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On Liouville Theorem and Apriori Estimates for the Scalar Curvature Equations

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Abstract. In this article, we study the problem of nonexistence of positive solutions of the equation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{1}$$

where $K(x)$ is a C^1 homogeneous function of degree $l > 0$. Suppose that $K(x)$ satisfies the nondegenerate condition,

$$c_1|x|^{l-1} \le |\nabla K(x)| \le c_2|x|^{l-1} \tag{2}$$

for $x \in \mathbb{R}^n \backslash \{0\}$, where c_i are positive constants. We prove that equation (1) admits no positive solutions. This Liouville Theorem allows us to derive apriori estimates for positive solutions of scalar curvature equations in the region where the scalar curvature function is nonpositive. Various apriori estimates are also derived under different circumstances.

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1. – Introduction

Let (M, g_0) be a Riemannian manifold of dimension $n \ge 3$. For a given smooth function R on M, one would like to find a metric g conformal to g_0 such that R is the scalar curvature of the new metric g. Set $g = v^{\frac{4}{n-2}} g_0$ for some positive function v, then the question above is equivalent to finding positive solutions of

$$\frac{4(n-1)}{n-2}\Delta_{g_0} v - kv + Rv^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad M\,, \tag{1.1}$$

where Δ_{g_0} is the Beltrami-Laplace operator of (M, g_0) and k is the scalar curvature of g_0. In recent years, there have been a lot of progress in understanding equation (1.1), in particular, when (M, g_0) is the standard n-dimensional

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sphere S^n. In this case, $k(x) \equiv n(n-1)$ and then equation (1.1) becomes

$$\Delta_0 v - \frac{n(n-2)}{4} v + \frac{n-2}{4(n-1)} R v^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{S}^n . \tag{1.2}$$

By using the stereographic projection π of $\mathbb{S}^n$ onto $\mathbb{R}^n$ and letting $u(x) = c_n(\frac{2}{1+|x|^2})^{\frac{n-2}{2}} v(y)$ with $x = \pi(y)$ for $y \in \mathbb{S}^n$ and for some suitable positive constant c_n, equation (1.2) reduces into

$$\Delta u + K u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n , \tag{1.3}$$

where $K(x) = R(\pi^{-1}(x))$ for $x \in \mathbb{R}^n$. In general, if (M, g_0) is a locally conformally flat manifold, a local flat metric can be chosen and then, equation (1.1) is reduced to equation (1.3) in an open set of $\mathbb{R}^n$.

It is well-known that, when $R(y)$ is a positive constant identically for $y \in \mathbb{S}^n$, equation (1.2) possesses a family of solutions, whose total energy could be concentrated in a small neighborhood of some point $y_0 \in \mathbb{S}^n$. Thus, it is of great interest from the viewpoint of PDE to study the blow-up behavior for a sequence of positive solutions of (1.2) when $R(y)$ is a nonconstant function. For works in this aspect, we refer the reader to [3], [6,7], [11,12], [20] and references therein. To do the blow up analysis, we first rescale solutions near a blow-up point. Since the solution structure of equation (1.3) was completely understood for the case $K(x) \equiv$ a positive constant, it is easier to describe the limit of the rescaling solutions when $K(x)$ is a positive function. This is the reason why most works have only considered the situation where K is positive.

In this paper, we will consider the problem of finding apriori estimates for the region where the curvature function R is nonpositive. In [5], Chen-Li studied equation (1.2) on $\mathbb{S}^n$ and proved some apriori bound for solutions in the region $\{y \in \mathbb{S}^n \mid R(y) \le 0\}$ by assuming $\nabla R(y) \ne 0$ whenever $R(y) = 0$. However, their argument seems to work only for *solutions globally defined on the whole space* $\mathbb{S}^n$. In this paper, we want to extend their result for *solutions defined locally*, which can be applied in the general case when M is locally conformally flat. We believe that the result in local nature should be more useful. For applications, we can prove the apriori bound without the condition $\nabla R(y) \ne 0$ whenever $R(y) = 0$, although a nondegenerate condition is still required.

It is quite well-known that the establishment of apriori bounds is closely related to the Liouville theorem. The Liouville theorem always plays an important role in the theory of elliptic equations. For the scalar curvature equations (1.3) in $\mathbb{R}^n$, we [14] have proved the Liouville theorem for several classes of functions K. One of them is that $K(x) \equiv K(x_1)$ depends only on the x_1 variable only. Assume $K(x_1)$ is nondecreasing in x_1, $K \equiv K_1$ for $x_1 \le b$ and $K \equiv K_2 > \max(0, K_1)$ for $x_1 \ge a$, we [14] proved that equation (1.3) possesses no positive solutions in $\mathbb{R}^n$. The Liouville theorem for this class of K was first observed by Y.Y. Li [10], where he proved that there are no positive solutions *with the decay rate $O(|x|^{2-n})$ at infinity.* In fact, he asked whether equation (1.3) possesses no positive solutions when a nonconstant $K(x_1, \dots, x_n)$ is bounded by two positive constants and $K(x_1, \dots, x_n)$ is nondecreasing in x_1. In general, this problem still remains open. Our first result is:

THEOREM 1.1. *Let $K(x_1, x_2, \dots, x_n) \equiv x_1^m$ for some positive integer m. Then equation* (1.3) *possesses no positive solutions in $\mathbb{R}^n$.*

Obviously, $K(x_1)$ is increasing in x_1 only when m is an odd positive integer. For the case of even positive integer, Theorem 1.1 can be extended to:

THEOREM 1.2. *Suppose that $K \geq 0$ in $\mathbb{R}^n$ and satisfies $K(t\xi) = t^l K(\xi)$ for $t > 0$ and $|\xi| = 1$, where $l > 0$ is a constant. Assume there exists an open set $\omega_0 \subset S^{n-1}$ such that $K(\xi) \geq c_0 > 0$ for some constant c_0 and $\xi \in \omega_0$. Then equation* (1.3) *possesses no positive solutions.*

In fact, for a homogeneous function K, we have more general result than Theorem 1.1. Note that in the following theorem, K could allow to change signs.

THEOREM 1.3. *Suppose that K is a C^1 homogeneous function of degree $l > 0$ and satisfies*

$$c_1 |x|^{l-1} \leq |\nabla K(x)| \leq c_2 |x|^{l-1} \tag{1.4}$$

for positive constants c_1 and c_2. Then equation (1.3) possesses no positive solutions in $\mathbb{R}^n$.

As mentioned before, the Liouville theorem is closely related to the existence of apriori bound of solutions. An immediate consequence of Theorem 1.1 is the following theorem. (Throughout the paper, B_r always denotes the open ball with center 0 and the radius r).

COROLLARY 1.4. *Suppose $K \in C^1(\bar{B}_1)$ satisfies $|K(x)| \leq c_2$ and*

$$0 < c_1 \leq |\nabla K(x)| \leq c_2$$

for $x \in \bar{B}_1$. Then there exists a constant c which depends on c_1, c_2 and the dimension n such that

$$u(x) \leq c \text{ for } |x| \leq 1/2 \tag{1.5}$$

holds for any solution u of

$$\Delta u + K(x) u^{\frac{n+2}{n-2}} = 0 \text{ for } |x| < 1\,. \tag{1.6}$$

DEFINITION. A function $K \in C^1(\bar{B}_1)$ with $K(0) = 0$ is said to satisfy the *nondegenerate condition at* 0 if in a neighborhood of 0, K can be written by

$$[\mathrm{NG}]\begin{cases} K(x)=Q(x)+R(x), \text{ where } Q(x) \text{ is a homogenous function of degree} \\ l>0 \text{ such that} \\ c_1|x|^{l-1} \le |\nabla Q(x)| \le c_2|x|^{l-1} \\ \text{holds for } x \in \mathbb{R}^n \text{ and both } |R(x)||x|^{-l} \text{ and } |\nabla R(x)||x|^{1-l} \text{ tend to} \\ \text{zero as } |x| \to 0. \end{cases}$$

COROLLARY 1.5. *Suppose* $K \in C^1(\bar{B}_1)$ *with* $K(0) = 0$ *satisfies* [NG] *at 0. Let* u_i *be a sequence of solutions of* (1.6). *Assume that* u_i *is uniformly bounded in any compact set of* $\bar{B}_1\backslash\{0\}$. *Then* u_i *is uniformly bounded in* $\bar{B}_1$.

