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Optimal Regularity for Mixed Parabolic Problems in Spaces of Functions Which Are Hölder Continuous with Respect to Space Variables

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Abstract. We give necessary and sufficient conditions in order that a general linear mixed parabolic problem have a solution in spaces of functions with derivatives which are Hölder continuous with respect to space variables. Autonomous and nonautonomous problems are considered.

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0. – Introduction

Consider a linear autonomous parabolic initial-boundary value problem of the form

$$(0.1) \qquad \begin{cases} \partial_t u(t,x) = \mathcal{A}(t,x,\partial_x)u(t,x) + f(t,x), t \in]0,T], x \in \overline{\Omega}, \\ \mathcal{B}_j(t,x',\partial)u(t,x') = g_j(t,x'), 1 \le j \le m, t \in]0,T], x' \in \partial\Omega, \\ u(0,x) = u_0(x), x \in \overline{\Omega}, \end{cases}$$

where for $t \in [0,T]$ $\mathcal{A}(t,x,\partial_x) = \sum_{|\alpha|\le 2m} a_\alpha(t,x)\partial_x^\alpha$ is strongly elliptic of order $2m$ and the boundary operators $\mathcal{B}_j(t,x',\partial_x)$ of order $\mu_j < 2m$ satisfy the usual requirements guaranteeing the existence of a resolvent in a suitable subset of the complex plane $\mathbb{C}$. We are interested in the existence and uniqueness of strict solutions of (0.1), that is, of solutions which are continuous in $[0,T]\times\overline{\Omega}$ together with their first derivative with respect to t and their derivatives of order less or equal to $2m$ with respect to x. Connected with this, there are well known theorems of optimal regularity, giving necessary and sufficient conditions (under suitable assumptions on Ω and the regularity of the coefficients) on the data in order to have a strict solution u with the first derivative with respect to

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t and the derivatives of order less or equal to $2m$ with respect to x which are hölder continuous with respect to the parabolic distance (see [13]).

More generally, we shall look for necessary and sufficient conditions on the data to have a solution with the mentioned derivatives which are hölder continuous only with respect to the space variables x; before quoting certain significant results in this direction, it is convenient to introduce some notations: we shall identify complex valued maps of domain $[0,T]\times\overline{\Omega}$ with corresponding maps of domain $[0,T]$ and values in functional spaces on $\overline{\Omega}$. So, a strict solution will be an element of $C^1([0,T];C(\overline{\Omega}))\cap C([0,T];C^{2m}(\overline{\Omega}))$; if E is a Banach space with norm $\|.\|$, we shall indicate with $B([0,T];E)$ $\{f:[0,T]\to E|t\to f(t)$ is bounded$\}$. If $u\in B([0,T];E)$, we set

$$\|u\|_{B([0,T];E)} := \sup_{0\le t\le T}\|u(t)\|;$$

$C([0,T];E)$ will inherit the norm of subspace of $B([0,T];E)$. If $u\in C^1([0,T];E)$, we set

$$\|u\|_{C^1([0,T];E)} := \|u\|_{B([0,T];E)} + \|u'\|_{B([0,T];E)}.$$

Let $\alpha\in]0,1[$; if $u\in C([0,T];E)$, we set

$$[u]_{C^\alpha([0,T];E)} := \sup_{0\le s<t\le T}(t-s)^{-\alpha}\|u(t)-u(s)\|$$

and

$$\|u\|_{C^\alpha([0,T];E)} := \|u\|_{B([0,T];E)} + [u]_{C^\alpha([0,T];E)}$$

and $C^\alpha([0,T];E) := \{u\in C([0,T];E)|[u]_{C^\alpha([0,T];E)} < +\infty\}$.

Coming back to our problem, first of all we mention the paper by Kruzkov, Castro, Lopez ([7]) where the problem in $\mathbb{R}^n$ without boundary conditions is treated: in essence, they show that, under suitable assumptions on the coefficients, if $f\in C([0,T];C(\mathbb{R}^n))\cap B([0,T];C^\theta(\mathbb{R}^n))\}$ and $u_0\in C^{2+\theta}(\mathbb{R}^n)$ for some $\theta\in]0,1[$, the parabolic problem without boundary conditions has a unique strict solution u belonging also to $B([0,T];C^{2+\theta}(\mathbb{R}^n))$, with $\partial_t u\in B([0,T];C^\theta(\mathbb{R}^n))$ (where our definition of $C^k(\mathbb{R}^n)$ requires the boundedness of the involved derivatives).

In the paper [10] A. Lunardi considers the case of a second order elliptic operator $\mathcal{A}(t,x,\partial_x)$ with a first order boundary operator $\mathcal{B}(t,x',\partial_x)$; under natural assumptions of regularity of the coefficients, she shows that (0.1) has a strict solution $u\in B([0,T];C^{2+\theta}(\overline{\Omega}))$ such that $u'\in B([0,T];C^\theta(\overline{\Omega}))$ for some $\theta\in]0,1[$ if and only if $f\in C([0,T];C(\overline{\Omega}))\cap B([0,T];C^\theta(\overline{\Omega}))$, g (the datum on the boundary) belongs to $B([0,T];C^{1+\theta}(\partial\Omega))\cap C^{\frac{1+\theta}{2}}([0,T];C(\partial\Omega))$, $u_0\in C^{2+\theta}(\overline{\Omega})$ and $\mathcal{B}(0,.,\partial_x)u_0 = g(0,.)$. The autonomous case with $\mathcal{A}(x,\partial_x)$ of second order and homogeneous Dirichlet boundary conditions is studied by E. Sinestrari and W. von Wahl (see [12]); they consider the assumptions $f\in C([0,T];C(\overline{\Omega}))\cap B([0,T];C^\theta(\overline{\Omega}))$ for some $\theta\in]0,1[$, $u_0\in C^{2+\theta}(\overline{\Omega})$ and, if γ is the trace operator on $\partial\Omega$, $\gamma u_0 = 0$ and $\gamma(\mathcal{A}(.,\partial_x)u_0 + f(0)) = 0$ and prove many properties of the solution u given by the variation of parameter formula using the semigroup naturally associated to the problem in $L^p(\Omega)$ for $p \in]1, +\infty[$, but do not prove the expected result that $u \in B([0,T]; C^{2+\theta}(\overline{\Omega}))$ and $u' \in B([0,T]; C^\theta(\overline{\Omega}))$. In fact, [12] contains a counterexample due to Wiegner showing that for any $\beta < \theta$, if $\gamma f \in C^{\frac{\beta}{2}}([0,T]; C(\partial\Omega))$ it can really happen that u has not the expected regularity. This suggests the possibility that a further necessary condition is $\gamma f \in C^{\frac{\theta}{2}}([0,T]; C(\partial\Omega))$. Some years later, using techniques of potential theory, M. López Morales ([8]) shows that this further assumption is sufficient to guarantee the existence of a strict solution belonging to $B([0,T]; C^{2+\theta}(\overline{\Omega}))$ with $u' \in B([0,T]; C^\theta(\overline{\Omega}))$; however, the necessity of the solution is not clear in his paper yet.

Recently the author ([6]) has given a new proof of López Morales' result, based on semigroup techniques, showing also the necessity of the foregoing condition. Such a proof was based on an estimate, due to Bolley, Camus, P. The Lai (see [3]) of the solution of the elliptic boundary value problem depending on a parameter obtained applying formally the Laplace transform with respect to t to the parabolic system.

The aim of this paper is to extend the methods and the results of [6]; in the autonomous case we shall consider general boundary value problems instead of only the Dirichlet problem, and $\theta \in]0, 2m[- \mathbb{Z}$, instead of simply $\theta \in]0, 1[$. We shall consider also nonautonomous general boundary value problems, but only in the case $\theta \in]0, 1[$.

We are now going to describe the content of the paper: the first section deals with elliptic boundary value problems in spaces of hölder continuos functions; the main result is an estimate depending on a parameter of the solution of a general elliptic boundary value problem (see 1.6), generalizing the already quoted estimate due to Bolley, Camus, P. The Lai in the case of a second order operator with Dirichlet boundary conditions and $\theta \in]0, 1[$. We recall again that the mentioned authors use techniques of pseudodifferential operators, while our estimate (see (1.2)) is obtained by functional analytic methods.

The second section is the core of the paper. It deals with general linear autonomous parabolic problems; the main result is Theorem 2.8, where necessary and sufficient conditions are given in order to have a strict solution of the autonomous version of (0.1) belonging to $B([0,T]; C^{2m+\theta}(\overline{\Omega}))$ with $u' \in B([0,T]; C^\theta(\overline{\Omega}))$ in case $\theta \in]0, 2m[, \theta \notin \mathbb{Z}$.

The third and final section deals with nonautomous problems; here we have been able to give a complete generalization of the results of the third section only in case $\theta \in]0, 1[$ (see 3.2).

We introduce now some notations we shall use in the sequel.

Let Ω be an open subset of $\mathbb{R}^n$ whose boundary $\partial\Omega$ is an submanifold of $\mathbb{R}^n$ of class C^1; for any $x' \in \partial\Omega$ we shall indicate with $T_{x'}(\partial\Omega)$ the set of vectors in $\mathbb{R}^n$ which are tangent in x' to $\partial\Omega$; if $x \in \mathbb{R}^n$, we shall indicate with dist(x, $\partial\Omega$) the distance of x from $\partial\Omega$.

We shall use Kronecker's symbol $\delta_{ij}(= 1$ if $i = j, = 0$ otherwise).

We shall indicate with $\mathbb{N}$ and $\mathbb{N}_0$ respectively the set of positive and non-negative integers.

If θ is a real number we shall indicate with $[\theta]$ the maximum integer less or equal to θ and with $\{\theta\}$ the difference $\theta - [\theta]$.

If λ is a complex number, $Arg(\lambda)$ will indicate the unique element of the argument of λ in the interval $]-\pi, \pi]$.

If $\alpha \in \mathbb{N}_0^n$, $\alpha = (\alpha_1, ..., \alpha_n)$, we set $|\alpha| := \alpha_1 + ... + \alpha_n$ and we shall use the notation ∂_x^α to indicate the differential operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$. As we have already done, we shall write u' or $\partial_t u$ instead of $\frac{\partial u}{\partial t}$.

Let A be a linear closed operator in a Banach space E with norm $\|.\|$; we shall equip the domain $D(A)$ of the natural norm $\|x\|_{D(A)} := \|x\| + \|Ax\|$ transforming $D(A)$ into a Banach space continuously embedded into E; we shall indicate with $\rho(A)$ its resolvent set.

If E and F are Banach spaces, we shall indicate with $\mathcal{L}(E, F)$ the Banach space of linear bounded operators from E to F; if $E = F$ we shall simply write $\mathcal{L}(E)$.

In the following we shall also use some elements of real interpolation theory (see for Example [9] ch. 1). Assume that E_0 and E_1 are Banachv spaces with norms $\|.\|_0$ and $\|.\|_1$ respectively and E_1 is continuously embedded into E_0; let $x \in E_0$ and $s > 0$; we put $K(s, x) := \inf\{\|x - y\|_0 + s\|y\|_1 | y \in E_1\}$; if $\alpha \in]0, 1[$, we set, for $x \in E_0$, $\|x\|_{\alpha,\infty} := \sup_{s>0} s^{-\alpha} K(s, x)$ and $(E_0, E_1)_{\alpha,\infty} := \{x \in E_0 | \|x\|_{\alpha,\infty} < +\infty\}$. It is well known that $(E_0, E_1)_{\alpha,\infty}$ with the norm $\|.\|_{\alpha,\infty}$ is a Banach space containing E_1 (with continuous embedding) and continuously embedded into E_0. Such space is of interpolation between E_0 and E_1; if $E_1 = D(A)$ with $[0, +\infty[\subseteq \rho(A)$ and $\|\xi(\xi - A)^{-1}\|_{\mathcal{L}(E)} \leq M$ with M independent of $\xi > 0$, one can show that $(E_0, E_1)_{\alpha,\infty} = \{x \in E_0 | \sup_{\xi>0} \xi^\alpha \|A(\xi - A)^{-1}x\|_0 < +\infty\}$ and $\|.\|_{\alpha,\infty}$ is equivalent (as a norm) to $x \to \sup_{\xi>0} \xi^\alpha \|A(\xi - A)^{-1}x\|_0$ (see [9] Proposition 2.2.6). We shall also consider certain intermediate spaces which are not necessarily of interpolation: let E be a Banach space, $\alpha \in]0, 1[$, $E_1 \subseteq E \subseteq E_0$ with continuous embeddings; we shall say that $E \in J_\alpha(E_0, E_1)$ if there exists $C > 0$ such that for every $x \in E_1$ $\|x\|_E \leq C\|x\|_0^{1-\alpha}\|x\|_1^\alpha$. An equivalent definition is the following: there exists $C > 0$ such that for any $x \in E_1$ and any $\rho > 0$

$$\|x\|\|_E \leq C[\rho\|x\|_1 + \rho^{\frac{\alpha}{\alpha-1}}\|x\|_0]. \tag{0.2}$$

Finally, some indications about constants in estimates: we shall use quite loosely the symbol C to indicate a constant that we are not interested to specify and may be different from time to time, even in the same sequence of computations; we shall indicate with $C(a, b, ...)$ a constant depending on $a, b, ...$. In general we shall explicitly declare our interest in specifying the dependence (or independence) of a constant on the involved variables.

