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Monotonicity and Symmetry of Solutions of p-Laplace Equations, $1 < p < 2$, via the Moving Plane Method

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Abstract. In this paper we prove some monotonicity and symmetry properties of positive solutions of the equation $-\operatorname{div}(|Du|^{p-2} Du) = f(u)$ satisfying an homogenuous Dirichlet boundary condition in a bounded domain Ω. We assume $1 < p < 2$ and f locally Lipschitz continuous and we do not require any hypothesis on the critical set of the solution. In particular we get that if Ω is a ball then the solutions are radially symmetric and strictly radially decreasing.

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1. - Introduction

In this paper we consider the problem

$$
\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}
$$

where Δ_p denotes the p-Laplacian operator $\Delta_p u = \operatorname{div}(|Du|^{p-2} Du)$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and f is a locally Lipschitz continuous function.

We are interested in studying monotonicity and symmetry properties of solutions of (1.1) in dependence of the geometry of the domain Ω.

In the case $p = 2$ several results have been obtained starting with the famous paper [GNN] by Gidas, Ni and Nirenberg where, among other things, it is proved that, if Ω is a ball and $p = 2$, solutions of (1.1) are radially symmetric and strictly radially decreasing. This paper had a big impact not only in virtue of the several monotonicity and symmetry results that it contains,

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but also because it brought to attention the moving plane method which, since then, has been largely used in many different problems. This method, which is essentially based on maximum principles, goes back to Alexandrov [H] and was first used by Serrin in [S]. Quite recently the moving plane method has been improved and simplified by Berestycki and Nirenberg in [BN] with the aid of the maximum principle in small domains.

Very little is known about the monotonicity and symmetry of solutions of (1.1) when $p \neq 2$. In this case the solutions can only be considered in a weak sense since, generally, they belong to the space $C^{1,\alpha}(\Omega)$ (see [Di] and [T]). Anyway this is not a difficulty because the moving plane method can be adapted to weak solutions of strictly elliptic problems in divergence form (see [D] and [Da1]).

The real difficulty with problem (1.1), for $p \neq 2$, is that the p-Laplacian operator is degenerate in the critical points of the solutions, so that comparison principles (which could substitute the maximum principles in order to use the moving plane method when the operator is not linear) are not available in the same form as for $p = 2$. Actually counterexamples both to the validity of comparison principles and to the symmetry results are available (see [GKPR], [Br]) for any p with different degrees of regularity of f.
Before stating our main theorems let us recall some known results about (1.1).

When Ω is a ball in [BaNa] the symmetry of the solutions of (1.1) is obtained assuming that their gradient vanishes only at the center.

In [GKPR] by a suitable approximation procedure is shown that isolated solutions with nonzero index, in suitable function spaces, are symmetric.

A different approach is used in [KP] where, using symmetrization techniques, is proved that if $p = N$, Ω is a ball and f is only continuous, but $f(s) > 0$ for $s > 0$, then u is radially symmetric and strictly radially decreasing.

While we were completing this paper F. Brock told us that in [Br] he gets the symmetry result in the ball in the case $1 < p < 2$ or $p > 2$ but f monotone. For other symmetric domains he shows that solutions are "locally symmetric" in a suitable sense defined in [Br]. His method does not use comparison principles but the so called "continuous Steiner symmetrization".

A first step towards extending the moving plane method to solutions of problems involving the p-Laplacian operator has been done in [Da2]. In this paper the author mainly proves some weak and strong comparison principles for solutions of differential inequalities involving the p-Laplacian. Using these principles he adapts the moving plane method to solutions of (1.1) getting some monotonicity and symmetry results in the case $1 < p < 2$. Although the comparison principles of [Da2] are quite powerful for $1 < p < 2$, the symmetry result is not complete and relies on the assumption that the set of the critical points of u does not disconnect the caps which are constructed by the moving plane method.

In this paper we use the results of [Da2] to get monotonicity and symmetry for solutions u of (1.1) in smooth domains in the case $1 < p < 2$ without extra-assumptions on u.

To state our results we need some notations.

Let ν be a direction in $\mathbb{R}^N$, i.e. $\nu \in \mathbb{R}^N$ and $|\nu| = 1$. For a real number λ we define

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\} \tag{1.2}$$

$$\Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\} \tag{1.3}$$

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2\,(\lambda - x \cdot \nu)\,\nu\,, \qquad x \in \mathbb{R}^N \tag{1.4}$$

(i.e. R_λ^ν is the reflection through the hyperplane T_λ^ν)

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu\,. \tag{1.5}$$

If $\lambda > a(\nu)$ then Ω_λ^ν is nonempty, thus we set

$$\left(\Omega_\lambda^\nu\right)' = R_\lambda^\nu\left(\Omega_\lambda^\nu\right). \tag{1.6}$$

Following [GNN] we observe that if Ω is smooth and $\lambda > a(\nu)$, with $\lambda - a(\nu)$ small, then the reflected cap $\left(\Omega_\lambda^\nu\right)'$ is contained in Ω and will remain in it, at least until one of the following occurs:

(i) $\left(\Omega_\lambda^\nu\right)'$ becomes internally tangent to $\partial\Omega$ at some point not on T_λ^ν

(ii) T_λ^ν is orthogonal to $\partial\Omega$ at some point.

Let $\Lambda_1(\nu)$ be the set of those $\lambda > a(\nu)$ such that for each $\mu \in (a(\nu), \lambda)$ none of the conditions (i) and (ii) holds and define

$$\lambda_1(\nu) = \sup \Lambda_1(\nu)\,. \tag{1.7}$$

The main result of the paper is the following.

THEOREM 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and $u \in C^1(\overline{\Omega})$ a weak solution of* (1.1) *with $1 < p < 2$. For any direction ν and for λ in the interval* $(a(\nu)\,,\, \lambda_1(\nu)]$ *we have*

$$u(x) \leq u\left(x_\lambda^\nu\right) \qquad \forall\, x \in \Omega_\lambda^\nu\,. \tag{1.8}$$

Moreover

$$\frac{\partial u}{\partial \nu}(x) > 0 \qquad \forall\, x \in \Omega_{\lambda_1(\nu)}^\nu \setminus Z \tag{1.9}$$

where $Z = \{x \in \Omega : Du(x) = 0\}$.