DEFINITION. A function $K \in C^1(\bar{B}_1)$ is said to satisfy *condition* [M] if
(i) $K \ge 0$ in $\bar{B}_1$ and $\nabla K(x) \ne 0$ whenever $K(x) \ne 0$.
(ii) The zero set $\Lambda = \{x \in B_1 \mid K(x) = 0\}$ is a k-dimensional submanifold with $0 \le k < n$. Let N be a tube neighborhood of Λ and let π denote the orthorgonal projection from N onto Λ such that $|x - \pi(x)| = d(x)$, where $d(x) = d(x, \Lambda)$ is the distance of x to Λ. For $x \in N$, K satisfies

$$c_1\ d(x)^{l-1} \le |\nabla K(x)| \le c_2\ d(x)^{l-1}, \text{ and}$$

$$\lim_{x\to x_0} d(x)^{-l}K(x) = Q_{x_0}(\xi)\,,$$

where $x_0 = \pi(x)$, $\xi = d(x)^{-1}(x - \pi(x)) \in S^{n-1}$, $Q_{x_0}(\xi)$ is a positive homogeneous function of degree l in $(T_{x_0}\Lambda)^\perp$, the orthogonal complement of the tangent subspace of Λ at x_0.

It is easy to see that notions of [NG] and [M] can be defined in a locally confomally flat manifold, and their definitions are independent of the choice of the local flat metrics.

COROLLARY 1.6. *Suppose* $K \in C^1(\bar{B}_1)$ *with* $K(0) = 0$ *and satisfies the assumption* [M]. *Then there exists a constant c such that*

$$u(x) \le c \text{ for } |x| \le 1/2$$

holds for any solution of (1.6).

Together with Corollary 1.4 through Corollary 1.6, we have the following apriori estimates for equation (1.1). To state our result, we let $\Omega^- = \{p_0 \in M \mid R(p) \le 0$ in a neighborhood of $p_0\}$ and $\Omega^0 = \{p_0 \in M \mid R(p_0) = 0$ and there exists a neighborhood U of p_0 such that R is nonnegative in U, but does not vanish identically in $U\}$. Obviously, Ω^- is an open subset in M, and Ω^- is disjoint from Ω^0.

THEOREM 1.7. *Let (M, g_0) be a locally conformally flat manifold and R be a C^1 function on M. Assume that* (i) *If $\Omega^0 \neq \emptyset$, then R has only a finite set of critical points $\{p_1, \dots, p_m\}$ on $\partial\Omega^-$ and satisfies the nondegenerate condition at each p_j, and* (ii) *If $\Omega^0 \neq \emptyset$, then Ω^0 is a union of a finite number of submanifold and R satisfies [M] on each component of Ω^0. Then there exists $\varepsilon_0 > 0$ and $c > 0$ such that if u is a positive solution of* (1.1), *then*

$$u(p) \leq c$$

for any $p \in M$ and $R(p) \leq \varepsilon_0$.

We note that if a C^1 function R has no critical points with vanishing critical values, then R satisfies the assumption of Theorem 1.7. In this case, Theorem 1.7 was previously proved in [5] when (M, g_0) is the standard n-dimensional sphere. However, Theorem 1.7 is much stronger than the one in [5] even when (M, g_0) is the n-dimensional sphere. For example, if at each critical point p of R with $R(p) = 0$, the Hessian of R at p is non-singular, then R satisfies the assumption of Theorem 1.7. Thus, we extend the previous result of [5] to allow R to have critical points with zero critical value, of course, the nondegenerate condition is still needed.

As mentioned before, the apriori bound for the part where R is positive was obtained in a number of recent works, e.g., see [3], [6,7], [10,11] and references therein. Together with Theorem 1.7, one could obtain apriori bound for solutions for some class of R where R is not assumed to be positive. For example, we have the following result on $\mathbb{S}^3$.

THEOREM 1.8. *Let $(\mathbb{S}^3, g_0)$ be the standard 3-sphere and R be a given Morse function on $\mathbb{S}^3$. Assume R satisfies $\Delta R(y) \neq 0$ whenever $R(y) > 0$ and $\nabla R(y) = 0$. Then there exists a constant $c > 0$ such that for any solution v of* (1.2), *one has*

$$c^{-1} \leq v(y) \leq c \text{ for } y \in \mathbb{S}^3 .$$

Furthermore, the Leray-Schauder degree d for equation (1.2) is given by

$$d = -\left(1 + \sum_{p \in \Lambda^+} (-1)^{i(p)}\right) \tag{1.7}$$

where $\Lambda^+ = \{p \in \mathbb{S}^3 \mid p \text{ is a critical point of } R \text{ with } R(p) > 0 \text{ and } \Delta R(p) < 0\}$, and $i(p)$ is the Morse index of R at $p \in \Lambda^+$.

We note that the Leray-Schauder degree d is defined as the standard topological degree of the nonlinear map $v + \frac{1}{8}(\Delta_0 - \frac{3}{4})^{-1} R v^5$ from $C^{2,\alpha}(\mathbb{S}^3)$ to itself for $0 < \alpha < 1$. For reference, please see [15].

For a locally conformally flat manifold which is not conformally equivalent to $\mathbb{S}^n$, we can apply the positive mass theorem and obtain:

THEOREM 1.9. *Let (M, g_0) be a locally conformally flat n-manifold and R be a C^{n-2} function which for $n \geq 4$, in addition assume that for any $\varepsilon > 0$, there exists a neighborhood U of the critical set $\{p \in M \mid \nabla R(p) = 0$ and $R(p) > 0\}$ such that*

$$|\nabla^s R(q)| \leq \varepsilon |\nabla R(q)|^{\frac{l-s}{l-1}} \text{ for } q \in U, \tag{1.8}$$

where $l = n - 2$. Suppose R also satisfies the assumption of Theorem 1.7 on the part $\partial\Omega^-$ and Ω^0 if they are not empty. Then there exists a positive constant $c > 0$ such that

$$v(p) \leq c$$

for $p \in M$ and for any solution v of (1.1). Furthermore, if the first eigenvalue of the conformal Laplacian operator is positive and $\max_M R > 0$. Then the Leray-Schauder degree is -1.

There are two alternative methods to compute the Leray-Schauder degree when (M, g_0) is the standard n-sphere. One is to deform R to a function which is close to a positive constant. And then we have to prove an uniform bound for solutions during the deformation. This can be done for the case when R is positive in $\mathbb{S}^3$. Please see [3]. But, when R changes signs, the uniform bound fails to exist. Another method is to approach equation (1.2) by subcritical exponents as done in [20]. It is not difficult to see that Theorem 1.7 still holds while (1.2) is approached from the subcritical exponents. Thus, a blow-up point should occur at most the critical points of R with a positive critical value and $\Delta R < 0$. Therefore, the work of [20] can still be applied and the degree-counting formulas (1.7) can be obtained. This is the proof of Theorem 1.8. Theorem 1.9 can be proved similarly. In fact, an uniform bound will be derived when equation (1.1) is approached by subcritical exponents. We give a proof in Section 4.

Finally, we would like to remark that Corollary 1.6 is not optimal when the dimension of Λ is equal to 0. In this case, the assumption of Corollary 1.6 simply means that K has a nondegenerate minimum at 0, namely, $Q(x) > 0$ for $x \neq 0$. Thus, it can not be applied to the degenerate case, say, $K(x) = \sum_{j=1}^{n-1} x_j^2 + x_n^4$ for x near 0. In this case, we have the following.

