah functional are well studied in mathematical literature (cf. [5], [6], [7], [10], [11], [18]), mainly because of the possible numerical applications.

To be more precise, our strategy is the following.

- (1) For  $\varepsilon > 0$  we define

$$(1.1) \quad \hat{F}_\varepsilon(u) = \frac{1}{\varepsilon} \int_{\mathbb{R}} \arctan \left( \frac{(u(x+\varepsilon) - u(x))^2}{\varepsilon} \right) dx$$

for all  $u \in L^2(\mathbb{R})$  which are constant on each interval  $[z\varepsilon, (z+1)\varepsilon[$ , with  $z$  integer. In Section 3 we prove some results of convergence of  $\hat{F}_\varepsilon$  to  $\mathcal{F}$  (up to multiplicative constants).

- (2) Given an initial datum  $u_0 \in L^\infty(\mathbb{R}) \cap SBV_{\text{loc}}(\mathbb{R})$ , with  $\mathcal{F}(u_0) < +\infty$ , and a suitable family  $\{u_{0\varepsilon}\}$  of piecewise constant approximations of  $u$

(cf. Section 4.1 for the details), we consider for all  $\varepsilon > 0$  the evolution problem

$$(1.2) \quad u'_\varepsilon(t) = -[\nabla \hat{F}_\varepsilon](u_\varepsilon(t)), \quad u_\varepsilon(0) = u_{0\varepsilon}.$$

Since  $\hat{F}_\varepsilon$  is regular, the standard theory of ODEs in Hilbert spaces provides a unique solution  $u_\varepsilon(t)$  of (1.2), defined for all  $t \geq 0$ .

- (3) We show that there exist a sequence  $\{\varepsilon_n\} \rightarrow 0^+$ , and a continuous function  $u : [0, +\infty[ \rightarrow L^2_{\text{loc}}(\mathbb{R})$  such that

$$(1.3) \quad \{u_{\varepsilon_n}(t)\} \longrightarrow u(t) \text{ in } L^2_{\text{loc}}(\mathbb{R})$$

for all  $t \geq 0$  (Theorem 4.4).

- (4) We show that for large classes of initial data, the whole family  $\{u_\varepsilon(t)\}$  converges to a limit  $u(t)$ , which does not depend on  $\{u_{0\varepsilon}\}$  (cf. Theorem 5.4). This unique limit  $u$  is the only candidate to be the gradient flow for the Mumford-Shah functional with initial datum  $u_0$ .

Roughly speaking,  $u(t)$  can be obtained by evolving  $u_0$ , out of its singular set, according to the (rescaled) heat equation with homogeneous Neumann boundary conditions, and restarting the evolution (with the new initial datum) whenever a singularity “disappears” (see Section 5.1 for the details of this construction).

Finally, we prove that the Mumford-Shah energy is decreasing along the trajectory.

- (5) We show that for some special choices of  $u_0$  there is a continuum of possible limit points in (1.3), depending on the sequence  $\{\varepsilon_n\}$  and on the family  $\{u_{0\varepsilon}\}$  (cf. Theorem 5.10). However, *only one* of these limit points has the property that the function  $t \rightarrow \mathcal{F}(u(t))$  is non-increasing, and this limit can be characterized as in the regular case (cf. Theorem 5.9).

We hope that similar techniques will provide a definition of gradient flow for the Mumford-Shah functional also in the  $n$ -dimensional case. For example, using the finite difference approximations introduced in [18], it is possible to repeat word-by-word in any dimension the steps (1)-(3) described above; however a precise characterization of the possible limits seems to be a challenging problem.

We hope also to approach in a similar way evolution problems (with free discontinuities) involving second order time derivatives.

We finally remark the analogy between our construction and the approximation of motion by mean curvature (which may be thought as the gradient flow of the area functional) by the Allen-Cahn equation (which is the (rescaled) gradient flow of an approximation of the area functional): the interested reader is referred to [20].

This paper is organized as follows: in Section 2 we give notations and preliminaries; in Section 3 we prove some convergence results for the family of functionals  $\{\hat{F}_\varepsilon\}$  introduced in (1.1); in Section 4 we study the evolution problems (1.2), then we prove (3) and some general properties of the possible limits; in Section 5 we prove the results sketched in (4) and (5).

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## 2. – Preliminaries

In this section we fix notations and we recall basic definitions from the theory of  $SBV$  functions and  $\Gamma$ -convergence.

For all  $\alpha \in \mathbb{R}$ , the integer part of  $\alpha$  is denoted by  $[\alpha] = \sup\{z \in \mathbb{Z} : z \leq \alpha\}$ . Given  $A, B \subseteq \mathbb{R}$ , we write  $A \subset\subset B$  if the closure of  $A$ , denoted by  $\bar{A}$ , is compact and contained in  $B$ .

The Lebesgue measure and the 0-dimensional Hausdorff measure of a set  $B \subseteq \mathbb{R}$  are denoted by  $|B|$  and  $\mathcal{H}^0(B)$  respectively. We recall that  $\mathcal{H}^0(B)$  coincides with the number of elements of  $B$ , sometimes denoted also by  $\#B$ . The characteristic function of a set  $B \subseteq \mathbb{R}$  is denoted by  $\chi_B$ . We use standard notations for the Banach spaces  $L^p(\mathbb{R})$ , and for the metrizable spaces  $L^p_{\text{loc}}(\mathbb{R})$ , whose metric is denoted by  $d_{L^p_{\text{loc}}(\mathbb{R})}$ . All the functionals introduced in this paper, and also all the operations of  $\lim$ ,  $\liminf$ ,  $\limsup$ , are intended with range in the extended real line  $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ .

Even if the functions we consider depend only on one space variable, we use vector notations such as  $\nabla u$ ,  $\Delta u$ , for differential operators with respect to this space variable.

### 2.1. – Special functions of bounded variation

For the general theory of functions with bounded variation we refer to [17], [22]; here we just recall some definitions and some basic results for the one-dimensional case.

Let  $\Omega \subseteq \mathbb{R}$  be an open set. We say that  $u$  is a *function of bounded variation* in  $\Omega$ , and we write  $u \in BV(\Omega)$ , if  $u \in L^1(\Omega)$  and its distributional derivative is a real-valued measure  $Du$  with finite total variation  $|Du|(\Omega)$ . We say that  $u \in BV_{\text{loc}}(\Omega)$  if  $u \in BV(A)$  for every open set  $A \subset\subset \Omega$ .

If  $u \in BV_{\text{loc}}(\mathbb{R})$ , then for all  $x \in \mathbb{R}$  there exist

$$u^-(x) := \lim_{h \rightarrow 0^+} \frac{1}{h} \int_{x-h}^x u(s) ds, \quad u^+(x) := \lim_{h \rightarrow 0^+} \frac{1}{h} \int_x^{x+h} u(s) ds.$$

We denote by  $S_u$  the *discontinuity set* of  $u$ , i.e.

$$S_u := \{x \in \Omega : u^+(x) \neq u^-(x)\}.$$

It turns out that  $S_u$  is a countable set. Moreover,  $u^+$  and  $u^-$  have a derivative (in the classical sense) for a.e.  $x \in \Omega$ . These two derivatives coincide

for a.e.  $x \in \Omega$ , and their common value is called the *approximate gradient* of  $u$ , which is denoted by  $\nabla u$ .

Now let us write  $Du = D^a u + D^s u$ , where  $D^a u$  is absolutely continuous with respect to the Lebesgue measure  $\mathcal{L}^1$ , and  $D^s u$  is singular with respect to  $\mathcal{L}^1$ . We call  $D^a u$  the “*absolutely continuous part*” of  $Du$ . Moreover, we call the restriction  $D^j u$  of  $D^s u$  to  $S_u$  the “*jump part*” of  $Du$ , and the restriction  $D^c u$  of  $D^s u$  to  $\Omega \setminus S_u$  the “*Cantor part*” of  $Du$ .

With these notations we have the following decomposition:

$$Du = D^a u + D^j u + D^c u.$$

Moreover, it turns out that

$$\begin{aligned} D^a u &= \nabla u(x) \mathcal{L}^1, \\ D^j u &= \sum_{y \in S_u} (u^+(y) - u^-(y)) \delta_y, \end{aligned}$$

where  $\delta_y(A) = 1$  if  $y \in A$ , and  $\delta_y(A) = 0$  if  $y \notin A$ .

**DEFINITION 2.1.** Let  $\Omega \subseteq \mathbb{R}$  be an open set, and let  $u \in BV(\Omega)$ . We say that  $u$  is a *special function of bounded variation*, and we write  $u \in SBV(\Omega)$ , if  $D^c u = 0$ .

We say that  $u \in SBV_{\text{loc}}(\Omega)$  if  $u \in SBV(A)$  for all  $A \subset\subset \Omega$ .

For all positive real numbers  $\lambda, \mu$ , let us set

$$(2.1) \quad MS_{\lambda, \mu}(u) := \begin{cases} \lambda \int_{\mathbb{R}} |\nabla u(x)|^2 dx + \mu \mathcal{H}^0(S_u) & \text{if } u \in L^2_{\text{loc}}(\mathbb{R}) \cap SBV_{\text{loc}}(\mathbb{R}), \\ +\infty & \text{if } u \in L^2_{\text{loc}}(\mathbb{R}) \setminus SBV_{\text{loc}}(\mathbb{R}). \end{cases}$$

It turns out that  $MS_{\lambda, \mu}(u) < +\infty$  if and only if  $u$  has a finite set of discontinuity points  $S_u$ , and  $u \in H^1(\mathbb{R} \setminus S_u)$ .

The following semicontinuity and compactness result may be simply deduced from the general theory of [2].

**THEOREM 2.2.** For all  $\lambda > 0, \mu > 0$ , the functional  $MS_{\lambda, \mu}(u)$  defined in (2.1) is lower semicontinuous in  $L^2_{\text{loc}}(\mathbb{R})$ .

Moreover, if  $\{u_n\} \subseteq SBV_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$  is a sequence such that

$$\sup_{n \in \mathbb{N}} \{MS_{\lambda, \mu}(u_n) + \|u_n\|_\infty\} < +\infty,$$

then there exists a subsequence  $\{u_{n_k}\}$  converging in  $L^2_{\text{loc}}(\mathbb{R})$  to some  $u \in SBV_{\text{loc}}(\mathbb{R})$ . Moreover

$$\begin{aligned} \{\nabla u_{n_k}\} &\rightharpoonup \nabla u && \text{weakly in } L^2(\mathbb{R}); \\ \{D^j u_{n_k}\} &\xrightarrow{*} D^j u && \text{as Radon measures,} \end{aligned}$$

and

$$\begin{aligned} \liminf_{k \rightarrow \infty} \int_{\mathbb{R}} |\nabla u_{n_k}(x)|^2 dx &\geq \int_{\mathbb{R}} |\nabla u(x)|^2 dx; \\ \liminf_{k \rightarrow \infty} \mathcal{H}^0(S_{u_{n_k}}) &\geq \mathcal{H}^0(S_u). \end{aligned} \quad \square$$

### 2.2. – $\Gamma$ -convergence

DEFINITION 2.3. Let  $X$  be a metric space, let  $\{F_n\}$  be a sequence of functions defined in  $X$  with values in  $\overline{\mathbb{R}}$ , and let  $F : X \rightarrow \overline{\mathbb{R}}$ . We say that  $\{F_n\}$   $\Gamma$ -converges to  $F$ , and we write

$$F(x) = \Gamma - \lim_{n \rightarrow \infty} F_n(x),$$

if the following two conditions are satisfied:

- (i) for every  $x \in X$ , and every sequence  $\{x_n\}$  converging to  $x$  we have that

$$F(x) \leq \liminf_{n \rightarrow \infty} F_n(x_n);$$

- (ii) for every  $x \in X$ , there exists a sequence  $\{x_n\}$  converging to  $x$  such that

$$F(x) \geq \limsup_{n \rightarrow \infty} F_n(x_n).$$

The  $\Gamma$ -limit, when it exists, is unique, and stable under subsequences. Finally, we say that a family  $\{F_\varepsilon\}_{\varepsilon > 0}$  of functions  $\Gamma$ -converges to  $F$  as  $\varepsilon \rightarrow 0^+$ , if  $\{F_{\varepsilon_n}\}$   $\Gamma$ -converges to  $F$  for every sequence  $\{\varepsilon_n\} \rightarrow 0^+$ .

The reader interested to variational properties of  $\Gamma$ -convergence is referred to [13], [15].

### 2.3. – The heat equation

We recall some results about the heat equation that will be frequently used in Section 5.

Let  $I = [a, b]$  be an interval, and let  $v_0 \in H^1(I)$ . Then the problem

$$\begin{cases} v_t(t, x) = 2 v_{xx}(t, x) & \text{in } [0, +\infty[ \times I \\ v_x(t, a) = v_x(t, b) = 0 & \forall t \geq 0 \\ v(0, x) = v_0(x) \end{cases}$$

has a unique solution which depends continuously on  $v_0$  (with respect to  $L^2$  convergence).

Moreover, for each  $y \in [a, b]$ , the function  $t \rightarrow v(t, y)$  is continuous in  $[0, +\infty[$ , analytic in  $]0, +\infty[$ , and depends continuously (with its derivatives of any order) on  $v_0$ .

Furthermore, the following comparison result hold true: if  $u$  is a solution of the same PDE, with the same initial datum, and boundary conditions  $u_x(t, a) \geq 0$ ,  $u_x(t, b) \leq 0$  for all  $t \geq 0$ , then  $u(t, x) \leq v(t, x)$  in  $[0, +\infty[ \times I$ .

Completely analogous results hold true if  $I$  as a half line, up to replace the lost boundary condition by  $u(t) \in L^\infty$ , and  $L^2$  and  $H^1$ , respectively, by  $L^2_{\text{loc}}$  and  $H^1_{\text{loc}}$ .

### 3. – Approximation results

In this section we approximate the one-dimensional Mumford-Shah functional by a family  $\{\hat{F}_\varepsilon\}$  of regular functionals which are finite only in suitable spaces of piecewise constant functions. We prove both the  $\Gamma$ -convergence (Theorem 3.10), and some *higher order* convergence results (Theorem 3.12, 3.13, and 3.15) which will be crucial in Section 4-5.

#### 3.1. – Definitions

In order to simplify many formulas we introduce some notations.

DEFINITION 3.1. For every  $\rho \in \mathbb{R}$ , and every function  $u : \mathbb{R} \rightarrow \mathbb{R}$ , we set

$$D^\rho u(x) := \begin{cases} \frac{u(x + \rho) - u(x)}{\rho}, & \text{if } \rho \neq 0, \\ 0, & \text{if } \rho = 0. \end{cases}$$

The following properties of the operator  $D^\rho$  are an easy consequence of the above definition (the simple proofs are left to the interested reader).

PROPOSITION 3.2 (Properties of  $D^\rho$ ). *For all  $\rho \in \mathbb{R}$  the following hold true.*

- (i) *Linearity: given two real functions  $u, v : \mathbb{R} \rightarrow \mathbb{R}$ , and two real numbers  $\alpha, \beta$ , we have that*

$$(3.1) \quad D^\rho(\alpha u + \beta v)(x) = \alpha D^\rho u(x) + \beta D^\rho v(x), \quad \forall x \in \mathbb{R}.$$

- (ii) *Chain-rule: given two real functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  we have that*

$$(3.2) \quad D^\rho(f \circ g)(x) = (D^{g(x+\rho)-g(x)} f)(g(x)) \cdot D^\rho g(x), \quad \forall x \in \mathbb{R}.$$

- (iii) *Integration by parts: let  $\Omega \subseteq \mathbb{R}$  be an open set, and let  $u, v \in L^2(\Omega)$ . If either  $u$  or  $v$  has compact support in  $\Omega$ , and  $|\rho|$  is small enough, then we have that:*

$$(3.3) \quad \int_{\Omega} u(x) D^\rho v(x) dx = - \int_{\Omega} v(x) D^{-\rho} u(x) dx.$$

*If  $\Omega = \mathbb{R}$ , equality (3.3) holds without restrictions on  $\rho$ , and on the support of  $u$  and  $v$ .*

- (iv) *Approximation of derivatives: for all  $\Phi \in C_0^1(\mathbb{R})$  we have that*

$$(3.4) \quad D^\rho \Phi \longrightarrow \nabla \Phi \text{ uniformly on } \mathbb{R}$$

*as  $\rho \rightarrow 0$ .*

- (v) *Lipschitz continuity: for all  $p \in [1, +\infty]$  the operator  $u \rightarrow D^\rho u$  is well defined and Lipschitz continuous in  $L^p(\mathbb{R})$ .  $\square$*



Properties (i)-(iv) show the analogy between the operator  $D^\rho$  and the derivation operator.

For each  $\varepsilon > 0$  we consider the function

$$(3.5) \quad \varphi_\varepsilon(r) := \frac{1}{\varepsilon} \arctan(\varepsilon r^2), \quad \forall r \in \mathbb{R},$$

and the space

$$(3.6) \quad \text{PC}_\varepsilon^2 := \{u \in L^2(\mathbb{R}) : u(x) = u(\varepsilon[x/\varepsilon]), \forall x \in \mathbb{R}\},$$

so that each  $u \in \text{PC}_\varepsilon^2$  is constant in the interval  $[z\varepsilon, (z+1)\varepsilon[$  for all  $z \in \mathbb{Z}$ .

Finally, we consider the functional

$$(3.7) \quad \begin{cases} \int_{\mathbb{R}} \varphi_\varepsilon(D^\varepsilon u(x)) dx & \text{if } u \in \text{PC}_\varepsilon^2, \\ +\infty & \text{if } u \in L^2_{\text{loc}}(\mathbb{R}) \setminus \text{PC}_\varepsilon^2, \end{cases}$$

defined for every  $\varepsilon > 0$ ,  $u \in L^2_{\text{loc}}(\mathbb{R})$ , with values in  $\mathbb{R} \cup \{+\infty\}$ .