1. – Hölder continuous functions and elliptic boundary value problems

Let Ω be an open set in $\mathbb{R}^n$, $s \in \mathbb{N}_0$; we indicate with $C^s(\overline{\Omega})$ the set of elements u of $C^s(\Omega)$ whose derivatives of order less or equal to s are uniformly continuous and bounded in Ω. It is well known that any element of $C^s(\overline{\Omega})$ is continuously extensible, together with its derivatives of order less or equal to s, to $\overline{\Omega}$. We shall identify the function of domain Ω with its extension to $\overline{\Omega}$. If $\Omega = \mathbb{R}^n$, we shall write $C^s(\mathbb{R}^n)$ instead of $C^s(\overline{\mathbb{R}^n})$. This should cause no confusion, as we shall have no occasion to consider functions which are not (at least) uniformly continuous and bounded.

We set

$$\|u\|_{s,\overline{\Omega}} := \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq s} \|\partial^\alpha u\|_{\infty,\Omega}.$$

Next, let $s \in]0,1[$; if u is a complex valued function of domain $\overline{\Omega}$, we set

$$[u]_{s,\overline{\Omega}} := \sup_{x,y \in \overline{\Omega}, x \neq y} |x-y|^{-s}|u(x)-u(y)|$$

and, if $s \in \mathbb{R}^+ - \mathbb{Z}$ and $u \in C^{[s]}(\overline{\Omega})$,

$$\|u\|_{s,\overline{\Omega}} := \|u\|_{[s],\overline{\Omega}} + \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = [s]} [\partial^\alpha u]_{\{s\},\overline{\Omega}}.$$

It is easily seen that for any $s \geq 0$ $C^s(\overline{\Omega}) := \{u \in C^{[s]}(\overline{\Omega}) | \|u\|_{s,\overline{\Omega}} < +\infty\}$ with the norm $\|.\|_{s,\overline{\Omega}}$ is a Banach space; let now Ω be an open bounded set in $\mathbb{R}^n$ lying on one side of its boundary $\partial\Omega$ which is a submanifold of $\mathbb{R}^n$ of class $C^{m+\theta}$, with $m \in \mathbb{N}$ and $\theta \geq 0$. We shall briefly say that Ω is of class $C^{m+\theta}$. We have

PROPOSITION 1.1. *Let Ω be an open bounded set in $\mathbb{R}^n$ of class $C^{m+\theta}$ or $\Omega = \mathbb{R}^n$, s_0, s, s_1 real numbers with $0 \leq s_0 < s < s_1 \leq m+\theta$; we have:*

(a) $C^s(\overline{\Omega}) \in J_{\frac{s-s_0}{s_1-s_0}}(C^{s_0}(\overline{\Omega}), C^{s_1}(\overline{\Omega}))$;

(b) if $s \notin \mathbb{Z}$,

$$C^s(\overline{\Omega}) = (C^{s_0}(\overline{\Omega}), C^{s_1}(\overline{\Omega}))_{\frac{s-s_0}{s_1-s_0},\infty}$$

with equivalent norms;

(c) if $s_1 \notin \mathbb{Z}$, closed balls in $C^{s_1}(\overline{\Omega})$ are also closed in $C^{s_0}(\overline{\Omega})$.

PROOF. (a) and (b) are well known; for proofs see [9], ch. 1 or [15], in particular 2.5.7 and 3.3.6.

We show (c); it is clearly sufficient to show that the closed unit ball in $C^{s_1}(\overline{\Omega})$ is closed in $C^{s_0}(\overline{\Omega})$; let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $C^{s_1}(\overline{\Omega})$ such that $\|u_k\|_{s_1,\overline{\Omega}} \leq 1$ for any k and $u_0 \in C^{s_0}(\overline{\Omega})$ such that $\|u_k - u_0\|_{s_0,\overline{\Omega}} \to 0$ as $k \to +\infty$. Owing to (a), it is easily seen that $(u_k)_{k\in\mathbb{N}}$ converges in $C^s(\overline{\Omega})$ for any $s < s_1$, so that $u_0 \in C^s(\overline{\Omega})$ for any $s < s_1$; in particular, $u_0 \in C^{[s_1]}(\overline{\Omega})$

and $\|u_k - u_0\|_{[s_1],\overline{\Omega}} \to 0$, as $k \to +\infty$. Take for any multiindex α such that $|\alpha| = [s_1]$ two different points x_α and y_α in $\overline{\Omega}$; we have for any $k \in \mathbb{N}$

$$\sum_{|\alpha|=[s_1]} |x_\alpha - y_\alpha|^{-\{s_1\}} |\partial^\alpha u_k(x_\alpha) - \partial^\alpha u_k(y_\alpha)| \leq 1 - \|u_k\|_{[s_1],\overline{\Omega}};$$

passing to the limit with k we obtain

$$\sum_{|\alpha|=[s_1]} |x_\alpha - y_\alpha|^{-\{s_1\}} |\partial^\alpha u_0(x_\alpha) - \partial^\alpha u_0(y_\alpha)| \leq 1 - \|u_0\|_{[s_1],\overline{\Omega}},$$

which implies $u_0 \in C^{s_1}(\overline{\Omega})$ and $\|u_0\|_{s_1,\overline{\Omega}} \leq 1$.

In case Ω is a bounded open subset of $\mathbb{R}^n$ of class $C^{m+\theta}$, we shall need also spaces $C^s(\partial\Omega)$ $(0 \leq s \leq m+\theta)$ which can be defined by local charts. We assume that on each of them a natural norm $\|.\|_{s,\partial\Omega}$ (obtained using some system of local charts) is fixed. We limit ourselves to recall the (almost obvious) fact that if $u \in C^s(\overline{\Omega})$ for some $s \in [0, m+\theta]$ and if $\beta \in \mathbb{N}_0^n$ and $|\beta| \leq s$, the restriction of $\partial^\beta u$ to $\partial\Omega$ belongs to $C^{s-|\beta|}(\partial\Omega)$. Moreover, we state without proof the following

PROPOSITION 1.2. *Let Ω be an open bounded set in $\mathbb{R}^n$ of class $C^{m+\theta}$, s_0, s, s_1 real numbers with $0 \leq s_0 < s < s_1 \leq m+\theta$; we have:*
(a) $C^s(\partial\Omega) \in J_{\frac{s-s_0}{s_1-s_0}}(C^{s_0}(\partial\Omega), C^{s_1}(\partial\Omega))$;
(b) if $s \notin \mathbb{Z}$,

$$C^s(\partial\Omega) = (C^{s_0}(\partial\Omega), C^{s_1}(\partial\Omega))_{\frac{s-s_0}{s_1-s_0},\infty}$$

with equivalent norms.

For a discussion of spaces in $\partial\Omega$ see [15] ch. 3.

We consider now $m+1$ partial differential operators B_0, ..., B_m with coefficients in $C(\partial\Omega)$; they form a Dirichlet system of order m if for any $j = 0, ..., m$ the order of B_j equals j and $\partial\Omega$ is never characteristic with respect to each of them.

Given $m+1$ complex valued functions $g_0, ..., g_m$ defined on $\partial\Omega$, we are interested in the existence of some suitably regular function v defined on $\overline{\Omega}$ such that for every $j = 0, ..., m$ $B_j v = g_j$. The following result is proved in [11], 6:

THEOREM 1.3. *Let Ω be an open subset of $\mathbb{R}^n$ of class $C^{m+\theta}$, with $m \in \mathbb{N}_0$ and $\theta \geq 0$. Let $\{B_0, \dots, B_m\}$ be a Dirichlet system of order m on $\partial\Omega$. Assume that the coefficients of B_j are of class $C^{m-j+\theta}(\partial\Omega)$. Then, there exists $N \in \mathcal{L}(\prod_{j=0}^m C^{m-j}(\partial\Omega); C^m(\overline{\Omega}))$ such that for any $j = 0, \dots, m$, $(f_0, \dots, f_m) \in \prod_{j=0}^m C^{m-j}(\partial\Omega)$ one has $B_j N(f_0, \dots, f_j, \dots f_m) = f_j$. Moreover, for any $s \in [0, \theta]$ the restriction of N to $\prod_{j=0}^m C^{m-j+s}(\partial\Omega)$ is a linear bounded operator from $\prod_{j=0}^m C^{m-j+s}(\partial\Omega)$ to $C^{m+s}(\overline{\Omega})$ and for every $k = 0, \dots, m$ $(f_0, \dots, f_k) \to N(f_0, \dots, f_k, 0, \dots, 0)$ can be extended to an element of $\mathcal{L}(\prod_{j=0}^k C^{k-j+s}(\partial\Omega), C^{k+s}(\overline{\Omega})$ for every $s \in [0, m-k+\theta]$.*

We pass now to elliptic boundary value problems; we consider the following assumptions: let $m \in \mathbb{N}$, $\theta \in \mathbb{R}, \theta \geq 0$; we shall say that the assumptions (H_θ) are satisfied if:

(a) Ω is an open bounded subset of $\mathbb{R}^n$ of class $C^{2m+\theta}$;

(b) $\mathcal{A} = \sum_{|\alpha| \leq 2m} a_\alpha(x)\partial^\alpha$ is a properly elliptic linear partial differential operator of order $2m$ with coefficients in $C^\theta(\overline{\Omega})$;

(c) for $1 \leq j \leq m$ $\mathcal{B}_j = \sum_{|\beta| \leq \mu_j} b_{j,\beta}(x')\partial^\beta$ is a linear partial differential operator of order $\mu_j \leq 2m-1$, with coefficients in $C^{2m-\mu_j+\theta}(\partial\Omega)$ such that $\partial\Omega$ is never characteristic with respect to it; we assume also that if $1 \leq j_1 < j_2 \leq m$, $\mu_{j_1} \neq \mu_{j_2}$;

(d) let $\mathcal{A}^\#(x,\xi) := \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$ $(x \in \overline{\Omega}, \xi \in \mathbb{C}^n)$; then, $r^{2m}e^{i\theta} - \mathcal{A}^\#(x, i\xi) \neq 0$ if $r \in [0,+\infty[, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], x \in \overline{\Omega}, \xi \in \mathbb{R}^n$ and $(r,\xi) \neq (0,0)$;

(e)(complementing condition) let $\lambda \in \mathbb{C}$ with $Re(\lambda) \geq 0$, $x' \in \partial\Omega$, $\xi' \in T_{x'}(\partial\Omega)$ with $(\lambda,\xi') \neq (0,0)$; then the O. D. E. problem

$$\begin{cases} \lambda v(t) - \mathcal{A}^\#(x', i\xi' + \nu(x')\partial_t)v(t) = 0, t \in \mathbb{R}; \\ \mathcal{B}_j^\#(x', i\xi' + \nu(x')\partial_t)v(0) = 0, 1 \leq j \leq m, \\ v \text{ bounded in } \mathbb{R}^+ \end{cases}$$

has only the trivial solution.