THEOREM 1.10. *Let K be a nonnegative C^1 function in $\bar{B}_1$ such that $K(0) = 0$ and $\nabla K(x) \neq 0$ for $x \neq 0$. In addition, K satisfies*

$$x \cdot \nabla K(x) \geq c_0 K(x) \tag{1.9}$$

for x near 0 and for some positive constant $c_0 > 0$. Suppose u_i is a sequence of solutions of (1.6) for $|x| < 1$. Then $u_i(x)$ is uniformly bounded for $|x| \leq 1/2$.

This paper is organized as follows. In Section 2, Theorem 1.1 is proved for the case when m is an odd positive integer. Corollary 1.4 is proved as a consequence of Theorem 1.1. In Section 3, we will give the proofs of Theorem 1.2, Theorem 1.3 and their corollaries. Finally, Theorem 1.9 and Theorem 1.10 are proved in Section 4. Also in Section 4, we will give an alternative proof of the result of Chen-Li, based on the Pohozaev identity.

2. – The method of moving planes

In this section, we will prove Theorem 1.1 when m is an odd positive integer. For the case when m is an even positive integer, Theorem 1.1 is a special case of Theorem 1.2, whose proof will be given in the next section. Our proof for Theorem 1.1 with odd m will use the well-known method of moving planes. The method of moving planes was invented by A.D. Alexandrov and has been developed further to study the problem of radial symmetry for positive solutions of elliptic equations by Serrin [17], Gidas-Ni-Nirenberg [9], Caffarelli-Gidas-Spruck [2], Chen-Li [4] and others. It was also used to study the behavior of a blow-up sequence of solutions of (1.3) near a blow-up point by Chen-Lin [6,7]. Another ingredient in our proof is a blow-up argument, which originally was due to R. Schoen [16]. The argument is important, because it enables us to reduce the general case of Theorem 1.1 to the situation when solutions is bounded in $\mathbb{R}^n$. First we begin with a lemma.

LEMMA 2.1. *Assume that both $K_i(x)$ and $|\nabla K_i(x)|$ are bounded between by two positive constants for $|x| \leq 1$. Let u_i be a sequence of positive solutions of*

$$\Delta u_i + K_i(x) u_i^{\frac{n+2}{n-2}} = 0 \; for \; |x| \leq 1$$

Then there exists a constant C independent of i such taht

$$u_i(x) \leq C \tag{2.1}$$

holds for $|x| \leq 1/2$.

Lemma 2.1 is a special case of Corollary 1.4 in [6]. For a proof, we refer the reader to [6].

LEMMA 2.2. *Suppose $K_i(x) \in C(\bar{B}_1)$ uniformly converges to $K(x)$ in $\bar{B}_1$ and $K(x) \leq -c_0$ for $x \in \bar{B}_1$ and for some positive constant c_0. Let u_i be a sequence of solution of*

$$\Delta u_i + K_i(x) u_i^{\frac{n+2}{n-2}} = 0 \; for \; x \in B_1 \, .$$

The $u_i(x)$ is unifomly bounded for $|x| \leq 1/2$.

It is easy to see that Lemma 2.2 is an immediate consequence of the Liouville theorem, which states that equation (1.3) *possesses no positive solutions when* $K \equiv$ a negative constant.

PROOF OF THEOREM 1.1. Suppose that u is a positive solution of (1.3) with $K(x) \equiv x_1^m$. *Throughout the section, $K(x)$ always denotes the particular function x_1^m.* We first claim that.

LEMMA 2.3. *Assume that there exists a positive solution of* (1.3). *Then* (1.3) *possesses a bounded positive solution.*

PROOF. Assume $\sup_{\mathbb{R}^n} u(x) = +\infty$. Otherwise, Lemma 2.3 holds automatically. We divide the proof into two steps.

STEP 1. We want to prove

$$u(x) \leq C \ |x_1|^{-\beta}$$

for some constant C where $\beta = \frac{(n-2)(2+m)}{4} > 0$.

We first consider the case $x_1 > 0$. Let

$$v(y) = x_1^{\beta} u(x + x_1 y) \tag{2.2}$$

where x_1 denotes the x_1-coordinate of x. Then v satisfies

$$\Delta v + K_1(y) v^{\frac{n+2}{n-2}} = 0\,, \tag{2.3}$$

where $K_1(y) = (1 + y_1)^m$. For $|y| \leq 1/2$, it is easy to see that $K_1(y)$ satisfies the hypothesis of Lemma 2.1. Thus by (2.1),

$$x_1^{\beta} u(x) = v(0) \leq C \tag{2.4}$$

for some constant $C > 0$.

For the case $x_1 < 0$, we use the similar scaling $\tilde{v}(y) = |x_1|^{\beta} u(x + x_1 y)$ for $|y| \leq 1/2$. Then by applying Lemma 2.2, (2.4) holds also for the case $x_1 < 0$.

STEP 2. In order to construct a bounded solution, we will apply a blow-up argument which originally was due to R. Schoen. Let $x^j = (x_1^j, \tilde{x}^j)$ be a sequeuse of points such that $u(x^j) \to +\infty$ as $j \to +\infty$. By Step 1, we have $x_1^j \to 0$ as $j \to +\infty$. Set $u_j(x) = u(\tilde{x}^j + x)$ and

$$S_j(x) = u_j^{\frac{1}{\beta}}(x)(1 - |x|) \text{ for } |x| \leq 1\,.$$

Let τ_j satisfy

$$S_j(\tau_j) = \max_{|x| \leq 1} S_j(x)\,. \tag{2.5}$$

Then $S_j(\tau_j) \geq u^{\frac{1}{\beta}}(x^j)(1 - |x_1^j|) \to +\infty$ as $j \to +\infty$. Let

$$v_j(y) = M_j^{-1} u_j(\tau_j + M_j^{-\frac{1}{\beta}} y)\,,$$

where $M_j = u_j(\tau_j)$. By Step 1,

$$|\tau_{j,1}| M_j^{\frac{1}{\beta}} \leq C_0$$

Let $\xi_j = \tau_{j,1} M_j^{\frac{1}{\beta}}$. We may assume

$$\xi_0 = \lim_{j\to+\infty} \tau_{j,1} M_j^{\frac{1}{\beta}} \tag{2.6}$$

Thus, v_j satisfies

$$\Delta v_j + (\xi_j + y_1)^m v_j^{\frac{n+2}{n-2}} = 0 . \tag{2.7}$$

For any fixed $R > 0$ and $|y| \le R$, we let $x = \tau_j + M_j^{-\frac{1}{\beta}} y$. Then

$$v_j^{\frac{1}{\beta}} = \left(\frac{u_j(x)}{u_j(\tau_j)}\right)^{\frac{1}{\beta}} = \frac{S_j(x)}{S_j(\tau_j)} \left(\frac{1-|\tau_j|}{1-|x|}\right) .$$

Since

$$\begin{aligned} 1-|x| &\ge 1-|\tau_j| - M_j^{-\frac{1}{\beta}}|y| \\ &= M_j^{-\frac{1}{\beta}}(S_j(\tau_j)-|y|) \\ &\ge \frac{1}{2} M_j^{-\frac{1}{\beta}} S_j(\tau_j) \\ &= \frac{1}{2}(1-|\tau_j|) \end{aligned}$$

for large j, we have .

$$v_j(y) \le \left(\frac{1-|\tau_j|}{1-|x|}\right)^{\beta} \le 2^{\beta} .$$

Since $v_j(y)$ is uniformly bounded in any compact set of $\mathbb{R}^n$, by elliptic estimates, there exists a subsequence (still denoted by v_j) which converges to v in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. By (2.6), v satisfies

$$\Delta v + (\xi_0 + y_1)^m v^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n .$$

Since $v(0)=1$, $v(y)>0$ in $\mathbb{R}^n$. Therefore the proof of Lemma 2.3 is finished. □

Now we are back to the proof of Theorem 1.1. By Lemma 2.3, we assume that u is bounded. Following conventional notations, we let for any λ, $T_\lambda = \{x \mid x_1 = \lambda\}$, $\Sigma_\lambda = \{x \mid x_1 < \lambda\}$ and $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$ be the reflection point of x with respect to T_λ. Set

$$w_\lambda(x) = u(x^\lambda) - u(x)$$

for $x \in \Sigma_\lambda$. We want to claim

$$w_\lambda(x) > 0 \text{ for } x \in \Sigma_\lambda \text{ and } \lambda \in \mathbb{R} . \tag{2.8}$$

We divide two steps to prove (2.8).