PROPOSITION 3.3 (Properties of  $\hat{F}_\varepsilon$ ). *For all  $\varepsilon > 0$  we have that:*

- (i)  $\text{PC}_\varepsilon^2$  is a closed vector subspace of  $L^2(\mathbb{R})$ , hence it is a Hilbert space with respect to the usual scalar product of  $L^2(\mathbb{R})$ ;
- (ii) the restriction of  $\hat{F}_\varepsilon$  to  $\text{PC}_\varepsilon^2$  is a real valued function of class  $C^\infty$ ;
- (iii) for every  $u \in \text{PC}_\varepsilon^2$  the gradient of  $\hat{F}_\varepsilon$  in  $u$  is given by

$$(3.8) \quad [\nabla \hat{F}_\varepsilon(u)](x) = -D^{-\varepsilon}(\varphi'_\varepsilon(D^\varepsilon u(x)));$$

- (iv)  $\nabla \hat{F}_\varepsilon$  is a Lipschitz continuous function on  $\text{PC}_\varepsilon^2$ .

PROOF. Statements (i) and (ii) follow immediately from the definitions of  $\text{PC}_\varepsilon^2$ ,  $D^\varepsilon$ , and  $\varphi_\varepsilon$ .

In order to prove (iii), we use the standard relation between the differential and the Gateaux derivative along a direction  $v \in \text{PC}_\varepsilon^2$ :

$$\begin{aligned} \langle \nabla \hat{F}_\varepsilon(u), v \rangle_{L^2(\mathbb{R})} &= \lim_{h \rightarrow 0} \frac{\hat{F}_\varepsilon(u + hv) - \hat{F}_\varepsilon(u)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_{\mathbb{R}} (\varphi_\varepsilon(D^\varepsilon u(x) + hD^\varepsilon v(x)) - \varphi_\varepsilon(D^\varepsilon u(x))) dx \\ &= \int_{\mathbb{R}} \varphi'_\varepsilon(D^\varepsilon u(x)) D^\varepsilon v(x) dx \\ &= - \int_{\mathbb{R}} D^{-\varepsilon} \varphi'_\varepsilon(D^\varepsilon u(x)) v(x) dx, \end{aligned}$$

where the last inequality follows from (3.3). This proves (iii).

Therefore the function  $\nabla \hat{F}_\varepsilon$  is the composition of the three maps

$$u \longrightarrow D^\varepsilon u, \quad v \longrightarrow \varphi'_\varepsilon(v), \quad w \longrightarrow D^{-\varepsilon}(w),$$

which are Lipschitz continuous. This proves (iv).  $\square$

For every  $\varepsilon > 0$ , we now write  $D^\varepsilon$  as the sum of two operators, which may be thought as the “absolutely continuous part” and the “jump part” of  $D^\varepsilon$ .

DEFINITION 3.4. For every  $\varepsilon > 0$ , and every function  $u : \mathbb{R} \rightarrow \mathbb{R}$ , we set

$$D^{\varepsilon,-}u(x) := \begin{cases} D^\varepsilon u(x) & \text{if } |D^\varepsilon u(x)| < 3^{-1/4}\varepsilon^{-1/2}, \\ 0 & \text{otherwise,} \end{cases}$$

and we define  $D^{\varepsilon,+}u$  in such a way that

$$D^\varepsilon u(x) = D^{\varepsilon,+}u(x) + D^{\varepsilon,-}u(x), \quad \forall x \in \mathbb{R}.$$

Moreover we set

$$I_\varepsilon^+(u) := \{x \in \mathbb{R} : D^{\varepsilon,+}u(x) \neq 0\}.$$

We remark that  $3^{-1/4}\varepsilon^{-1/2}$  is the maximum point of  $\varphi'_\varepsilon$ .

The following estimates are a crucial tool in many proofs.

PROPOSITION 3.5. Let  $\hat{F}_\varepsilon$  be the functional introduced in (3.7), and let  $D^{\varepsilon,+}$ ,  $D^{\varepsilon,-}$ ,  $I_\varepsilon^+$  be as in Definition 3.4.

Then for all  $u \in L^2_{\text{loc}}(\mathbb{R})$  we have that

$$(3.9) \quad |I_\varepsilon^+(u)| \leq \frac{6}{\pi} \varepsilon \hat{F}_\varepsilon(u),$$

$$(3.10) \quad \int_{\mathbb{R} \setminus I_\varepsilon^+(u)} (D^\varepsilon u(x))^2 dx = \int_{\mathbb{R}} (D^{\varepsilon,-}u(x))^2 dx \leq \frac{6}{\pi} \hat{F}_\varepsilon(u).$$

Moreover

$$(3.11) \quad \int_{\Omega} |D^\varepsilon u(x)| dx \leq \frac{12}{\pi} \|u\|_\infty \hat{F}_\varepsilon(u) + \left\{ \frac{6}{\pi} |\Omega| \hat{F}_\varepsilon(u) \right\}^{1/2}$$

for every open set  $\Omega \subset \subset \mathbb{R}$ .

PROOF. Since  $\arctan r \geq \frac{\pi}{6}$  for  $r \geq 3^{-1/2}$ , we have that

$$\hat{F}_\varepsilon(u) \geq \int_{I_\varepsilon^+(u)} \varphi_\varepsilon(D^\varepsilon u(x)) dx \geq \frac{\pi}{6} \frac{1}{\varepsilon} |I_\varepsilon^+(u)|,$$

which is equivalent to (3.9). Moreover, since  $\arctan r \geq \frac{\pi}{6} r$  for  $0 \leq r \leq 3^{-1/2}$ , we have that

$$\hat{F}_\varepsilon(u) \geq \int_{\mathbb{R} \setminus I_\varepsilon^+(u)} \varphi_\varepsilon(D^\varepsilon u(x)) dx \geq \frac{\pi}{6} \int_{\mathbb{R} \setminus I_\varepsilon^+(u)} (D^\varepsilon u(x))^2 dx,$$

which is equivalent to (3.10). Finally, by Hölder’s inequality we have that

$$\begin{aligned} \int_{\Omega} |D^\varepsilon u(x)| dx &\leq \int_{I_\varepsilon^+(u)} |D^\varepsilon u(x)| dx + \int_{\Omega \setminus I_\varepsilon^+(u)} |D^\varepsilon u(x)| dx \\ &\leq \frac{2\|u\|_\infty}{\varepsilon} |I_\varepsilon^+(u)| + |\Omega|^{1/2} \left( \int_{\mathbb{R} \setminus I_\varepsilon^+(u)} |D^\varepsilon u(x)|^2 dx \right)^{1/2}, \end{aligned}$$

so that (3.11) follows from (3.9) and (3.10). □

### 3.2. – Some lemmata

We now prove three general lemmata.

LEMMA 3.6. *Let  $\{\varepsilon_n\} \rightarrow 0^+$ , let  $\{g_n\}$  be a sequence such that  $g_n \in \text{PC}_{\varepsilon_n}^2$  for all  $n \in \mathbb{N}$ , and let  $g \in L^2(\mathbb{R})$ . Let us assume that*

$$(3.12) \quad \{g_n\} \rightharpoonup g \quad \text{weakly in } L^2(\mathbb{R});$$

$$(3.13) \quad \sup_{n \in \mathbb{N}} \|D^{-\varepsilon_n} g_n\|_{L^2(\mathbb{R})} < +\infty.$$

Then  $g \in H^1(\mathbb{R})$ , and moreover

$$(3.14) \quad \begin{array}{ll} \{g_n\} \longrightarrow g & \text{uniformly on compact sets;} \\ \{D^{-\varepsilon_n} g_n\} \rightharpoonup \nabla g & \text{weakly in } L^2(\mathbb{R}). \end{array}$$

PROOF. By (3.12) we have that

$$M := \sup_{n \in \mathbb{N}} \|g_n\|_{L^2(\mathbb{R})} < +\infty.$$

STEP 1. By (3.13) there exist a sequence  $\{n_k\} \rightarrow 0^+$ , and a function  $h \in L^2(\mathbb{R})$  such that

$$(3.15) \quad \{D^{-\varepsilon_{n_k}} g_{n_k}\} \rightharpoonup h \quad \text{weakly in } L^2(\mathbb{R}).$$

Now let  $\Phi \in C_0^\infty(\mathbb{R})$ . By (3.3) we have that

$$\int_{\mathbb{R}} g_{n_k}(x) D^{\varepsilon_{n_k}} \Phi(x) dx = - \int_{\mathbb{R}} D^{-\varepsilon_{n_k}} g_{n_k}(x) \Phi(x) dx,$$

so that, passing to the limit as  $k \rightarrow \infty$ , and using (3.4), (3.12), and (3.15), we obtain

$$\int_{\mathbb{R}} g(x) \nabla \Phi(x) dx = - \int_{\mathbb{R}} h(x) \Phi(x) dx.$$

Since  $\Phi$  is arbitrary, this proves that  $g \in H^1(\mathbb{R})$  (therefore  $g$  is continuous), and  $h = \nabla g$ . Since this limit does not depend on the sequence  $\{n_k\}$ , (3.14) is proved.

STEP 2. We show that

$$(3.16) \quad |g_n(x) - g_n(y)| \leq M(|x - y| + \varepsilon_n)^{1/2},$$

for every  $n \in \mathbb{N}$ , and every  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ .

To this end, let us set for simplicity  $h = [x/\varepsilon]$ ,  $k = [y/\varepsilon]$ , and let us assume that  $h > k$ . Then by Cauchy Schwarz inequality we have that

$$\begin{aligned} |g_n(x) - g_n(y)| &= |g_n(h\varepsilon_n) - g_n(k\varepsilon_n)| \\ &= \left| \sum_{i=k+1}^h \varepsilon_n D^{-\varepsilon_n} g_n(i\varepsilon_n) \right| \\ &\leq \varepsilon_n^{1/2} |h - k|^{1/2} \left\{ \sum_{i=k+1}^h \varepsilon_n |D^{-\varepsilon_n} g_n(i\varepsilon_n)|^2 \right\}^{1/2} \\ &\leq (|x - y| + \varepsilon_n)^{1/2} M. \end{aligned}$$

STEP 3. We show that  $\{g_n\}$  pointwise converges to  $g$ .

To this end, let us fix  $x_0 \in \mathbb{R}$ , and  $\delta > 0$ . By (3.16) we have that

$$\left| \int_{x_0}^{x_0+\delta} (g_n(x) - g_n(x_0)) dx \right| \leq \int_{x_0}^{x_0+\delta} |g_n(x) - g_n(x_0)| dx \leq \frac{2}{3} M |\delta + \varepsilon_n|^{3/2},$$

hence

$$\delta g_n(x_0) - \frac{2}{3} M |\delta + \varepsilon_n|^{3/2} \leq \int_{x_0}^{x_0+\delta} g_n(x) dx \leq \delta g_n(x_0) + \frac{2}{3} M |\delta + \varepsilon_n|^{3/2}.$$

Taking the  $\liminf$  as  $n \rightarrow \infty$ , and dividing by  $\delta$  we obtain

$$\liminf_{n \rightarrow \infty} g_n(x_0) - \frac{2}{3} M |\delta|^{1/2} \leq \frac{1}{\delta} \int_{x_0}^{x_0+\delta} g(x) dx \leq \liminf_{n \rightarrow \infty} g_n(x_0) + \frac{2}{3} M |\delta|^{1/2}.$$

Passing to the limit as  $\delta \rightarrow 0^+$ , and exploiting the continuity of  $g$ , we finally obtain that

$$\liminf_{n \rightarrow \infty} g_n(x_0) = g(x_0).$$

Since we can argue in the same way with the  $\limsup$ , pointwise convergence is proved.

STEP 4. Uniform convergence follows in a standard way from pointwise convergence and from (3.16).  $\square$

LEMMA 3.7. Let  $\{g_\varepsilon\} \subseteq L^2(\mathbb{R})$ , and let  $g \in L^2(\mathbb{R})$  be such that

$$\{g_\varepsilon\} \rightharpoonup g \quad \text{weakly in } L^2(\mathbb{R}).$$

Then

$$(3.17) \quad \liminf_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \varphi_\varepsilon(g_\varepsilon(x)) dx \geq \int_{\mathbb{R}} |g(x)|^2 dx,$$

where  $\varphi_\varepsilon$  is the function defined in (3.5).

PROOF. Since it is enough to prove (3.17) for every sequence  $\{\varepsilon_n\} \rightarrow 0^+$ , we can assume that

$$M := \sup_{\varepsilon > 0} \|g_\varepsilon\|_{L^2(\mathbb{R})} < +\infty.$$

Now for all  $A > 0$  let us set

$$I_{A,\varepsilon}^+ := \{x \in \mathbb{R} : |g_\varepsilon(x)| \geq A\varepsilon^{-1/2}\}.$$

Since

$$M^2 \geq \int_{I_{A,\varepsilon}^+} |g_\varepsilon(x)|^2 dx \geq \frac{A^2}{\varepsilon} |I_{A,\varepsilon}^+|,$$

it follows that  $|I_{A,\varepsilon}^+| \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$ , hence

$$g_\varepsilon \chi_{I_{A,\varepsilon}^+} \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{R}),$$

and therefore

$$(3.18) \quad g_\varepsilon \chi_{\mathbb{R} \setminus I_{A,\varepsilon}^+} \rightharpoonup g \quad \text{weakly in } L^2(\mathbb{R}).$$

Moreover, since  $\varphi_\varepsilon(r) \geq \frac{\arctan A^2}{A^2} r$  if  $0 \leq r \leq A\varepsilon^{-1/2}$ , then we have that

$$\begin{aligned} \int_{\mathbb{R}} \varphi_\varepsilon(g_\varepsilon(x)) dx &\geq \int_{\mathbb{R} \setminus I_{A,\varepsilon}^+} \varphi_\varepsilon(g_\varepsilon(x)) dx \\ &\geq \frac{\arctan A^2}{A^2} \int_{\mathbb{R} \setminus I_{A,\varepsilon}^+} |g_\varepsilon(x)|^2 dx \\ &= \frac{\arctan A^2}{A^2} \|g_\varepsilon \chi_{\mathbb{R} \setminus I_{A,\varepsilon}^+}\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

hence by (3.18)

$$\liminf_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \varphi_\varepsilon(g_\varepsilon(x)) dx \geq \frac{\arctan A^2}{A^2} \|g\|_{L^2(\mathbb{R})}^2.$$

Taking the limit as  $A \rightarrow 0^+$ , (3.17) is proved.  $\square$

LEMMA 3.8. Let  $\{g_\varepsilon\} \subseteq L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , and let  $g \in L^2(\mathbb{R})$ . Let us assume that

$$(3.19) \quad \|g_\varepsilon\|_\infty \leq \varepsilon^{-1/2},$$

$$(3.20) \quad g_\varepsilon \rightharpoonup g \quad \text{weakly in } L^2(\mathbb{R}).$$

Then

$$(3.21) \quad \varphi'_\varepsilon(g_\varepsilon) \rightharpoonup 2g \quad \text{weakly in } L^2(\mathbb{R}).$$

where  $\varphi_\varepsilon$  is the function defined in (3.5).

PROOF. By assumption (3.20), thesis is equivalent to show that

$$(3.22) \quad \varphi'_\varepsilon(g_\varepsilon) - 2g_\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{R}).$$

To this end (looking at the explicit expression for  $\varphi'_\varepsilon$ ) it is enough to prove that

$$(3.23) \quad \varepsilon^2 g_\varepsilon^4 \longrightarrow 0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}).$$

Since it is enough to prove (3.23) for any sequence  $\{\varepsilon_n\} \rightarrow 0^+$ , thanks to (3.20) we may assume that

$$M := \sup_{\varepsilon > 0} \|g_\varepsilon\|_{L^2(\mathbb{R})} < +\infty.$$

Now let  $I \subset\subset \mathbb{R}$  be a bounded interval, and let  $A > 0$ . Let us set

$$I_{A,\varepsilon}^+ := \{x \in I : |g_\varepsilon(x)| \geq A\varepsilon^{-1/2}\}.$$

Then we have that

$$\|g_\varepsilon\|_{L^2(\mathbb{R})}^2 \geq \|g_\varepsilon\|_{L^2(I_{A,\varepsilon}^+)}^2 \geq \frac{A^2}{\varepsilon} |I_{A,\varepsilon}^+|,$$

hence

$$(3.24) \quad |I_{A,\varepsilon}^+| \leq \frac{M^2 \varepsilon}{A^2}.$$

By (3.19) and (3.24) we have that

$$\int_I (\varepsilon^2 g_\varepsilon^4)^2 = \int_{I_{A,\varepsilon}^+} (\varepsilon^2 g_\varepsilon^4)^2 + \int_{I \setminus I_{A,\varepsilon}^+} (\varepsilon^2 g_\varepsilon^4)^2 \leq |I_{A,\varepsilon}^+| + A^8 |I| \leq \frac{M^2 \varepsilon}{A^2} + A^8 |I|,$$

hence

$$\limsup_{\varepsilon \rightarrow 0^+} \|\varepsilon^2 g_\varepsilon^4\|_{L^2(I)}^2 \leq A^8 |I|.$$

Since  $A$ , and  $I$ , are arbitrary, (3.23) is proved.  $\square$

### 3.3. – $\Gamma$ -convergence and compactness

We now study the convergence of the family  $\{\hat{F}_\varepsilon\}$  to the Mumford-Shah functional. The following approximation lemma will be crucial in the sequel.

LEMMA 3.9. *Let  $u_0 \in L^\infty(\mathbb{R})$  with  $MS_{1, \frac{\pi}{2}}(u_0) < +\infty$ .*

*Then there exists a family  $\{u_{0\varepsilon}\} \subseteq L^\infty(\mathbb{R})$  such that*

- (i)  $\{u_{0\varepsilon}\} \rightarrow u_0$  in  $L^2_{\text{loc}}(\mathbb{R})$ ;
- (ii)  $\|u_{0\varepsilon}\|_\infty \leq \|u_0\|_\infty$  for all  $\varepsilon > 0$ ;
- (iii)  $u_{0\varepsilon} \in \text{PC}_\varepsilon^2$  for all  $\varepsilon > 0$ ;
- (iv)  $\limsup_{\varepsilon \rightarrow 0^+} \hat{F}_\varepsilon(u_{0\varepsilon}) \leq MS_{1, \frac{\pi}{2}}(u_0)$  for all  $\varepsilon > 0$ .