We consider the following elliptic boundary value problem:

$$\begin{cases} \lambda u - \mathcal{A}u = f, \\ \mathcal{B}_j u = g_j, 1 \leq j \leq m \end{cases} \tag{1.1}$$

where f is defined in $\overline{\Omega}$ and, for $1 \leq j \leq m$, g_j is defined in $\partial\Omega$. We have the following result, the proof of which can be obtained using a well known method due to Agmon (a proof is given in [4] Lemma 2.12; to eliminate the norms of the intermediate spaces use (0.2)):

PROPOSITION 1.4. *Assume that* $\theta \in \mathbb{R}^+ - \mathbb{Z}$ *and the assumptions* (H_θ) *are satisfied; then, there exist* $R \geq 0$ *and* $\phi_0 \in]\frac{\pi}{2}, \pi]$ *such that for any* $\lambda \in \mathbb{C}$ *with* $|Arg(\lambda)| \leq \phi_0$ *and* $|\lambda| \geq R$, $f \in C^\theta(\overline{\Omega})$, $(g_j)_{1 \leq j \leq m} \in \prod_{1 \leq j \leq m} C^{2m+\theta-\mu_j}(\partial\Omega)$ *problem* (1.1) *has a unique solution u in* $C^{2m+\theta}(\overline{\Omega})$. *Moreover there exists* $M > 0$ *independent of* λ, f, $(g_j)_{1 \leq j \leq m}$ *such that*

$$\sum_{r=0}^{2m+[\theta]} |\lambda|^{1+\frac{\theta-r}{2m}} \|u\|_{r,\overline{\Omega}} + \sum_{r=0}^{2m+[\theta]} |\lambda|^{1+\frac{[\theta]-r}{2m}} \|u\|_{r+\{\theta\},\overline{\Omega}}$$
$$\leq M \left[\|f\|_{\theta,\overline{\Omega}} + |\lambda|^{\frac{\theta}{2m}} \|f\|_{0,\overline{\Omega}} + \sum_{j=1}^m \|g_j\|_{2m-\mu_j+\theta,\partial\Omega} + \sum_{j=1}^m |\lambda|^{\frac{2m-\mu_j+\theta}{2m}} \|g_j\|_{0,\partial\Omega} \right].$$

In the treatment of parabolic boundary value problems we shall need a more refined estimate of u; we start by recalling some well known definitions and facts: assume that the assumptions (H_0) are satisfied; we define the following operator A:

$$D(A) := \left\{ u \in \bigcap_{1 \le p < +\infty} W^{2m,p}(\Omega) | \mathcal{A}u \in C(\overline{\Omega}) \text{ and } \mathcal{B}_j u = 0 \text{ for } j = 1, ..., m \right\},$$

$$Au := \mathcal{A}u.$$

We think of A as a linear unbounded operator in $C(\overline{\Omega})$. We recall some facts concerning the operator A which are in large part well known (see [14],[1] and [5]):

THEOREM 1.5. *There exist $R \ge 0$, $\phi_0 \in]\frac{\pi}{2}, \pi]$ and $M > 0$ such that $\{\lambda \in \mathbb{C} \| Arg(\lambda)| \le \phi_0, |\lambda| \ge R\} \subseteq \rho(A)$. Moreover, for λ in the declared set, $\|(\lambda - A)^{-1}\|_{\mathcal{L}(C(\overline{\Omega}))} \le M|\lambda|^{-1}$. If $0 < \theta < 1$ and $2m\theta \notin \mathbb{Z}$, we have*

$$(C(\overline{\Omega}), D(A))_{\theta,\infty} = \{f \in C^{2m\theta}(\overline{\Omega}) | \mathcal{B}_j f = 0 \text{ if } \mu_j < 2m\theta\}.$$

After these preliminaries we give the following refinement of the estimate in 1.4 (for the case of Dirichlet boundary conditions and $m = 1$ see [3]):

THEOREM 1.6. *Let $0 < \theta < 2m$ with $\theta \notin \mathbb{Z}$ and assume that the assumptions (H_θ) are satisfied; let $\lambda \in \mathbb{C}$ with $|Arg(\lambda)| \le \phi_0$ and $|\lambda| \ge R$ (see 1.4), $f \in C^\theta(\overline{\Omega})$, $(g_j)_{1 \le j \le m} \in \prod_{j=1}^m C^{2m-\mu_j+\theta}(\partial\Omega)$; then there exists $M > 0$ independent of λ, f, $(g_j)_{1 \le j \le m}$ such that, if u solves (1.1),*

$$\begin{aligned} |\lambda| \|u\|_{\theta,\overline{\Omega}} + \|u\|_{2m+\theta,\overline{\Omega}} \le M \Bigg[& \|f\|_{\theta,\overline{\Omega}} + \sum_{\mu_j < \theta} |\lambda|^{\frac{\theta-\mu_j}{2m}} \|\mathcal{B}_j f\|_{0,\partial\Omega} \\ & + \sum_{j=1}^m \|g_j\|_{2m-\mu_j+\theta,\partial\Omega} + \sum_{j=1}^m |\lambda|^{\frac{2m-\mu_j+\theta}{2m}} \|g_j\|_{0,\partial\Omega} \Bigg]. \end{aligned} \tag{1.2}$$

PROOF. Owing to 1.4, it is clearly sufficient to consider the case $g_j \equiv 0$ for any $j \in \{1, ..., m\}$. We start by assuming that $\mathcal{B}_j f \equiv 0$ for any j such that $\mu_j < \theta$ (for convenience we shall indicate with $\mathcal{I}$ the set $\{j \in \{1, ..., m\} | \mu_j < \theta\}$). By 1.5, $f \in (C(\overline{\Omega}), D(A))_{\frac{\theta}{2m},\infty}$. Again by 1.5, we have $\|(\lambda - A)^{-1}\|_{\mathcal{L}(D(A))} \le M|\lambda|^{-1}$ for $\lambda \in \mathbb{C}$, $|Arg(\lambda)| \le \phi_0$, $|\lambda| \ge R$, so that, by interpolation, for the same values of λ

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}((C(\overline{\Omega}), D(A))_{\frac{\theta}{2m},\infty})} \le M|\lambda|^{-1}.$$

As $(C(\overline{\Omega}), D(A))_{\frac{\theta}{2m},\infty}$ is a closed subspace of $C^\theta(\overline{\Omega})$, we have, for a suitable $C_1 > 0$ independent of λ and f,

$$\|u\|_{\theta,\overline{\Omega}} \le C_1 M|\lambda|^{-1} \|f\|_{\theta,\overline{\Omega}}.$$

Moreover,by 1.4 (or [2]) for a suitable $C_2 > 0$

$$\begin{aligned}\|u\|_{2m+\theta,\overline{\Omega}} &\le C_2\left[\|\mathcal{A}u\|_{\theta,\overline{\Omega}} + \|u\|_{\theta,\overline{\Omega}}\right]\\ &= C_2\left[\|\lambda u - f\|_{\theta,\overline{\Omega}} + \|u\|_{\theta,\overline{\Omega}}\right]\\ &\le M'\|f\|_{\theta,\overline{\Omega}},\end{aligned}$$

for a certain $M' > 0$.

We consider now the general case; let $(B_l)_{0\le l\le[\theta]}$ be a Dirichlet system of order $[\theta]$ in Ω with coefficients of B_l of class $C^{2m-l+\theta}(\partial\Omega)$ and such that for any $j \in \mathcal{I}$ $B_{\mu_j} = \mathcal{B}_j$. We set also for $l = 0, ..., m$

$$g_l = \begin{cases} \mathcal{B}_j f & \text{if } l = \mu_j \text{ for some } j \in \mathcal{I},\\ 0 & \text{otherwise} .\end{cases}$$

Next we consider an operator N of the type described in 1.3, with $[\theta]$ replacing m. Finally we take a family of function $(\chi_\epsilon)_{0<\epsilon\le\epsilon_0}$ in $C^{2m+\theta}(\overline{\Omega})$ such that

(a) $\chi_\epsilon(x) = 1$ if dist(x, $\partial\Omega$) $\le \frac{\epsilon}{2}$;

(b) $\chi_\epsilon(x) = 0$ if dist(x, $\partial\Omega$) $\ge \epsilon$;

(c) for any $s \in [0, 2m+\theta]$ there exists $C(s) > 0$ such that for any $\epsilon \in]0, \epsilon_0]$ one has $\|\chi_\epsilon\|_{s,\overline{\Omega}} \le C(s)\epsilon^{-s}$. A family of functions with these properties can be easily constructed by local charts (if $\Omega = \{x \in \mathbb{R}^n | x_n > 0\}$, one can take $\chi_\epsilon(x) = \phi(\frac{x_n}{\epsilon})$, with $\phi \in C^\infty([0, +\infty[)$, $\phi(t) = 1$ if $0 \le t \le \frac{1}{2}$, $\phi(t) = 0$ if $t \ge 1$). Let now $\lambda \in \mathbb{C}$, such that $|Arg(\lambda)| \le \phi_0$ and $|\lambda| \ge R$ (see 1.4) and $0 < \epsilon \le \epsilon_0$. Let $u_1 \in C^{2m+\theta}(\overline{\Omega})$ be the solution of

$$\begin{cases} (\lambda - \mathcal{A})u_1 = f - \chi_\epsilon N(g_0, ..., g_{[\theta]}),\\ \mathcal{B}_j u_1 = 0 \text{ for } j = 1, ..., m.\end{cases}$$

Then, as $\mathcal{B}_j(f - \chi_\epsilon N(g_0, ..., g_{[\theta]})) = 0$ for any $j \in \mathcal{I}$, we have

$$|\lambda|\|u_1\|_{\theta,\overline{\Omega}} + \|u_1\|_{2m+\theta,\overline{\Omega}} \le M''\|f - \chi_\epsilon N(g_0, ..., g_{[\theta]})\|_{\theta,\overline{\Omega}},$$

for some $M'' > 0$. Let $u_2 \in C^{2m+\theta}(\overline{\Omega})$ be the solution of

$$\begin{cases} (\lambda - \mathcal{A})u_2 = \chi_\epsilon N(g_0, ..., g_{[\theta]}),\\ \mathcal{B}_j u_2 = 0 \text{ for } j = 1, ..., m.\end{cases}$$

Then, by 1.4, we have

$$\begin{aligned}|\lambda|\|u_2\|_{\theta,\overline{\Omega}} + \|u_2\|_{2m+\theta,\overline{\Omega}} \le M\Big[&\|\chi_\epsilon N(g_0, ..., g_{[\theta]})\|_{\theta,\overline{\Omega}}\\ &+|\lambda|^{\frac{\theta}{2m}}\|\chi_\epsilon N(g_0, ..., g_{[\theta]})\|_{0,\overline{\Omega}}\Big].\end{aligned}$$

Putting together the two estimates, we have for some $C > 0$

$$
\begin{aligned}
&|\lambda| \|u\|_{\theta,\overline{\Omega}} + \|u\|_{2m+\theta,\overline{\Omega}} \\
&\le C \left[\|f\|_{\theta,\overline{\Omega}} + \|\chi_\epsilon N(g_0, ..., g_{[\theta]})\|_{\theta,\overline{\Omega}} + |\lambda|^{\frac{\theta}{2m}} \|\chi_\epsilon N(g_0, ..., g_{[\theta]})\|_{0,\overline{\Omega}} \right]
\end{aligned}
$$

with C independent of λ, f, ϵ. Reducing oneself, by local charts, to the case $\Omega = \{x \in \mathbb{R}^n | x_n > 0\}$ and employing Taylor's formula, one can verify that, if $0 < \text{dist}(\mathrm{x}, \partial\Omega) \le \epsilon \le \epsilon_0$ and $|\beta| \le \mu_j$,

$$
|\partial^\beta N(0, ..., g_{\mu_j}, .., 0)(x)| \le C\epsilon^{\mu_j - |\beta|} \|g_{\mu_j}\|_{0,\partial\Omega}
$$

for some constant C independent of x, ϵ, g_{μ_j}. It follows

$$
\begin{aligned}
&\|\chi_\epsilon N(g_0, ..., g_{[\theta]})\|_{0,\overline{\Omega}} \le \sum_{j \in \mathcal{I}} \|\chi_\epsilon N(0, ..., g_{\mu_j}, ..., 0)\|_{0,\overline{\Omega}} \\
&\le C \sum_{j \in \mathcal{I}} \sup_{\text{dist}(\mathrm{x},\partial\Omega) \le \epsilon} |N(0, ..., g_{\mu_j}, ..., 0)(x)| \\
&\le C \sum_{j \in \mathcal{I}} \epsilon^{\mu_j} \|g_{\mu_j}\|_{0,\partial\Omega}.
\end{aligned}
$$

Moreover, $\|\chi_\epsilon N(g_0, ..., g_{[\theta]})\|_{\theta,\overline{\Omega}} \le \sum_{j \in \mathcal{I}} \|\chi_\epsilon N(0, ..., g_{\mu_j}, ..., 0)\|_{\theta,\overline{\Omega}}$ and for any $j \in \mathcal{I}$

$$
\begin{aligned}
&\|\chi_\epsilon N(0, ..., g_{\mu_j}, ..., 0)\|_{\theta,\overline{\Omega}} \\
&= \|\chi_\epsilon N(0, ..., g_{\mu_j}, ..., 0)\|_{[\theta],\overline{\Omega}} + \sum_{|\alpha| = [\theta]} [\partial^\alpha (\chi_\epsilon N(0, ..., g_{\mu_j}, ..., 0))]_{\{\theta\},\overline{\Omega}}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\|\chi_\epsilon N(0, ..., g_{\mu_j}, ..., 0)\|_{[\theta],\overline{\Omega}} \\
&\le C \sum_{l=0}^{[\theta]} \|\chi_\epsilon\|_{[\theta]-l,\overline{\Omega}} \sum_{|\beta|=l} \sup_{\text{dist}(\mathrm{x},\partial\Omega) \le \epsilon} |\partial^\beta N(0, ..., g_{\mu_j}, ..., 0)(x)| \\
&\le C \left(\epsilon^{\mu_j - [\theta]} \|g_{\mu_j}\|_{0,\partial\Omega} + \sum_{\mu_j < l \le [\theta]} \epsilon^{l - [\theta]} \|g_{\mu_j}\|_{l-\mu_j,\partial\Omega} \right).
\end{aligned}
$$