STEP 1. (2.8) holds for $y \in \Sigma_\lambda$ and $\lambda \le 0$.

To prove Step 1, we want to show that

$$\Delta w_\lambda(x) < 0 \text{ whenever } w_\lambda(x) \le 0 \text{ for } x \in \Sigma_\lambda \text{ and } l \le 0. \tag{2.9}$$

Suppose $u(x) \ge u(x^\lambda)$ for some $x \in \Sigma_\lambda$. If $x_1 \le 2\lambda$, then $K(x^\lambda) \ge 0$. Hence

$$\Delta w_\lambda(x) = K(x)u^{\frac{n+2}{n-2}} - K(x^\lambda)(u^\lambda)^{\frac{n+2}{n-2}} < 0.$$

If $x_1 > 2\lambda$, then $|x_1^\lambda| \le |x_1|$ and

$$\Delta w_\lambda(x) = -|K(x_1)|u^{\frac{n+2}{n-2}} + |K(x_1^\lambda)|(u^\lambda)^{\frac{n+2}{n-2}} < 0$$

provided that $u(x) \ge u^\lambda(x)$. Hence, (2.9) is proved.

Assume

$$\inf_{x\in\Sigma_\lambda} w_\lambda(x) < 0 \tag{2.10}$$

for some $\lambda \le 0$. Let

$$g_\lambda(x) = \sum_{j=2}^{n} \log[(\lambda - x_1 + 2)^2 + x_j^2]$$

for $y \in \sum_\lambda$. Clearly, $g_\lambda(x) > 0$ and is a harmonic function in Σ_λ. Set

$$\bar{w}_\lambda(x) = \frac{w_\lambda(x)}{g_\lambda(x)}. \tag{2.11}$$

Since $g_\lambda(y) \to +\infty$ as $|y| \to +\infty$, we have by (2.10)

$$\bar{w}_\lambda(x_0) = \inf_{x\in\Sigma_\lambda} \bar{w}_\lambda(x) < 0.$$

Applying the maximum principle at x_0 and by (2.9), we have

$$\begin{aligned} 0 &\le \Delta\bar{w}_\lambda(x_0) + 2\,\nabla\log(g_\lambda)(x_0)\cdot\nabla\bar{w}_\lambda(x_0) \\ &= g_\lambda^{-1}(x_0)\Delta w_\lambda(x_0) \\ &< 0, \end{aligned}$$

which yields a contradiction obviously. Thus, Step 1 is proved.

Let

$$\lambda_0 = \sup\{\lambda \mid w_\mu(x) > 0 \text{ in } Sigma_\mu \text{ for } \mu \le \lambda\}.$$

By Step 1, $\lambda_0 \ge 0$. Clearly, (2.8) follows if $\lambda_0 = +\infty$.

STEP 2. $\lambda_0 = +\infty$.

Suppose $\lambda_0 < +\infty$. By continuity, we have

$$w_{\lambda_0}(y) > 0 \text{ in } \Sigma_{\lambda_0} \text{ and } \frac{\partial w_{\lambda_0}(y)}{\partial y_1} > 0 \text{ for } y \in T_{\lambda_0} \tag{2.12}$$

by the maximum principle and the Hopf boundary point Lemma. By the definition of λ_0, there exists a sequence of $\lambda_j > \lambda_0$ such that $\lambda_0 = \lim_{j\to+\infty} \lambda_j$ and

$$\inf_{\Sigma_{\lambda_j}} w_j(x) < 0\,.$$

where $w_j \equiv w_{\lambda_j}$. As in Step 1, we want to show for large j,

$$\Delta w_j(x) < 0 \text{ whenever } w_j(x) \le 0 \text{ for } x \in \Sigma_{\lambda_j}\,. \tag{2.13}$$

We prove (2.13) by contradiction. Assume that there exists a sequence of $x^j \in \Sigma_{\lambda_j}$ such that $w_j(x^j) \le 0$ and $\Delta w_j(x^j) \ge 0$. Let $x^j = (x_1^j, \tilde{x}^j) \in \mathbb{R}\times\mathbb{R}^{n-1}$. Since $K(x^{j\lambda}) \ge 0$ and

$$0 \le \Delta w_j(x^j) = K(x^j)u^{\frac{n+2}{n-2}}(x^j) - K(x^{j\lambda})(u^\lambda)^{\frac{n+2}{n-2}}(x^j)\,,$$

we have $K(x^j) \ge 0$. Thus, $0 \le x_1^j \le \lambda_j$.

If $\tilde{x}^j$ is bounded, we may assume $\lim_{j\to+\infty} x^j = x^0$ exists. If $x^0 \in \Sigma_{\lambda_0}$, then $w_{\lambda_0}(x^0) \le 0$ which yields a contradiction to (2.12). If $x^0 \in \partial\Sigma_{\lambda_0}$, then $\frac{\partial w_{\lambda_0}}{\partial x_1}(x^0) = 0$ which contradicts to (2.12) again. Thus, we may assume $|\tilde{x}^j| \to +\infty$ as $j \to +\infty$.

Let

$$\bar{u}_j(x) = u(x + \tilde{x}^j)\,.$$

By elliptic estimates, $\bar{u}_j$ is bounded in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\alpha > 0$. Thus, a subsequence of $\bar{u}_j$ (still denoted by $\bar{u}_j$) converges to a nonnegative function $\bar{u}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$, where $\bar{u}$ satisfies

$$\Delta\bar{u} + x_1^m\ \bar{u}^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n\,.$$

We claim that $\bar{u} \equiv 0$ in $\mathbb{R}^n$. To see it, two cases are discussed seperately. If $\lim_{j\to+\infty} x_1^j = x_1^0 < \lambda_0$, then by continuity, (2.12) implies

$$0 \le \bar{w}_{\lambda_0}(x) \equiv \bar{u}^\lambda(x) - \bar{u}(x) \text{ in } \Sigma_{\lambda_0}\,.$$

On the other hand, $\bar{w}_{\lambda_0}(x^0) = \lim_{j\to+\infty} w_j(x^j) \le 0$. By the maximum principle, $\bar{w}_{\lambda_0}(x) \equiv 0$ in Σ_{λ_0}, which implies $\bar{u} \equiv 0$ in $\mathbb{R}^n$. If $\lim_{j\to+\infty} x_i^j = \lambda_0$, then there exists $\eta^j \in (x_1^j, \lambda_j)$ such that $\frac{\partial \bar{w}_j}{\partial x_1}(\eta^j, 0) \geq 0$, which implies $\frac{\partial \bar{w}_{\lambda_0}}{\partial x_1}(\lambda_0, 0) = 0$. Thus $\bar{u}(x) \equiv 0$ by the Hopf boundary point Lemma.