PROOF.

STEP 1. Let us assume that  $u_0$  has compact support. Modifying  $u$  on a negligible set (if necessary), we can assume that  $u$  is left continuous. Let us set

$$u_{0\varepsilon}(x) := u_0(\varepsilon[x/\varepsilon]).$$

It is clear that  $\{u_{0\varepsilon}\}$  satisfies (ii) and (iii). Moreover, since  $u$  is left continuous,  $\{u_{0\varepsilon}\}$  pointwise converges to  $u$ , hence (i) follows from Lebesgue's theorem.

In order to prove (iv), let us write

$$\hat{F}_\varepsilon(u_{0\varepsilon}) = \int_{A_\varepsilon} \varphi_\varepsilon(D^\varepsilon u_{0\varepsilon}(x)) dx + \int_{\mathbb{R} \setminus A_\varepsilon} \varphi_\varepsilon(D^\varepsilon u_{0\varepsilon}(x)) dx,$$

where

$$A_\varepsilon := \bigcup_{y \in S_{u_0}} [\varepsilon[y/\varepsilon], \varepsilon([y/\varepsilon] + 1)],$$

and let us estimate separately the two summands. The first one can be trivially estimated by

$$(3.25) \quad \int_{A_\varepsilon} \varphi_\varepsilon(D^\varepsilon u_{0\varepsilon}(x)) dx \leq \frac{\pi}{2} \frac{|A_\varepsilon|}{\varepsilon} \leq \frac{\pi}{2} \mathcal{H}^0(S_u).$$

In order to estimate the second summand, we note that if  $x \notin A_\varepsilon$ , then  $u_0$  is absolutely continuous in  $[\varepsilon[x/\varepsilon], \varepsilon([x/\varepsilon] + 1)]$ , hence, by Hölder's inequality:

$$\begin{aligned} (u_{0\varepsilon}(x + \varepsilon) - u_{0\varepsilon}(x))^2 &= (u_0(\varepsilon[x/\varepsilon] + \varepsilon) - u_0(\varepsilon[x/\varepsilon]))^2 \\ &\leq \left( \int_{\varepsilon[x/\varepsilon]}^{\varepsilon([x/\varepsilon] + 1)} |\nabla u_0(\tau)| d\tau \right)^2 \\ &\leq \varepsilon \int_{\varepsilon[x/\varepsilon]}^{\varepsilon([x/\varepsilon] + 1)} |\nabla u_0(\tau)|^2 d\tau. \end{aligned}$$

Thus, since  $\arctan(x) \leq x$  for all  $x \geq 0$ :

$$\begin{aligned} \int_{\mathbb{R} \setminus A_\varepsilon} \varphi_\varepsilon(D^\varepsilon u_{0\varepsilon}(x)) \, dx &\leq \frac{1}{\varepsilon} \int_{\mathbb{R}} \int_{\varepsilon[x/\varepsilon]}^{\varepsilon([x/\varepsilon]+1)} |\nabla u_0(\tau)|^2 \, d\tau \, dx \\ &= \frac{1}{\varepsilon} \sum_{z \in \mathbb{Z}} \varepsilon \int_{\varepsilon z}^{\varepsilon(z+1)} |\nabla u_0(\tau)|^2 \, d\tau = \int_{\mathbb{R}} |\nabla u_0(\tau)|^2 \, dx. \end{aligned}$$

By (3.25) and this last estimate, (iv) easily follows.

STEP 2. If  $u_0$  has not compact support, then we set

$$u_{0n}(x) := \begin{cases} u_0(x) & \text{if } x \in [-n, n], \\ \left(2 - \frac{x}{n}\right) u_0(n) & \text{if } x \in [n, 2n], \\ \left(2 + \frac{x}{n}\right) u_0(-n) & \text{if } x \in [-2n, -n], \\ 0 & \text{if } |x| \geq 2n. \end{cases}$$

It is clear that  $u_{0n}$  has compact support,  $\{u_{0n}\} \rightarrow u_0$  in  $L^2_{\text{loc}}(\mathbb{R})$ , and  $\|u_{0n}\|_\infty \leq \|u_0\|_\infty$ . Moreover,  $S_{u_{0n}} = S_{u_0 \cap ]-n, n[}$ , and

$$\int_{\mathbb{R}} |\nabla u_{0n}(x)|^2 \, dx = \int_{-n}^n |\nabla u_0(x)|^2 \, dx + \frac{1}{n} (|u_0(n)|^2 + |u_0(-n)|^2),$$

hence  $\{MS_{1, \frac{\pi}{2}}(u_{0n})\} \rightarrow MS_{1, \frac{\pi}{2}}(u_0)$ .

Since each  $u_{0n}$  may be approximated as in Step 1, by a diagonal argument thesis is proved also in the general case.  $\square$

We are now ready to show that  $MS_{1, \frac{\pi}{2}}$  is the  $\Gamma$ -limit of  $\{\hat{F}_\varepsilon\}$  with respect to the metric of  $L^2_{\text{loc}}(\mathbb{R})$ .

**THEOREM 3.10.** *Let  $\hat{F}_\varepsilon$  and  $MS_{\lambda, \mu}$  be the functionals defined in (3.7) and (2.1) respectively.*

*Then*

$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} \hat{F}_\varepsilon(u) = MS_{1, \frac{\pi}{2}}(u),$$

where the  $\Gamma$ -limit is computed with respect to the usual metric of  $L^2_{\text{loc}}(\mathbb{R})$ .

Moreover, if  $\{u_\varepsilon\} \subseteq L^\infty(\mathbb{R})$  is any family such that

$$(3.26) \quad \sup_{\varepsilon > 0} \left\{ \hat{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty \right\} < +\infty,$$

then  $\{u_\varepsilon\}$  is relatively compact in  $L^2_{\text{loc}}(\mathbb{R})$ , and every limit point belongs to  $SBV_{\text{loc}}(\mathbb{R})$ .



PROOF.

*Liminf part.* In [18] it is proved that

$$(3.27) \quad \liminf_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \varphi_{\varepsilon}(D^{\varepsilon} u_{\varepsilon}(x)) dx \geq MS_{1, \frac{\pi}{2}}(u),$$

whenever  $\{u_{\varepsilon}\} \rightarrow u$  in  $L^1_{\text{loc}}(\mathbb{R})$ . Since convergence in  $L^2_{\text{loc}}(\mathbb{R})$  implies convergence in  $L^1_{\text{loc}}(\mathbb{R})$ , and

$$\hat{F}_{\varepsilon}(v) \geq \int_{\mathbb{R}} \varphi_{\varepsilon}(D^{\varepsilon} v(x)) dx$$

for all  $v \in L^2_{\text{loc}}(\mathbb{R})$ , by (3.27) we have that

$$\liminf_{\varepsilon \rightarrow 0^+} \hat{F}_{\varepsilon}(u_{\varepsilon}) \geq MS_{1, \frac{\pi}{2}}(u),$$

whenever  $\{u_{\varepsilon}\} \rightarrow u$  in  $L^2_{\text{loc}}(\mathbb{R})$ .

*Limsup part.* Follows in a standard way from Lemma 3.9, and from the density of  $L^{\infty}(\mathbb{R})$  in  $L^2_{\text{loc}}(\mathbb{R})$ .

*Compactness.* For every  $\varepsilon > 0$ , and every open set  $\Omega \subset\subset \mathbb{R}$ , we have that  $u_{\varepsilon} \in BV(\Omega)$ , and

$$|Du_{\varepsilon}|(\Omega) = \sum_{\substack{z \in \mathbb{Z} \\ \varepsilon z \in \Omega}} \varepsilon |D^{\varepsilon} u_{\varepsilon}(\varepsilon z)| \leq \int_{\Omega_{\varepsilon}} |D^{\varepsilon} u_{\varepsilon}(x)| dx,$$

where  $\Omega_{\varepsilon} := \{x \in \mathbb{R} : |x - y| < \varepsilon \text{ for some } y \in \Omega\}$ .

By (3.11), and assumption (3.26), it follows that

$$\sup_{\varepsilon \in [0, 1]} \left\{ \int_{\Omega} |u_{\varepsilon}(x)| dx + |Du_{\varepsilon}|(\Omega) \right\} < +\infty.$$

Therefore, by the standard compactness theorem for  $BV$  functions, there exist a sequence  $\{\varepsilon_n\} \rightarrow 0^+$ , and  $u \in BV_{\text{loc}}(\mathbb{R})$  such that

$$\{u_{\varepsilon_n}\} \longrightarrow u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}).$$

Up to subsequences we can assume that the convergence is pointwise a.e.; thanks to the bound in the  $L^{\infty}$ -norm and Lebesgue's theorem, this implies that up to subsequences

$$\{u_{\varepsilon_n}\} \longrightarrow u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}).$$

Finally, by the “liminf part”, it follows that  $MS_{1, \frac{\pi}{2}}(u) < +\infty$ , hence  $u \in SBV_{\text{loc}}(\mathbb{R})$ .  $\square$

### 3.4. – Convergence of gradients

Given a family  $\{u_\varepsilon\} \rightarrow u$ , we now study the convergence of  $\{D^\varepsilon u_\varepsilon\}$  to  $\nabla u$ . A first result is the following.

LEMMA 3.11. *Let  $\{u_\varepsilon\} \subseteq L^\infty(\mathbb{R})$ , and let  $u \in L^\infty(\mathbb{R})$ . Let us assume that*

$$\begin{aligned} u_\varepsilon &\longrightarrow u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}), \\ \sup_{\varepsilon > 0} \left\{ \hat{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty \right\} &< +\infty. \end{aligned}$$

*Then  $u \in SBV_{\text{loc}}(\mathbb{R})$  and*

$$D^\varepsilon u_\varepsilon(x) dx \xrightarrow{*} Du \quad \text{weakly } * \text{ as Radon measures.}$$

PROOF. By (3.11), and our assumptions, it follows that

$$\sup_{\varepsilon > 0} \int_{\Omega} |D^\varepsilon u_\varepsilon(x)| dx < +\infty$$

for every open set  $\Omega \subset \subset \mathbb{R}$ . Therefore, by the compactness theorem for Radon measures, there exist a sequence  $\{\varepsilon_n\} \rightarrow 0^+$ , and a Radon measure  $\mu$  such that

$$D^{\varepsilon_n} u_{\varepsilon_n}(x) dx \xrightarrow{*} \mu \quad \text{weakly } *.$$

Now let  $\Phi \in C_0^1(\mathbb{R})$ , and let  $I \subset \subset \mathbb{R}$  be a bounded open interval which contains the support of  $\Phi$ . By (3.4) and (3.3) we have that

$$\begin{aligned} \int_{\mathbb{R}} u(x) \nabla \Phi(x) dx &= \int_I u(x) \nabla \Phi(x) dx = \lim_{n \rightarrow \infty} \int_I u_{\varepsilon_n}(x) D^{-\varepsilon_n} \Phi(x) dx \\ &= - \lim_{n \rightarrow \infty} \int_I D^{\varepsilon_n} u_{\varepsilon_n}(x) \Phi(x) dx = - \int_I \Phi(x) d\mu \\ &= - \int_{\mathbb{R}} \Phi(x) d\mu. \end{aligned}$$

Since  $\Phi$  is arbitrary, this proves that  $\mu = Du$ . By a standard argument, it follows that also the whole family  $\{D^\varepsilon u_\varepsilon\}$  weakly \* converges to  $Du$ .  $\square$

The following theorem improves both Lemma 3.11, and the “liminf part” of Theorem 3.19, showing that assumption (3.26) forces the separate convergence of  $\{D^{\varepsilon, -} u_\varepsilon\}$  and  $\{D^{\varepsilon, +} u_\varepsilon\}$ , respectively, to  $D^a u$  and  $D^j u$ .

THEOREM 3.12. Let  $\{u_\varepsilon\} \subseteq L^\infty(\mathbb{R})$ , and let  $u \in L^\infty(\mathbb{R})$ . Let us assume that

$$u_\varepsilon \longrightarrow u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}),$$

$$\sup_{\varepsilon > 0} \left\{ \hat{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty \right\} < +\infty.$$

Then  $u \in SBV_{\text{loc}}(\mathbb{R})$  and

$$(3.28) \quad D^{\varepsilon, -} u_\varepsilon \rightharpoonup \nabla u \quad \text{weakly in } L^2(\mathbb{R});$$

$$(3.29) \quad D^{\varepsilon, +} u_\varepsilon(x) dx \xrightarrow{*} D^j u \quad \text{weakly } * \text{ as Radon measures.}$$

Moreover for every  $y \in S_u$ , and every  $\delta > 0$  we have that

$$(3.30) \quad \liminf_{\varepsilon \rightarrow 0^+} \varepsilon \max_{x \in [y-\delta, y+\delta]} |D^\varepsilon u_\varepsilon(x)| > 0.$$

Finally

$$(3.21) \quad \liminf_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \varphi_\varepsilon(D^{\varepsilon, -} u_\varepsilon(x)) dx \geq \int_{\mathbb{R}} |\nabla u(x)|^2 dx;$$

$$(3.32) \quad \liminf_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \varphi_\varepsilon(D^{\varepsilon, +} u_\varepsilon(x)) dx \geq \frac{\pi}{2} \mathcal{H}^0(S_u).$$

PROOF.

STEP 1. Let us prove (3.28). By (3.10) we have that

$$\sup_{\varepsilon > 0} \int_{\mathbb{R}} (D^{\varepsilon, -} u_\varepsilon(x))^2 dx \leq \frac{6}{\pi} \sup_{\varepsilon > 0} \hat{F}_\varepsilon(u_\varepsilon) < +\infty.$$

Therefore there exist a sequence  $\{\varepsilon_n\} \rightarrow 0^+$ , and a function  $g \in L^2(\mathbb{R})$  such that

$$(3.33) \quad D^{\varepsilon_n, -} u_{\varepsilon_n} \rightharpoonup g \quad \text{weakly in } L^2(\mathbb{R}).$$

If we prove that  $g = \nabla u$ , (3.28) will follow by a standard argument.

To this end, let us observe that by (3.9) we have that

$$\sup_{\varepsilon > 0} |I_\varepsilon^+(u_\varepsilon)| \leq \frac{6}{\pi} \varepsilon \sup_{\varepsilon > 0} \hat{F}_\varepsilon(u_\varepsilon) =: M\varepsilon;$$

hence, for each  $n \in \mathbb{N}$ , the set  $I_{\varepsilon_n}^+(u_{\varepsilon_n})$  is the union of at most  $M$  intervals of length  $\varepsilon_n$ . Therefore, up to subsequences, there exists a finite set

$$F = \{x_1, \dots, x_k\} \subseteq \mathbb{R}$$

such that for all  $\delta > 0$

$$(3.34) \quad I_{\varepsilon_n}^+(u_{\varepsilon_n}) \subseteq F_\delta := \{x \in \mathbb{R} : |x - y| < \delta \text{ for some } y \in F \text{ or } |x| > \delta^{-1}\}$$

for  $n$  large enough. Without loss of generality, we can assume that  $F \supseteq S_u$ . By (3.34) for  $n$  large enough we have that

$$(3.35) \quad D^{\varepsilon_n} u_{\varepsilon_n}|_{\mathbb{R} \setminus F_\delta} = D^{\varepsilon_n, -} u_{\varepsilon_n}|_{\mathbb{R} \setminus F_\delta}.$$

Thanks to (3.33) the right hand side of (3.35) weakly converges to  $g|_{\mathbb{R} \setminus F_\delta}$ , while by Lemma 3.11 the left hand side of (3.35) weakly  $*$  converges to  $Du|_{\mathbb{R} \setminus F_\delta}$ , i.e. to  $\nabla u(x) dx$ . This proves that  $g$  coincides with  $\nabla u$  for a.e.  $x \in \mathbb{R} \setminus F_\delta$ . Since  $\delta$  is arbitrary and  $F$  is finite, it follows that  $g = \nabla u$  for a.e.  $x \in \mathbb{R}$ , and therefore (3.28) is proved.

STEP 2. Since

$$\varphi_\varepsilon(D^\varepsilon u_\varepsilon) = \varphi_\varepsilon(D^{\varepsilon, -} u_\varepsilon) + \varphi_\varepsilon(D^{\varepsilon, +} u_\varepsilon),$$

(3.29) follows from (3.28) and Lemma 3.11

STEP 3. In order to prove (3.30) let us set

$$M_1 := \sup_{\varepsilon > 0} \hat{F}_\varepsilon(u_\varepsilon), \quad M_2 := \min_{y \in S_u} |D^j u(\{y\})| > 0.$$

By (3.29) and (3.9) for all  $\delta > 0$  we have that

$$\begin{aligned} M_2 &\leq |D^j u|([y - \delta, y + \delta]) \\ &\leq \liminf_{\varepsilon \rightarrow 0^+} \int_{-\delta}^{\delta} |D^{\varepsilon, +} u_\varepsilon(x)| dx \\ &\leq \liminf_{\varepsilon \rightarrow 0^+} |I_\varepsilon^+(u_\varepsilon)| \max_{x \in [y - \delta, y + \delta]} |D^\varepsilon u_\varepsilon(x)| \\ &\leq \frac{6M_1}{\pi} \liminf_{\varepsilon \rightarrow 0^+} \varepsilon \max_{x \in [y - \delta, y + \delta]} |D^\varepsilon u_\varepsilon(x)|. \end{aligned}$$

This proves (3.30).

STEP 4. By (3.28) we can apply Lemma 3.7 with  $g_\varepsilon = D^{\varepsilon, -} u_\varepsilon$ , and  $g = \nabla u$ . This proves (3.31).