Next, if $|\alpha| = [\theta]$, we have

$$
\left[\partial^\alpha (\chi_\epsilon N(0, ..., g_{\mu_j}, ..., 0))\right]_{\{\theta\},\overline{\Omega}} \le C \sum_{\beta \le \alpha} \left[\partial^{\alpha-\beta} \chi_\epsilon \partial^\beta N(0, ..., g_{\mu_j}, ..., 0)\right]_{\{\theta\},\overline{\Omega}}.
$$

If $|\beta| < \mu_j$ we have by 1.1(a)

$$
\begin{aligned}
&\left[\partial^{\alpha-\beta}\chi_\epsilon \partial^\beta N(0, ..., g_{\mu_j}, ..., 0)\right]_{\{\theta\},\overline{\Omega}} \\
&\le C\left\{\|\partial^{\alpha-\beta}\chi_\epsilon \partial^\beta N(0, ..., g_{\mu_j}, ..., 0)\|_{0,\overline{\Omega}}\right\}^{1-\{\theta\}} \\
&\times\left\{\|\partial^{\alpha-\beta}\chi_\epsilon \partial^\beta N(0, ..., g_{\mu_j}, ..., 0)\|_{1,\overline{\Omega}}\right\}^{\{\theta\}} \\
&\le C\epsilon^{\mu_j-\theta}\|g_{\mu_j}\|_{0,\partial\Omega}.
\end{aligned}
$$

If $|\beta| \ge \mu_j$,

$$
\begin{aligned}
&\left[\partial^{\alpha-\beta}\chi_\epsilon \partial^\beta N(0, ..., g_{\mu_j}, ..., 0)\right]_{\{\theta\},\overline{\Omega}} \le \|\partial^{\alpha-\beta}\chi_\epsilon\|_{0,\overline{\Omega}}\left[\partial^\beta N(0, ..., g_{\mu_j}, ..., 0)\right]_{\{\theta\},\overline{\Omega}} \\
&+\left[\partial^{\alpha-\beta}\chi_\epsilon\right]_{\{\theta\},\overline{\Omega}}\left\|\partial^\beta N\left(0, ..., g_{\mu_j}, ..., 0\right)\right\|_{0,\overline{\Omega}} \\
&\le C\left[\epsilon^{|\beta|-[\theta]}\|g_{\mu_j}\|_{|\beta|+\{\theta\}-\mu_j,\partial\Omega} + \epsilon^{|\beta|-\theta}\|g_{\mu_j}\|_{|\beta|-\mu_j,\partial\Omega}\right].
\end{aligned}
$$

Summing up, we obtain

$$
\begin{aligned}
&\|\chi_\epsilon N\left(0, ..., g_{\mu_j}, ..., 0\right)\|_{\theta,\overline{\Omega}} \\
&\le C\left[\sum_{r=0}^{[\theta]-\mu_j}\epsilon^{r+\mu_j-\theta}\|g_{\mu_j}\|_{r,\partial\Omega} + \sum_{r=0}^{[\theta]-\mu_j}\epsilon^{r+\mu_j-[\theta]}\|g_{\mu_j}\|_{r+\{\theta\},\partial\Omega}\right]
\end{aligned}
$$

so that

$$
\begin{aligned}
&\|\chi_\epsilon N(g_0, ..., g_{[\theta]})\|_{\theta,\overline{\Omega}} + |\lambda|^{\frac{\theta}{2m}}\|\chi_\epsilon N(g_0, ..., g_{[\theta]})\|_{0,\overline{\Omega}} \\
&\le C\sum_{j\in\mathcal{I}}\left(\sum_{r=0}^{[\theta]-\mu_j}\epsilon^{r+\mu_j-\theta}\|\mathcal{B}_j f\|_{r,\partial\Omega} + \sum_{r=0}^{[\theta]-\mu_j}\epsilon^{r+\mu_j-[\theta]}\|\mathcal{B}_j f\|_{r+\{\theta\},\partial\Omega}\right. \\
&\left.+|\lambda|^{\frac{\theta}{2m}}\epsilon^{\mu_j}\|\mathcal{B}_j f\|_{0,\partial\Omega}\right),
\end{aligned}
$$

with C independent of λ, f, ϵ. Choosing $\epsilon = |\lambda|^{-\frac{1}{2m}}$, which is possible if $|\lambda|$ is sufficiently large, we obtain

$$
\begin{aligned}
&|\lambda|\|u\|_{\theta,\overline{\Omega}} + \|u\|_{2m+\theta,\overline{\Omega}} \\
&\le C\left[\|f\|_{\theta,\overline{\Omega}} + \sum_{j\in\mathcal{I}}\left(\sum_{r=0}^{[\theta]-\mu_j}|\lambda|^{\frac{\theta-\mu_j-r}{2m}}\|\mathcal{B}_j f\|_{r,\partial\Omega}\right.\right. \\
&\left.\left.+\sum_{r=0}^{[\theta]-\mu_j}|\lambda|^{\frac{[\theta]-\mu_j-r}{2m}}\|\mathcal{B}_j f\|_{r+\{\theta\},\partial\Omega}\right)\right]
\end{aligned}
$$

which implies the desired estimate, using 1.2(a) and (0.2).

We conclude this section considering the stability of constant M in estimate (1.2) under small perturbations of the coefficients:

COROLLARY 1.7. *Assume that the assumptions* (H_θ) *are satisfied for some* $\theta \in]0,1[$; *let* $\mathcal{C} = \sum_{|\alpha|\le 2m} c_\alpha(x)\partial^\alpha$ *with coefficients* c_α *in* $C^\theta(\overline{\Omega})$ *and, for* $1 \le j \le m$, $\mathcal{D}_j = \sum_{|\beta|\le\mu_j} d_{j,\beta}(x')\partial^\beta$ *a linear partial differential operator of order less or equal to* μ_j, *with coefficients in* $C^{2m-\mu_j+\theta}(\partial\Omega)$; *assume that for a certain* $\delta > 0$

$$\sum_{|\alpha|\le 2m} \|c_\alpha\|_{\theta,\overline{\Omega}} + \sum_{j=1}^{m}\sum_{|\beta|\le\mu_j} \|d_{j,\beta}\|_{2m-\mu_j+\theta,\overline{\Omega}} \le \delta;$$

consider problem

$$\begin{cases} \lambda u - \mathcal{A}u - \mathcal{C}u = f, \\ \mathcal{B}_j u + \mathcal{D}_j u = g_j,\ 1 \le j \le m \end{cases} \tag{1.3}$$

with $f \in C^\theta(\overline{\Omega})$ *and, for* $1 \le j \le m$, $g_j \in C^{2m-\mu_j+\theta}(\partial\Omega)$; *then there exist* $M > 0, R > 0, \phi_0 \in]\frac{\pi}{2}, \pi[$ *independent of* $\mathcal{C}$ *and* $(\mathcal{D}_j)_{1\le j\le m}$ *such that, if* δ *is sufficiently small,* (1.3) *has a unique solution* $u \in C^{2m+\theta}(\overline{\Omega})$ *for any* $\lambda \in \mathbb{C}$ *with* $|Arg\lambda| \le \phi_0$, $|\lambda| \ge R$, $f \in C^\theta(\overline{\Omega})$, $(g_j)_{1\le j\le m} \in \prod_{j=1}^m C^{2m-\mu_j+\theta}(\partial\Omega)$.
Moreover,

$$|\lambda|\|u\|_{\theta,\overline{\Omega}} + \|u\|_{2m+\theta,\overline{\Omega}} \le M\Bigg[\|f\|_{\theta,\overline{\Omega}} + \sum_{\mu_j=0}|\lambda|^{\frac{\theta}{2m}}\|(\mathcal{B}_j+\mathcal{D}_j)f\|_{0,\partial\Omega}$$
$$+\sum_{j=1}^m \|g_j\|_{2m-\mu_j+\theta,\partial\Omega} + \sum_{j=1}^m |\lambda|^{\frac{2m-\mu_j+\theta}{2m}}\|g_j\|_{0,\partial\Omega}\Bigg].$$

PROOF. If δ is sufficiently small, it is easily seen that all the assumptions (H_θ) are satisfied replacing $\mathcal{A}$ with $\mathcal{A}+\mathcal{C}$ and, for $j = 1, \ldots, m$, $\mathcal{B}_j$ with $\mathcal{B}_j + \mathcal{D}_j$. By a simple perturbation argument, it is also easily seen that 1.4 can be extended to the perturbed system with R, ϕ_0 and M independent of $\mathcal{C}$ and $(\mathcal{D}_j)_{1\le j\le m}$. Consider now the proof of 1.6; it is easy to see that the Dirichlet system $\{\mathcal{B}_j + \mathcal{D}_j | \mu_j = 0\}$ admits a corresponding operator N (see 1.3) satisfying

$$\|N\|_{\mathcal{L}(C^s(\partial\Omega);C^s(\overline{\Omega}))} \le C(s)$$

for any $s \in [0, 2m+\theta]$, with $C(s)$ independent of $\mathcal{C}$ and $(\mathcal{D}_j)_{1\le j\le m}$; so the estimate of u_2 in 1.6 is uniform with respect to them. The only nontrivial point is the estimate of u_1; one has to show the following: there exist $M > 0, R > 0, \phi_0 \in]\frac{\pi}{2}, \pi[$ such that, if δ is sufficiently small, the solution u of

$$\begin{cases} \lambda u - \mathcal{A}u - \mathcal{C}u = f, \\ \mathcal{B}_j u + \mathcal{D}_j u = 0,\ 1 \le j \le m \end{cases} \tag{1.4}$$

with $\lambda \in \mathbb{C}$, $|Arg\lambda| \leq \phi_0, |\lambda| \geq R$, $f \in C^\theta(\overline{\Omega})$, $(\mathcal{B}_j + \mathcal{D}_j)f = 0$ if $\mu_j = 0$ satisfies

$$(1.5) \qquad |\lambda| \|u\|_{\theta,\overline{\Omega}} + \|u\|_{2m+\theta,\overline{\Omega}} \leq M\|f\|_{\theta,\overline{\Omega}}.$$

It is immediately seen that $\mathcal{B}_j u + \mathcal{D}_j u = 0$ is equivalent to $\mathcal{B}_j u = 0$. So, applying 1.6 to the unperturbed problem and 1.1(a), one has

$$\begin{aligned}
&|\lambda| \|u\|_{\theta,\overline{\Omega}} + \|u\|_{2m+\theta,\overline{\Omega}} + |\lambda|^{\frac{\theta}{2m}} \|u\|_{2m,\overline{\Omega}} \\
&\quad \leq M \left[\|f\|_{\theta,\overline{\Omega}} + \|\mathcal{C}u\|_{\theta,\overline{\Omega}} + |\lambda|^{\frac{\theta}{2m}} \sum_{\mu_j=0} \|\mathcal{B}_j \mathcal{C}u\|_{0,\partial\Omega} \right] \\
&\quad \leq M \left[\|f\|_{\theta,\overline{\Omega}} + C\delta \left(\|u\|_{2m+\theta,\overline{\Omega}} + |\lambda|^{\frac{\theta}{2m}} \|u\|_{2m,\overline{\Omega}} \right) \right]
\end{aligned}$$

implying the desired estimate if δ is sufficiently small.

REMARK 1.8. The result of 1.7 can be extended to every $\theta \in]0, 2m[- \mathbb{Z}$, but the general case is more complicated and will be given elsewhere.

2. – Autonomous parabolic problems

Assume that the assumptions (H_θ) are satisfied, for some $\theta \in \mathbb{R}^+$, $\theta \notin \mathbb{Z}$. We consider the following problem:

$$(2.1) \qquad \begin{cases} \partial_t u(t) - \mathcal{A}u(t) = f(t), 0 \leq t \leq T, \\ \mathcal{B}_j u(t) = g_j(t), 0 \leq t \leq T, 1 \leq j \leq m, \\ u(0) = u_0 \end{cases}$$

with $f \in C([0,T]; C(\overline{\Omega}))$, for $1 \leq j \leq m$ $g_j \in C([0,T]; C^{2m-\mu_j}(\overline{\Omega}))$, $u_0 \in C^{2m}(\overline{\Omega})$.

A *strict solution* of (2.1) is by definition an element $u \in C^1([0,T]; C(\overline{\Omega})) \cap C([0,T]; C^{2m}(\overline{\Omega}))$ satisfying the conditions (2.1).