Let $v_j(x) = \frac{\bar{u}_j(x)}{\bar{u}_j(0)}$. By the Harnack inequality, $v_j(x)$ is bounded in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. Thus, we may assume v_j converges to v in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. Then, v is harmonic and $v \equiv 1$ in $\mathbb{R}^n$. In particular, we have for any $\varepsilon > 0$, there exists $j_0 = j_0(\varepsilon)$ such that

$$\sup_{B(x_1^j, 1+\lambda_0)} |\nabla \bar{u}_j(x)| \leq \varepsilon \inf_{B(x_1^j, 1+\lambda_0)} \bar{u}_j(x), \tag{2.14}$$

where $B(x_1^j, 1+\lambda_0)$ is the ball with center $(x_1^j, 0, \dots, 0)$ and the radius $1+\lambda_0$. Thus, for $x = (t, \tilde{x}^j)$ where $t \in [x_1^j, \lambda_j]$,

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\left(x_1^m u_j(x)^{\frac{n+2}{n-2}} - (x_1^\lambda)^m u_j^{\frac{n+2}{n-2}}(x^\lambda)\right) \\
=&x_1^{m-1} u_j^{\frac{4}{n-2}}\left(m u_j(x) + \frac{n+2}{n-2} x_1 \frac{\partial u_j}{\partial x_1}\right) \\
&+ (x_1^\lambda)^{m-1} (u_j^\lambda)^{\frac{4}{n-2}}\left(m u_j^\lambda(x) + \frac{n+2}{n-2} x_1^\lambda \frac{\partial u_j^\lambda}{\partial x_1}\right) \\
>&0
\end{aligned}$$

by (2.14). Hence

$$\Delta w_j(x_j) = K(x_1^j) u_j(x_j)^{\frac{n+2}{n-2}} - K(x_1^{j\lambda_j}) u^{\lambda_j}(x^j)^{\frac{n+2}{n-2}} < 0,$$

which yields a contradiction. Then (2.13) is proved.

Let $g_\lambda(x) = \sum_{i=2}^n \log[(2+\lambda_j - x_1)^2 + x_i^2)$. Applying the maximum principle to $w^*(x) = \frac{w_{\lambda_j}(x)}{g_j(x)}$ as in Step 1, (2.13) yields a contradiction. Thus, Step 2 is proved.

Let $\Lambda_R = \{x = (x_1, \tilde{x}) \mid R \leq x_1 \leq 2R \text{ and } \tilde{x}| \leq 1\}$, and $m_0 = \inf_{\substack{x_1=0 \\ |x| \leq 1}} u(x) > 0$. Rewrite equation (1.3) into

$$\Delta u(x) + C(x) u(x) = 0 \text{ for } x \in \Lambda_R,$$

where

$$C(x) = x_1^m u^{\frac{4}{n-2}}(x) \geq m_0 R^m \text{ for } x \in \Lambda_R.$$

Since the first eigenvalue of Δ with zero boundary value for Λ_R is bounded, by using the comparison theorem of the first eigenvalue, we have obtained a contradiction provided that R is large enough. Therefore, the proof of Theorem 1.1 is completely finished. □

PROOF OF COROLLARY 1.4. Assume that there exists a sequence of $K_i \in C^1(\bar{B}_1)$ satisfying the hypothesis of Corollary 1.4, and a sequence of positive solutions u_i of

$$\Delta u_i + K_i(x) u_i^{\frac{n+2}{n-2}} = 0 \text{ in } B_1$$

such that

$$\max_{|x| \le \frac{1}{2}} u_i(x) \to +\infty\,. \tag{2.15}$$

By Lemma 2.1 and Lemma 2.2, the blow-up set of $\{u_i\}$ must be contained in the zero set of K, where K is a limit function of a subsequence of K_i. Thus, $\Gamma_i = \{x \in B_1 \mid K_i(x) = 0\} \neq \emptyset$ and is $(n-1)$-dimensional manifold. Let $d_i(x) = d(x, \Gamma_i)$ denote the distance of x to Γ_i. For $|x| \le 3/4$, let

$$v_i(y) = d_i(x)^{\frac{3(n-2)}{4}} u_i(x + d(x)y)$$

for $|y| \le 1/2$. Then v_i satisfies

$$\Delta v_i + \bar{K}_i(y) v_i^{\frac{n+2}{n-2}} = 0$$

where $\bar{K}_i(y) = d_i(x)^{-1} K_i(x + d_i(x)y)$. By (1.4), both $|\bar{K}_i(y)|$ and $|\bigtriangledown \bar{K}_i(y)|$ are bounded between by two positive constants which depend on c_1 and c_2. Thus, by Lemma 2.1 and Lemma 2.2,

$$d_i(x)^{\frac{3(n-2)}{4}} u_i(x) \le c_3 \tag{2.16}$$

for some constant c_3 and for $|x| \le 3/4$.

By the assumption (2.15), we have $\lim_{i \to +\infty} u_i(x_i) \to +\infty$ for a sequence of $|x_i| \le 1/2$. As before, set

$$S_i(x) = u_i^{\frac{1}{\beta}}(x)(\delta_0 - |x - x_i|)$$

for $x \in B(x_i, \delta_0)$ where $0 < \delta_0 \le 1/8$ and $\beta = 3(n-2)/4$. Let τ_i satisfies

$$S_i(\tau_i) = \max_{B(x_i, \delta_0)} S_i(x) \ge \delta_0 u_i^{\frac{1}{\beta}}(x_i) \to +\infty \tag{2.17}$$

as $i \to +\infty$. Let

$$\tilde{v}_i(y) = M_i^{-1} u_i(\tau_i + M_i^{-\frac{1}{\beta}} y)\,,$$

where $M_i = u_i(\tau_i)$. Obviously, $\tilde{v}_i$ satisfies

$$\Delta \tilde{v}_i + \tilde{K}_i(y) \tilde{v}_i^{\frac{n+2}{n-2}} = 0\,,$$

where $\tilde{K}_i(y) = M_i^{\frac{4}{n-2}} M_i^{-\frac{2}{\beta}} K_i(\tau_i + M_i^{-\frac{1}{\beta}} y)$. Let $\xi_i \in \Gamma_i$ such that $d_i(\tau_i) = |\tau_i - \xi_i|$. By (2.16), we may assume $\lim_{i\to+\infty} M_i^{\frac{1}{\beta}}(\tau_i - \xi_i) = \eta_0 \in \mathbb{R}^n$. Since $K_i(\xi_i) = 0$,

$$\begin{aligned}\tilde{K}_i(y) &= (1+o(1)) \nabla K_i(\xi_i) \cdot (M_i^{\frac{1}{\beta}}(\tau_i - \xi_i) + y) \\ &\to y_1 + \eta_{0,1}\end{aligned}$$

as $i \to +\infty$, where $\nabla K_i(\xi_i)$ is assumed to converge to $(1, 0, \dots, 0)$. By the argument in the proof of Lemma 2.3, $\tilde{v}_i$ is bounded in any compact set in $\mathbb{R}^n$ by (2.17). Let $\tilde{v}$ be the limit function of $\tilde{v}_i$. Then $\tilde{v}(0) = 1$ and $\tilde{v}$ satisfies

$$\Delta \tilde{v} + (y_1 + \eta_{0,1}) \tilde{v}^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n,$$

which yields a contradiction to Theorem 1.1. □

3. – Proofs of Theorem 1.2 and Theorem 1.3

In this section, we begin with a proof of Theorem 1.3 which is a nice applicaiton of Theorem 1.1 and Corollary 1.4. To begin with, we need a lemma first.

LEMMA 3.1. *Suppose K satisfies the hypothesis of Theorem 1.3 and u is a positive solution of (1.3). Then there exists a constant C_0 such that*

$$u(x) \le C_0 |x|^{-\beta} \tag{3.1}$$

where $\beta = \frac{(n-2)(2+l)}{4}$.

PROOF. Fix $x \in \mathbb{R}^n$ and let $r = |x|$. Set

$$v(y) = r^\beta u(ry) \tag{3.2}$$

for $1/2 \le |y| \le 2$. Then v satisfies

$$\Delta v(y) + K(y) v^{\frac{n+2}{n-2}} = 0 \text{ for } 1/2 \le |y| \le 2.$$

Since $K(y)$ has no critical point in the annulus $\{y \mid 1/2 \le |y| \le 2\}$, we have by Lemma 2.1 and Corollary 1.4,

$$v(y) \le C_0 \text{ for } |y| = 1$$

for some positive constant C_0. Thus, (3.1) follows immediately. □

PROOF OF THEOREM 1.3. Rewrite equation (1.3) into

$$\Delta u(x) + C(x)u(x) = 0 \text{ in } \mathbb{R}^n ,$$

where $C(x) = K(x)u^{\frac{4}{n-2}}$. By (3.1), we have

$$|C(x)| \leq c_1 |x|^{-2} \tag{3.3}$$

for $x \in \mathbb{R}^n$. Thus, by the Harnack inequality and the gradient estimate, we have

$$\max_{|x|=r} u \leq c_2 \min_{|x|=r} u , \tag{3.4}$$

and

$$| \nabla u(x)| \leq c_2 |x|^{-1} u(x) \tag{3.5}$$

for some positive constant c_2.