STEP 5. Let us prove (3.32). To this end, let  $\delta > 0$  be such that

$$(S_{u_0})_\delta := \{x \in \mathbb{R} : |x - y| < \delta \text{ for some } y \in S_{u_0}\}$$

has exactly  $\mathcal{H}^0(S_{u_0})$  connected components (this is true for every  $\delta$  small enough), and let  $y \in S_u$ . By (3.30) there exists  $\alpha_y > 0$  such that for  $\varepsilon$  small enough

$$\max_{x \in [y - \frac{\delta}{2}, y + \frac{\delta}{2}]} |D^\varepsilon u_\varepsilon(x)| \geq \alpha_y \varepsilon^{-1} > 3^{-1/4} \varepsilon^{-1/2}.$$

Therefore for  $\varepsilon$  small enough,  $I_\varepsilon^+(u_\varepsilon) \cap ]y - \delta, y + \delta[$  contains at least one interval  $J_{y,\varepsilon}$  of length  $\varepsilon$ .

Therefore

$$\begin{aligned} \int_{\mathbb{R}} \varphi_\varepsilon(D^{\varepsilon,+}u_\varepsilon(x)) dx &\geq \sum_{y \in S_u} \int_{J_{y,\varepsilon}} \varphi_\varepsilon(D^{\varepsilon,+}u_\varepsilon(x)) dx \\ &\geq \sum_{y \in S_{u_0}} \arctan\left(\frac{\alpha_y^2}{\varepsilon}\right). \end{aligned}$$

Taking the limit as  $\varepsilon \rightarrow 0^+$ , inequality (3.32) is proved.  $\square$

The following result will be crucial in Subsection 3.5 and Subsection 4.4.

**THEOREM 3.13.** *Let  $\{u_\varepsilon\} \subseteq L^\infty(\mathbb{R})$ , and let  $u \in L^\infty(\mathbb{R})$ . Let us assume that*

$$\begin{aligned} u_\varepsilon &\longrightarrow u \quad \text{in } L_{\text{loc}}^2(\mathbb{R}), \\ \sup_{\varepsilon > 0} \left\{ \hat{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty \right\} &< +\infty. \end{aligned}$$

Then  $u \in SBV_{\text{loc}}(\mathbb{R})$  and

$$(3.36) \quad \varphi'_\varepsilon(D^\varepsilon u_\varepsilon) \rightharpoonup 2 \nabla u \quad \text{weakly in } L^2(\mathbb{R}).$$

**PROOF.** Since  $\|\varphi'_\varepsilon\|_\infty \leq 2\varepsilon^{-1/2}$ , by (3.9) we have that

$$\begin{aligned} \left| \int_{\mathbb{R}} \varphi'_\varepsilon(D^{\varepsilon,+}u_\varepsilon(x)) \Phi(x) dx \right| &\leq \frac{2}{\sqrt{\varepsilon}} \|\Phi\|_\infty |I_\varepsilon^+(u_\varepsilon)| \\ &\leq \frac{12}{\pi} \|\Phi\|_\infty \sqrt{\varepsilon} \sup_{\varepsilon > 0} \hat{F}_\varepsilon(u_\varepsilon), \end{aligned}$$

hence

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \varphi'_\varepsilon(D^{\varepsilon,+}u_\varepsilon(x)) \Phi(x) dx = 0$$

for every  $\Phi \in C_0^0(\mathbb{R})$ . By a density argument this proves that

$$(3.37) \quad \varphi'_\varepsilon(D^{\varepsilon,+}u_\varepsilon) \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{R}).$$

Moreover, applying Lemma 3.8 with  $g_\varepsilon = D^{\varepsilon,-}u_\varepsilon$ , we have that

$$(3.38) \quad \varphi'_\varepsilon(D^{\varepsilon,-}u_\varepsilon) \rightharpoonup 2 \nabla u \quad \text{weakly in } L^2(\mathbb{R}).$$

Since

$$\varphi'_\varepsilon(D^\varepsilon u_\varepsilon) = \varphi'_\varepsilon(D^{\varepsilon,-}u_\varepsilon) + \varphi'_\varepsilon(D^{\varepsilon,+}u_\varepsilon),$$

thesis follows from (3.37) and (3.38).  $\square$

### 3.5. – Convergence of slopes

We now study the convergence of the family  $\{\|\nabla \hat{F}_\varepsilon\|_{L^2(\mathbb{R})}\}$ .

DEFINITION 3.14. For all  $u \in L^2_{\text{loc}}(\mathbb{R})$  we set

$$|\nabla MS_{1, \frac{\pi}{2}}(u)| := \begin{cases} 2 \|\Delta u\|_{L^2(\mathbb{R})} & \text{if } MS_{1, \frac{\pi}{2}}(u) < +\infty, \nabla u \in H^1(\mathbb{R}), \\ & \text{and } \nabla u(y) = 0 \text{ for every } y \in S_u, \\ +\infty & \text{otherwise,} \end{cases}$$

where  $\Delta u$  denotes the weak derivative of the approximate gradient  $\nabla u$ . We recall that if  $\nabla u \in H^1(\mathbb{R})$ , then  $\nabla u(x)$  is well defined for all  $x \in \mathbb{R}$ .

THEOREM 3.15. Let  $\{u_\varepsilon\} \subseteq L^\infty(\mathbb{R})$ , and let  $u \in L^\infty(\mathbb{R})$ . Let us assume that

$$\begin{aligned} u_\varepsilon &\longrightarrow u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}), \\ \sup_{\varepsilon > 0} \left\{ \hat{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty \right\} &< +\infty. \end{aligned}$$

Then

$$(3.39) \quad \liminf_{\varepsilon \rightarrow 0^+} \|\nabla \hat{F}_\varepsilon(u_\varepsilon)\|_{L^2(\mathbb{R})} \geq |\nabla MS_{1, \frac{\pi}{2}}(u)|.$$

PROOF. If

$$\liminf_{\varepsilon \rightarrow 0^+} \|\nabla \hat{F}_\varepsilon(u_\varepsilon)\|_{L^2(\mathbb{R})} = +\infty,$$

then thesis is trivial. We can therefore assume that there exists a sequence  $\{\varepsilon_n\} \rightarrow 0^+$  such that

$$(3.40) \quad \lim_{n \rightarrow \infty} \|\nabla \hat{F}_{\varepsilon_n}(u_{\varepsilon_n})\|_{L^2(\mathbb{R})} = \liminf_{\varepsilon \rightarrow 0^+} \|\nabla \hat{F}_\varepsilon(u_\varepsilon)\|_{L^2(\mathbb{R})} < +\infty,$$

Now let us set

$$g_n := \varphi'_{\varepsilon_n}(D^{\varepsilon_n} u_{\varepsilon_n}).$$

Since  $D^{-\varepsilon_n} g_n = -\nabla \hat{F}_{\varepsilon_n}(u_{\varepsilon_n})$ , by (3.40) and Theorem 3.13 we have that  $g_n$  satisfies the assumptions of Lemma 3.6 with  $g = 2 \nabla u$ . It follows that  $\nabla u \in H^1(\mathbb{R})$ , and

$$(3.41) \quad \varphi'_{\varepsilon_n}(D^{\varepsilon_n} u_{\varepsilon_n}) \longrightarrow 2 \nabla u \quad \text{uniformly on compact sets;}$$

$$(3.42) \quad \nabla \hat{F}_{\varepsilon_n}(u_{\varepsilon_n}) \rightharpoonup \nabla(2 \nabla u) = 2 \Delta u \quad \text{weakly in } L^2(\mathbb{R}).$$

By (3.40) and (3.42), it follows that

$$\liminf_{\varepsilon \rightarrow 0^+} \|\nabla \hat{F}_\varepsilon(u_\varepsilon)\|_{L^2(\mathbb{R})} \geq 2 \|\Delta u\|_{L^2(\mathbb{R})}.$$

It remains to show that  $\nabla u(y) = 0$  for all  $y \in S_u$ . To this end, let us fix  $\delta > 0$ . By (3.30) there exists a constant  $\alpha_y > 0$  such that

$$\max_{x \in [y-\delta, y+\delta]} |D^{\varepsilon_n} u_{\varepsilon_n}| \geq \frac{\alpha_y}{\varepsilon_n},$$

for  $n$  large enough, hence

$$\min_{x \in [y-\delta, y+\delta]} |\varphi'_{\varepsilon_n}(D^{\varepsilon_n} u_{\varepsilon_n}(x))| \leq \varphi'_{\varepsilon_n}\left(\frac{\alpha_y}{\varepsilon_n}\right) \leq \frac{2\varepsilon_n}{\alpha_y^3}.$$

Passing to the limit as  $n \rightarrow \infty$ , by (3.41) we have that

$$\min_{x \in [y-\delta, y+\delta]} |\nabla u(x)| = 0.$$

Since  $\delta$  is arbitrary, and  $\nabla u$  is continuous, it follows that necessarily  $\nabla u(y) = 0$ .  $\square$

**REMARK 3.16.** For the reader familiar with the notion of descending slope of a function defined in a metric space (cf. [16]), we observe that if  $u \in L^2(\mathbb{R})$ , then  $\|\nabla \hat{F}_\varepsilon(u)\|_{L^2(\mathbb{R})}$  and  $|\nabla MS_{1, \frac{\pi}{2}}(u)|$  are the descending slopes in  $u$ , respectively, of the restrictions to  $L^2(\mathbb{R})$  of  $\hat{F}_\varepsilon$  and  $MS_{1, \frac{\pi}{2}}$ . It is also possible to prove that

$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} \left\{ \hat{F}_\varepsilon(u) + \|\nabla \hat{F}_\varepsilon(u)\|_{L^2(\mathbb{R})} \right\} = MS_{1, \frac{\pi}{2}}(u) + |\nabla MS_{1, \frac{\pi}{2}}(u)|.$$

Indeed, the “liminf part” is a straightforward consequence of Theorem 3.10 and Theorem 3.15, while the “limsup part” may be proved refining the construction of Lemma 3.9.

#### 4. – Convergence of approximate gradient flows

In this section we study the convergence of the gradient flows relative to the functionals  $\{\hat{F}_\varepsilon\}$  introduced in Section 3.

In order to simplify the notations, we set

$$\mathbb{X} := \{u \in L^\infty(\mathbb{R}) \cap SBV_{\text{loc}}(\mathbb{R}) : MS_{1, \frac{\pi}{2}}(u) < +\infty\},$$

and we define on  $\mathbb{X}$  the metric

$$d_{\mathbb{X}}(u, v) := d_{L^2_{\text{loc}}(\mathbb{R})}(u, v) + \|\|u\|_\infty - \|v\|_\infty\| + |MS_{1, \frac{\pi}{2}}(u) - MS_{1, \frac{\pi}{2}}(v)|,$$

where  $d_{L^2_{\text{loc}}(\mathbb{R})}$  is any metric in  $L^2_{\text{loc}}(\mathbb{R})$ .

#### 4.1. – The approximate problem

In the sequel we assume that  $u_0 \in \mathbb{X}$  is a given function. Our aim is to define the gradient flow relative to the Mumford-Shah functional, with  $u_0$  as initial datum.

To this end, we consider a family  $\{u_{0\varepsilon}\} \subseteq L^\infty(\mathbb{R})$  satisfying the following conditions:

- (P1)  $\{u_{0\varepsilon}\} \longrightarrow u_0$  in  $L^2_{\text{loc}}(\mathbb{R})$ ;
- (P2)  $\lim_{\varepsilon \rightarrow 0^+} \|u_{0\varepsilon}\|_\infty = \|u_0\|_\infty$ ;
- (P3)  $u_{0\varepsilon} \in \text{PC}^2_\varepsilon$  for all  $\varepsilon > 0$ ;
- (P4)  $\lim_{\varepsilon \rightarrow 0^+} \hat{F}_\varepsilon(u_{0\varepsilon}) = MS_{1, \frac{\pi}{2}}(u_0)$ .

Existence of families with the above properties follows from Lemma 3.9 and Theorem 3.10. Moreover, since we are always interested to limits as  $\varepsilon \rightarrow 0^+$ , we may assume, without loss of generality, that

$$(4.1) \quad \sup_{\varepsilon > 0} \left\{ \hat{F}_\varepsilon(u_{0\varepsilon}) + \|u_{0\varepsilon}\|_\infty \right\} < +\infty.$$

Then we consider the evolution problems

$$(4.2) \quad \begin{cases} u'_\varepsilon(t) = -[\nabla \hat{F}_\varepsilon](u_\varepsilon(t)), & t \geq 0, \\ u_\varepsilon(0) = u_{0\varepsilon}. \end{cases}$$

Since  $\nabla \hat{F}_\varepsilon$  is a Lipschitz continuous operator (cf. statement (iv) of Proposition 3.3), the following result is a straightforward consequence of the standard Cauchy-Lipschitz-Picard theorem for ODEs (cf. [9, Theorem VII.3]).

**THEOREM 4.1.** *For every  $\varepsilon > 0$ , problem (4.2) has a unique solution  $u_\varepsilon \in C^1([0, +\infty[; \text{PC}^2_\varepsilon)$ , which depends continuously on the initial datum.  $\square$*

In the sequel we sometimes denote by  $GF(\hat{F}_\varepsilon, u_{0\varepsilon})$  the unique solution of (4.2).

#### 4.2. – Basic estimates

**PROPOSITION 4.2.** *For every  $\varepsilon > 0$ , let  $u_\varepsilon$  be the solution of (4.2). Then the following hold true.*

- (i)  $\hat{F}_\varepsilon$ -Energy equality. *For all  $0 \leq t_1 \leq t_2$  we have that*

$$(4.3) \quad \begin{aligned} \hat{F}_\varepsilon(u_\varepsilon(t_1)) - \hat{F}_\varepsilon(u_\varepsilon(t_2)) &= \int_{t_1}^{t_2} \|\nabla \hat{F}_\varepsilon(u_\varepsilon(\tau))\|_{L^2(\mathbb{R})}^2 d\tau \\ &= \int_{t_1}^{t_2} \|u'_\varepsilon(\tau)\|_{L^2(\mathbb{R})}^2 d\tau. \end{aligned}$$

*In particular, the function  $t \rightarrow \hat{F}_\varepsilon(u_\varepsilon(t))$  is non-increasing for  $t \geq 0$ .*



(ii)  $\hat{F}_\varepsilon$ -Energy estimate. For all  $t \geq 0$  we have that

$$(4.4) \quad \hat{F}_\varepsilon(u_\varepsilon(t)) \leq \hat{F}_\varepsilon(u_{0\varepsilon}).$$

(iii) Maximum principle. For all  $t \geq 0$  we have that  $u_\varepsilon(t) \in L^\infty(\mathbb{R})$ , and

$$(4.5) \quad \|u_\varepsilon(t)\|_\infty \leq \|u_{0\varepsilon}\|_\infty.$$

(iv) Hölder estimate. For all  $0 \leq t_1 \leq t_2$  we have that

$$(4.6) \quad \|u_\varepsilon(t_1) - u_\varepsilon(t_2)\|_{L^2(\mathbb{R})} \leq (\hat{F}_\varepsilon(u_{0\varepsilon}))^{1/2} |t_1 - t_2|^{1/2}.$$

PROOF. (i)  $\hat{F}_\varepsilon$ -Energy equality. It is enough to integrate in  $[t_1, t_2]$  the equality

$$\begin{aligned} \frac{d}{dt}(\hat{F}_\varepsilon \circ u_\varepsilon)(t) &= \langle \nabla \hat{F}_\varepsilon(u_\varepsilon(t)), u'_\varepsilon(t) \rangle_{L^2(\mathbb{R})} \\ &= -\|u'_\varepsilon(t)\|_{L^2(\mathbb{R})}^2 = -\|\nabla \hat{F}_\varepsilon(u_\varepsilon(t))\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

(ii)  $\hat{F}_\varepsilon$ -Energy estimate. Trivial consequence of (4.3).

(iii) Maximum principle. Let  $G : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that

(G1)  $G(r) = 0$  for  $r \leq 0$ , and  $G(r) > 0$  for  $r > 0$ ;

(G2)  $G$  is convex;

(G3)  $G$  is of class  $C^2$ , and  $G''$  is bounded.

Let  $K = \|u_{0\varepsilon}\|_\infty$ , and let us consider the function  $\psi_\varepsilon : [0, +\infty[ \rightarrow \mathbb{R}$  defined by

$$\psi_\varepsilon(t) := \int_{\mathbb{R}} G(u_\varepsilon(t) - K) dx.$$

By (G1) and (G3), it follows that

$$G(r - K) \leq \|G''\|_\infty r^2,$$

and therefore  $\psi_\varepsilon(t)$  is well defined. Moreover, by (3.3) and (3.2) we have that

$$\begin{aligned} \psi'_\varepsilon(t) &= \int_{\mathbb{R}} G'(u_\varepsilon(t) - K) u'_\varepsilon(t) dx \\ &= \int_{\mathbb{R}} G'(u_\varepsilon(t) - K) D^{-\varepsilon} \varphi'_\varepsilon(D^\varepsilon u'_\varepsilon(t)) dx \\ &= - \int_{\mathbb{R}} D^\varepsilon G'(u_\varepsilon(t) - K) \varphi'_\varepsilon(D^\varepsilon u'_\varepsilon(t)) dx \\ &= - \int_{\mathbb{R}} D^{u_\varepsilon(x+\varepsilon)-u_\varepsilon(x)} G'(u_\varepsilon(t) - K) D^\varepsilon u'_\varepsilon(t) \varphi'_\varepsilon(D^\varepsilon u'_\varepsilon(t)) dx \leq 0, \end{aligned}$$

where in the last inequality we exploited that  $D^\rho G'(r) \geq 0$  for every  $\rho$  and every  $r$  (by (G2)), and that the function  $r \rightarrow r \varphi'_\varepsilon(r)$  is non-negative.

Since by (G1) we have that  $\psi_\varepsilon(0) = 0$ , and  $\psi_\varepsilon(t) \geq 0$  for  $t \geq 0$ , then necessarily  $\psi_\varepsilon(t) = 0$  for every  $t \geq 0$ , hence  $u_\varepsilon(t) \leq K$ .

In an analogous way it is possible to prove that  $u_\varepsilon(t) \geq -K$  for every  $t \geq 0$ . This completes the proof of the maximum principle.