We start by remarking that (2.1) has at most one strict solution:

LEMMA 2.1. *Problem* (2.1) *has at most one strict solution.*

PROOF. Let $\{T(t)|t \geq 0\}$ be the semigroup (not necessarily strongly continuous in 0) generated by A in $C(\overline{\Omega})$; let u be a strict solution of (2.1) with all the data $(u_0, f, (g_j)_{1\leq j\leq m}$ vanishing; as $\{u \in C^2(\overline{\Omega})|\mathcal{B}_j u = 0$ for $j = 1, ..., m\} \subseteq D(A)$, we have for every $t \in [0,T]$

$$u(t) = T(t)u_0 = 0.$$

We are now going to look for necessary conditions on f,$(g_j)_{1\leq j\leq m}$ and u_0 assuring that (2.1) has a strict solution u belonging to $B([0,T]; C^{2m+\theta}(\overline{\Omega}))$, with $u' \in B([0,T]; C^\theta(\overline{\Omega}))$, some $\theta \in]0, 2m[$, $\theta \notin \mathbb{Z}$; we shall use the following

LEMMA 2.2. *Let $u \in C^1([0,T];C(\overline{\Omega}))$ be such that $\partial_t u \in B([0,T];C^\theta(\overline{\Omega}))$ for some $\theta > 0$, $\theta \notin \mathbb{Z}$; then, if $0 \le s \le t \le T$, $u(t)-u(s) \in C^\theta(\overline{\Omega})$ and*

$$\|u(t)-u(s)\|_{C^\theta(\overline{\Omega})} \le \sup_{s\le\tau\le t} \|u'(\tau)\|_{C^\theta(\overline{\Omega})}(t-s).$$

PROOF. We have

$$u(t)-u(s) = \int_s^t u'(\tau)d\tau,$$

with the integral converging in $C(\overline{\Omega})$. So, we have that in this space

$$u(t)-u(s) = \lim_{k\to\infty} \frac{t-s}{k}\sum_{j=1}^k u'\left(s+\frac{j}{k}(t-s)\right).$$

But

$$\left\|\frac{t-s}{k}\sum_{j=1}^k u'\left(s+\frac{j}{k}(t-s)\right)\right\|_{C^\theta(\overline{\Omega})} \le \sup_{s\le\tau\le t}\|u'(\tau)\|_{C^\theta(\overline{\Omega})}(t-s).$$

So the result follows from 1.1(c).

LEMMA 2.3. *Assume that the assumptions (H_θ) are satisfied for some $\theta \in]0,2m[$, $\theta \notin \mathbb{Z}$ and (2.1) has a strict solution u such that $u \in B([0,T];C^{2m+\theta}(\overline{\Omega}))$ and $u' \in B([0,T];C^\theta(\overline{\Omega}))$; then, necessarily:*

(*a*) $u_0 \in C^{2m+\theta}(\overline{\Omega})$;

(*b*) $f \in C([0,T];C(\overline{\Omega})) \cap B([0,T];C^\theta(\overline{\Omega}))$;

(*c*) *if* $\theta < \mu_j$, $g_j \in B([0,T];C^{2m-\mu_j+\theta}(\partial\Omega)) \cap C^{1-\frac{\mu_j-\theta}{2m}}([0,T];C(\partial\Omega))$;

(*d*) *if* $\mu_j < \theta$, $g_j \in B([0,T];C^{2m-\mu_j+\theta}(\partial\Omega)) \cap C([0,T];C^{2m-\mu_j}(\partial\Omega)) \cap C^1([0,T];C(\partial\Omega))$, $g_j' \in B([0,T];C^{\theta-\mu_j}(\partial\Omega))$ *and* $g_j' - \mathcal{B}_j f \in C^{\frac{\theta-\mu_j}{2m}}([0,T];C(\partial\Omega))$;

(*e*) *for every* $j \in \{1,\ldots,m\}$ $\mathcal{B}_j u_0 = g_j(0)$;

(*f*) *if* $\mu_j < \theta$, $(g_j' - \mathcal{B}_j f)(0) = \mathcal{B}_j \mathcal{A} u_0$.

PROOF. (a) and (b) are obvious.

We show (c); it is clear that $g_j = \mathcal{B}_j u \in B([0,T];C^{2m-\mu_j+\theta}(\partial\Omega))$. On the other hand, by 2.2 and 1.1(a), as $\mu_j > \theta$, $u \in C^{1-\frac{\mu_j-\theta}{2m}}([0,T];C^{\mu_j}(\overline{\Omega}))$ which implies (c).

We show(d); if $\mu_j < \theta$, as $u' \in B([O,T];C^\theta(\overline{\Omega}))$, $\mathcal{B}_j u'$ is well defined and belongs to $B([0,T];C^{\theta-\mu_j}(\partial\Omega))$; as $u' \in C([0,T];C(\overline{\Omega})) \cap B([0,T];C^\theta(\overline{\Omega}))$, it follows from 1.1($a$) that $u' \in C([0,T];C^{\theta'}(\overline{\Omega}))$ for any $\theta' < \theta$. As a consequence, we have that $u \in C^1([0,T]; C^{\theta'}(\overline{\Omega}))$ for any $\theta' < \theta$, which implies that

$$\mathcal{B}_j u' = (\mathcal{B}_j u)' = g_j',$$

so that $g \in C^1([0,T]; C(\partial\Omega))$ and $g_j' \in B([0,T]; C^{\theta-\mu_j}(\partial\Omega))$. Next,

$$g_j' - \mathcal{B}_j f = \mathcal{B}_j \mathcal{A} u.$$

From 1.1(*a*) and 2.2, we have that $u \in C^{\frac{\theta-\mu_j}{2m}}([0,T]; C^{2m+\mu_j}(\overline{\Omega}))$, implying $\mathcal{B}_j \mathcal{A} u \in$
$C^{\frac{\theta-\mu_j}{2m}}([0,T]; C(\partial\Omega))$. So, (*d*) is proved.

(*e*) is obvious.

Finally, (*f*) follows again from $g_j' - \mathcal{B}_j f = \mathcal{B}_j \mathcal{A} u$ if $\mu_j < \theta$.

Our next target is to show that the conditions (*a*) – (*f*) in 2.3 are also sufficient to have a strict solution u such that $u \in B([0,T]; C^{2m+\theta}(\overline{\Omega}))$ with $u' \in B([0,T];$
$C^{\theta}(\overline{\Omega}))$. We start with the following lemma, which will be useful in certain estimates:

LEMMA 2.4. *Let c, α, β be real numbers, $c > 0, \alpha > 0, \alpha > \beta > \alpha - 1$. Then, there exists $C > 0$, depending on c, α, β such that for any $\xi > 0, t > 0$*

$$\xi^{\beta+1-\alpha} \int_0^t \left(\int_0^{+\infty} e^{-cr(t-s)} (\xi + r)^{-1} r^{\alpha} (t-s)^{\beta} dr \right) ds \le C.$$

PROOF. We have

$$\begin{aligned}
&\xi^{\beta+1-\alpha} \int_0^t \left(\int_0^{+\infty} e^{-cr(t-s)} (\xi + r)^{-1} r^{\alpha} (t-s)^{\beta} dr \right) ds \\
&= \xi^{\beta+1-\alpha} \int_0^t (t-s)^{\beta-\alpha} \left(\int_0^{+\infty} e^{-cr} [(t-s)\xi + r]^{-1} r^{\alpha} dr \right) ds \\
&= (t\xi)^{\beta+1-\alpha} \int_0^1 s^{\beta-\alpha} \left(\int_0^{+\infty} e^{-cr} (ts\xi + r)^{-1} r^{\alpha} dr \right) ds \\
&= \phi(t\xi),
\end{aligned}$$

with $\phi(\tau) = \tau^{\beta+1-\alpha} \int_0^1 s^{\beta-\alpha} (\int_0^{+\infty} e^{-cr} (\tau s + r)^{-1} r^{\alpha} dr) ds$. To conclude, it suffices to show that ϕ is bounded in $\mathbb{R}^+$. We have

$$\int_0^{+\infty} e^{-cr} (\tau s + r)^{-1} r^{\alpha} dr \le C_1 (\tau s)^{-1}$$

with $C_1 = \int_0^{+\infty} e^{-cr} r^{\alpha} dr$ and also

$$\int_0^{+\infty} e^{-cr} (\tau s + r)^{-1} r^{\alpha} dr \le C_2$$

with $C_2 = \int_0^{+\infty} e^{-cr} r^{\alpha-1} dr$. It follows

$$\int_0^{+\infty} e^{-cr}(\tau s + r)^{-1} r^{\alpha} dr \leq C_3 \min\{1, (\tau s)^{-1}\},$$

with $C_3 = \max\{C_1, C_2\}$. So,

$$\phi(\tau) \leq C_3 \tau^{\beta+1-\alpha} \int_0^1 s^{\beta-\alpha} \min\{1, (\tau s)^{-1}\} ds$$

which implies $\phi(\tau) \leq C_3 \tau^{\beta+1-\alpha} \int_0^1 s^{\beta-\alpha} ds$ if $0 < \tau \leq 1$.

If $\tau > 1$, we have

$$\phi(\tau) \leq C_3 \left(\tau^{\beta+1-\alpha} \int_0^{\frac{1}{\tau}} s^{\beta-\alpha} ds + \tau^{\beta-\alpha} \int_{\frac{1}{\tau}}^1 s^{\beta-\alpha-1} ds \right) \leq C_4$$

with C_4 depending only on C_3, α, β.

We assume that the assumptions (H_θ) are satisfied, for some $\theta \in]0, 2m[- \mathbb{Z}$. We consider again the operator A in $C(\overline{\Omega})$; we recall that $\{T(t) | t \geq 0\}$ is the semigroup (not necessarily strongly continuous in 0) generated by A. It is well known that, for $t > 0$,

$$T(t) = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} (\lambda - A)^{-1} d\lambda,$$

where γ is the clockwise oriented boundary of $\{\lambda \in \mathbb{C} | |Arg \lambda| \leq \phi_0, |\lambda| \geq R\}$ (see 1.4 for ϕ_0 and R). From (1.2) we have that there exists $M > 0$ such that, for $0 < t \leq T$, for any $f \in C^\theta(\overline{\Omega})$ we have

$$\text{(2.2)} \quad \|T(t)f\|_{\theta,\overline{\Omega}} + t\|T(t)f\|_{2m+\theta,\overline{\Omega}} \leq C \left(\|f\|_{\theta,\overline{\Omega}} + \sum_{\mu_j < \theta} t^{-\frac{\theta-\mu_j}{2m}} \|\mathcal{B}_j f\|_{0,\partial\Omega} \right).$$

We set $T^{(-1)}(t) := \int_0^t T(s) ds = \frac{1}{2\pi i} \int_\gamma \frac{e^{\lambda t}}{\lambda} (\lambda - A)^{-1} d\lambda$ and we have, again for $0 < t \leq T$,

$$\text{(2.3)} \quad \begin{aligned} &\|T^{(-1)}(t)f\|_{\theta,\overline{\Omega}} + t\|T^{(-1)}(t)f\|_{2m+\theta,\overline{\Omega}} \\ &\leq C \left(t\|f\|_{\theta,\overline{\Omega}} + \sum_{\mu_j < \theta} t^{1-\frac{\theta-\mu_j}{2m}} \|\mathcal{B}_j f\|_{0,\partial\Omega} \right). \end{aligned}$$

Remark also that $AT^{(-1)}(t) = T(t) - 1$. We have:

LEMMA 2.5. *Assume that the assumptions* (H_θ) *are satisfied for some* $\theta \in]0, 2m[-\mathbb{Z}$; *let* $f \in C([0,T]; C(\overline{\Omega})) \cap B([0,T]; C^\theta(\overline{\Omega}))$; *assume also that, if* $\mu_j < \theta$, $\mathcal{B}_j f \in C^{\frac{\theta-\mu_j}{2m}}([0,T]; C(\partial\Omega))$ *and* $\mathcal{B}_j f(0) = 0$. *Set*

$$u(t) := \int_0^t T(t-s)f(s)ds.$$

Then, u is a strict solution of (2.1) *with* $g_j \equiv 0$ *for any* $j = 1, \ldots, m$ *and* $u_0 = 0$. *Moreover,* u' *and* $\mathcal{A}u$ *belong to* $B([0,T]; C^\theta(\overline{\Omega}))$. *Finally, for any* $T_0 > 0$ *there exists* $C > 0$ *such that, if* $T \leq T_0$, *for any f with the declared properties*

$$\|u'\|_{B([0,T];C^\theta(\overline{\Omega}))} + \|\mathcal{A}u\|_{B([0,T];C^\theta(\overline{\Omega}))} \leq C \Bigg[\|f\|_{B([0,T];C^\theta(\overline{\Omega}))} + \sum_{\mu_j<\theta} [\mathcal{B}_j f]_{C^{\frac{\theta-\mu_j}{2m}}([0,T];C(\partial\Omega))} \Bigg].$$

PROOF. We start by remarking that the assumptions of 2.5 are exactly the conditions $(a)-(f)$ in 2.3 in case $u_0 = 0$ and $g_j(t) \equiv 0$ for any $j = 1, \ldots, m$.