Set $w(t, \theta) = r^{\beta} u(x)$ where $r = e^t$ and $\theta = r^{-1}x$. By a straightforward computation, w satisfies

$$w_{tt} - \frac{l}{2} w_t + \tau w + \Delta_0 w + K(\theta) w^{\frac{n+2}{n-2}} = 0 , \tag{3.6}$$

where $\tau = l^2 - 4(n-2)^2/16$, Δ_0 is the Beltrami-Laplace operator of the standard $\mathbb{S}^{n-1}$, w_t and w_{tt} denote the first derivative and the second derivative with respect to t respectively. By (3.1) and (3.5), both w and w_t are uniformly bounded in $(-\infty, \infty) \times \mathbb{S}^{n-1}$. Applying estimates of the linear elliptic equations, $|\partial_t^{\alpha} \partial_{\theta}^{\beta} w|$ is uniformly bounded on $(-\infty, \infty) \times \mathbb{S}^{n-1}$ for $|\alpha| + |\beta| \leq 3$. Thus, for any sequence $T_i \to +\infty$ there exists a subsequence (still denoted by T_i) such that $w(T_i, \theta)$ converges in $C^{2,\alpha}(\mathbb{S}^{n-1})$ as $i \to +\infty$.

Let

$$E(t) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} (w_t^2 - | \nabla_0 w|^2 + \tau w^2) + \frac{n-2}{2n} \int_{\mathbb{S}^{n-1}} K(\theta) w^{\frac{2n}{n-2}} , \tag{3.7}$$

where $\nabla_0 w$ is the gradient of w on $\mathbb{S}^{n-1}$. By (3.6), we have

$$E'(t) = \frac{l}{2} \int_{\mathbb{S}^{n-1}} |w_t|^2 > 0 \tag{3.8}$$

Hence $E(t)$ is increasing in t and $E(t) > E(-\infty) = 0$ for $t \in \mathbb{R}$. Since $E(t)$ is bounded, we have $\|w_t\| \in L^2(\mathbb{R})$, where $\|w_t\|$ denote the L^2 norm of w_t on $\mathbb{S}^{n-1}$. Applying (3.6) once more,

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} w_{tt}^2 d\sigma = &\int_{\mathbb{S}^{n-1}} \nabla_0 w \, \nabla_0 w_{tt} - \tau \int_{\mathbb{S}^{n-1}} w w_{tt} + \frac{l}{4} \int_{\mathbb{S}^{n-1}} (w_t^2)' \\ &- \int_{\mathbb{S}^{n-1}} K(\theta) w^{\frac{n+2}{n-2}} w_{tt} d\sigma .\end{aligned}$$

Integrating along t, we have

$$
\begin{aligned}
&\int_{-T}^{T}\int_{\mathbb{S}^{n-1}} w_{tt}^2 d\sigma dt \\
&= -\int_{-T}^{T}\int_{\mathbb{S}^{n-1}} |\nabla_0 w_t|^2 d\sigma dt - \int_{\mathbb{S}^{n-1}} \Delta_0 w(T) w_t(T) d\sigma \\
&\quad + \int_{\mathbb{S}^{n-1}} \Delta_0 w(-T) w_t(-T) d\sigma + \tau \int_{-T}^{T}\int_{\mathbb{S}^{n-1}} w_t^2 d\sigma dt + \tau \int_{\mathbb{S}^{n-1}} w(T) w_t(T) d\sigma \\
&\quad - \tau \int_{\mathbb{S}^{n-1}} w(-T) w_t(-T) d\sigma + \frac{l}{4}\int_{\mathbb{S}^{n-1}} (w_t^2(T) - w_t^2(-T)) d\sigma \\
&\quad + \frac{n+2}{n-2}\int_{-T}^{T}\int_{\mathbb{S}^{n-1}} K w^{\frac{4}{n-2}} w_t^2 d\sigma dt + \int_{\mathbb{S}^{n-1}} K(\theta) w^{\frac{n+2}{n-2}}(-T) w_t(-T) d\sigma \\
&\quad - \int_{\mathbb{S}^{n-1}} K(\theta) w^{\frac{n+2}{n-2}}(T) w_t(T) d\sigma .
\end{aligned}
$$

Since $\|w_t\|_{L^2} \in L^2(\mathbb{R})$ and w is uniformly bounded, there exists a sequence of $T_i \to +\infty$ such that $\|w_t(T_i)\| \to 0$ and $w(T_i)$ converges in $C^{2,\alpha}(\mathbb{S}^{n-1})$ as $i \to +\infty$. Thus, $w_{tt} \in L^2(\mathbb{R} \times \mathbb{S}^{n-1})$. Without loss of generality, we may also assume $\|w_{tt}(T_i)\| \to 0$ as $i \to +\infty$. Let $w_\infty(\theta) = \lim_{i\to+\infty} w(T_i)$. Hence w_∞ satisfies

$$
\Delta_0 w_\infty + \tau w_\infty + K(\theta) w_\infty^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{S}^{n-1} . \tag{3.9}
$$

Therefore by (3.9),

$$
\begin{aligned}
0 &< \lim_{i\to+\infty} E(T_i) \\
&= -\frac{1}{2}\int_{\mathbb{S}^{n-1}} (|\nabla_0 w|^2 - \tau w^2) d\sigma + \frac{n-2}{2n}\int_{\mathbb{S}^{n-1}} K(\theta) w^{\frac{2n}{n-2}} d\sigma \\
&= \left(\frac{n-2}{2n} - \frac{1}{2}\right)\int_{\mathbb{S}^{n-1}} K(\theta) w^{\frac{2n}{n-2}} d\sigma \\
&\le 0 ,
\end{aligned}
$$

which yields a contradiction. And the proof of Theorem 1.3 is finished. □

PROOF OF THEOREM 1.2. Since K is nonnegative, $K(t\xi)$ is nondecreasing along any ray issuing from the origin. Thus, from the proof the of Theorem 1.1 in [14], we conclude that the quantity $u(x)|x|^{\frac{n-2}{2}}$ is also increasing along any ray from the origin. Let $\Omega_R = \{x \in \mathbb{R}^n \mid x = t\xi$ for some $\xi \in \omega_0$ and $R \le t \le 2R\}$. Rewrite equtaion (1.1) into

$$
\Delta u(x) + C(x) u(x) = 0 ,
$$

where $C(x) = K(x)u^{\frac{4}{n-2}}(x)$. Set

$$m = \min_{|x|=1} u(x) > 0\,.$$

Then, the inequality

$$u(x) \geq m|x|^{-\frac{n-2}{2}} \tag{3.10}$$

holds for $|x| \geq 1$. Particularly, for $x \in \Omega_R$ one has

$$C(x) \geq mc_0(2R)^{l-2}\,, \tag{3.11}$$

since the first eigenvalue of the Laplace operator on Ω_R with the zero boundary value is bounded by a constant $\cdot R^{-2}$, (3.11) contradicts to the comparison theorem of the first eigenvalue. □

By applying Theorem 1.3, we can give a proof of Corollary 1.5.