(iv) *Hölder estimate.* By Hölder’s inequality and (4.3) we have that

$$\begin{aligned} \|u_\varepsilon(t_1) - u_\varepsilon(t_2)\|_{L^2(\mathbb{R})} &\leq \int_{t_1}^{t_2} \|u'_\varepsilon(\tau)\|_{L^2(\mathbb{R})} d\tau \\ &\leq \left\{ \int_{t_1}^{t_2} \|u'_\varepsilon(\tau)\|_{L^2(\mathbb{R})}^2 d\tau \right\}^{1/2} |t_1 - t_2|^{1/2} \\ &\leq (\hat{F}_\varepsilon(u_{0\varepsilon}))^{1/2} |t_1 - t_2|^{1/2}. \end{aligned} \quad \square$$

### 4.3. – Passing to the limit

From now on we consider the space  $C^0([0, +\infty[; L^2_{loc}(\mathbb{R}))$  endowed with the topology of uniform convergence on compact sets. For the convenience of the reader, we recall the following compactness result, which is a particular case of the standard Ascoli theorem (proof is omitted).

LEMMA 4.3. *Let  $\{u_\varepsilon\} \subseteq C^0([0, +\infty[; L^2_{loc}(\mathbb{R}))$ . Let us assume that*

- (i) *for every  $t \geq 0$ , the family  $\{u_\varepsilon(t)\}$  is relatively compact in  $L^2_{loc}(\mathbb{R})$ ;*
- (ii) *there exists a constant  $M \in \mathbb{R}$  such that*

$$\|u_\varepsilon(t_1) - u_\varepsilon(t_2)\|_{L^2(\mathbb{R})} \leq M|t_1 - t_2|^{1/2}$$

*for every  $\varepsilon > 0$ , and every  $0 \leq t_1 \leq t_2$ .*

*Then  $\{u_\varepsilon\}$  is relatively compact in  $C^0([0, +\infty[; L^2_{loc}(\mathbb{R}))$ .* □

We are now ready to prove our main compactness result.

THEOREM 4.4. *For every  $\varepsilon > 0$ , let  $u_\varepsilon$  be the solution of (4.2).*

*Then there exist a sequence  $\{\varepsilon_n\} \rightarrow 0^+$ , and a continuous function  $u : [0, +\infty[ \rightarrow L^2_{loc}(\mathbb{R})$  such that*

$$\{u_{\varepsilon_n}\} \longrightarrow u \quad \text{in } C^0([0, +\infty[; L^2_{loc}(\mathbb{R})).$$

PROOF. It is enough to show that  $\{u_\varepsilon\}$  satisfies the assumptions of Lemma 4.3.

By the energy estimate (4.4), the maximum principle (4.5), and (4.1), it follows that

$$\sup_{\varepsilon > 0} \{ \hat{F}_\varepsilon(u_\varepsilon(t)) + \|u_\varepsilon(t)\|_\infty \} \leq \sup_{\varepsilon > 0} \{ \hat{F}_\varepsilon(u_{0\varepsilon}) + \|u_{0\varepsilon}\|_\infty \} < +\infty$$

for all  $t \geq 0$ . Therefore, by Theorem 3.10, assumption (i) of Lemma 4.3 is satisfied.

On the other hand, assumption (ii) of Lemma 4.3, with

$$M = \sup_{\varepsilon > 0} (\hat{F}_\varepsilon(u_{0\varepsilon}))^{1/2}$$

follows from the Hölder estimate (4.6). □

The above theorem provides the candidates to be the gradient flow relative to the Mumford-Shah functional with  $u_0$  as initial datum, and justifies the following notation.

DEFINITION 4.5. For every  $u_0 \in \mathbb{X}$  we denote by  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  the set of all possible limits in Theorem 4.4.

To be more precise, we say that a function  $u \in C^0([0, +\infty[; L^2_{\text{loc}}(\mathbb{R}))$  belongs to  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  if there exist a sequence  $\{\varepsilon_n\} \rightarrow 0^+$ , and a sequence  $\{u_{0\varepsilon_n}\} \subseteq L^\infty(\mathbb{R})$  such that

$$\left\{ d_{L^2_{\text{loc}}(\mathbb{R})}(u_{0\varepsilon_n}, u_0) + \left| \|u_{0\varepsilon_n}\|_\infty - \|u_0\|_\infty \right| + |\hat{F}_{\varepsilon_n}(u_{0\varepsilon_n}) - MS_{1, \frac{\pi}{2}}(u_0)| \right\} \longrightarrow 0, \\ \{GF(\hat{F}_{\varepsilon_n}, u_{0\varepsilon_n})\} \longrightarrow u \quad \text{in } C^0([0, +\infty[; L^2_{\text{loc}}(\mathbb{R})).$$

#### 4.4. – First properties of the limit

PROPOSITION 4.6. *Let  $u_0 \in \mathbb{X}$ . Then every  $u \in GF(MS_{1, \frac{\pi}{2}}, u_0)$  has the following properties.*

- (i)  $u(0) = u_0$ .
- (ii) For all  $0 \leq t_1 \leq t_2$  we have that

$$\|u(t_1) - u(t_2)\|_{L^2(\mathbb{R})} \leq (MS_{1, \frac{\pi}{2}}(u_0))^{1/2} |t_1 - t_2|^{1/2}.$$

- (iii) For all  $t \geq 0$  we have that  $u(t) \in \mathbb{X}$ , and

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \\ MS_{1, \frac{\pi}{2}}(u(t)) \leq MS_{1, \frac{\pi}{2}}(u_0).$$

- (iv) The function  $u$  (as a function of  $(t, x)$ ) is a distributional solution in  $]0, +\infty[ \times \mathbb{R}$  of the equation

$$\frac{\partial u}{\partial t} = 2D(\nabla u),$$

where  $D(\nabla u)$  is the distributional derivative (with respect to the  $x$ -variable) of the approximate gradient  $\nabla u$ .

- (v) For a.e.  $t \geq 0$  we have that  $\nabla(u(t)) \in H^1(\mathbb{R})$  (as a function of the  $x$ -variable), and

$$(\nabla u(t))(x) = 0, \quad \forall x \in S_{u(t)}.$$

PROOF. Let  $\{\varepsilon_n\}$  and  $\{u_{0\varepsilon_n}\}$  be as in Definition 4.5, and let us set for simplicity  $u_{\varepsilon_n} := GF(\hat{F}_{\varepsilon_n}, u_{0\varepsilon_n})$ .

- (i) Trivial.

(ii) Since the  $L^2$ -norm is lower semicontinuous with respect to  $L^2_{\text{loc}}$  convergence, by (4.6) and (P4) we have that

$$\begin{aligned} \|u(t_1) - u(t_2)\|_{L^2(\mathbb{R})} &\leq \liminf_{n \rightarrow \infty} \|u_{\varepsilon_n}(t_1) - u_{\varepsilon_n}(t_2)\|_{L^2(\mathbb{R})} \\ &\leq |t_1 - t_2|^{1/2} \liminf_{n \rightarrow \infty} \left( \hat{F}_{\varepsilon_n}(u_{0\varepsilon_n}) \right)^{1/2} \\ &= |t_1 - t_2|^{1/2} \left( MS_{1, \frac{\pi}{2}}(u_0) \right)^{1/2}. \end{aligned}$$

(iii) By (4.4), (P4), and the  $\Gamma$ -convergence of  $\{\hat{F}_\varepsilon\}$ , we have that

$$\begin{aligned} MS_{1, \frac{\pi}{2}}(u(t)) &\leq \liminf_{n \rightarrow \infty} \hat{F}_{\varepsilon_n}(u_{\varepsilon_n}(t)) \\ &\leq \liminf_{n \rightarrow \infty} \hat{F}_{\varepsilon_n}(u_{0\varepsilon_n}) = MS_{1, \frac{\pi}{2}}(u_0). \end{aligned}$$

In an analogous way, since the  $L^\infty$ -norm is lower semicontinuous with respect to  $L^2_{\text{loc}}$  convergence, by (4.5) and (P2) it follows that

$$\begin{aligned} \|u(t)\|_\infty &\leq \liminf_{n \rightarrow \infty} \|u_{\varepsilon_n}(t)\|_\infty \\ &\leq \liminf_{n \rightarrow \infty} \|u_{0\varepsilon_n}\|_\infty = \|u_0\|_\infty. \end{aligned}$$

(iv) Let  $\Phi \in C_0^\infty(]0, +\infty[ \times \mathbb{R})$ . By (3.8) and (3.3) we have that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} u_{\varepsilon_n} \frac{\partial \Phi}{\partial t} dx dt &= - \int_0^\infty \int_{\mathbb{R}} u'_{\varepsilon_n} \Phi dx dt \\ &= - \int_0^\infty \int_{\mathbb{R}} D^{-\varepsilon_n} \varphi'_{\varepsilon_n}(D^{\varepsilon_n} u_{\varepsilon_n}) \Phi dx dt \\ &= + \int_0^\infty \int_{\mathbb{R}} \varphi'_{\varepsilon_n}(D^{\varepsilon_n} u_{\varepsilon_n}) D^{\varepsilon_n} \Phi dx dt. \end{aligned}$$

Passing to the limit, by (3.4) and (3.36) we obtain

$$\int_0^\infty \int_{\mathbb{R}} u \frac{\partial \Phi}{\partial t} dx dt = 2 \int_0^\infty \int_{\mathbb{R}} \nabla u \nabla \Phi dx dt, \quad \forall \Phi \in C_0^\infty(]0, +\infty[ \times \mathbb{R}).$$

This identity is equivalent to statement (iv).

(v) By Fatou's lemma and (4.3) it follows that

$$\begin{aligned} \int_0^\infty \liminf_{n \rightarrow \infty} \|\nabla \hat{F}_{\varepsilon_n}(u_{\varepsilon_n}(\tau))\|_{L^2(\mathbb{R})}^2 d\tau &\leq \\ &\leq \liminf_{n \rightarrow \infty} \int_0^\infty \|\nabla \hat{F}_{\varepsilon_n}(u_{\varepsilon_n}(\tau))\|_{L^2(\mathbb{R})}^2 d\tau \leq MS_{1, \frac{\pi}{2}}(u_0). \end{aligned}$$

Therefore

$$\liminf_{n \rightarrow \infty} \|\nabla \hat{F}_{\varepsilon_n}(u_{\varepsilon_n}(t))\|_{L^2(\mathbb{R})} < +\infty$$

for a.e.  $t \geq 0$ . By Theorem 3.15, statement (v) is proved.  $\square$

REMARK 4.7. In general there are infinitely many functions  $u(t)$  satisfying conditions (i)-(v) of Proposition 4.6 with a given  $u_0$ . The heuristic reason of this non-uniqueness is that no information on the evolving discontinuity set  $S_{u(t)}$  is provided by Proposition 4.6. The evolution of the discontinuity set will be analyzed in Subsection 4.5.

We now prove a semicontinuity property of the map

$$u_0 \longrightarrow GF(MS_{1, \frac{\pi}{2}}, u_0).$$

PROPOSITION 4.8. *Let  $\{u_{0n}\} \subseteq \mathbb{X}$ , and let  $u_0 \in \mathbb{X}$  be such that*

$$\lim_{n \rightarrow \infty} d_{\mathbb{X}}(u_{0n}, u_0) = 0.$$

*Moreover, let  $u_n \in GF(MS_{1, \frac{\pi}{2}}, u_{0n})$  for each  $n \in \mathbb{N}$ .*

*Then:*

- (i)  $\{u_n\}$  is relatively compact in  $C^0([0, +\infty[; L_{\text{loc}}^2(\mathbb{R}))$ ;
- (ii) every limit point of  $\{u_n\}$  belongs to  $GF(MS_{1, \frac{\pi}{2}}, u_0)$ .

PROOF. By statements (ii) and (iii) of Proposition 4.6, the sequence  $\{u_n\}$  satisfies the assumptions of Lemma 4.3. This proves (i).

In order to prove (ii) we can assume, up to subsequences, that

$$\{u_n\} \longrightarrow u \quad \text{in } C^0([0, +\infty[; L_{\text{loc}}^2(\mathbb{R})).$$

Let  $d_Y$  be any metric on  $C^0([0, +\infty[; L_{\text{loc}}^2(\mathbb{R}))$  which induces the topology of uniform convergence on compact sets. Then for each  $k \in \mathbb{N}$ , there exists  $n_k \in \mathbb{N}$  such that

$$(4.7) \quad d_{\mathbb{X}}(u_{0n_k}, u_0) \leq \frac{1}{2k}, \quad d_Y(u_{n_k}, u) \leq \frac{1}{2k}.$$

Moreover, since  $u_{n_k} \in GF(MS_{1, \frac{\pi}{2}}, u_{0n_k})$ , by Definition 4.5 there exists  $\varepsilon_k$  and  $v_{0\varepsilon_k} \in \text{PC}_{\varepsilon_k}^2$  such that

$$0 < \varepsilon_k \leq \frac{1}{k};$$

$$d_{L_{\text{loc}}^2(\mathbb{R})}(v_{0\varepsilon_k}, u_{0n_k}) + |||v_{0\varepsilon_k}|||_{\infty} - |||u_{0n_k}|||_{\infty} + |\hat{F}_{\varepsilon_k}(v_{0\varepsilon_k}) - MS_{1, \frac{\pi}{2}}(u_{0n_k})| \leq \frac{1}{2k};$$

$$d_Y(GF(\hat{F}_{\varepsilon_k}, v_{0\varepsilon_k}), u_{n_k}) \leq \frac{1}{2k}.$$

By these inequalities and (4.7) it follows that  $\{\varepsilon_k\} \rightarrow 0^+$ , and  $\{v_{0\varepsilon_k}\}$  satisfies

$$d_{L_{\text{loc}}^2(\mathbb{R})}(v_{0\varepsilon_k}, u_0) + |||v_{0\varepsilon_k}|||_{\infty} - |||u_0|||_{\infty} + |\hat{F}_{\varepsilon_k}(v_{0\varepsilon_k}) - MS_{1, \frac{\pi}{2}}(u_0)| \leq \frac{1}{k};$$

$$d_Y(GF(\hat{F}_{\varepsilon_k}, v_{0\varepsilon_k}), u) \leq \frac{1}{k}.$$

By Definition 4.5, this proves that  $u \in GF(MS_{1, \frac{\pi}{2}}, u_0)$ .  $\square$

**Caution!** The following very attractive properties are in general *false* for elements of  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  (cf. the discussion in Section 5):

- FALSE 1: the function  $t \rightarrow MS_{1, \frac{\pi}{2}}(u(t))$  is non-increasing;
- FALSE 2: for all  $t > 0$

$$\lim_{n \rightarrow \infty} \hat{F}_{\varepsilon_n}(u_{\varepsilon_n}(t)) = MS_{1, \frac{\pi}{2}}(u(t));$$

- FALSE 3: the function  $v(t) = u(T + t)$  belongs to  $GF(MS_{1, \frac{\pi}{2}}, u(T))$  for all  $T > 0$ .

**4.5. – Evolution of discontinuities**

In order to study the evolving discontinuity set  $S_{u(t)}$ , we now analyze the map  $t \rightarrow I_\varepsilon^+(u_\varepsilon(t))$ . First of all, we examine the case  $t = 0$ , showing that (P1)-(P4) force the set  $I_\varepsilon^+(u_{0\varepsilon})$  to approximate  $S_{u_0}$ .

PROPOSITION 4.9. *Let  $u_0 \in \mathbb{X}$ , and let  $\{u_{0\varepsilon}\}$  be any family satisfying properties (P1)-(P4) of Section 4.1. Let  $\delta > 0$  be such that*

$$(S_{u_0})_\delta := \{x \in \mathbb{R} : |x - y| < \delta \text{ for some } y \in S_{u_0}\}$$

*has exactly  $\mathcal{H}^0(S_{u_0})$  connected components (this is true for every  $\delta$  small enough). Then for every  $\varepsilon > 0$  small enough we have that:*

- (i)  $I_\varepsilon^+(u_{0\varepsilon})$  is the union of exactly  $\mathcal{H}^0(S_{u_0})$  intervals of length  $\varepsilon$ ;
- (ii) each connected component of  $(S_{u_0})_\delta$  contains exactly one of these intervals;
- (iii) for every  $y \in S_{u_0}$ , the sign of  $D^{\varepsilon,+}u_{0\varepsilon}$  in the component of  $I_\varepsilon^+(u_{0\varepsilon})$  contained in  $]y - \delta, y + \delta[$  coincides with the sign of  $D^j u_0(\{y\})$ .

PROOF.

STEP 1. Let  $\{\varepsilon_n\} \rightarrow 0^+$  be any sequence. By (3.31), (3.32), (P1), and (P4), we have that

$$\begin{aligned} MS_{1, \frac{\pi}{2}}(u_0) &= \int_{\mathbb{R}} |\nabla u_0(x)|^2 dx + \frac{\pi}{2} \mathcal{H}^0(S_{u_0}) \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_{\varepsilon_n}(D^{\varepsilon_n,-}u_{0\varepsilon_n}(x)) dx + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_{\varepsilon_n}(D^{\varepsilon_n,+}u_{0\varepsilon_n}(x)) dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_{\varepsilon_n}(D^{\varepsilon_n}u_{0\varepsilon_n}(x)) dx \\ &= MS_{1, \frac{\pi}{2}}(u_0), \end{aligned}$$

hence

$$(4.8) \quad \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_{\varepsilon_n}(D^{\varepsilon_n,+}u_{0\varepsilon_n}(x)) dx = \frac{\pi}{2} \mathcal{H}^0(S_{u_0}).$$

for every sequence  $\{\varepsilon_n\} \rightarrow 0^+$ .