Easy consequences of Theorem 1.1 are the following.

COROLLARY 1.1. *If, for a direction* ν, *the domain* Ω *is symmetric with respect to the hyperplane* $T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$ *and* $\lambda_1(\nu) = \lambda_1(-\nu) = 0$, *then* u *is symmetric,* i.e. $u(x) = u(x_0^\nu)$ *for any* $x \in \Omega$, *and decreasing in the* ν *direction in* Ω_0^ν. *Moreover* $\frac{\partial u}{\partial \nu} > 0$ *in* $\Omega_0^\nu \setminus Z$.

COROLLARY 1.2. *Suppose that* Ω *is the ball* $B_R(0)$ *in* $\mathbb{R}^N$ *with center at the origin and radius* R. *Then* u *is radially symmetric and* $\frac{\partial u}{\partial r} < 0$ *for* $0 < r < R$.

Note that the previous theorem implies also a regularity result since from $Du \neq 0$ in $B_R(0) \setminus \{0\}$, by standard regularity results, we deduce that u belongs to $C^2(B_R(0) \setminus \{0\})$.

The proof of Theorem 1.1 is long and technically quite complicated, therefore we would like to illustrate the main ideas beyond it, so to clarify also the role of the smoothness of $\partial\Omega$.

The starting point is Theorem 1.5 of [Da2] which is presented and extended in Section 3 (Theorem 3.1). This theorem asserts that, once we start the moving plane procedure, we must reach the maximal possible position (see the Definition 3.1 of $\lambda_2(\nu)$ in Section 3) unless the set Z of the critical points of u creates a connected component C of the set where $Du \neq 0$, which is symmetric with respect to a certain hyperplane $T^\nu_{\lambda_0(\nu)}$ ($\lambda_0(\nu)$ is defined in (3.5)) and where u coincides with the symmetric function $u^\nu_{\lambda_0(\nu)}$. Therefore all the subsequent efforts are in the direction of proving that such a set C cannot exist.

A first result, deduced from Proposition 3.1, is that if u is constant on a connected subset of critical points of ∂C whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains an open subset of $T^\nu_{\lambda_0(\nu)}$ then such a set C cannot exist. This is proved by a careful use of the Hopf's lemma and gives a property of the critical set of a solution u of (1.1) which is interesting in itself (see Proposition 3.1).

As explained in Remark 3.1 the hypothesis that u is constant on a connected set of critical points (which could appear obviously satisfied by any C^1 function) is not unnecessary, since could not hold if the critical set of u is very singular. Therefore, if the critical set Z of u is not very bad the assertion of Theorem 1.1 is a consequence of Theorem 3.1 and Proposition 3.1, without exploiting any smoothness assumption on $\partial\Omega$ (see Remark 3.1).

But, of course, one cannot in general have "a priori" informations on the critical set Z. Hence, to prove that u is constant on a connected subset of critical points of ∂C whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains an open subset of the hyperplane, some extra work is needed.

To do that a new argument is presented in Section 4 and consists in moving hyperplanes orthogonal to directions close to ν in order to prove that the "bad" set C is also symmetric with respect to nearby hyperplanes and hence (see Step 3 in the proof of Theorem 1.1 in Section 4) on its boundary there is at least one connected piece where u is constant, $Du = 0$, and whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains an open subset of the hyperplane.

This procedure of moving nearby hyperplanes to be efficient, needs a certain continuity of the minimal and maximal positions of the hyperplanes T^ν_λ with respect to ν. It is this very continuity property that is ensured whenever $\partial\Omega$ is smooth and fails for simple domains with corners, as for example the square in the plane. We think that this method of looking at directions close to a fixed direction ν could be useful also in other problems.

The paper is organized as follows. In Section 2 we state the maximum and comparison principles needed in the sequel. In Section 3 we recall and extend Theorem 1.5 of [Da2] and present some results for general domains. In Section 4 we give the proof of Theorem 1.1 and its corollaries.

2. – Preliminaries

In this section we recall some known results about solutions of equations involving the p-Laplacian operator. We begin with a version of the strong maximum principle and of the Hopf's lemma for the p-Laplacian. It is a particular case of a result proved in [V].

THEOREM 2.1 (Strong Maximum Principle and Hopf's Lemma). *Let Ω be a domain in $\mathbb{R}^N$ and suppose that $u \in C^1(\Omega)$, $u \geq 0$ in Ω, weakly solves*

$$-\Delta_p u + cu^q = g \geq 0 \qquad \text{in } \Omega \tag{2.1}$$

with $1 < p < \infty$, $q \geq p-1$, $c \geq 0$ and $g \in L^\infty_{\mathrm{loc}}(\Omega)$. If $u \not\equiv 0$ then $u > 0$ in Ω. Moreover for any point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied, and such that $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$ we have that $\frac{\partial u}{\partial s} > 0$ for any inward directional derivative (this means that if y approaches x_0 in a ball $B \subseteq \Omega$ that has x_0 on its boundary then $\lim_{y\to x_0} \frac{u(y)-u(x_0)}{|y-x_0|} > 0$).

Next we recall some weak and strong comparison principles, whose proofs can be found in [Da2] (see Theorem 1.2, Theorem 1.4 therein and the remarks that follow).

Suppose that Ω is a bounded domain in $\mathbb{R}^N$ and that $u, v \in C^1(\overline{\Omega})$ weakly solve

$$\begin{cases} -\Delta_p u \leq f(u) & \text{in } \Omega \\ -\Delta_p v \geq f(v) & \text{in } \Omega \end{cases} \tag{2.2}$$

with $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous. For any set $A \subseteq \Omega$ we define

$$M_A = M_A(u, v) = \sup_A (|Du| + |Dv|) \tag{2.3}$$

and denote by $|A|$ its Lebesgue measure.