We show that $u \in C([0,T]; C^{2m}(\overline{\Omega}))$: we set $u(t) = v(t) + z(t)$, with $v(t) = \int_0^t T(t-s)[f(s) - f(t)]ds$ and $z(t) = T^{(-1)}(t)f(t)$. First of all, we have that $f \in C([0,T]; C^{\theta'}(\overline{\Omega}))$ for any $\theta' < \theta$ and this implies that $(t,s) \to T(t-s)[f(s) - f(t)]$ belongs to $C(\{(t,s) \in \mathbb{R}^2 | 0 < s < t \leq T\}; C^{2m+\theta'}(\overline{\Omega}))$ for any $\theta' < \theta$. Moreover, by (2.2) and 1.1(a), we have that

$$\|T(t-s)[f(s) - f(t)]\|_{2m,\overline{\Omega}} \leq C(t-s)^{\frac{\theta}{2m}-1} \Bigg[\|f(s) - f(t)\|_{\theta,\overline{\Omega}} + \sum_{\mu_j<\theta} (t-s)^{-\frac{\theta-\mu_j}{2m}} \|\mathcal{B}_j(f(s) - f(t))\|_{0,\partial\Omega} \Bigg] \leq C(t-s)^{\frac{\theta}{2m}-1}.$$

Therefore, $v \in C([0,T]; C^{2m}(\overline{\Omega}))$. Clearly, $z \in C(]0,T]; C^{2m+\theta'}(\overline{\Omega}))$ for any $\theta' < \theta$. In force of (2.3),

$$\|z(t)\|_{2m,\overline{\Omega}} \leq C \left(t^{\frac{\theta}{2m}} \|f(t)\|_{\theta,\overline{\Omega}} + \sum_{\mu_j<\theta} t^{\frac{\mu_j}{2m}} \|\mathcal{B}_j f(t)\|_{0,\partial\Omega} \right)$$

which converges to 0 as $t \to O^+$ because, if $\mu_j = 0$, $\mathcal{B}_j f(0) = 0$. So, $u \in C([0,T]; C^{2m}(\overline{\Omega}))$ and, for any $t \in [0,T]$,

$$\mathcal{A}u(t) = \int_0^t AT(t-s)[f(s) - f(t)]ds + (T(t) - 1)f(t).$$

Let now $\epsilon \in]0, T[$. Set, for $\epsilon \leq t \leq T$,

$$u_\epsilon(t) = \int_0^{t-\epsilon} T(t-s)f(s)ds.$$

Then, $u_\epsilon \in C^1([\epsilon, T]; C(\overline{\Omega}))$ and, for $\epsilon \leq t \leq T$,

$$\begin{aligned} u'_\epsilon(t) &= T(\epsilon)f(t-\epsilon) + \int_0^{t-\epsilon} AT(t-s)f(s)ds \\ &= T(\epsilon)[f(t-\epsilon) - f(t)] + T(t)f(t) + \int_0^{t-\epsilon} AT(t-s)[f(s) - f(t)]ds. \end{aligned}$$

We have that $\lim_{\epsilon\to 0^+} \|u_\epsilon(t) - u(t)\|_{0,\overline{\Omega}} = 0$ uniformly in $[\delta, T]$ for any $\delta \in]0, T]$. Moreover, $\lim_{\epsilon\to 0^+} \|u'_\epsilon(t) - T(t)f(t) - \int_0^t AT(t-s)[f(s) - f(t)]ds\|_{0,\overline{\Omega}} = 0$ uniformly in $[\delta, T]$ for any $\delta \in]0, T]$. It follows that $u \in C^1(]0, T]; C(\overline{\Omega}))$, $u'(t) = T(t)f(t) + \int_0^t AT(t-s)[f(s) - f(t)]ds = \mathcal{A}u(t) + f(t)$ for any $t \in]0, T]$ and $\lim_{t\to 0} u'(t) = \lim_{t\to 0}(\mathcal{A}u(t) + f(t)) = f(0)$ in $C(\overline{\Omega})$). It follows that $u \in C^1([0, T]; C(\overline{\Omega}))$ and solves the first equation in (2.1). Next, we have that, for $0 < t \leq T$, for $1 \leq j \leq m$,

$$\mathcal{B}_j u(t) = \int_0^t \mathcal{B}_j T(t-s)[f(s) - f(t)]ds + \mathcal{B}_j T^{(-1)}(t)f(t) = 0.$$

So, u is a strict solution of (2.1) with $g_j \equiv 0$ for any $j = 1, ..., m$ and $u_0 = 0$. It remains to show that $\mathcal{A}u$ and (so, by difference) u' belong to $B([0, T]; C^\theta(\overline{\Omega}))$. We have that, for $0 < t \leq T$, $\mathcal{A}u(t) = \mathcal{A}v(t) + \mathcal{A}z(t)$. Now, $\mathcal{A}z(t) = T(t)f(t) - f(t)$ and from (2.2)

$$\begin{aligned} \|T(t)f(t)\|_{\theta,\overline{\Omega}} &\leq C\left[\|f(t)\|_{\theta,\overline{\Omega}} + \sum_{\mu_j<\theta} t^{-\frac{\theta-\mu_j}{2m}} \|\mathcal{B}_j f(t)\|_{0,\partial\Omega}\right]. \\ &\leq C\left(\|f\|_{B_\mu([0,T];C^\theta(\overline{\Omega}))} + \sum_{\mu_j<\theta} [\mathcal{B}_j f]_{C^{\frac{\theta-\mu_j}{2m}-\mu}([0,T];C(\partial\Omega))}\right) \end{aligned}$$

using the fact that $\mathcal{B}_j f(0) = 0$ if $\mu_j < \theta$. Observe that C is here independent of $T \leq T_0$.

It remains to show that also $\mathcal{A}v \in B([0, T]; C^\theta(\overline{\Omega}))$. To this aim, observe that by 1.5 $(C(\overline{\Omega}), D(A))_{\frac{\theta}{2m},\infty}$ is a closed subspace of $C^\theta(\overline{\Omega})$. We shall in fact show that $\mathcal{A}v \in B([0, T]; (C(\overline{\Omega}), D(A))_{\frac{\theta}{2m},\infty})$. We have for $0 < t \leq T$

$$\begin{aligned} \mathcal{A}v(t) &= \int_0^t AT(t-s)[f(s) - f(t)]ds \\ &= \frac{1}{2\pi i}\int_0^t \left(\int_{\gamma_0+2R} e^{\lambda(t-s)}A(\lambda - A)^{-1}[f(s) - f(t)]d\lambda\right) ds, \end{aligned}$$

where we have indicated with γ_0 the clockwise oriented boundary of $\{\lambda \in \mathbb{C} | |Arg(\lambda)| \leq \phi_0\}$ and with $\gamma_0 + 2R$ its shift to the right of $2R$ (here we have assumed, and this is obviously not restrictive, that $sin(\phi_0) \geq \frac{1}{2}$, so that $\gamma_0+2R \subseteq \rho(A)$). Using Cauchy's theorem one has for any $\xi > 0$:

$$(2R + \xi - A)^{-1}\mathcal{A}v(t) = \frac{1}{2\pi i}\int_0^t \left(\int_{\gamma_0+2R} e^{\lambda(t-s)}(2R+\xi-\lambda)^{-1}A(\lambda - A)^{-1}d\lambda\right) \times [f(s) - f(t)]ds$$

and

$$(2R - A)(2R + \xi - A)^{-1}\mathcal{A}v(t) = \frac{1}{2\pi i}\int_0^t \left(\int_{\gamma_0+2R} e^{\lambda(t-s)}(2R-\lambda)(2R+\xi-\lambda)^{-1}A(\lambda - A)^{-1}d\lambda\right) \times [f(s) - f(t)]ds$$

Now, for a certain $C > 0$,

$$\|\mathcal{A}v(t)\|_{\theta,\overline{\Omega}} \leq C \sup_{\xi>0} \xi^{\frac{\theta}{2m}} \|(A - 2R)(2R + \xi - A)^{-1}\mathcal{A}v(t)\|_{C(\overline{\Omega})}$$

and

$$\xi^{\frac{\theta}{2m}} \|(A - 2R)(2R + \xi - A)^{-1}\mathcal{A}v(t)\|_{0,\overline{\Omega}} = \xi^{\frac{\theta}{2m}} \|\frac{1}{2\pi i}\int_0^t \left(\int_{\gamma_0} e^{(\lambda+2R)(t-s)}\frac{\lambda}{\xi-\lambda}A(\lambda + 2R - A)^{-1}d\lambda\right) \times [f(s) - f(t)]ds\|_{0,\overline{\Omega}} \leq C\xi^{\frac{\theta}{2m}}$$
$$\int_0^t \left(\int_{\gamma_0} e^{(t-s)Re\lambda}\frac{|\lambda|}{|\xi-\lambda|}\|(\lambda + R - A)^{-1}[f(s) - f(t)]\|_{2m,\overline{\Omega}}|d\lambda|\right) ds.$$

Using $1.1(a)$ and (1.2), this expression can be majorized by

$$C\left[\xi^{\frac{\theta}{2m}}\int_0^t\left(\int_{\gamma_0} e^{(t-s)Re\lambda}\frac{|\lambda|^{1-\frac{\theta}{2m}}}{|\xi-\lambda|}d|\lambda|\right)ds\|f\|_{B([0,T];C^\theta(\overline{\Omega}))} + \sum_{\mu_j<\theta}\xi^{\frac{\theta}{2m}}\int_0^t\left(\int_{\gamma_0} e^{(t-s)Re\lambda}\frac{|\lambda|^{1-\frac{\mu_j}{2m}}}{|\xi-\lambda|}d|\lambda|\right) \times (t-s)^{\frac{\theta-\mu_j}{2m}}ds[\mathcal{B}_jf]_{C^{\frac{\theta-\mu_j}{2m}}([0,T];C(\partial\Omega))}\right]$$
$$\leq C\left(\|f\|_{B([0,T];C^\theta(\overline{\Omega}))} + \sum_{\mu_j<\theta}[\mathcal{B}_jf]_{C^{\frac{\theta-\mu_j}{2m}}([0,T];C(\partial\Omega))}\right),$$

applying 2.4.

We continue to assume that the assumptions (H_θ) are satisfied; let $1 \leq i \leq m, \lambda \in \mathbb{C}$ with $|Arg\lambda| \leq \phi_0$ and $|\lambda| \geq R$. Let, for $g \in C^{2m-\mu_i+\theta}(\partial\Omega)$ $N_i(\lambda)g$ be the solution u of (1.1) with $f = 0$, $g_i = g$, $g_j = 0$ if $j \neq i$. For $t > 0$ we set

$$K_i(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t} N_i(\lambda) d\lambda.$$

Let $T_0 > 0$; by (1.2) there exists $C > 0$ independent of g and $t \in]0, T_0]$ such that

$$\|K_i(t)g\|_{\theta,\overline{\Omega}} + t\|K_i(t)g\|_{2m+\theta,\overline{\Omega}} \leq C\left[\|g\|_{2m-\mu_i+\theta,\partial\Omega} + t^{\frac{\mu_i-\theta-2m}{2m}}\|g\|_{0,\partial\Omega}\right]. \tag{2.4}$$

Consider the case $\mu_i > \theta$ and set, for $t > 0$,

$$K_i^{(-1)}(t) := \int_0^t K_i(s)ds = \frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t}}{\lambda} N_i(\lambda) d\lambda; \tag{2.5}$$

We have that for any $T_0 > 0$ there exists $C > 0$ independent of g and $t \in]0, T_0]$ such that

$$\begin{aligned} &\|K_i^{(-1)}(t)g\|_{\theta,\overline{\Omega}} + t\|K_i^{(-1)}(t)g\|_{2m+\theta,\overline{\Omega}} \\ &\leq C\left[t\|g\|_{2m-\mu_i+\theta,\partial\Omega} + t^{\frac{\mu_i-\theta}{2m}}\|g\|_{0,\partial\Omega}\right]. \end{aligned} \tag{2.6}$$

One can easily verify that, for $t > 0$, $\mathcal{A}K_i(t) = K_i'(t), \mathcal{A}K_i^{(-1)}(t) = K_i(t)$ and, applying Cauchy's theorem, for $1 \leq j \leq m$, $\mathcal{B}_j K_i(t) = 0$, $\mathcal{B}_j K_i^{(-1)}(t) = \delta_{ij}$ (Kronecker's symbol).