STEP 2. Let  $y \in S_{u_0}$ . Arguing as in Step 5 of the proof of Theorem 3.12, we can prove that, for  $\varepsilon > 0$  small enough,  $I_\varepsilon^+(u_{0\varepsilon}) \cap ]y - \delta, y + \delta[$  contains at least one interval  $J_{y,\varepsilon}$  of length  $\varepsilon$ , and

$$(4.9) \quad \liminf_{\varepsilon \rightarrow 0^+} \sum_{y \in S_{u_0}} \int_{J_{y,\varepsilon}} \varphi_\varepsilon(D^{\varepsilon,+}u_{0\varepsilon}(x)) dx \geq \frac{\pi}{2} \mathcal{H}^0(S_{u_0}).$$

In order to prove (i) and (ii), it will be enough to show that

$$I_\varepsilon^+(u_{0\varepsilon}) = \bigcup_{y \in S_{u_0}} J_{y,\varepsilon}$$

for all  $\varepsilon$  small enough. To this end, let us assume by contradiction that there exists a sequence  $\{\varepsilon_n\} \rightarrow 0^+$ , and that for each  $n \in \mathbb{N}$  there exists an interval  $I_n$  of length  $\varepsilon_n$  contained in

$$I_{\varepsilon_n}^+(u_{0\varepsilon_n}) \setminus \bigcup_{y \in S_{u_0}} J_{y,\varepsilon_n}.$$

Since  $\arctan r \geq \pi/6$  for  $r \geq 3^{-1/2}$ , for all  $n$  we have that

$$\int_{I_n} \varphi_{\varepsilon_n}(D^{\varepsilon_n,+}u_{0\varepsilon_n}(x)) dx \geq \frac{\pi}{6},$$

hence by (4.9)

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_{\varepsilon_n}(D^{\varepsilon_n,+}u_{0\varepsilon_n}(x)) dx \geq \frac{\pi}{2} \mathcal{H}^0(S_{u_0}) + \frac{\pi}{6},$$

which contradicts (4.8).

STEP 3. By Theorem 3.12 the family of Radon measures  $\{D^{\varepsilon,+}u_{0\varepsilon}(x) dx\}$  weakly \* converges to  $D^j u_0$ . Therefore, since  $|D^j u_0|(\{y - \delta, y + \delta\}) = 0$ , it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{J_{y,\varepsilon}} \varphi_\varepsilon(D^{\varepsilon,+}u_{0\varepsilon}(x)) dx = D^j u_0([y - \delta, y + \delta]) = D^j u_0(\{y\}).$$

This proves (iii). □

We now need an abstract lemma about ODEs.

LEMMA 4.10. *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a Lipschitz continuous function, and let  $C, V$  be real numbers such that  $f(C) = -f(-C) = V$ . Let  $y \in C^1([0, +\infty[; \mathbb{R})$  be a function such that*

$$|y'(t) + f(y(t))| \leq V, \quad \forall t \geq 0.$$

*Then for all  $T \geq 0$  we have that*

$$|y(T)| < C \implies |y(t)| < C, \quad \forall t \geq T.$$

PROOF. Since  $z(t) \equiv C$  is a solution of the ODE

$$z' = V - f(z),$$

and  $y$  is a sub-solution of the same equation, by the standard comparison theorems for ODEs, it follows that

$$y(T) < C \implies y(t) < C, \quad \forall t \geq T.$$

Moreover,  $w(t) \equiv -C$  is a solution of the ODE

$$w' = -V - f(w),$$

and  $y$  is a super-solution of the same equation. Therefore

$$y(T) > -C \implies y(t) > -C, \quad \forall t \geq T.$$

This completes the proof.  $\square$

We can now show that  $t \rightarrow I_\varepsilon^+(u_\varepsilon(t))$  is a non-increasing map (with respect to set inclusion).

PROPOSITION 4.11. *Let  $\varepsilon > 0$ , and let  $u_\varepsilon$  be the solution of (4.2). Then*

$$I_\varepsilon^+(u_\varepsilon(t_2)) \subseteq I_\varepsilon^+(u_\varepsilon(t_1)),$$

*whenever  $0 \leq t_1 \leq t_2$ .*

PROOF. It is enough to show that for every  $T > 0$ , and every  $x \in \mathbb{R}$ , we have that

$$(4.10) \quad \left| (D^\varepsilon u_\varepsilon(T))(x) \right| < 3^{-1/4} \varepsilon^{-1/2} \implies \left| (D^\varepsilon u_\varepsilon(t))(x) \right| < 3^{-1/4} \varepsilon^{-1/2},$$

$$\forall t \geq T.$$

With an abuse of notation, let us set

$$u_\varepsilon(t, x) = (u_\varepsilon(t))(x).$$



Then we have that

$$\begin{aligned} (D^\varepsilon u_\varepsilon(t, x))' &= D^\varepsilon (u'_\varepsilon(t, x)) \\ &= D^\varepsilon D^{-\varepsilon} \varphi'_\varepsilon(D^\varepsilon u_\varepsilon(t, x)) \\ &= \frac{1}{\varepsilon^2} \{ \varphi'_\varepsilon(D^\varepsilon u_\varepsilon(t, x + \varepsilon)) + \varphi'_\varepsilon(D^\varepsilon u_\varepsilon(t, x - \varepsilon)) \\ &\quad - 2\varphi'_\varepsilon(D^\varepsilon u_\varepsilon(t, x)) \}, \end{aligned}$$

hence

$$\left| (D^\varepsilon u_\varepsilon(t, x))' + \frac{2}{\varepsilon^2} \varphi'_\varepsilon(D^\varepsilon u_\varepsilon(t, x)) \right| \leq \frac{2}{\varepsilon^2} \|\varphi'_\varepsilon\|_\infty.$$

Therefore (4.10) follows applying Lemma 4.10 with

$$y(t) = D^\varepsilon u_\varepsilon(t, x), \quad C = 3^{-1/4} \varepsilon^{-1/2},$$

$$f(r) = \frac{2}{\varepsilon^2} \varphi'_\varepsilon(r), \quad V = f(C) = -f(-C) = \frac{2}{\varepsilon^2} \|\varphi'_\varepsilon\|_\infty. \quad \square$$

We can now prove some properties of the map  $t \rightarrow S_{u(t)}$ .

**PROPOSITION 4.12.** *Let  $u \in GF(MS_{1, \frac{\pi}{2}}, u_0)$  for some  $u_0 \in \mathbb{X}$ .*

*Then we have that*

- (i)  $S_{u(t)} \subseteq S_{u_0}$  for all  $t \geq 0$ ;
- (ii) the function  $t \rightarrow \mathcal{H}^0(S_{u(t)})$  is lower semicontinuous;
- (iii) the set  $\{t \geq 0 : y \in S_{u(t)}\}$  is an open subset of  $[0, +\infty[$  for every  $y \in S_{u_0}$ .

**PROOF.** Let  $\{\varepsilon_n\}$  and  $\{u_{0\varepsilon_n}\}$  be as in Definition 4.5, and let us set for simplicity  $u_{\varepsilon_n} := GF(\hat{F}_{\varepsilon_n}, u_{0\varepsilon_n})$ . Let  $t \geq 0$ , and let  $\Omega \subset\subset \mathbb{R} \setminus S_{u_0}$  be any open set. By Proposition 4.11 and Proposition 4.9 we have that, for all  $n$  large enough,

$$I_{\varepsilon_n}^+(u_{\varepsilon_n}(t)) \cap \Omega \subseteq I_{\varepsilon_n}^+(u_{0\varepsilon_n}) \cap \Omega = \emptyset,$$

hence

$$\int_{\Omega} |(D^{\varepsilon_n, +} u_{\varepsilon_n}(t))(x)| dx = 0.$$

Since  $\{D^{\varepsilon_n, +} u_{\varepsilon_n}(t) dx\} \xrightarrow{*} D^j u(t)$  weakly  $*$  as Radon measures, it follows that

$$(D^j u(t))(\Omega) = 0, \quad \forall \Omega \subset\subset \mathbb{R} \setminus S_{u_0}.$$

This is equivalent to statement (i).

Statement (ii) follows from the continuity of  $u$ , and Theorem 2.2.

In order to prove statement (iii), we show that for all  $y \in S_{u_0}$  the set

$$C_y := \{t \geq 0 : y \notin S_{u(t)}\}$$

is closed. Indeed, let  $\{t_n\} \subseteq C_y$  be a sequence converging to a certain  $T$ . By the continuity of  $u$  and Theorem 2.2, we have that  $\{D^j u(t_n)\} \xrightarrow{*} D^j u(T)$  weakly  $*$

as Radon measures. Now let  $[y - \delta, y + \delta] \cap S_{u_0} = \{y\}$  for some  $\delta > 0$ . By statement (i) and our definition of  $C_y$ , we have that  $S_{u(t_n)} \cap [y - \delta, y + \delta] = \emptyset$  for all  $n$ , hence

$$0 = \lim_{n \rightarrow \infty} \left| D^j u(t_n) \right| ([y - \delta, y + \delta]) = \left| D^j u(T) \right| ([y - \delta, y + \delta]).$$

This shows that  $T \in C_y$ . □

**Caution!** It is in general *false* that  $S_{u(t_2)} \subseteq S_{u(t_1)}$  for all  $0 \leq t_1 \leq t_2$  (cf. the discussion in Section 5).

### 5. – Characterizations

In this section we characterize the elements of  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  defined in Section 4 as the solutions of the (rescaled) heat equation with homogeneous Neumann boundary conditions in suitable variable domains.

#### 5.1. – Definition of a semigroup

We introduce the notion of regular and super-regular initial data. Moreover we introduce two functions

$$Reg : \mathbb{X} \longrightarrow ]0, +\infty], \quad SReg : \mathbb{X} \longrightarrow ]0, +\infty],$$

and, for every  $t \geq 0$ , we define a map

$$F_t : \mathbb{X} \longrightarrow \mathbb{X}.$$

To this end, we fix  $u_0 \in \mathbb{X}$ , and for all  $t \geq 0$  we define  $F_t(u_0)$  according to the following construction.

(Step 1) Let  $v : [0, +\infty[ \rightarrow L^\infty(\mathbb{R} \setminus S_{u_0})$  be the solution of the rescaled heat equation  $v_t = 2 \Delta v$  in the open set  $\mathbb{R} \setminus S_{u_0}$ , with homogeneous Neumann boundary conditions on  $S_{u_0}$ , and initial datum  $v(0) = u_0$ .

Since  $S_{u_0}$  is a finite set, then for all  $t \geq 0$  we can consider  $v(t)$  as a function defined for a.e.  $x \in \mathbb{R}$ . In this sense it turns out that  $v(t) \in \mathbb{X}$ .

(Step 2) For every  $y \in S_{u_0}$ , let us consider the function  $J_{u_0, y} : [0, +\infty[ \rightarrow \mathbb{R}$  defined by

$$J_{u_0, y}(t) := (D^j v(t))(\{y\}).$$

It turns out (cf. Subsection 2.3) that  $J_{u_0, y}$  is continuous in  $[0, +\infty[$ , analytic in  $]0, +\infty[$ , and depends continuously on  $u_0$ . Moreover,  $J_{u_0, y}(0) \neq 0$  for every  $y \in S_{u_0}$ .

(Step 3) Let us set

$$T_1 := \sup \{t \geq 0 : J_{u_0,y}(\tau) \neq 0, \forall y \in S_{u_0}, \forall \tau \in [0, t]\}.$$

Roughly speaking,  $T_1$  is “the first time in which a discontinuity of  $u_0$  disappears”. By the discussion in (Step 2) it follows that  $T_1 > 0$ .

If  $T_1 < +\infty$ , then  $J_{u_0,y}(T_1) = 0$  for some  $y \in S_{u_0}$ . In this case, we say that  $T_1$  is *regular* if there exists exactly one  $y \in S_{u_0}$  such that  $J_{u_0,y}(T_1) = 0$ , and furthermore the function  $J_{u_0,y}$  changes its sign in  $T_1$  (i.e. it is positive in a right neighborhood of  $T_1$ , and negative in a left neighborhood of  $T_1$ , or viceversa).

We say that  $T_1$  is *super-regular* if in addition  $J'_{u_0,y}(T_1) \neq 0$ .

(Step 4) We define

$$F_t(u_0) := v(t), \quad \forall t \in [0, T_1[.$$

(Step 5) If  $T_1 = +\infty$ , then the construction is complete. If  $T_1 < +\infty$ , then we reiterate the construction. This means that we consider the solution  $w : [T_1, +\infty[ \rightarrow L^\infty(\mathbb{R} \setminus S_{v(T_1)})$  of the equation  $w_t = 2 \Delta w$  in the open set  $\mathbb{R} \setminus S_{v(T_1)}$ , with homogeneous Neumann boundary conditions on  $S_{v(T_1)}$ , and initial datum  $w(T_1) = v(T_1)$ . As in (Step 3) we define  $T_2 > T_1$  as “the first time in which a discontinuity of  $v(T_1)$  disappears”, and then we set

$$F_t(u_0) := w(t), \quad \forall t \in [T_1, T_2[.$$

(Step 6) It is clear that after each reiteration the number of discontinuity points of  $F_t(u_0)$  strictly decreases. Since  $u_0$  has only a finite number of discontinuities, then the construction will be complete (i.e. we have defined  $F_t(u_0)$  for all  $t \geq 0$ ) after a finite number of reiterations.

(Step 7) We denote by  $\mathcal{R}eg(u_0)$  the first reiteration time which is not regular according to the definition given in (Step 3), with the convention that  $\mathcal{R}eg(u_0) = +\infty$  if all the reiteration times (if any) are regular. In a similar way we define  $\mathcal{S}\mathcal{R}eg(u_0)$  as the first reiteration time which is not super-regular.

We say that  $u_0$  is *regular* (resp. *super-regular*) if  $\mathcal{R}eg(u_0) = +\infty$  (resp.  $\mathcal{S}\mathcal{R}eg(u_0) = +\infty$ ).

## 5.2. – Properties of the semigroup

We state (without proof) some properties of  $\{F_t\}$  that follow from the given definitions, and from analogous properties of solutions of the heat equation with homogeneous Neumann boundary conditions.

PROPOSITION 5.1 (Properties of  $\{F_t\}$ ). *The family  $\{F_t\}$  satisfies the*

(i) semigroup property: *for all  $t \geq 0, s \geq 0$  we have that*

$$F_{t+s} = F_t \circ F_s.$$

*Moreover for all  $u_0 \in \mathbb{X}$  the function  $t \rightarrow F_t(u_0)$  has the following properties.*

- (ii) *All the properties of the elements of  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  stated in Proposition 4.6.*  
 (iii) *Volume-energy equality: for all  $0 \leq t_1 \leq t_2$  we have that*

$$\|\nabla(F_{t_1}(u_0))\|_{L^2(\mathbb{R})}^2 - \|\nabla(F_{t_2}(u_0))\|_{L^2(\mathbb{R})}^2 = 4 \int_{t_1}^{t_2} \|\Delta(F_\tau(u_0))\|_{L^2(\mathbb{R})}^2 d\tau,$$

*where “ $\Delta$ ” denotes the distributional derivative of the approximate gradient.*

- (iv) *Surface-energy monotonicity: the function  $t \rightarrow \mathcal{H}^0(S_{F_t}(u_0))$  is non-increasing.*  
 (v) *Mumford-Shah energy inequality: for all  $0 \leq t_1 \leq t_2$  we have that*

$$MS_{1, \frac{\pi}{2}}(F_{t_1}(u_0)) - MS_{1, \frac{\pi}{2}}(F_{t_2}(u_0)) \geq \int_{t_1}^{t_2} |\nabla MS_{1, \frac{\pi}{2}}(F_\tau(u_0))|^2 d\tau.$$

(vi) Hölder continuity: *for all  $0 \leq t_1 \leq t_2$  we have that*

$$\|F_{t_1}(u_0) - F_{t_2}(u_0)\|_{L^2(\mathbb{R})}^2 \leq \int_{t_1}^{t_2} |\nabla MS_{1, \frac{\pi}{2}}(F_\tau(u_0))| d\tau,$$

*and in particular*

$$(5.11) \quad \|F_{t_1}(u_0) - F_{t_2}(u_0)\|_{L^2(\mathbb{R})}^2 \leq |t_1 - t_2|^{1/2} \{MS_{1, \frac{\pi}{2}}(u_0)\}^{1/2}. \quad \square$$

REMARK 5.2. If  $u_0 \in L^2(\mathbb{R}) \cap \mathbb{X}$ , statements (v) and (vi) of Proposition 5.1 are equivalent to say that  $F_t(u_0)$  is a maximal slope curve (in the sense of [16]) for the Mumford-Shah functional in  $L^2(\mathbb{R})$ .

With the following result we motivate the definition of the semigroup, relating  $F_t(u_0)$  with the elements of  $GF(MS_{1, \frac{\pi}{2}}, u_0)$ .

THEOREM 5.3. *Let  $u \in GF(MS_{1, \frac{\pi}{2}}, u_0)$  for some  $u_0 \in \mathbb{X}$ , and let  $M \geq 0$ . Let us assume that*

$$(5.12) \quad S_{u(t_2)} \subseteq S_{u(t_1)}, \quad \text{whenever } 0 \leq t_1 \leq t_2 \leq M.$$

*Then  $u(t) = F_t(u_0)$ , for all  $t \in [0, M]$ .*

PROOF.

STEP 1. Let us set

$$\tilde{T}_1 := \sup\{t \in [0, M] : S_{u(\tau)} = S_{u_0} \quad \forall \tau \in [0, t]\}.$$

Since by assumption we have that  $S_{u(t)} \subseteq S_{u_0}$  for all  $t \in [0, M]$ , then from statement (iii) of Proposition 4.12 it follows that  $\tilde{T}_1 > 0$ .

STEP 2. By (iv) and (v) of Proposition 4.6 it follows that, for  $t \in [0, \tilde{T}_1]$ , the function  $u(t)$  is the solution in the open set  $\mathbb{R} \setminus S_{u_0}$  of the equation  $u_t = 2 \Delta u$  with homogeneous Neumann boundary conditions on  $S_{u_0}$ , and initial datum  $u(0) = u_0$ .