THEOREM 2.2 (Weak Comparison Principle). *Suppose that $1 < p < 2$, then there exist α, $M > 0$, depending on p, $|\Omega|$, M_Ω and the L^∞ norms of u and v such that: if an open set $\Omega' \subseteq \Omega$ satisfies $\Omega' = A_1 \cup A_2$, $|A_1 \cap A_2| = 0$, $|A_1| < \alpha$, $M_{A_2} < M$ then $u \leq v$ on $\partial\Omega'$ implies $u \leq v$ in Ω'.*

THEOREM 2.3 (Strong Comparison Principle). *Suppose that* $1 < p < \infty$ *and define* $Z_u^v = \{x \in \Omega : Du(x) = Dv(x) = 0\}$. *If* $u \leq v$ *in* Ω *and there exists* $x_0 \in \Omega \setminus Z_u^v$ *with* $u(x_0) = v(x_0)$, *then* $u \equiv v$ *in the connected component of* $\Omega \setminus Z_u^v$ *containing* x_0.

REMARK 2.1. A function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous if and only if for each $R > 0$ there exist $C_1(R)$, $C_2(R) \geq 0$ such that

(i) $f_1(s) = f(s) - C_1 s$ is nonincreasing in $[-R, R]$
(ii) $f_2(s) = f(s) + C_2 s$ is nondecreasing in $[-R, R]$.

In the proof of Theorem 2.2 only (i) is used, while the proof of Theorem 2.3 only exploits (ii).

3. – Results for general domains

Here we prove some monotonicity and symmetry results for bounded domains which do not need to be smooth. From now on, p will be a fixed number in the interval $(1, 2)$, Ω a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and $f : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz continuous function. For any direction ν let $a(\nu)$, Ω_λ^ν, $(\Omega_\lambda^\nu)'$ be as defined in Section 1. Next we define

$$\Lambda_2(\nu) = \left\{\lambda > a(\nu) : \left(\Omega_\mu^\nu\right)' \subset \Omega \text{ for any } \mu \in (a(\nu), \lambda]\right\}$$

and, if $\Lambda_2(\nu) \neq \emptyset$

$$\lambda_2(\nu) = \sup \Lambda_2(\nu) . \tag{3.1}$$

We observe that if Ω is smooth then

$$\emptyset \neq \Lambda_1(\nu) \subseteq \Lambda_2(\nu)$$

where $\Lambda_1(\nu)$ is defined as in Section 1.

If $a(\nu) < \lambda \leq \lambda_2(\nu)$, $x \in \Omega_\lambda^\nu$, $u \in C^1(\overline{\Omega})$ we set

$$u_\lambda^\nu(x) = u(x_\lambda^\nu) \tag{3.2}$$

where x_λ^ν is as in (1.4),

$$Z_\lambda^\nu = Z_\lambda^\nu(u) = \{x \in \Omega_\lambda^\nu : Du(x) = Du_\lambda^\nu(x) = 0\} \tag{3.3}$$

$$Z = Z(u) = \{x \in \Omega : Du(x) = 0\} . \tag{3.4}$$

Finally we define

$$\Lambda_0(\nu) = \left\{\lambda \in (a(\nu), \lambda_2(\nu)] : u \leq u_\mu^\nu \text{ in } \Omega_\mu^\nu \text{ for any } \mu \in (a(\nu), \lambda]\right\} .$$

If $\Lambda_0(\nu) \neq \emptyset$ we set

$$\lambda_0(\nu) = \sup \Lambda_0(\nu) . \tag{3.5}$$

Obviously we have $\lambda_0(\nu) \leq \lambda_2(\nu)$.

Let us now state a first result which is a different formulation and an extension of Theorem 1.5 in [Da2]. We present the details of the proof, since we need them in the proof of Theorem 1.1.

THEOREM 3.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), with $1 < p < 2$. For any direction ν such that $\Lambda_2(\nu) \neq \emptyset$ we have that $\Lambda_0(\nu) \neq \emptyset$ and, if $\lambda_0(\nu) < \lambda_2(\nu)$ then there exists at least one connected component C^ν of $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ such that $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν.*

For any such component C^ν we also get

$$Du(x) \neq 0 \qquad \forall\, x \in C^\nu \tag{3.6}$$

$$Du(x) = 0 \qquad \forall\, x \in \partial C^\nu \setminus \left(T^\nu_{\lambda_0(\nu)} \cup \partial\Omega\right). \tag{3.7}$$

Moreover for any λ with $a(\nu) < \lambda < \lambda_0(\nu)$ we have

$$u < u^\nu_\lambda \qquad \text{in } \Omega^\nu_\lambda \setminus Z^\nu_\lambda \tag{3.8}$$

and finally

$$\frac{\partial u}{\partial \nu}(x) > 0 \qquad \forall\, x \in \Omega^\nu_{\lambda_0(\nu)} \setminus Z\,. \tag{3.9}$$

PROOF. Since the proof is quite long we divide it in three steps.

STEP 1. Let ν be a direction such that $\Lambda_2(\nu) \neq \emptyset$. For $a(\nu) < \lambda \leq \lambda_2(\nu)$ we compare the functions u and u^ν_λ in the open set Ω^ν_λ using Theorem 2.2 and Theorem 2.3, since u^ν_λ satisfies in Ω^ν_λ the same equation $-\Delta_p u^\nu_\lambda = f(u^\nu_\lambda)$. In particular, since

$$\|u\|_{L^\infty(\Omega^\nu_\lambda)},\ \|u^\nu_\lambda\|_{L^\infty(\Omega^\nu_\lambda)} \leq \|u\|_{L^\infty(\Omega)}, \quad |Du| + |Du^\nu_\lambda| \leq 2\sup_\Omega |Du|, \quad |\Omega^\nu_\lambda| \leq |\Omega|$$

we can fix α, $M > 0$, indipendent from λ, so that Theorem 2.2 applies in Ω^ν_λ to u and $v = u^\nu_\lambda$.