We have:

LEMMA 2.6. *Assume that the assumptions (H_θ) are satisfied for some $\theta \in]0, 2m[-\mathbb{Z}$; let, for $1 \leq i \leq m$, $\mu_i > \theta$ and $g \in B([0,T]; C^{2m-\mu_i+\theta}(\partial\Omega)) \cap C^{1-\frac{\mu_i-\theta}{2m}}([0,T]; C(\partial\Omega))$; let, for $0 < t \leq T$*

$$v(t) = \int_0^t K_i(t-s)[g(s) - g(t)]ds.$$

Then, $v \in C([0,T]; C^{2m}(\overline{\Omega}))$ and $\mathcal{A}v \in B([0,T]; C^\theta(\overline{\Omega}))$. Finally, for any $T_0 > 0$ there exists $C > 0$ such that, if $0 < T \leq T_0$,

$$\|\mathcal{A}v\|_{B([0,T];C^\theta(\overline{\Omega}))} \leq C\left[\|g\|_{B([0,T];C^{2m-\mu_i+\theta}(\partial\Omega))} + [g]_{C^{1-\frac{\mu_i-\theta}{2m}}([0,T];C(\partial\Omega))}\right].$$

PROOF. As $g \in C([0,T]; C^{2m-\mu_i+\theta'}(\overline{\Omega}))$ for any $\theta' < \theta$, we have that $(t,s) \to K_i(t-s)[g(s)-g(t)] \in C(\{(t,s) \in \mathbb{R}^2 | 0 < s < t \le T\}; C^{2m+\theta'}(\overline{\Omega}))$ for any $\theta' < \theta$. Moreover, by (2.4) and 1.1(*a*),

$$\|K_i(t-s)[g(s)-g(t)]\|_{2m,\overline{\Omega}} \le C(t-s)^{\frac{\theta}{2m}-1}\Big[\|g(s)-g(t)\|_{2m-\mu_i+\theta,\partial\Omega} + (t-s)^{\frac{\mu_i-\theta-2m}{2m}}\|g(s)-g(t)\|_{0,\partial\Omega}\Big] \le C(t-s)^{\frac{\theta}{2m}-1}.$$

Therefore, $v \in C([0,T]; C^{2m}(\overline{\Omega}))$ and for any $t \in [0,T]$

$$\begin{aligned}\mathcal{A}v(t) &= \int_0^t \mathcal{A}K_i(t-s)[g(s)-g(t)]ds \\ &= \frac{1}{2\pi i}\int_0^t \left(\int_\gamma e^{\lambda(t-s)}\mathcal{A}N_i(\lambda)d\lambda\right)(g(s)-g(t))ds.\end{aligned}$$

It remains to show that $\mathcal{A}v \in B([0,T]; C^\theta(\overline{\Omega}))$; we shall in fact see that $\mathcal{A}v \in B([0,T]; (C(\overline{\Omega})), D(A))_{\frac{\theta}{2m},\infty})$. Let $\xi > 0$; using Cauchy's theorem, the fact that $\mathcal{A}N_i(\lambda) = \lambda N_i(\lambda)$ and formula

$$(2R+\xi-A)^{-1}N_i(\lambda) = (\lambda-2R-\xi)^{-1}[N_i(2R+\xi)-N_i(\lambda)]$$

we have that

$$\begin{aligned}&(2R+\xi-A)^{-1}\mathcal{A}v(t) \\ &= \frac{1}{2\pi i}\int_0^t \left(\int_{\gamma_0+2R} e^{\lambda(t-s)}(2R+\xi-\lambda)^{-1}\mathcal{A}N_i(\lambda)d\lambda\right)(g(s)-g(t))ds\end{aligned}$$

and

$$\begin{aligned}&(2R-A)(2R+\xi-A)^{-1}\mathcal{A}v(t) \\ &= \frac{1}{2\pi i}\int_0^t \left(\int_{\gamma_0+2R} e^{\lambda(t-s)}(2R-\lambda)(2R+\xi-\lambda)^{-1}\mathcal{A}N_i(\lambda)d\lambda\right)(g(s)-g(t))ds\end{aligned}$$

(Here γ_0 has the same meaning as in the proof of 2.5). Then the result follows with the same method of 2.5, using the assumptions on g, (1.2) and 2.4.

LEMMA 2.7. *Assume that all the assumptions of* 2.6 *are satisfied and, in addition,* $g(0) = 0$. *Set, for* $0 \le t \le T$,

$$u(t) := \int_0^t K_i(t-s)g(s)ds; \tag{2.7}$$

then, u is a strict solution of (2.1) with $f(t) \equiv 0$, $g_j(t) \equiv 0$ if $j \neq i$, $g_i = g$ and $u_0 = 0$. Moreover, u' and $\mathcal{A}u$ belong to $B([0,T]; C^\theta(\overline{\Omega}))$. Finally, for any $T_0 > 0$ there exists $C > 0$ such that, if $T \leq T_0$, for any g with the declared properties,

$$\|u'\|_{B([0,T];C^\theta(\overline{\Omega}))} + \|\mathcal{A}u\|_{B([0,T];C^\theta(\overline{\Omega}))} \leq C\Bigg[\|g\|_{B([0,T];C^{2m-\mu_i+\theta}(\partial\Omega))} + [g]_{C^{1-\frac{\mu_i-\theta}{2m}}([0,T];C(\partial\Omega))}\Bigg].$$

PROOF. We set $v(t) = \int_0^t K_i(t-s)[g(s)-g(t)]ds$, $z(t) = K_i^{(-1)}(t)g(t)$, in such a way that $u(t) = v(t) + z(t)$. We already know from 2.6 that $v \in C([0,T]; C^{2m}(\overline{\Omega}))$ and

$$\mathcal{A}v(t) = \int_0^t \mathcal{A}K_i(t-s)[g(s)-g(t)]ds$$

for every $t \in]0,T]$. From (2.6) and the condition $g(0) = 0$ we have also that $z \in C([0,T]; C^{2m}(\overline{\Omega})) \cap B([0,T]; C^{2m+\theta}(\overline{\Omega}))$, $\lim_{t\to 0+} \|K_i^{(-1)}(t)g(t)\|_{2m,\overline{\Omega}} = 0$ and for any $t \in]0,T]$

$$\mathcal{A}z(t) = \mathcal{A}K_i^{(-1)}(t)g(t) = K_i(t)g(t).$$

So, for any $t \in [0,T]$

$$\mathcal{A}u(t) = K_i(t)g(t) + \int_0^t \mathcal{A}K_i(t-s)[g(s)-g(t)]ds.$$

Let now $\epsilon \in]0,T[$. Set, for $\epsilon \leq t \leq T$,

$$u_\epsilon(t) = \int_0^{t-\epsilon} K_i(t-s)g(s)ds.$$

Clearly, $u_\epsilon \in C^1([\epsilon,T]; C(\overline{\Omega}))$ and, for $\epsilon \leq t \leq T$,

$$u'_\epsilon(t) = K_i(\epsilon)g(t-\epsilon) + \int_0^{t-\epsilon} \mathcal{A}K_i(t-s)g(s)ds.$$

We have

$$\begin{aligned} K_i(\epsilon)g(t-\epsilon) &+ \int_0^{t-\epsilon} \mathcal{A}K_i(t-s)g(s)ds \\ &= K_i(\epsilon)[g(t-\epsilon)-g(t)] + K_i(t)g(t) + \int_0^{t-\epsilon} \mathcal{A}K_i(t-s)[g(s)-g(t)]ds. \end{aligned}$$

Fix $\theta' \in]0, \theta[-\mathbb{Z}$; then,

$$\|K_i(\epsilon)[g(t-\epsilon)-g(t)]\|_{0,\overline{\Omega}} \leq \|K_i(\epsilon)[g(t-\epsilon)-g(t)]\|_{\theta',\overline{\Omega}}$$
$$\leq C\left(\|g(t-\epsilon)-g(t)\|_{\theta'+2m-\mu_i,\partial\Omega} + \epsilon^{\frac{\mu_i-\theta'-2m}{2m}}\|g(t-\epsilon)-g(t)\|_{0,\partial\Omega}\right).$$

We have that $\|g(t-\epsilon)-g(t)\|_{\theta'+2m-\mu_i,\partial\Omega} \to 0$ as $\epsilon \to 0^+$ uniformly in $[\delta, T]$ for any $\delta \in]0, T[$. Moreover,

$$\epsilon^{\frac{\mu_i-\theta'-2m}{2m}}\|g(t-\epsilon)-g(t)\|_{0,\partial\Omega} \leq C\epsilon^{\frac{\theta-\theta'}{2m}},$$

so that we conclude that, for any $\delta \in]0, T[$ $\lim_{\epsilon\to 0^+}\|u_\epsilon'(t) - \mathcal{A}u(t)\|_{0,\overline{\Omega}} = 0$, uniformly in $[\delta, T]$. As $\lim_{\epsilon\to 0^+}\|u_\epsilon(t) - u(t)\|_{0,\overline{\Omega}} = 0$, uniformly in $[\delta, T]$, we have that $u \in C^1(]0, T]; C(\overline{\Omega}))$ and for any $t \in]0, T]$ $u'(t) = \mathcal{A}u(t)$. As $u \in C([0, T]; C^{2m}(\overline{\Omega})) \cap B([0, T]; C^{2m+\theta}(\overline{\Omega}))$ and $\lim_{t\to 0}\|u(t)\|_{2m,\overline{\Omega}} = 0$, we conclude that $u \in C^1([0, T]; C$
$(\overline{\Omega}))$ and $u' \in B([0, T]; C^\theta(\overline{\Omega}))$.

Finally, let $j \in \{1, ..., m\}$; we have, for $0 < t \leq T$,

$$\begin{aligned}\mathcal{B}_j u(t) &= \int_0^t \mathcal{B}_j K_i(t-s)[g(s)-g(t)]ds + \mathcal{B}_j K_i^{(-1)}(t)g(t) \\ &= \delta_{ij}g(t).\end{aligned}$$

With this the proof is done.

THEOREM 2.8. *Assume that the assumptions* (H_θ) *are satisfied for some* $\theta \in]0, 2m[$, $\theta \notin \mathbb{Z}$; *then,* (2.1) *has a strict solution u such that* $u \in B([0, T]; C^{2m+\theta}(\overline{\Omega}))$ *and* $u' \in B([0, T]; C^\theta(\overline{\Omega}))$ *if and only if the following conditions are satisfied:*

(*a*) $u_0 \in C^{2m+\theta}(\overline{\Omega})$;

(*b*) $f \in C([0, T]; C(\overline{\Omega})) \cap B([0, T]; C^\theta(\overline{\Omega}))$;

(*c*) *if* $\theta < \mu_j$, $g_j \in B([0, T]; C^{2m-\mu_j+\theta}(\partial\Omega)) \cap C^{1-\frac{\mu_j-\theta}{2m}}([0, T]; C(\partial\Omega))$;

(*d*) *if* $\mu_j < \theta$, $g_j \in B([0, T]; C^{2m-\mu_j+\theta}(\partial\Omega)) \cap C([0, T]; C^{2m-\mu_j}(\partial\Omega)) \cap C^1([0, T]; C(\partial\Omega))$, $g_j' \in B([0, T]; C^{\theta-\mu_j}(\partial\Omega))$ *and* $g_j' - \mathcal{B}_j f \in C^{\frac{\theta-\mu_j}{2m}}(]0, T]; C(\partial\Omega))$;

(*e*) *for every* $j = 1, ..., m$ $\mathcal{B}_j u_0 = g_j(0)$.

(*f*) *if* $\mu_j < \theta$, $(g_j' - \mathcal{B}_j f)(0) = \mathcal{B}_j \mathcal{A} u_0$.

Finally, for any $T_0 > 0$ there exists $C > 0$ such that for any $T \in]0, T_0]$, if u is the solution of (2.1), *we have*

$$\begin{aligned} &\|u\|_{B([0,T];C^{2m+\theta}(\overline{\Omega}))} + \|u'\|_{B([0,T];C^{\theta}(\overline{\Omega}))} \\ &\leq C\Bigg[\|u_0\|_{2m+\theta,\overline{\Omega}} + \sum_{j=1}^{m} \|g_j\|_{B([0,T];C^{2m-\mu_j+\theta}(\partial\Omega))} \\ &+ \sum_{\mu_j<\theta} \|g_j'\|_{B([0,T];C^{\theta-\mu_j}(\partial\Omega))} + \|f\|_{B([0,T];C^{\theta}(\overline{\Omega}))} \\ &+ \sum_{\mu_j<\theta} [\mathcal{B}_j f - g_j']_{C^{\frac{\theta-\mu_j}{2m}}([0,T];C(\partial\Omega))}\Bigg]. \end{aligned} \tag{2.8}$$

PROOF. Let $(B_l)_{0\leq l\leq[\theta]}$ be a Dirichlet system of order $[\theta]$ in Ω with coefficients of B_l of class $C^{2m-l+\theta}(\overline{\Omega})$ and such that for $\mu_j < \theta$ $B_{\mu_j} = \mathcal{B}_j$; let, for $l = 0, ..., [\theta]$, $t \in]0, T]$, $h_l(t) = g_j(t) - \mathcal{B}_j u_0$ if $l = \mu_j$ for some $\mu_j < \theta$, $h_l(t) = 0$ otherwise. Set

$$\begin{aligned} u(t) &:= u_0 + N(h_0(t), ..., h_{[\theta]}(t)) \\ &+ \int_0^t T(t-s)[f(s) - (\partial_t - \mathcal{A})(u_0 + N(h_0, ..., h_{[\theta]}))(s)]ds \\ &+ \sum_{\mu_j>\theta} \int_0^t K_j(t-s)[g_j(s) - \mathcal{B}_j(u_0 + N(h_0(s), ..., h_{[\theta]}(s))]ds. \end{aligned} \tag{2.9}$$

Applying the assumptions, 2.5 and 2.7, it is not difficult to verify that u is a strict solution of (2.1) and $u \in B([0, T]; C^{2m+\theta}(\overline{\Omega}))$, $u' \in B([0, T]; C^{\theta}(\overline{\Omega}))$. Estimate (2.8) follows easily from 1.3, 2.5, 2.6, 2.7.