This implies that  $\tilde{T}_1 = \min\{T_1, M\}$ , where  $T_1$  is the first reiteration point in the construction described in Subsection 5.1, and

$$u(t) = F_t(u_0), \quad \forall t \in [0, \tilde{T}_1].$$

STEP 3. If  $\tilde{T}_1 = M$ , thesis is proved. If it is not, let us set

$$\tilde{T}_2 := \sup\{t \in [T_1, M] : S_{u(\tau)} = S_{u(T_1)} \quad \forall \tau \in [T_1, t]\}.$$

Arguing as in Step 1 and Step 2, we can show that  $\tilde{T}_2 = \min\{T_2, M\}$ , and that

$$u(t) = F_t(u_0), \quad \forall t \in [T_1, \tilde{T}_2].$$

Repeating this argument (if necessary), in a finite number of steps thesis is proved.  $\square$

### 5.3. – The regular case

The following result completely characterizes the elements of  $GF(MS_1, \frac{\pi}{2}, u_0)$  before the first non regular reiteration time.

**THEOREM 5.4.** *Let  $u_0 \in \mathbb{X}$ , and let  $u \in GF(MS_1, \frac{\pi}{2}, u_0)$ . Then*

$$u(t) = F_t(u_0) \quad \forall t \in \overline{[0, \mathcal{R}eg(u_0)]}.$$

**PROOF.** Since  $u$  and  $F_t(u_0)$  are continuous functions of time, it is enough to show that

$$u(t) = F_t(u_0), \quad \forall t \in [0, M],$$

for all  $M < \mathcal{R}eg(u_0)$ .

By Theorem 5.3, we have only to show that  $u$  satisfies assumption (5.12) for all  $M < \mathcal{R}eg(u_0)$ . To this end, since  $S_{u(t)} \subseteq S_{u_0}$  (statement (i) of Proposition 4.12), it is enough to show that

$$A_w := \{t \geq 0 : w \in S_{u(t)}\} \cap \overline{[0, \mathcal{R}eg(u_0)]}$$

is a connected set for every  $w \in S_{u_0}$ .

Let us assume by contradiction that this is not the case. Then, among all the points of  $S_{u_0}$  which do not satisfy this property, let  $y$  be the one which minimizes the length of the connected component of  $A_y$  containing  $t = 0$ , and let  $T$  be such a length (clearly  $T < \mathcal{R}eg(u_0)$ ).

STEP 1. We have that

- $A_y$  is an open subset of  $\overline{[0, \mathcal{R}eg(u_0)[}$  (by statement (iii) of Proposition 4.12).
- $T$  is the reiteration time in the construction of  $F_t(u_0)$  where the discontinuity in  $y$  disappears. Indeed, since  $A_y$  is an open set, by definition of  $T$  it follows that  $[0, T[\subseteq A_y$ , and  $T \notin A_y$ . Therefore  $y \in S_{u(t)}$  for every  $t \in [0, T[$ , and  $y \notin S_{u(T)}$ .
- $u(t) = F_t(u_0)$  for all  $t \in [0, T[$  (we can apply Theorem 5.3 with  $M = T$ ).
- There exists  $T_* > T$  such that  $y \in S_{u(T_*)}$ .

STEP 2. Let us assume, without loss of generality, that  $D^j u_0(\{y\}) > 0$ . We claim that

$$(5.13) \quad (\nabla u(t))(y) \geq 0, \quad \text{for a.e. } t \in [0, T_*]$$

(since  $y \in S_{u(t)}$  for  $t \in [0, T[$ , we already know that  $(\nabla u(t))(y) = 0$  in the interval  $[0, T[$ ).

Let  $\{\varepsilon_n\} \rightarrow 0^+$ , and  $\{u_{0\varepsilon_n}\} \subseteq L^\infty(\mathbb{R})$  be as in Definition 4.5. Let us set for simplicity  $u_{\varepsilon_n} = GF(\hat{F}_{\varepsilon_n}, u_{0\varepsilon_n})$ , and let  $\delta > 0$  be such that

$$(S_{u_0})_\delta := \{x \in \mathbb{R} : |x - y| < \delta \text{ for some } y \in S_{u_0}\}$$

has exactly  $\mathcal{H}^0(S_{u_0})$  connected components. Since  $y \in S_{u(T_*)}$ , and since  $\{D^{\varepsilon_n,+} u_{\varepsilon_n}(T_*) dx\} \xrightarrow{*} D^j u(T_*)$  weakly  $*$  as Radon measures, it follows that, for  $n$  large enough, the set

$$C_n := I_{\varepsilon_n}^+(u_{\varepsilon_n}(T_*)) \cap ]y - \delta, y + \delta[$$

contains at least one interval of length  $\varepsilon_n$ .

By Proposition 4.11 we have that

$$C_n \subseteq I_{\varepsilon_n}^+(u_{\varepsilon_n}(t)) \cap ]y - \delta, y + \delta[, \quad \forall t \in [0, T_*],$$

and in particular

$$C_n \subseteq I_{\varepsilon_n}^+(u_{0\varepsilon_n}) \cap ]y - \delta, y + \delta[.$$

By statement (ii) of Proposition 4.9 it turns out that  $C_n$  is the only component of  $I_{\varepsilon_n}^+(u_{0\varepsilon_n})$  contained in  $]y - \delta, y + \delta[$ . Since we have assumed that  $D^j u_0(\{y\}) > 0$ , by statement (iii) of Proposition 4.9 it follows that, for  $n$  large enough,  $D^{\varepsilon_n,+} u_{0\varepsilon_n}$  is positive in  $C_n$ , hence, by continuity,  $D^{\varepsilon_n,+} u_{\varepsilon_n}(t)$  is positive in  $C_n$  for all  $t \in [0, T_*]$ .

Therefore

$$(5.14) \quad \max_{x \in [y-\delta, y+\delta]} \varphi'_{\varepsilon_n} ((D^{\varepsilon_n} u_{\varepsilon_n}(t))(x)) \geq 0, \quad \forall t \in [0, T_*].$$

Since

$$\liminf_{n \rightarrow \infty} \|\nabla \hat{F}_{\varepsilon_n}(u_{\varepsilon_n}(t))\|_{L^2(\mathbb{R})} < +\infty$$

for a.e.  $t \geq 0$  (cf. the proof of statement (v) of Proposition 4.6), applying Lemma 3.6 with  $g_n = \varphi'_{\varepsilon_n}(D^{\varepsilon_n} u_{\varepsilon_n}(t))$  and  $g = 2 \nabla u(t)$ , up to subsequences we have that

$$\varphi'_{\varepsilon_n}(D^{\varepsilon_n} u_{\varepsilon_n}(t)) \longrightarrow 2 \nabla u(t)$$

uniformly on compact sets (as functions of the  $x$ -variable) for a.e.  $t \geq 0$ .

Therefore, passing to the limit in (5.14), we obtain that

$$\max_{x \in [y-\delta, y+\delta]} (\nabla u(t))(x) \geq 0$$

for a.e.  $t \geq 0$ . Since  $\delta$  is arbitrary, (5.13) is proved.

STEP 3. With the same assumptions of Step 2, i.e.  $D^j u_0(\{y\}) > 0$ , we claim that

$$(5.15) \quad (D^j u(t))(\{y\}) \geq 0, \quad \forall t \in [0, T_*].$$

Indeed, with the same notations as in the proof of Step 2 we have that

$$\begin{aligned} (D^j u(t))(\{y\}) &= (D^j u(t))([y - \delta, y + \delta]) \\ &= \lim_{n \rightarrow \infty} \int_{C_n} D^{\varepsilon_n, +} u_{\varepsilon_n}(t) dx \geq 0, \end{aligned}$$

where the last inequality follows since  $D^{\varepsilon_n, +} u_{\varepsilon_n}(t)$  is positive in  $C_n$  for all  $t \in [0, T_*]$ .

STEP 4. Let us denote by  $S$  the last reiteration time before  $T$  (if  $T$  is the first reiteration time, we set  $S = 0$ ), and let us choose  $U \in ]T, T_*]$  in such a way that  $S_{u(T)} \setminus \{y\} = S_{u(\tau)} \setminus \{y\}$  for all  $\tau \in [T, U]$ . Let  $I$  be the connected component of  $\mathbb{R} \setminus S_{u(S)}$  with  $y$  as infimum. Let  $v$  be the solution in  $[S, U[ \times I$  of the equation  $v_t = 2 \Delta v$  with homogeneous Neumann boundary conditions, and initial datum  $v(S) = u(S)|_I$ .

By (5.13) and by (iv-v) of Proposition 4.6, in  $[S, U[ \times I$  the function  $u$  (as a function of  $(t, x)$ ) is a solution of the same equation, with the same initial condition, but Neumann boundary condition equal to

$$(\nabla u(t))(y) \geq 0$$

in  $y$ , and equal to zero in the supremum of  $I$  (if different from  $+\infty$ ). By the comparison result recalled in Subsection 2.3, it follows that  $u \leq v$  in  $[S, U[ \times I$ , hence

$$\lim_{x \rightarrow y^+} (v(t))(x) \geq \lim_{x \rightarrow y^+} (u(t))(x), \quad \forall t \in [S, U[.$$

Arguing in an analogous way in the connected component of  $\mathbb{R} \setminus S_{u(S)}$  with  $y$  as supremum, it is possible to prove that

$$\lim_{x \rightarrow y^-} (v(t))(x) \leq \lim_{x \rightarrow y^-} (u(t))(x), \quad \forall t \in [S, U[.$$

Therefore by (5.15)

$$J_{u(S),y}(t) := \lim_{x \rightarrow y^+} (v(t))(x) - \lim_{x \rightarrow y^-} (v(t))(x) \geq \left( D^j u(t) \right) (\{y\}) \geq 0$$

for all  $t \in [S, U[$ . This is a contradiction, since by the regularity of  $u_0$  the function  $J_{u(S),y}$  must change its sign in  $T$ . The proof is thus complete.  $\square$

The above theorem shows that  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  consists of a single function, namely  $F_t(u_0)$ , provided that  $u_0$  is regular. Therefore  $F_t(u_0)$  is the only candidate to be the gradient flow relative to the Mumford-Shah functional with a regular initial datum  $u_0$ .

EXAMPLE 5.5. Let  $u_0 \in H^1(\mathbb{R}) \subseteq \mathbb{X}$ . Then  $u_0$  is super-regular, and the only element of  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  is the solution  $u : [0, +\infty[ \rightarrow H^1(\mathbb{R})$  of the equation  $u_t = 2 \Delta u$  with initial datum  $u_0$ .

EXAMPLE 5.6. Let  $u_0 \in \mathbb{X}$  with  $\nabla u_0 \equiv 0$ . Then  $u_0$  is super-regular and the only element of  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  is the constant function  $u(t) \equiv u_0$ . In this case we say that  $u_0$  is a *stationary point* for the Mumford-Shah functional. It is easy to prove that  $u_0 \in \mathbb{X}$  is stationary if and only if  $\nabla u_0 \equiv 0$ .

#### 5.4. – The general case

If  $u_0 \in \mathbb{X}$  is not regular, it may happen that  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  contains infinitely many functions (cf. Theorem 5.10). However, we now show that  $F_t(u_0)$  always plays a special role among these functions.

Before we state the precise result, we need two lemmata about the semigroup  $\{F_t\}$ . The first one may be considered as a lower semicontinuity property of the function  $SReg$ .

LEMMA 5.7. *Let  $\{u_{0n}\} \subseteq \mathbb{X}$ , and let  $u_0 \in \mathbb{X}$  be such that*

$$(5.16) \quad \begin{aligned} \{u_{0n}\} &\longrightarrow u_0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}); \\ S_{u_{0n}} &= S_{u_0} \quad \forall n \in \mathbb{N}. \end{aligned}$$



Then we have that

$$(5.17) \quad \liminf_{n \rightarrow \infty} \mathcal{SReg}(u_{0n}) \geq \mathcal{SReg}(u_0);$$

$$(5.18) \quad \{F_t(u_{0n})\} \longrightarrow F_t(u_0) \text{ in } C^0([0, \mathcal{SReg}(u_0)]; L^2_{\text{loc}}(\mathbb{R})).$$

PROOF. We argue by induction on  $k = \mathcal{H}^0(S_{u_0})$ .

If  $k = 0$ , then by (5.16) also  $\mathcal{H}^0(S_{u_{0n}}) = 0$ , hence  $u_{0n}$  is super-regular for each  $n \in \mathbb{N}$ . Moreover (5.18) follows from the continuous dependence on the initial datum for the solutions of the heat equation.

Now let us assume that thesis is proved for some  $k$ , and let  $\mathcal{H}^0(S_{u_0}) = k + 1$ . We distinguish two cases.

CASE 1. Let us assume that  $\mathcal{SReg}(u_0)$  is the first reiteration time in the construction of  $F_t(u_0)$ . Then for each  $M < \mathcal{SReg}(u_0)$  we have that (we use the notations introduced in Subsection 5.1)

$$J_{u_0,y}(t) \neq 0, \quad \forall t \in [0, M], \quad \forall y \in S_{u_0}.$$

Since  $\{J_{u_{0n},y}\} \rightarrow J_{u_0,y}$  in  $C^0([0, M]; \mathbb{R})$  for all  $y \in S_{u_0}$ , we have that

$$J_{u_{0n},y}(t) \neq 0, \quad \forall t \in [0, M], \quad \forall y \in S_{u_0}$$

for every  $n$  large enough. This proves that  $\mathcal{SReg}(u_{0n}) \geq M$  for  $n$  large enough. Moreover

$$\{F_t(u_{0n})\} \longrightarrow F_t(u_0) \text{ in } C^0([0, M]; L^2_{\text{loc}}(\mathbb{R})).$$

Since  $M$  is arbitrary, (5.17) and (5.18) are proved in this case.

CASE 2. Let  $T < \mathcal{SReg}(u_{0n})$  be the first reiteration time in the construction of  $F_t(u_0)$ , and let  $y \in S_{u_0}$  be the only jump point which disappears in  $T$ . Since  $T$  is super-regular we have that

$$J_{u_0,y}(T) = 0, \quad J'_{u_0,y}(T) \neq 0.$$

Since we have that  $\{J_{u_{0n},y}\} \rightarrow J_{u_0,y}$  in  $C^0([0, T]; \mathbb{R})$ , and  $\{J'_{u_{0n},y}\} \rightarrow J'_{u_0,y}$  in  $C^0(]0, T]; \mathbb{R})$ , it follows that, for  $n$  large enough, the first reiteration time in the construction of  $F_t(u_{0n})$  is a super-regular time  $T_{n,y}$  in which  $y$  disappears. Moreover  $\{T_{n,y}\} \rightarrow T$ , and

$$(5.19) \quad \{F_{T_{n,y}}(u_{0n})\} \longrightarrow F_T(u_0) \text{ in } L^2_{\text{loc}}(\mathbb{R}).$$

Since

$$\mathcal{SReg}(u_{0n}) = T_{n,y} + \mathcal{SReg}(F_{T_{n,y}}(u_{0n})),$$

and since both  $F_T(u_0)$  and  $F_{T_{n,y}}(u_{0n})$  have the same  $k$  discontinuity points, by (5.19) and the inductive hypothesis we have that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathcal{SReg}(u_{0n}) &= T + \liminf_{n \rightarrow \infty} \mathcal{SReg}(F_{T_{n,y}}(u_{0n})) \\ &\geq T + \mathcal{SReg}(F_T(u_0)) = \mathcal{SReg}(u_0). \end{aligned}$$

This proves (5.17). Finally, (5.18) may be proved combining the convergence in  $[0, T]$  (due to the continuous dependence on the initial datum for solutions of the heat equation), and the convergence in  $[T, \mathcal{SReg}(u_0)[$  (due to the inductive hypothesis).

Therefore thesis is proved also in this second case. □

In the second lemma we show that any trajectory  $F_t(u_0)$  may be approximated by “asymptotically super-regular” trajectories.

LEMMA 5.8. *For all  $u_0 \in \mathbb{X}$  there exists a sequence  $\{u_{0n}\} \subseteq \mathbb{X}$  such that*

$$(5.20) \quad \{d_{\mathbb{X}}(u_{0n}, u_0)\} \longrightarrow 0;$$

$$(5.21) \quad \{\mathcal{SReg}(u_{0n})\} \longrightarrow +\infty;$$

$$(5.22) \quad \{F_t(u_{0n})\} \longrightarrow F_t(u_0) \quad \text{in } C^0([0, +\infty)[; L^2_{\text{loc}}(\mathbb{R})).$$

PROOF. By a diagonal argument, it is enough to show that for each  $M > 0$ , and each  $u_0 \in \mathbb{X}$ , there exists a sequence  $\{u_{0n}\} \subseteq \mathbb{X}$  satisfying (5.20) and

$$(5.23) \quad \limsup_{n \rightarrow \infty} \mathcal{SReg}(u_{0n}) \geq M ;$$

$$\{F_t(u_{0n})\} \longrightarrow F_t(u_0) \quad \text{in } C^0([0, M]; L^2_{\text{loc}}(\mathbb{R})).$$

To this end, let us denote by  $B(u_0, M)$  the set of reiteration times in the construction of  $F_t(u_0)$  that are not super-regular, and contained in  $[0, M]$ .

If  $\#B(u_0, M) = 0$ , thesis is trivial. Arguing by induction on  $\#B(u_0, M)$ , it is enough to show that every  $u_0$  with  $\#B(u_0, M) = k + 1$  may be approximated by a sequence  $\{u_{0n}\} \subseteq \mathbb{X}$  satisfying (5.20), (5.23), and  $\#B(u_{0n}, M) = k$  for  $n$  large enough.

Therefore let us assume that  $\#B(u_0, M) = k + 1$ , and let us set

$$T := \max B(u_0, M), \quad S := \max B(u_0, M) \setminus \{T\},$$

with the convention that  $S = 0$  if  $B(u_0, M) = \{T\}$ .

We distinguish two cases.

CASE 1. We assume that there exists exactly one jump point  $y \in S_{u_0}$  which disappears in  $T$ . In this case we can assume, without loss of generality, that  $D^j u_0(\{y\}) > 0$ .