If $\lambda > a(\nu)$ and $\lambda - a(\nu)$ is small then $|\Omega^\nu_\lambda|$ is small. Moreover we have on $\partial\Omega^\nu_\lambda$ the inequality $u \leq u^\nu_\lambda$, since $0 = u \leq u^\nu_\lambda$ on $\partial\Omega \cap \partial\Omega^\nu_\lambda$, while $u = u^\nu_\lambda$ on $\partial\Omega \cap T^\nu_\lambda$ by definition. Therefore, by Theorem 2.2 we have that $u \leq u^\nu_\lambda$ in Ω^ν_λ for $\lambda > a(\nu)$, $\lambda - a(\nu)$ small, so that $\Lambda_0(\nu) \neq \emptyset$.

STEP 2. In the cap $\Omega^\nu_{\lambda_0(\nu)}$, by continuity, the inequality $u \leq u^\nu_{\lambda_0(\nu)}$ holds. Moreover,by Theorem 2.3 we have that if C^ν is a connected component of $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ then either $u < u^\nu_{\lambda_0(\nu)}$ in C^ν or $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν.

Suppose now that $\lambda_0(\nu) < \lambda_2(\nu)$ and assume, arguing by contradiction, that $u < u^\nu_{\lambda_0(\nu)}$ in $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$. Let us choose an open set A, with $Z^\nu_{\lambda_0(\nu)} \subset A \subset \Omega^\nu_{\lambda_0(\nu)}$, such that $M_{A,\lambda_0} = \sup_A(|Du| + |Du^\nu_{\lambda_0(\nu)}|) < \frac{M}{2}$ (this is possible since $Du = Du^\nu_{\lambda_0(\nu)} = 0$ in $Z^\nu_{\lambda_0(\nu)}$). We also fix a compact set $K \subset \Omega^\nu_{\lambda_0(\nu)}$ such that $|\Omega^\nu_{\lambda_0(\nu)} \setminus K| < \frac{\alpha}{2}$ (α and M being the numbers fixed in Step 1). If $K \setminus A \neq \emptyset$, by our assumption the function $u^\nu_{\lambda_0(\nu)} - u$ is positive there, and since $K \setminus A$ is compact we have that $\min_{K\setminus A}\ (u^\nu_{\lambda_0(\nu)} - u) = m > 0$. By continuity there exists

$\epsilon > 0$ such that $\lambda_0(\nu) + \epsilon < \lambda_2(\nu)$ and such that for $\lambda_0(\nu) < \lambda < \lambda_0(\nu) + \epsilon$ we have

$$|\Omega_\lambda^\nu \setminus K| < \alpha, \quad M_{A,\lambda} = \sup_A \left(|Du| + |Du_\lambda^\nu|\right) < M$$

and, if $K \setminus A \neq \emptyset$,

$$u_\lambda^\nu - u > \frac{m}{2} > 0 \qquad \text{in } K \setminus A$$

(in particular $u_\lambda^\nu - u > 0$ on $\partial(K \setminus A)$). Moreover for such values of λ we have that $u \leq u_\lambda^\nu$ on $\partial\left(\Omega_\lambda^\nu \setminus (K \setminus A)\right)$ because if x_0 is a point on that boundary either $x_0 \in \partial\Omega_\lambda^\nu$ where $u \leq u_\lambda^\nu$ as already observed in Step 1, or $x_0 \in \partial(K \setminus A)$ where $u_\lambda^\nu - u$ is positive.

Since $\Omega_\lambda^\nu \setminus (K \setminus A)$ is the disjoint union of $\Omega_\lambda^\nu \setminus K = A_1$ (with small measure) and $K \cap A = A_2$ (where the gradients are small), from Theorem 2.2 we get that $u \leq u_\lambda^\nu$ in $\Omega_\lambda^\nu \setminus (K \setminus A)$. Since, if $K \setminus A \neq \emptyset$, we have that $u_\lambda^\nu - u > 0$ in $K \setminus A$, we obtain that $u \leq u_\lambda^\nu$ in Ω_λ^ν for $\lambda_0(\nu) < \lambda < \lambda_0(\nu) + \epsilon$. This contradicts the definition of $\lambda_0(\nu)$ and shows that if $\lambda_0(\nu) < \lambda_2(\nu)$ then it is not possible that $u < u_{\lambda_0(\nu)}^\nu$ in $\Omega_{\lambda_0(\nu)}^\nu \setminus Z_{\lambda_0(\nu)}^\nu$, so that there exists at least one connected component C^ν of $\Omega_{\lambda_0(\nu)}^\nu \setminus Z_{\lambda_0(\nu)}^\nu$ such that $u \equiv u_{\lambda_0(\nu)}^\nu$ in C^ν.

If C^ν is such a connected component, by definition $|Du| + |Du_{\lambda_0(\nu)}^\nu| > 0$ in C^ν, but since $u \equiv u_{\lambda_0(\nu)}^\nu$ we also have that $Du \neq 0$ in C^ν, i.e. (3.6).

Finally, by the very definition of C^ν, we have (3.7) since $\partial C^\nu \subset T_{\lambda_0(\nu)}^\nu \cup \partial\Omega \cup Z$ (Z is defined in (3.4)).

STEP 3. To prove (3.8) is enough to show that

$$u < u_\lambda \qquad \text{in } \Omega_\lambda^\nu \setminus Z \qquad \text{if} \quad a(\nu) < \lambda < \lambda_0(\nu)\,. \tag{3.10}$$

In fact if (3.8) is false and $u(x_0) = u_\lambda^\nu(x_0)$ for a point $x_0 \in \Omega_\lambda^\nu \setminus Z_\lambda^\nu$, then $u \equiv u_\lambda^\nu$ in the component of $\Omega_\lambda^\nu \setminus Z_\lambda^\nu$ to which x_0 belongs, and this implies that both $|Du(x_0)|$ and $|Du_{\lambda_0(\nu)}^\nu(x_0)|$ are not zero, i.e. $x_0 \in \Omega_\lambda^\nu \setminus Z$ so that (3.10) does not hold.

Let us now prove (3.10) and assume, for simplicity of notations, that $\nu = e_1 = (1, 0, \dots, 0)$. We write coordinates in $\mathbb{R}^N$ as $x = (y, z)$ with $y \in \mathbb{R}$, $z \in \mathbb{R}^{N-1}$ and we omit the superscript $\nu = e_1$ in Ω_λ^ν, u_λ^ν, etc.