Proof of Corollary 1.5. Suppose that u_i is a sequence of solutions of (1.6) such that $u_i(x_i) = \max_{|x|\leq\frac{1}{2}} u_i(x) \to +\infty$ as $i \to +\infty$. By the assumption, $x_i \to 0$ as $i \to +\infty$. Let

$$v_i(y) = |x_i|^{\beta}u_i(x_i + |x_i|y)\,,$$

where $\beta = \frac{(n-2)(2+l)}{4}$. Then v_i satisfies

$$\Delta v_i + \bar{K}_i(y)v_i^{\frac{n+2}{n-2}} = 0\,,$$

where $\bar{K}_i(y) = |x_i|^{-\frac{4\beta}{n-2}+2}K(x_i + |x_i|y)$. For $|y| \leq \frac{1}{2}$,

$$c_1(1-|y|)^l \leq |\bigtriangledown \bar{K}_i(y)| \leq c_2(1+|y|)^l\,.$$

Thus, applying Corollary 1.4,

$$u_i(0) = |x_i|^{\beta}u_i(x_i) \leq c_3 \tag{3.12}$$

After (3.12) is proved, we set

$$\tilde{v}_i(y) = M_i^{-1}u(x_i + M_i^{-\frac{1}{\beta}}y)\,,$$

where $M_i = u_i(x_i)$. Obviously, $\tilde{v}_i(y) \leq 1$ for $|y| \leq \frac{1}{4}M_i^{\frac{1}{\beta}}$ and $\tilde{v}_i$ satisfies

$$\Delta\tilde{v}_i + \tilde{K}_i(y)\tilde{v}_i^{\frac{n+2}{n-2}} = 0\,,$$

where $\tilde{K}_i(y) = M_i^{\frac{4}{n-2}-\frac{2}{\beta}} K(x_i + M_i^{-\frac{1}{\beta}} y) \to Q(\xi_0 + y)$, where

$$\xi_0 = \lim_{i \to +\infty} M_i^{\frac{1}{\beta}} x_i .$$

By elliptic estimates, a subsequence of $\tilde{v}_i$ converges to v in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, where $v(0) = 1$ and satisfies

$$\Delta v + Q(\xi_0 + y) v^{\frac{n+2}{n-2}}(y) = 0 ,$$

which yields a contradiction to Theorem 1.3. □

It is not difficult to see that by applying Theorem 1.2, Corollary 1.6 can be proved by the same arguments as the proofs of Corollary 1.4 and Corollary 1.5. Thus, we omit its proof.

4. – Apriori estimates

In this section, we first give a proof of Theorem 1.10.

Proof of Theorem 1.10. Let u_j be a sequence of solution of (1.6). By the assumption and Lemma 2.1, the blow up could occur at 0 at most. Thus there exists a constant $c_1 = c_1(r) > 0$ such that

$$u_j(x) \le c_1 \text{ for } r \le |x| \le \frac{1}{2} . \tag{4.1}$$

Applying the Pohozaev identity and (1.9), one has for small r_0

$$\begin{aligned} c_0 \int_{B_{r_0}} K(x) u_j^{\frac{2n}{n-2}}(x) dx &\le \int_{B_{r_0}} (x \cdot \nabla K(x)) u_j^{\frac{2n}{n-2}}(x) dx \\ &= \int_{\partial B_{r_0}} \left(\frac{n-2}{2} u_j \left(\frac{\partial u_j}{\partial r} \right) - \frac{r_0}{2} | \nabla u_j |^2 \right. \\ &\quad \left. + r_0 \left| \frac{\partial u_j}{\partial r} \right|^2 + \frac{n-2}{2n} r_0 K u_j^{\frac{2n}{n-2}} \right) d\sigma \\ &\le c_2 . \end{aligned} \tag{4.2}$$

Hence by (4.1),

$$\int_{B_{1/2}} K(x) u_j^{\frac{2n}{n-2}}(x) dx \le c_3 .$$

Let $\varphi \in C_0^\infty(B_{\frac{1}{2}})$ with $\varphi(x) \equiv 1$ for $|x| \le \frac{1}{4}$. Multiplying $\varphi^2 u_j$ on both sides of equation (1.6), one has

$$\begin{aligned}\int_{B_{1/2}} |\nabla(\varphi u_j)|^2 &\le \int_{B_{1/2}} u_j^2(\varphi|\Delta\varphi| + |\nabla\varphi|^2)dx \\ &\quad + \int_{B_{1/2}} K u_j^{\frac{2n}{n-2}}\varphi^2 dx \\ &\le c_4 .\end{aligned}$$

By the Sobolev embedding theorem, the inequality above implies

$$\int_{B_{1/2}} u_j^{\frac{2n}{n-2}} dx \le c_5 \tag{4.3}$$

for some constant c_5 and for all j. Rewrite equation (1.6):

$$\Delta u_j(x) + c_j(x)u_j = 0\,, \tag{4.4}$$

where $c_j(x) = K_j(x)u_j^{\frac{4}{n-2}}$. Multipying (4.4) by $\varphi^2 u_j^{2\alpha-1}(x)$ for $\alpha > 1$ where $\varphi \in C_0^\infty(B_{r_0})$ for small $r_0 > 0$, we have

$$\begin{aligned}\frac{2\alpha-1}{\alpha^2}\int_{B_{r_0}} |\nabla(\varphi_j u_j^\alpha)|^2 dx &= \int_{B_{r_0}} \varphi^2 c_j(x) u_j^{2\alpha} dx + \text{ boundary terms} \\ &\le \left(\int_{B_{r_0}} \varphi^{\frac{n}{2}}|c_j(x)|^{\frac{n}{2}}dx\right)^{\frac{2}{n}}\left(\int_{B_{r_0}} (\varphi u_j^\alpha)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + c_6 .\end{aligned}$$

Noting that by (4.3)

$$\begin{aligned}\int_{B_{r_0}} |c_j(x)|^{n/2}dx &= \int_{B_{r_0}} K(x)^{n/2} u_j^{\frac{2n}{n-2}}(x)dx \\ &\le \max_{B_{r_0}}(K(x))^{n/2} c_5 \\ &\le \frac{1}{2}\end{aligned}$$

if r_0 is chosen small. Thus, by the Sobolev embedding, we have

$$\int_{B_{r_0}} |u_j^{\frac{2\alpha n}{n-2}}| dx \le c_7(\alpha)$$

for any $\alpha > 1$. Thus, $c_j \in L^p(B_{\frac{1}{2}})$ for any $p > n/2$. Therefore, by the Harnack inequality, there exists a $c_8 > 0$ such that

$$\max_{B_{1/2}} u_j \leq c_8 \inf_{B_{1/2}} u_j \leq c_9 ,$$

which is the conclusion of Theorem 1.10. □

By the same reasoning, we can give an alternative proof of Chen-Li's result, based on Lemma 2.1 and the Pohozaev identity. Suppose u is a solution of

$$\Delta u + K(x) u^{\frac{n+2}{n-2}} = 0 \text{ in } \Omega \subseteq \mathbb{R}^n , \tag{4.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$. Assume K satisfies

$$\nabla K(x) \neq 0 \text{ for } x \in \Gamma , \tag{4.6}$$

where $\Gamma = \{x \mid K(x) = 0\}$, and

$$\Gamma \text{ is compact in } \Omega . \tag{4.7}$$

Then the blow-up for any sequence of solutions u_j of (4.5) *never occurs at points in* Γ.

To prove the claim, we let $T(x)$ be a differential operator of first order such that

$$TK(x) \geq c_0 > 0$$

for $x \in \Omega$ such that $|K(x)| \leq \varepsilon_0$. Multiplying (4.5) by Tu and by the integration by parts, we have

$$\begin{aligned} c_0 \int_N u^{\frac{2n}{n-2}}(x)dx &\leq \int_N (TK) u^{\frac{2n}{n-2}}(x)dx \\ &\leq c_1 \int_N |K(x)| u^{\frac{2n}{n-2}} dx + \text{ boundary terms } , \end{aligned} \tag{4.8}$$

where N is a tube neighborhood of Γ. If N is sufficiently small and the boundary terms are always bounded from above, we have by (4.8),

$$\int_N u^{\frac{2n}{n-2}}(x)dx \leq c_2$$

for some constant c_2 independent of u. Thus, applying the same argument as the final stage in the proof of Theorem 1.10, the Harnack inequality can be applied to u in $\bar{N}$. Thus, the claim follows immediately.