Setting  $J_{u_0,y}(t) := (D^j F_t(u_0))(\{y\})$ , this implies that

$$(5.24) \quad J_{u_0,y}(T) = 0, \quad J_{u_0,y}(t) > 0, \quad \forall t \in [0, T[ ,$$

hence, since  $J_{u_0,y}$  is analytic in  $]S, T[$ , there exists  $\delta > 0$  such that  $T - \delta > S$ , and

$$(5.25) \quad J'_{u_0,y}(t) < 0, \quad \forall t \in ]T - \delta, T[ .$$

Now let us set

$$u_{0n} := u_0 + \frac{1}{n} \chi_{]-\infty, y[} .$$

It is clear that  $\{u_{0n}\}$  satisfies (5.20). We claim that  $\{u_{0n}\}$  satisfies also (5.22), and that

$$(5.26) \quad \#B(u_{0n}, M) = k$$

for every  $n$  large enough.

Indeed, since

$$J_{u_{0n},y}(t) = J_{u_0,y}(t) - \frac{1}{n}$$

for all  $t \geq 0$  such that  $J_{u_{0n},y}(t) > 0$ , then by (5.24) we have that, for  $n$  large enough,  $J_{u_{0n},y}$  vanishes for some  $T_n \in ]T - \delta, T[$ . By (5.25),  $T_n$  is a super-regular point in the construction of  $F_t(u_{0n})$ . Moreover  $\{T_n\} \rightarrow T^-$ , and

$$(5.27) \quad F_t(u_{0n}) = F_t(u_0) + \frac{1}{n} \chi_{]-\infty, y[} \quad \forall t \in [0, T_n],$$

hence

$$B(u_{0n}, M) \cap [0, T_n] = B(u_0, M) \setminus \{T\}.$$

Therefore (5.26) will be proved if we show that  $\mathcal{S}Reg(u_{0n}) > M - T$  for  $n$  large enough. To this end, by Lemma 5.7, it is enough to prove that

$$(5.28) \quad \{F_{T_n}(u_{0n})\} \longrightarrow F_T(u_0) \quad \text{in } L^2_{loc}(\mathbb{R}).$$

Since

$$F_{T_n}(u_{0n}) - F_T(u_0) = (F_{T_n}(u_{0n}) - F_{T_n}(u_0)) + (F_{T_n}(u_0) - F_T(u_0)),$$

(5.28) follows from (5.27) and the time continuity of  $F_t(u_0)$ . This completes the proof of (5.26). Moreover (5.22) easily follows from (5.27), (5.28), and Lemma 5.7

CASE 2. There exist at least two jump points which disappear in  $T$ . In this case, let us assume by simplicity that there are only two such points  $y < z$ , and that

$$D^j u_0(\{y\}) > 0, \quad D^j u_0(\{z\}) > 0.$$

As in Case 1 we have that

$$\begin{aligned} J_{u_0,y}(T) &= J_{u_0,z}(T) = 0, \\ J_{u_0,y}(t) &> 0, \quad J_{u_0,z}(t) > 0, \quad \forall t \in [0, T[, \end{aligned}$$

and there exists  $\delta > 0$  such that  $T - \delta > S$ , and

$$J'_{u_0,y}(t) < 0, \quad J'_{u_0,z}(t) < 0, \quad \forall t \in ]T - \delta, T[.$$

We claim that there exists a sequence  $\{\alpha_n\} \rightarrow 0^+$  such that the sequence defined by

$$u_{0n} := u_0 + \frac{1}{n} \chi_{] - \infty, y[} + \alpha_n \chi_{] - \infty, z[}$$

satisfies (5.20), (5.23), and (5.26) for  $n$  large enough.

Indeed, arguing as in Case 1, we have that  $J_{u_{0n}, y}$  vanishes for some  $T_{n, y} \in ]T - \delta, T[$  which does *not* depend on  $\alpha_n$ . Now we have that

$$J_{u_{0n}, z}(t) = J_{u_0, z}(t) - \alpha_n,$$

hence

$$J'_{u_{0n}, z}(t) = J'_{u_0, z}(t) < 0$$

for all  $t \in ]T - \delta, T[$  such that  $J_{u_{0n}, z}(t) > 0$ . Therefore, if we choose  $\alpha_n = J_{u_0, z}(T_{n, y}) - \sigma_n$ , with  $\sigma_n$  small enough, then the function  $J_{u_{0n}, z}$  vanishes for some  $T_{n, z} \in ]T_{n, y}, T[$  with  $J'_{u_{0n}, z}(T_{n, z}) < 0$ . In such a way we have that

$$B(u_{0n}, M) \cap [0, T_{n, z}] = B(u_0, M) \setminus \{T\}.$$

Arguing as in Case 1, it remains to show that

$$\{F_{T_{n, z}}(u_{0n})\} \longrightarrow F_T(u_0) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}).$$

To this end, let us write

$$\begin{aligned} F_{T_{n, z}}(u_{0n}) - F_T(u_0) &= (F_{T_{n, z}}(u_{0n}) - F_{T_{n, y}}(u_{0n})) \\ &\quad + (F_{T_{n, y}}(u_{0n}) - F_{T_{n, y}}(u_0)) \\ &\quad + (F_{T_{n, y}}(u_0) - F_T(u_0)). \end{aligned}$$

The third summand is infinitesimal by the time continuity of the map  $F_t(u_0)$ ; the second one is infinitesimal since it is equal to

$$\frac{1}{n} \chi_{] - \infty, y[} + \alpha_n \chi_{] - \infty, z[};$$

the first one is infinitesimal since by (5.11)

$$\begin{aligned} \|F_{T_{n, z}}(u_{0n}) - F_{T_{n, y}}(u_{0n})\|_{L^2(\mathbb{R})} &\leq |T_{n, z} - T_{n, y}|^{1/2} \left\{ MS_{1, \frac{\pi}{2}}(u_{0n}) \right\}^{1/2} \\ &= |T_{n, z} - T_{n, y}|^{1/2} \left\{ MS_{1, \frac{\pi}{2}}(u_0) \right\}^{1/2}. \end{aligned}$$

A similar construction works with an arbitrary number of points which disappear in  $T$ . For example, if there are three such points  $y < z < w$ , then there exist two sequences  $\{\alpha_n\} \rightarrow 0^+$ ,  $\{\beta_n\} \rightarrow 0^+$ , such that

$$u_{0n} := u_0 \pm \frac{1}{n} \chi_{] - \infty, y[} \pm \alpha_n \chi_{] - \infty, z[} \pm \beta_n \chi_{] - \infty, w[}$$

(the sign  $\pm$  of each perturbation depending on the sign of  $D^J u_0$  in the corresponding jump point) has the required properties.  $\square$

We are now ready to prove the special role of  $F_t(u_0)$  among all the elements of  $GF(MS_{1, \frac{\pi}{2}}, u_0)$ .

**THEOREM 5.9.** *For all  $u_0 \in \mathbb{X}$  we have that*

- (i)  $F_t(u_0) \in GF(MS_{1, \frac{\pi}{2}}, u_0)$ ;
- (ii) if  $v \in GF(MS_{1, \frac{\pi}{2}}, u_0)$ , and the function  $t \rightarrow MS_{1, \frac{\pi}{2}}(v(t))$  is non-increasing, then  $v(t) = F_t(u_0)$  for all  $t \geq 0$ .

**PROOF OF (i).** Let  $\{u_{0n}\} \rightarrow u_0$  be a sequence as in Lemma 5.8, and for each  $n \in \mathbb{N}$ , let  $u_n \in GF(MS_{1, \frac{\pi}{2}}, u_{0n})$ . By Theorem 5.4 we have that

$$u_n(t) = F_t(u_{0n}), \quad \forall t \in [0, SReg(u_{0n})[ ,$$

and therefore, by (5.21) and (5.22), it follows that

$$\{u_n(t)\} \longrightarrow F_t(u_0) \quad \text{in } C^0([0, +\infty[; L^2_{loc}(\mathbb{R})).$$

By the second statement of Proposition 4.8, this proves (i).

**PROOF OF (ii).**

**STEP 1.** We show that the function  $t \rightarrow \mathcal{H}^0(S_{v(t)})$  is right continuous.

Indeed, let  $T \geq 0$ , and let  $\{t_n\} \rightarrow T^+$  be any sequence. By the monotonicity of  $MS_{1, \frac{\pi}{2}}(v(t))$  and Theorem 2.2, we have that

$$\begin{aligned} MS_{1, \frac{\pi}{2}}(v(T)) &\geq \liminf_{n \rightarrow \infty} MS_{1, \frac{\pi}{2}}(v(t_n)) \\ &\geq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} |\nabla v(t_n)|^2 + \frac{\pi}{2} \liminf_{n \rightarrow \infty} \mathcal{H}^0(S_{v(t_n)}) \\ &\geq \int_{\mathbb{R}} |\nabla v(T)|^2 + \frac{\pi}{2} \mathcal{H}^0(S_{v(T)}) = MS_{1, \frac{\pi}{2}}(v(T)), \end{aligned}$$

hence

$$\liminf_{n \rightarrow \infty} \mathcal{H}^0(S_{v(t_n)}) = \mathcal{H}^0(S_{v(T)}).$$

Since the sequence  $\{t_n\}$  is arbitrary, the right continuity of  $\mathcal{H}^0(S_{v(t)})$  is proved.

**STEP 2.** We show that for every  $T \geq 0$  there exists  $\delta > 0$  such that

$$(5.29) \quad S_{v(t)} = S_{v(T)}, \quad \forall t \in [T, T + \delta[ .$$

Indeed, by statement (iii) of Proposition 4.12, we have that  $\{t \geq 0 : S_{v(t)} \supseteq S_{v(T)}\}$  is an open set which contains  $T$ . Therefore (5.29) follows from the right continuity of the integer valued function  $t \rightarrow \mathcal{H}^0(S_{v(t)})$ .

**STEP 3.** In order to complete the proof, it is enough to show that  $v$  satisfies the assumption (5.12) of Theorem 5.3 for all  $M \geq 0$ .

To this end, let us assume by contradiction that there exist  $0 \leq T_1 \leq T_2$ , and  $y \in S_{v_0}$  such that  $y \in S_{v(T_2)}$ , but  $y \notin S_{v(T_1)}$ . Let us set

$$T_y := \inf \{t \geq T_1 : y \in S_{v(t)}\} .$$

By (5.29) it follows that  $y \in S_{v(T_y)}$ , hence  $T_y > T_1$ . This contradicts the minimality of  $T_y$ , since by statement (iii) of Proposition 4.12 we have that  $\{t \geq 0 : y \in S_{v(t)}\}$  is an open set which contains  $T_y$ .  $\square$

### 5.5. – A pathological initial datum

We finally give an explicit example of an initial datum  $u_0 \in \mathbb{X}$  such that  $F_t(u_0)$  is not the unique element of  $GF(MS_{1, \frac{\pi}{2}}, u_0)$ .

**THEOREM 5.10.** *There exists  $u_0 \in \mathbb{X}$  such that  $GF(MS_{1, \frac{\pi}{2}}, u_0)$  has the cardinality of continuum.*

**PROOF.**

**STEP 1.** In order to define  $u_0$ , let  $w : [0, +\infty[ \times [0, 1] \rightarrow \mathbb{R}$  be the solution of the problem

$$\begin{cases} w_t = 2 \Delta w & \text{in } [0, +\infty[ \times [0, 1], \\ w(0, x) = 8x^3 - 3x^2 & \forall x \in [0, 1], \\ w_x(t, 0) = w_x(t, 1) = 0 & \forall t \geq 0. \end{cases}$$

Let us consider the function  $\psi : [0, +\infty[ \rightarrow \mathbb{R}$  defined by

$$\psi(t) := w(t, 0).$$

Then we have that  $\psi(0) = 0$ , and

$$\begin{aligned} \psi'(t) &= 2 \left. \frac{d^2}{dx^2} (8x^3 - 3x^2) \right|_{x=0} < 0, \\ \lim_{t \rightarrow +\infty} \psi(t) &= \int_0^1 (8x^3 - 3x^2) dx > 0. \end{aligned}$$

Therefore there exist

$$C := \min_{t \geq 0} \psi(t) < 0, \quad T_* := \min \{t \geq 0 : \psi(t) = C\}.$$

Now let us set

$$u_0(x) := \begin{cases} C & \text{if } x \leq 0, \\ 8x^3 - 3x^2 & \text{if } x \in [0, 1], \\ 10 & \text{if } x > 1. \end{cases}$$

It is clear that  $u_0 \in \mathbb{X}$ , and  $S_{u_0} = \{0, 1\}$ .

**STEP 2.** Let  $u(t) = F_t(u_0)$ , and let  $v : [0, +\infty[ \rightarrow L^\infty(\mathbb{R} \setminus S_{u_0})$  be the solution of  $v_t = 2 \Delta v$  with homogeneous Neumann boundary conditions on  $S_{u_0}$ , and initial datum  $v(0) = u_0$ . We claim that in the construction of  $F_t(u_0)$  there is exactly one reiteration time, namely  $T_*$ , and that this time is not regular.

Indeed, setting  $J_{u_0, y}(t) := (D^j v(t))(\{y\})$  as in Subsection 5.1, we have that

$$\begin{aligned} J_{u_0, 0}(t) &= \psi(t) - C \geq 0, & \forall t \geq 0, \\ J_{u_0, 0}(T_*) &= 0, \end{aligned}$$

while by the maximum principle it follows that

$$J_{u_0,1}(t) \geq 10 - \max_{x \in [0,1]} \{8x^3 - 3x^2\} > 0, \quad \forall t \geq 0.$$

Therefore  $T_*$  is the first reiteration time in the construction of  $F_t(u_0)$ , and  $T_*$  is not regular. By the maximum principle it follows also that  $T_*$  is the only reiteration time.

STEP 3. We claim that there exists a sequence  $\{\varepsilon_n\} \rightarrow 0^+$  such that, setting

$$v_{0\varepsilon_n}(x) := u_0(\varepsilon_n[x/\varepsilon_n]) + \frac{1}{n} \chi_{[0,1]},$$

$$u_{0\varepsilon_n}(x) := u_0(\varepsilon_n[x/\varepsilon_n]) - \frac{1}{n} \chi_{[0,1]},$$

and

$$v_n := GF(\hat{F}_{\varepsilon_n}, v_{0\varepsilon_n}), \quad u_n := GF(\hat{F}_{\varepsilon_n}, u_{0\varepsilon_n}),$$

we have that

$$(5.30) \quad \{v_n\} \longrightarrow v, \quad \{u_n\} \longrightarrow u \quad \text{in } C^0([0, +\infty[; L^2_{\text{loc}}(\mathbb{R})),$$

and, in particular,  $\{u, v\} \subseteq GF(MS_{1, \frac{\pi}{2}}, u_0)$ .

Indeed, let us set

$$v_0^n := u_0 + \frac{1}{n} \chi_{[0,1]}.$$

Then  $\mathcal{SReg}(v_0^n) = +\infty$ , and

$$F_t(v_0^n) = v(t) + \frac{1}{n} \chi_{[0,1]} \quad \forall t \geq 0.$$

In particular

$$\{F_t(v_0^n)\} \longrightarrow v \quad \text{in } C^0([0, +\infty[; L^2_{\text{loc}}(\mathbb{R})).$$

Moreover, for each  $n \in \mathbb{N}$ , we have that

$$\{GF(\hat{F}_{\frac{1}{k}}, v_{0\frac{1}{k}}^n)\} \longrightarrow F_t(v_0^n)$$

as  $k \rightarrow \infty$ , where

$$v_{0\frac{1}{k}}^n(x) := u_0([kx]/k) + \frac{1}{n} \chi_{[0,1]}.$$

In an analogous way, if we set

$$u_0^n := u_0 - \frac{1}{n} \chi_{[0,1]}.$$

then, for  $n$  large enough, we have that  $\mathcal{SReg}(v_0^n) = +\infty$ , and

$$\{F_t(u_0^n)\} \longrightarrow u \quad \text{in } C^0([0, +\infty[; L_{\text{loc}}^2(\mathbb{R})).$$

Moreover, for each  $n$  large enough, we have that

$$\left\{GF(\hat{F}_{\frac{1}{k}}, u_{0\frac{1}{k}}^n)\right\} \longrightarrow F_t(u_0^n)$$

as  $k \rightarrow \infty$ , where

$$u_{0\frac{1}{k}}^n(x) := u_0([kx]/k) - \frac{1}{n} \chi_{[0,1]}.$$

Therefore our claim follows with a diagonal argument.

STEP 4. Since  $u(t) \neq v(t)$  for  $t > T_*$ , then there exist  $T > T_*$ , and  $I \subset \subset \mathbb{R}$  such that

$$\int_I u(T) dx \neq \int_I v(T) dx.$$

Let us assume, without loss of generality, that the left hand side is less than the right hand side. Then we claim that for all  $\lambda$  such that

$$\int_I u(T) dx < \lambda < \int_I v(T) dx,$$

there exists  $w \in GF(MS_{1, \frac{\pi}{2}}, u_0)$  such that

$$(5.31) \quad \int_I w(T) dx = \lambda.$$

Indeed, by (5.30) we have that

$$\int_I u_n(T) dx < \lambda < \int_I v_n(T) dx$$

for  $n$  large enough. Therefore, since the solution of (4.2) depends continuously on the initial datum, there exists  $\alpha_n \in [\frac{1}{n}, \frac{1}{n}]$  such that, setting

$$\begin{aligned} w_{0\varepsilon_n} &:= u_0(\varepsilon_n[x/\varepsilon_n]) + \alpha_n \chi_{[0,1]}, \\ w_n &:= GF(\hat{F}_{\varepsilon_n}, w_{0\varepsilon_n}), \end{aligned}$$

we have that

$$(5.32) \quad \int_I w_n(T) dx = \lambda.$$



Arguing as in Subsection 4.3, up to subsequences we can assume that  $\{w_n\} \rightarrow w$  in  $C^0([0, +\infty[; L^2_{\text{loc}}(\mathbb{R}))$ , hence  $w \in GF(MS_{1, \frac{\pi}{2}}, u_0)$ . Passing to the limit in (5.32), we prove that  $w$  satisfies (5.31). This completes the proof of the theorem.  $\square$

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