3. – Nonautonomous problems

In this third and final section we extend the result of 2.8 to the nonautonomous case for $\theta \in]0, 1[$. So, $0 < \theta < 1$, $m \in \mathbb{N}$ and Ω an open bounded subset of $\mathbb{R}^n$ of class $C^{2m+\theta}$; let, for $T > 0$, $t \in [0, T]$, $x \in \overline{\Omega}$ $\mathcal{A}(t) = \sum_{|\alpha|\leq 2m} a_\alpha(t, x)\partial_x^\alpha$ and, for $1 \leq j \leq m$, $t \in [0, T]$, $x' \in \partial\Omega$ $\mathcal{B}_j(t) = \sum_{|\beta|\leq\mu_j} b_{j,\beta}(t, x')\partial_x^\beta$; we assume that the following assumptions are satisfied:

(h_1) for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2m$ $a_\alpha \in C([0, T]; C^{\theta}(\overline{\Omega}))$;

(h_2) for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2m$ $a_{\alpha|\partial\Omega} \in C^{\frac{\theta}{2m}}([0, T]; C(\partial\Omega))$;

(h_3) for any $j = 1, ..., m$, $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq \mu_j$ $b_{j\beta} \in C([0, T]; C^{2m-\mu_j+\theta}(\partial\Omega))$;

(h_4) if $\mu_j \geq 1$, $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq \mu_j$, $b_{j\beta} \in C^{1-\frac{\mu_j-\theta}{2m}}([0,T]; C(\partial\Omega))$;

(h_5) if $\mu_j = 0$ $\mathcal{B}_j = \gamma$;

(h_6) for any $t \in [0,T]$ $\mathcal{A}(t)$ with the boundary operators $\{\mathcal{B}_j(t)|1 \leq j \leq m\}$ satisfies the conditions (a)-(e) in Section 1.

Owing to 1.7 and the compactness of $[0,T]$ we have that there exist $M > 0$, $R > 0$ and $\phi_0 \in]\frac{\pi}{2}, \pi[$ independent of $t \in [0,T]$ such that if $\lambda \in \mathbb{C}$, with $|Arg\lambda| \leq \phi_0$ and $|\lambda| \geq R$, for any $f \in C^\theta(\overline{\Omega})$, for any $(g_j)_{1\leq j\leq m} \in \prod_{j=1}^m C^{2m-\mu_j+\theta}(\partial\Omega)$ the problem

$$\left\{ \begin{array}{l} \lambda u - \mathcal{A}(t)u = f, \\ \mathcal{B}_j(t)u = g_j, \, 1 \leq j \leq m \end{array} \right. \tag{3.1}$$

has a unique solution $u \in C^{2m+\theta}(\overline{\Omega})$ and

$$\begin{aligned} |\lambda| \|u\|_{\theta,\overline{\Omega}} + \|u\|_{2m+\theta,\overline{\Omega}} \leq M & \Bigg[\|f\|_{\theta,\overline{\Omega}} + \alpha|\lambda|^{\frac{\theta}{2m}} \|f_{|\partial\Omega}\|_{0,\partial\Omega} \\ & + \sum_{j=1}^m \|g_j\|_{2m-\mu_j+\theta,\partial\Omega} + \sum_{j=1}^m |\lambda|^{\frac{2m-\mu_j+\theta}{2m}} \|g_j\|_{0,\partial\Omega} \Bigg], \end{aligned} \tag{3.2}$$

with $\alpha = 0$ if $\min_{1\leq j\leq m}\mu_j \geq 1$, $\alpha = 1$ otherwise.

Let now $\delta > 0$; we set

$$\begin{aligned} X_\delta := \{ & u \in C([0,\delta]; C^{2m}(\overline{\Omega})) \cap C^1([0,\delta]; C(\overline{\Omega})) \cap B([0,\delta]; C^{2m+\theta}(\overline{\Omega})) \\ & | u' \in B([0,\delta]; C^\theta(\overline{\Omega})) \text{ and if there exists } j \text{ such that } \mu_j = 0, u_{|\partial\Omega} = 0 \}. \end{aligned}$$

with norm

$$\|u\|_{X_\delta} := \|u\|_{B([0,\delta];C^{2m+\theta}(\overline{\Omega}))} + \|u'\|_{B([0,\delta];C^\theta(\overline{\Omega}))}.$$

With this norm X_δ is a Banach space; let now $u_0 \in C^{2m+\theta}(\overline{\Omega})$; we set: $X_\delta(u_0) := \{u \in X_\delta | u(0) = u_0\}$. $X_\delta(u_0)$ is a closed subset of X_δ.

We have

LEMMA 3.1. *Assume that Ω is an open subset of $\mathbb{R}^n$ of class $C^{2m+\theta}$ and the assumptions $(h_1) - (h_6)$ are satisfied; let $u \in X_\delta$; then,*

(a) $\mathcal{A}(.)u \in C([0,\delta]; C(\overline{\Omega})) \cap B([0,\delta]; C^\theta(\overline{\Omega}))$;

(b) $(\mathcal{A}(.)u)_{|\partial\Omega} \in C^{\frac{\theta}{2m}}([0,\delta]; C(\partial\Omega))$;

(c) for $j = 1, ..., m$ $\mathcal{B}_j(.)u \in C([0,\delta]; C^{2m-\mu_j}(\partial\Omega)) \cap B([0,\delta]; C^{2m-\mu_j+\theta}(\partial\Omega))$;

(d) if $\mu_j \geq 1$, $\mathcal{B}_j(.)u \in C^{1-\frac{\mu_j-\theta}{2m}}([0,\delta]; C(\partial\Omega))$;

(e) *there exists* $\omega :]0, T] \to]0, +\infty[$ *such that* $\lim_{\delta \to 0} \omega(\delta) = 0$ *and for every* $u \in X_\delta(0)$

$$\|[\mathcal{A}(.) - \mathcal{A}(0)]u\|_{B([0,\delta];C^\theta(\overline{\Omega}))} + \sum_{j=1}^{m} \|[\mathcal{B}_j(0) - \mathcal{B}_j(.)]u\|_{B([0,\delta];C^{2m-\mu_j+\theta}(\partial\Omega))}$$
$$+ \sum_{\mu_j \geq 1} [(\mathcal{B}_j(0) - \mathcal{B}_j(.))u]_{C^{1-\frac{\mu_j-\theta}{2m}}([0,\delta];C(\partial\Omega))} + [\gamma(\mathcal{A}(.) - \mathcal{A}(0))u]_{C^{\frac{\theta}{2m}};C(\partial\Omega))}$$
$$\leq \omega(\delta)\|u\|_{X_\delta}.$$

Proof. To prove $(a) - (d)$, one has only to apply again the fact that, if $u \in X_\delta$, $u \in C^{1-\frac{s-\theta}{2m}}([0, \delta]; C^s(\overline{\Omega}))$ for every $s \in]\theta, 2m + \theta[$,.

To prove (e), remark that, by 2.2, if $u \in X_\delta(0)$, for $t \in]0, \delta]$ $\|u(t)\|_{\theta,\overline{\Omega}} \leq t\|u\|_{X_\delta}$, so that, if $\theta \leq s \leq 2m + \theta, \|u(t)\|_{s,\overline{\Omega}} \leq C(s)t^{1-\frac{s-\theta}{2m}}\|u\|_{X_\delta}$ for every $t \in]0, \delta]$.

We are now in position to state and prove the main result of this section:

Theorem 3.2. *Assume that the assumptions* $(h1) - (h6)$ *are satisfied, with* $\theta \in]0, 1[$*; assume also that* u_0, f, $g_j (1 \leq j \leq m)$ *satisfy the conditions*

(a) $u_0 \in C^{2m+\theta}(\overline{\Omega})$;

(b) $f \in C([0, T]; C(\overline{\Omega})) \cap B([0, T]; C^\theta(\overline{\Omega}))$;

(c) *if* $\mu_j \geq 1$, $g_j \in B([0, T]; C^{2m-\mu_j+\theta}(\partial\Omega)) \cap C^{1-\frac{\mu_j-\theta}{2m}}([0, T]; C(\partial\Omega))$;

(d) *if* $\mu_j = 0$, $g_j \in B([0, T]; C^{2m+\theta}(\partial\Omega)) \cap C([0, T]; C^{2m}(\partial\Omega)) \cap C^1([0, T]; C(\partial\Omega))$, $g_j' \in B([0, T]; C^\theta(\partial\Omega))$ *and* $g_j' - f_{|\partial\Omega} \in C^{\frac{\theta}{2m}}([0, T]; C(\partial\Omega))$;

(e) *for every* $j \in \{1, ..., m\}$ $\mathcal{B}_j u_0 = g_j(0)$;

(f) *if* $\mu_j = 0$, $(g_j' - f_{|\partial\Omega})(0) = \mathcal{A}u_{0|\partial\Omega}$.

Then the following problem

$$\begin{cases} \partial_t u(t) - \mathcal{A}(t)u(t) = f(t), 0 < t \leq T, \\ \mathcal{B}_j(t)u(t) = g_j(t), 0 < t \leq T, 1 \leq j \leq m, \\ u(0) = u_0. \end{cases} \tag{3.3}$$

has a unique strict solution $u \in B([0, T]; C^{2m+\theta}(\overline{\Omega}))$ *with* $u' \in B([0, T]; C^\theta(\overline{\Omega}))$.

Proof. Subtracting, in case there exists $j \in \{1, ..., m\}$ such that $\mu_j = 0$, to u $u_0 + \alpha N(g_j - u_{0|\partial\Omega})$, where N is an operator of the type discribed in 1.3 with the Dirichlet system $\{\gamma\}$, we can assume $u_0 = 0$ and $g_j \equiv 0$ if $\mu_j = 0$. Let $\delta \in]0, T]$; for $v \in X_\delta(0)$ consider the problem

$$\begin{cases} \partial_t u(t) = \mathcal{A}(0)u(t) + [\mathcal{A}(t) - \mathcal{A}(0)]v(t) + f(t), 0 < t \leq \delta, \\ \mathcal{B}_j(0)u(t) = [\mathcal{B}_j(0) - \mathcal{B}_j(t)]v(t) + g_j(t), 0 < t \leq \delta, 1 \leq j \leq m, \\ u(0) = 0. \end{cases} \tag{3.4}$$

Owing to 3.1 and 2.8, (3.4) has a unique strict solution $u \in B([0,\delta]; C^{2m+\theta}(\overline{\Omega}))$ with $u' \in B([0,\delta]; C^{\theta}(\overline{\Omega}))$. Put now $Rv := u$; then, R is an operator of $X_\delta(0)$ in itself. Now we show that, if δ is sufficiently small, R is a contraction, so that it has a unique fixed point; in fact, let $v_j (j = 1, 2)$ belong to $X_\delta(0)$; then $Rv_1 - Rv_2$ solves

$$\begin{cases} \partial_t u(t) = \mathcal{A}(0)u(t) + [\mathcal{A}(t) - \mathcal{A}(0)][v_1(t) - v_2(t)], 0 < t \leq \delta, \\ \mathcal{B}_j(0)u(t) = [\mathcal{B}_j(0) - \mathcal{B}_j(t)][v_1(t) - v_2(t)], 0 < t \leq \delta, 1 \leq j \leq m, \\ u(0) = 0. \end{cases} \tag{3.5}$$

We can apply 3.1(e), so that, by (2.8), we have

$$\begin{aligned} \|R(v_1) - R(v_2)\|_{B([0,\delta];C^{2m+\theta}(\overline{\Omega}))} + \|R(v_1)' - R(v_2)'\|_{B([0,\delta];C^{\theta}(\overline{\Omega}))} \\ \leq C\omega(\delta)\|v_1 - v_2\|_{X_\delta} \end{aligned}$$

with C independent of δ and $\omega(\delta) \to 0$ as $\delta \to 0$. Therefore, if δ is sufficiently small, R is a contraction in X_δ. Remark that δ can be chosen independently of $f, (g_j)_{1\leq j\leq m}$. So, we have a unique strict solution of (3.3) in X_δ. We observe finally that the choice of δ can be done independently of the initial point $t_0 \in [0, T[$ (assuming to start from a point in $[0, T[$ not necessarily coinciding with 0), with the only restriction that $t_0 + \delta \leq T$. So, iterating the foregoing method starting, step after step, from $\delta, 2\delta, ...$, it is possible to construct a strict solution in $[0, T]$.

REMARK 3.3. It is not difficult to verify that, under the assumptions $(h_1) - (h_6)$, the conditions $(a) - (f)$ in 3.2 are also necessary in order to get a strict solution $u \in B([0,T]; C^{2m+\theta}(\overline{\Omega}))$ with $u' \in B([0,T]; C^{\theta}(\overline{\Omega})$.

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