Proof of Theorem 1.9. We shall prove Theorem 1.9 by deriving an uniform bound when equation (1.1) is approached from the subcritical exponents. Let v_j be a sequence of solutions of

$$L v_j + R v_j^p = 0 \quad \text{in} \quad M , \tag{4.9}$$

where $L = \Delta_{g_0} - \frac{n-2}{4(n-1)}k_0$ is the conformal Laplacian and $p = p_j \le \frac{n+2}{n-2}$ with $\lim_{j\to+\infty} p_j = n + 2/n - 2$. We may assume $\max_M R > 0$. Otherwise, the apriori bound is easy to derive. Suppose that $\max_M v_j \to +\infty$ as $j \to +\infty$. Let $p_0 \in M$ be a blow-up point of v_j. Let $g_0 = h^{\frac{4}{n-2}}|dx|^2$ for some positive function h in neighborhood of p_0. (We may assume $p_0 = 0$ in this coordinate of the chart at p_0). Let $u_j = v_j h$. Then u_j satisfies

$$\Delta u_j + Rh^{\tau_j} u_j^{p_j} = 0 \text{ in } B_{r_0} \tag{4.10}$$

for some $r_0 > 0$ and $\tau_j = \frac{n+2}{n-2} - p_j$. Applying Theorem 1.7 and Theorem 1.3 in [6] which states that 0 is a simple blow-up for u_j, the sequence v_j has at most a finite set of blow-up points $\{p_1, \dots, p_N\}$, where each p_j is a critical point of K with positive critical values. Let $M_j = \max_{B_{r_0}} v_j$. The same result quoted above implies that a subsequence of $M_j v_j$ converges to v_∞ in $M\backslash\{p_1, \dots, p_N\}$, and v_∞ is a sum of Green's function of L with singularities at $\{p_1, \dots, p_N\}$, i.e.,

$$v_\infty(p) = \sum_{j=1}^{N} c_j G(p, p_j) \tag{4.11}$$

with $c_j > 0$ and $N \ge 1$.

Let $u_j = v_j h$ as before and let y_j satisfy $u_j(y_j) = \max_{B_{r_0}} u_j(x) = M_j$. Applying the Pohozaev identity and its variant, we have

$$\int_{B_{r_0}} \nabla(Rh^{\tau_j})(y_j + x) u_j^{p_j+1}(y_j + x)dx = B_j\,, \tag{4.12}$$

and

$$\int_{B_{r_0}} x \cdot \nabla(Rh)^{\tau_j}(y_j + x) u_j^{p_j+1}(y_j + x)dx + \tau_j \int Rh^{\tau_j} u_j^{p_j+1} dx = D_j\,. \tag{4.13}$$

where B_j and D_j are both boundary terms. Since both B_j and D_j are involved with quadratic terms of u_j and its first derivatives at the boundary, by a straighforward computation and (4.11), we have

$$B_j = o(1)M_j^{-2} \quad \text{and} \quad D_j = -d_0 M_j^{-2} \tag{4.14}$$

for some $d_0 > 0$. Here, $d_0 > 0$ is due to the positivity of the regular part of the Green function, a consequence of the Positive Mass Theorem. We first consider the case $n = 3$. By Taylor expansion,

$$\nabla(Rh)^{\tau_j}(y_j + x) = \nabla(Rh)^{\tau_j}(y_j) + x \cdot \nabla^2(Rh)^{\tau_j}(y_j) + O(|x|^2)\,.$$

Since 0 is a simple blow up point, we have for $1 \le k \le 3$,

$$\begin{aligned}
&\int_{B_{r_0}} x_k u_j^{\frac{2n}{n-2}}(y_j + x)dx = o(1)M_j^{-2}, \\
&\int_{B_{r_0}} |x|^2 u_j^{\frac{2n}{n-2}}(y_j + x)dx = O(M_j^{-4})
\end{aligned} \tag{4.15}$$

where $o(1) \to 0$ as $i \to +\infty$. Thus, (4.12) and (4.14) implies

$$| \nabla (Rh)^{\tau_j}(y_j)| = o(1)M_j^{-2}. \tag{4.16}$$

Putting (4.13), (4.14) and (4.16) together, we have

$$\begin{aligned}
d_0 M_j^{-2} &\le \left| \int_{B_{r_0}} (x \cdot \nabla (Rh)^{\tau_j}(x + y_j)) u_j^{\frac{2n}{n-2}}(x + y_j)dx \right| \\
&\le o(1)M_j^{-2} | \nabla (Rh)^{\tau_j}(y_j)| + c_1 \int_{B_{r_0}} |x|^2 u_j^{\frac{2n}{n-2}}(y_j + x)dx \\
&= O(M_j^{-4}),
\end{aligned}$$

which obviously yields a contradiction.

For $n \ge 4$, one has by Taylor expansion,

$$\nabla R(x + y_j) = \nabla R(y_j) + \sum_{k=1}^{l-1} \nabla^k R(y_j)x^k + o(1)|x|^{n-3}, \tag{4.17}$$

where $l = n - 2$. Thus, by (4.12)

$$\begin{aligned}
o(1)M_j^{-2} + c_1\tau_j \ge | \nabla R(y_j)| - c_2 &\left(\sum_{k=1}^{l-1} | \nabla^{k+1} R(y_j)| M_j^{-\frac{2k}{n-2}} \right) \\
&- o(1)M_j^{-\frac{2(n-3)}{n-2}}.
\end{aligned} \tag{4.18}$$

By the assumption (1.8), one has

$$\begin{aligned}
| \nabla^{k+1} R(y_j)| M_j^{-\frac{2k}{n-2}} &\le o(1)\left[| \nabla R(y_j)|^{\frac{l-k-1}{l-1}} M_j^{-\frac{2k}{n-2}} \right] \\
&= o(1)(| \nabla R(y_j)|)^{\frac{l-k-1}{l-1}} \left(M_j^{-\frac{2(l-1)}{n-2}} \right)^{\frac{k}{l-1}} \\
&\le o(1)\left(| \nabla R(y_j)| + M_j^{-\frac{2(l-1)}{n-2}} \right).
\end{aligned}$$

Together with (4.18), the inequality implies that

$$| \nabla R(y_j)| \le c_1\tau_j + o(1)M_j^{-\frac{2(l-1)}{n-2}}. \tag{4.19}$$

By (4.13) and (4.17), one has

$$c_3(\tau_j + M_j^{-2}) \le o(1) M_j^{-\frac{2}{n-2}} |\nabla R(y_j)| + \sum_{k=2}^{l} \left(|\nabla R(y_j)|^{\frac{l-k}{l-1}} M_j^{-\frac{2k}{n-2}} \right).$$

Since

$$\begin{aligned} |\nabla R(y_j)|^{\frac{l-k}{l-1}} M_j^{\frac{-2k}{n-2}} &= (|\nabla R(y_j)| M_j^{\frac{-2}{n-2}})^{\frac{l-k}{l-1}} (M_j^{\frac{-2l}{n-2}})^{\frac{k-1}{l-1}} \\ &\le \frac{l-k}{l-1} |\nabla R(y_j)| M_j^{\frac{-2}{n-2}} + \frac{k-1}{l-1} M_j^{\frac{-2l}{n-2}}, \end{aligned}$$

one has by (4.19),

$$\begin{aligned} c_3(\tau_j + M_j^{-2}) &\le o(1) \big(M_j^{-\frac{2}{n-2}} |\nabla R(y_j)| + M_j^{-2} \big) \\ &\le o(1) \big(M_j^{\frac{-2}{n-2}} \tau_j + M_j^{-2} \big), \end{aligned}$$

which obviously leads to a contradiction. Hence, for any $p_0 > 1$, the uniform boundedness for solutions of (4.9) for $p_0 \le p \le \frac{n+2}{n-2}$ is established.

Now suppose $\lambda_1(L) > 0$,then it is not difficult to prove that the degree of all solutions of (4.9) with $p < n + 2/n - 2$ is equal to -1. Thus, the counting of the Leray-Schauder degree for the critical exponent follows from the uniform bound. □

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