Suppose, by contradiction, that there exists μ, with $a(e_1) < \mu < \lambda_0(e_1)$ and $x_0 = (y_0, z_0) \in \Omega_\mu \setminus Z$ such that $u(x_0) = u_\mu(x_0)$. By Theorem 2.3 we have that $u \equiv u_\mu$ in the component C of $\Omega_\mu \setminus Z_\mu$ to which x_0 belongs. If $\lambda > \mu$ and $\lambda - \mu$ is small we have that $(x_0)_\lambda = (x)_\mu$, where $x = (y, z_0)$ is a point of C with $y < y_0$, so that $u(x) = u((x)_\mu) = u((x_0)_\lambda) \geq u(x_0)$, since for $\lambda > \mu$, $\lambda - \mu$ small (more precisely for $\lambda \leq \lambda_0$) the inequality $u \leq u_\lambda$ holds in Ω_λ. So if $y < y_0$, with $y_0 - y$ small we get the inequality $u(y, z_0) \geq u(y_0, z_0)$, which implies that $u(y, z_0) = u(y_0, z_0)$ because u is nondecreasing in the e_1-direction in Ω_{λ_0}. Therefore the set

$$U = \left\{y < y_0 : (y, z_0) \in \Omega \quad \text{and } u(y, z_0) = u(y_0, z_0)\right\}$$

is nonempty. Let us now define

$$y_1 = \inf U\,. \tag{3.11}$$

We claim that $x_1 = (y_1, z_0) \in \partial\Omega$. In fact suppose that $x_1 \in \Omega$ and set $\lambda_1 = \frac{y_1+y_0}{2}$. By continuity $u(x_1) = u(x_0)$ and $x_1 \in \Omega_{\lambda_1} \setminus Z_{\lambda_1}$, since $(x_1)_{\lambda_1} = x_0$ and $Du(x_0) \neq 0$. Moreover from Theorem 2.3 we obtain that $u \equiv u_{\lambda_1}$ in the component of $\Omega_{\lambda_1} \setminus Z_{\lambda_1}$ to which x_1 belongs, which in turn implies that $Du(x_1) \neq 0$, as before. Repeating the previous arguments, with μ and x_0 substituted by λ_1 and x_1, we obtain that $u((y, z_0)) = u((y_0, z_0))$ for $y < y_1$, $y_1 - y$ small, and this contradicts the definition of y_1. So $x_1 \in \partial\Omega$ and $0 = u(x_1) = u(x_0) > 0$. This contradiction proves (3.10) and hence (3.8).

Finally (3.9) is a consequence of (3.8) and the usual Hopf's lemma for strictly elliptic operators. In fact let $x = (\lambda, z) \in \Omega_{\lambda_0} \setminus Z$, i.e. $\lambda < \lambda_0$ and $Du(x) \neq 0$. In a ball $B = B_r(x)$ we have that $|Du| \geq \epsilon > 0$, so that $|Du|, |Du_\lambda| \geq \epsilon > 0$ in $B \cap \Omega_\lambda$. This implies by standard results that $u \in C^2(B)$ and that the difference $u_\lambda - u$ satisfies a linear strictly elliptic equation $L(u_\lambda - u) = 0$ (see [S] and also [BaNa]). On the other hand we have, by (3.8), that $u_\lambda - u > 0$ in $B \cap \Omega_\lambda$ while $u(x) = u_\lambda(x)$ because x belongs to T_λ. Hence, by the usual Hopf's lemma we get $0 > \frac{\partial(u_\lambda - u)}{\partial x_1}(x) = -2\frac{\partial u}{\partial x_1}(x)$ i.e. (3.9) holds. □

Now we prove a proposition which gives a useful information on how the set Z of the critical points of u can intersect the cap $\Omega^\nu_{\lambda_0(\nu)}$.

PROPOSITION 3.1. *Suppose that $u \in C^1(\overline{\Omega})$ is a weak solution of* (1.1), *with $1 < p < 2$. For any direction ν the cap $\Omega^\nu_{\lambda_0(\nu)}$ does not contain any subset Γ of Z on which u is constant and whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains an open subset of $T^\nu_{\lambda_0(\nu)}$ (relatively to the induced topology).*

PROOF. For simplicity we take ν as the x_1-direction and denote a point x in $\mathbb{R}^N$ as $x = (y, z)$ with $y \in \mathbb{R}$, $z \in \mathbb{R}^{N-1}$. As usual we omit the superscript $\nu = e_1$ in Ω^ν_λ, u^ν_λ, etc. Arguing by contradiction we assume that Ω_{λ_0} contains a set Γ with the properties:

(i) there exist $\gamma > 0$ and $z_0 \in \mathbb{R}^{N-1}$ such that, for each point $(\lambda_0, z) \in T_{\lambda_0}$ with $|z - z_0| < \gamma$ there exists $y < \lambda_0$ with $(y, z) \in \Gamma$
(ii) $Du(x) = 0$ for all $x \in \Gamma$
(iii) $u(x) = m > 0$ for all $x \in \Gamma$.

Note that $\overline{\Gamma}$ satisfies the same properties as Γ and that by (iii)

$$\overline{\Gamma} \cap \partial\Omega = \emptyset\,. \tag{iv}$$

Let $\omega = \omega_\gamma$ be the $(N-1)$ dimensional ball centered at z_0 with radius γ. We consider the cylinder $\mathbb{R} \times \omega$ and denote by Σ the intersection of this cylinder with the cap Ω_{λ_0}. Now we distinguish two cases.

CASE 1. $f(m) \leq 0$.

In this case we consider the "left" part Σ_l of Σ with respect to the set Γ, i.e.

$$\Sigma_l = \{(y, z) : z \in \omega, y \in (\sigma_1(z), \sigma_2(z))\}$$

where

$$\begin{aligned}\sigma_1(z) &= \inf\left\{y \text{ such that } (y, z) \in \Omega_{\lambda_0}\right\}\\ \sigma_2(z) &= \sup\left\{y \text{ such that } (y', z) \notin \overline{\Gamma} \text{ for any } y' < y\right\}.\end{aligned}$$

Note that by (i) and (iv) the definition of Σ_l makes sense and $\sigma_1(z) < \sigma_2(z) < \lambda_0$. Moreover we have

$$u(x) \le m \quad \forall\, x \in \Sigma_l \quad \text{and} \quad u \not\equiv m \text{ in } \Sigma_l\,. \tag{3.12}$$

In fact if $(y, z) \in \Sigma_l$ then $u((y, z)) \le u((\sigma_2(z), z)) = m$ since u is nondecreasing in the x_1 direction in Ω_{λ_0} and the point $(\sigma_2(z), z)$ belongs to $\overline{\Gamma}$ (otherwise it would have a positive distance from $\overline{\Gamma}$ and we would have that $(y, z) \notin \overline{\Gamma}$ for $y > \sigma_2(z)$ and close to $\sigma_2(z)$).

Moreover $u \not\equiv m$ in Σ_l since $u = 0$ on $\partial\Omega$ and $\partial\Omega \cap \overline{\Sigma_l} \neq \emptyset$.

Since $f(m) \le 0$ we have

$$\begin{aligned}-\Delta_p(u - m) + \Lambda(u - m) &= f(u) + \Lambda u - \Lambda m\\ &\le f(u) + \Lambda u - (f(m) + \Lambda m)\end{aligned} \tag{3.13}$$

for any $\Lambda \ge 0$. On the other side since f is locally Lipschitz continuous there exists $\Lambda \ge 0$ depending on $\|u\|_{L^\infty(\Omega)}$ such that $f(s) + \Lambda s$ is nondecreasing for $s \in \left[0, \|u\|_{L^\infty(\Omega)}\right]$ (see Remark 2.1). For such a value of Λ (3.13) gives

$$-\Delta_p(u - m) + \Lambda(u - m) \le 0 \qquad \text{in } \Sigma_l\,. \tag{3.14}$$

Now let us observe that for some point x' on $\partial\Sigma_l \cap \overline{\Gamma}$ the interior sphere condition is satisfied. In fact let us take $y_0 \in \mathbb{R}$ with $\sigma_1(z_0) < y_0 < \sigma_2(z_0)$. Since dist $\left((y_0, z_0), \overline{\Gamma}\right) > 0$ there exists $\epsilon > 0$ such that $0 < \epsilon < \gamma$, $B_\epsilon((y_0, z_0)) \subset \Sigma_l$. For $y \in \mathbb{R}$ let $B(y)$ be the ball centered at (y, z_0) with radius ϵ and let us define

$$y_1 = \sup\left\{y > y_0 : \forall\, y' \in (y_0, y) \qquad \overline{B(y')} \cap \overline{\Gamma} = \emptyset\right\}.$$

Since if $\overline{B(y_1)} \cap \overline{\Gamma} = \emptyset$ then dist $(\overline{B(y_1)}, \overline{\Gamma}) > 0$, it is clear that $\overline{B(y_1)} \cap \overline{\Gamma} \neq \emptyset$. Moreover $\overline{B(y_1)} \cap \overline{\Gamma} \subset \partial B(y_1)$ as it is easy to check from the definition of y_1.

So $B(y_1) \subset \Sigma_l$ and there exists $x' \in \partial B(y_1) \cap \overline{\Gamma}$ with $u(x') = m$ by (iii). By (3.12), (3.14) and the Hopf's lemma (Theorem 2.1) we obtain that $\frac{\partial u}{\partial s}(x') < 0$ for any interior (with respect to Σ_l) directional derivative, which contradicts (ii).

Case 2. $f(m) > 0$.

In this case we consider the "right" part $\Sigma_r = \Sigma \setminus \overline{\Sigma_l}$ of Σ with respect to Γ, i.e.

$$\Sigma_r = \left\{(y, z) : z \in \omega, \sigma_2(z) < y < \lambda_0\right\}.$$

Again Σ_r is well defined and by the monotonicity of u in Ω_{λ_0} we have

$$(3.15) \qquad u(x) \geq m \quad \forall\, x \in \Sigma_r \quad \text{and} \quad u \not\equiv m \text{ in } \Sigma_r$$

because if u were flat in Σ_r then $f(m)$ should be zero, against our assumptions. As before, exploiting the lipschitzianity of f and the fact that $f(m)$ is positive we get

$$(3.16) \qquad -\Delta_p(u-m) + \Lambda(u-m) > f(u) + \Lambda u - \big(f(m) + \Lambda m\big) \geq 0$$

for some $\Lambda \geq 0$. From (3.15) and (3.16) applying the Hopf's lemma (Theorem 2.1) in a point $x' \in \partial\Sigma_r \cap \overline{\Gamma}$ we get (being $u(x') = m = \min_{\overline{\Sigma_r}} u$) $\frac{\partial u}{\partial s} > 0$ for any interior (with respect to Σ_r) directional derivative, which contradicts (ii). □

REMARK 3.1. A first consequence of Theorem 3.1 and Proposition 3.1 is the following. Suppose that u is a solution of (1.1) whose critical set Z is quite regular and such that u is constant on any connected component of the set Z of its critical points. Then, for any direction ν, we have that $\lambda_0(\nu) = \lambda_2(\nu)$.

In fact, if for a direction ν it happens that $\lambda_0(\nu) < \lambda_2(\nu)$ then, by Theorem 3.1 there exists a connected component C^ν of $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ such that $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν. Thus ∂C^ν would contain a set Γ on which $Du = 0$ (by definition, see (3.7)), u is constant (by assumption) and whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains an open subset of $T^\nu_{\lambda_0(\nu)}$; this would be impossible by Proposition 3.1.

Note that in what we have just stated we have used the assumption that u is constant on any connected component of the set Z of its critical points.

At a superficial glance one could think that this hypothesis is always satisfied by any C^1 function: but this is not true and actually the question of finding sufficient conditions on a connected set of critical points of a C^1 function g which ensure that g is constant there, is very deep and complicated. As a matter of fact there is a famous counterexample due to Whitney ([W]) on this subject as well as many subsequent research papers (see [N]) which show that this question is strictly related to Sard's lemma and the theory of fractal sets. Roughly speaking in our case one could say that if u is not constant on the connected components of Z, then Z is geometrically very complicated and contains fractal subsets.

Also in view of some counterexamples ([GKPR], [Br]) it could be reasonable to think that solutions u of (1.1) do not have such bad sets of critical points, so that the equality $\lambda_0(\nu) = \lambda_2(\nu)$ should hold for any solution of (1.1) in the case $1 < p < 2$, for general domains.

4. – Proof of Theorem 1.1 and its corollaries

We begin with a simple topological lemma that will be used later.

LEMMA 4.1. *Let A, B be nonempty open sets in a topological space such that $A \cap B \neq \emptyset$, $B \not\subseteq A$. If B is connected then $\partial A \cap B \neq \emptyset$.*

Proof. By the hypothesis $A \cap B$ is a nonempty proper open subset of B. Arguing by contradiction suppose that $\partial A \cap B = \emptyset$. Since B is open we also have that $\partial B \cap B = \emptyset$, and being $\partial(A \cap B) \subseteq \partial A \cup \partial B$ we obtain that $\partial(A \cap B) \cap B = \emptyset$.

This implies that $B \setminus (A \cap B) = B \setminus \left(\overline{A \cap B}\right)$ is open, nonempty and disjoint from $A \cap B$, contradicting the assumption that B is connected. □

Corollary 4.1. *Let A, B open connected sets in a topological space and assume that $A \cap B \neq \emptyset$, $A \not\equiv B$. Then $(\partial A \cap B) \cup (\partial B \cap A) \neq \emptyset$.*

Now we prove Theorem 1.1 and its corollaries. So, from now on, Ω will be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$. Let ν be a direction and $\lambda_1(\nu)$, $\lambda_2(\nu)$, $\lambda_0(\nu)$ as defined in Section 1 and Section 3. Since Ω is smooth we have that $\emptyset \neq \Lambda_1(\nu) \subseteq \Lambda_2(\nu)$, so that $\Lambda_0(\nu) \neq \emptyset$ by Theorem 3.1, and $a(\nu) < \lambda_1(\nu), \lambda_0(\nu) \leq \lambda_2(\nu)$. In view of the definition of $\lambda_0(\nu)$ and of Theorem 3.1, Theorem 1.1 will be proved if we show that $\lambda_0(\nu) \geq \lambda_1(\nu)$.

Since the proof is technically quite complicated we would like to help the reader, spending a few words about it. The idea of the proof is to show that if $\lambda_0(\nu) < \lambda_1(\nu)$ then there exists a small set Γ of critical points of u in the cap $\Omega^\nu_{\lambda_0(\nu)}$ on which u is constant and whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains an open subset of $T^\nu_{\lambda_0(\nu)}$. Once we show this we reach a contradiction with the statement of Proposition 3.1. As observed in Remark 3.1, if the critical set Z of u is not very "bad" then $\lambda_0(\nu) = \lambda_2(\nu)$. From this would follow $\lambda_0(\nu) \geq \lambda_1(\nu)$ and Theorem 1.1 would be proved. But, since a priori the critical set Z of u could be so nasty that u is not constant on its connected components (see Remark 3.1), to prove the existence of the set Γ we use Theorem 3.1 and a new method which consists in moving hyperplanes orthogonal to directions "close to ν" and which requires the smoothness of $\partial\Omega$ as observed in the introduction.

As usual let ν be a direction and define $\mathcal{F}_\nu$ as the collection of the connected components C^ν of $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ such that $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν, $Du \neq 0$ in C^ν, $Du = 0$ on $\partial C^\nu \setminus (T^\nu_{\lambda_0(\nu)} \cup \partial\Omega)$.

If $\lambda_0(\nu) < \lambda_1(\nu)$ then $\lambda_0(\nu) < \lambda_2(\nu)$ so that $\mathcal{F}_\nu \neq \emptyset$ by Theorem 3.1. If this is the case and $C^\nu \in \mathcal{F}_\nu$ we also have $u \equiv u^\nu_{\lambda_0(\nu)}$ in $\overline{C}^\nu$, so that $(\overline{C}^\nu \cap \partial\Omega) \setminus T^\nu_{\lambda_0(\nu)} = \emptyset$ since $u = 0$ on $\partial\Omega$, while $u^\nu_{\lambda_0(\nu)} > 0$ in $\overline{C}^\nu \setminus T^\nu_{\lambda_0(\nu)}$, because by the definition of $\lambda_1(\nu)$ (see Section 1) we have that $(\overline{\Omega^\nu_{\lambda_0(\nu)}} \setminus T^\nu_{\lambda_0(\nu)})' \subset \Omega$.

Hence there are two alternatives: either $Du(x) = 0$ for all $x \in \partial C^\nu$, in which case we define $\widetilde{C}^\nu = C$, or there are points $x \in \partial C^\nu \cap T^\nu_{\lambda_0(\nu)}$ such that $Du(x) \neq 0$. In this latter case we define $\widetilde{C}^\nu = C^\nu \cup C^\nu_1 \cup C^\nu_2$ where $C^\nu_1 = R^\nu_{\lambda_0(\nu)}(C^\nu)$, $C^\nu_2 = \{x \in \partial C^\nu \cap T^\nu_{\lambda_0(\nu)} : Du(x) \neq 0\}$. It is easy to check that $\widetilde{C}^\nu$ is open and connected, with $Du \neq 0$ in $\widetilde{C}^\nu$, $Du = 0$ on $\partial\widetilde{C}^\nu$.

Let us finally define the collection $\widetilde{\mathcal{F}}_\nu = \{\widetilde{C}^\nu : C^\nu \in \mathcal{F}_\nu\}$.

Remark 4.1. At this point a crucial remark for the sequel is the following: if ν_1, ν_2 are directions and $C^{\nu_1} \in \mathcal{F}_{\nu_1}$, $C^{\nu_2} \in \mathcal{F}_{\nu_2}$, then either $\widetilde{C}^{\nu_1} \cap \widetilde{C}^{\nu_2} = \emptyset$

or $\widetilde{C}^{\nu_1} \equiv \widetilde{C}^{\nu_2}$. In fact if $\widetilde{C}^{\nu_1} \cap \widetilde{C}^{\nu_2} \neq \emptyset$ and $\widetilde{C}^{\nu_1} \not\equiv \widetilde{C}^{\nu_2}$, then by Corollary 4.1 either $\partial\widetilde{C}^{\nu_1} \cap \widetilde{C}^{\nu_2}$ or $\partial\widetilde{C}^{\nu_2} \cap \widetilde{C}^{\nu_1}$ is nonempty, and this is not possible since $Du \neq 0$ in $\widetilde{C}^{\nu_i}$, $Du = 0$ on $\partial\widetilde{C}^{\nu_i}$, $i = 1, 2$.

PROOF OF THEOREM 1.1. If ν is a direction and $\delta > 0$ let us denote by $I_\delta(\nu)$ the set

$$I_\delta(\nu) = \left\{\mu \in \mathbb{R}^N : \ |\mu| = 1, |\mu - \nu| < \delta\right\}.$$

As already observed the theorem will be proved if we show that $\lambda_0(\nu) \geq \lambda_1(\nu)$ for any direction ν.

As in the proof of Theorem 3.1 we can fix α, $M > 0$ so that Theorem 2.2 applies, i.e. for any direction ν, any $\lambda \in (a(\nu), \lambda_1(\nu)]$ and any open subset Ω' of Ω_λ^ν, the inequality $u \leq u_\lambda^\nu$ on $\partial\Omega'$ implies the inequality $u \leq u_\lambda^\nu$ in Ω' provided $\Omega' = A_1 \cup A_2$ with $|A_1| < \alpha$, $M_{A_2} = \sup_{A_2}\left(|Du| + |Du_\lambda^\nu|\right) < M$.

Suppose now that ν_0 is a direction such that $\lambda_0(\nu_0) < \lambda_1(\nu_0)$. Since $\lambda_1(\nu_0) \leq \lambda_2(\nu_0)$, it follows that $\lambda_0(\nu_0) < \lambda_2(\nu_0)$, so that from Theorem 3.1 we get $\mathcal{F}_{\nu_0} \neq \emptyset$. Since $\mathbb{R}^N$ is a separable metric space and every component is open, $\mathcal{F}_{\nu_0}$ contains at most countably many components of $\Omega^{\nu_0}_{\lambda_0(\nu_0)} \setminus Z^{\nu_0}_{\lambda_0(\nu_0)}$, so $\mathcal{F}_{\nu_0} = \left\{C_i^{\nu_0}, \ i \in I \subseteq \mathbb{N}\right\}$. In case I is infinite, since the components are disjoints, we have that $\sum_{i=1}^\infty |C_i^{\nu_0}| \leq |\Omega| < \infty$, so that we can choose $n_0 \geq 1$ for which

$$\sum_{i=n_0+1}^{\infty} |C_i^{\nu_0}| < \frac{\alpha}{6}.$$

If I is finite let n_0 be its cardinality.

Let us then choose a compact $K_0 \subset (\Omega^{\nu_0}_{\lambda_0(\nu_0)} \setminus \cup_{i\in I} C_i^{\nu_0})$ so that

$$\left|\left(\Omega^{\nu_0}_{\lambda_0(\nu_0)} \setminus \cup_{i\in I} C_i\right) \setminus K_0\right| < \frac{\alpha}{6}.$$

Finally we take n_0 compact sets $K_i \subset C_i^{\nu_0}$, $i = 1, \ldots, n_0$, such that

$$\left|C_i^{\nu_0} \setminus K_i\right| < \frac{\alpha}{6n_0} \qquad i = 1, \ldots, n_0.$$

So we have decomposed $\Omega^{\nu_0}_{\lambda_0(\nu_0)}$ in the sets K_0, $K_1, \ldots, K_{n_0}$ and in a remaining part with measure

$$\left|\Omega^{\nu_0}_{\lambda_0(\nu_0)} \setminus \cup_{i=0}^{n_0} K_i\right| < 3\,\frac{\alpha}{6} = \frac{\alpha}{2}.$$

If $A = \{x \in \Omega^{\nu_0}_{\lambda_0(\nu_0)} : |Du(x)| + |Du^{\nu_0}_{\lambda_0(\nu_0)}| < \frac{M}{2}\}$, since $K_0 \setminus A$ is compact and $u < u^{\nu_0}_{\lambda_0(\nu_0)}$ in $K_0 \setminus A$ by Theorem 2.3, there exists $m > 0$ such that

$$u^{\nu_0}_{\lambda_0(\nu_0)} - u \geq m > 0 \qquad \text{in} \quad K_0 \setminus A.$$

Since Ω is smooth the functions $a(\nu)$ and $\lambda_1(\nu)$ defined in Section 1 are continuous with respect to ν. By continuity there exist ϵ_0, $\delta_0 > 0$ such that if $|\lambda - \lambda_0(\nu_0)| \leq \epsilon_0$ and $|\nu - \nu_0| \leq \delta_0$ then

$$\lambda_1(\nu) > \lambda_0(\nu_0) + \epsilon_0 \tag{4.1}$$

$$|Du| + |Du_\lambda^\nu| < M \qquad \text{in } A \tag{4.2}$$

$$\begin{cases} K_i \subset \Omega_\lambda^\nu & i = 0, \dots, n_0 \\ K_i' \subset (C_i^{\nu_0})' & i = 1, \dots, n_0 \end{cases} \tag{4.3}$$

where K_i' is the reflection $R_\lambda^\nu\ (K_i)$

$$\left| \Omega_\lambda^\nu \setminus \bigcup_{i=0}^{n_0} K_i \right| < \alpha \tag{4.4}$$

$$u_\lambda^\nu - u \geq \frac{m}{2} > 0 \qquad \text{in } K_0 \setminus A\,. \tag{4.5}$$

We now proceed in several steps in order to show that there exists $i_1 \in \{1, \dots, n_0\}$ and a direction $\nu_1 \in I_{\delta_0}(\nu_0)$ such that $\widetilde{C}_{i_1}^{\nu_0} \in \widetilde{\mathcal{F}}_\nu$ for any direction in a suitable neighbourhood $I_\delta(\nu_1)$ of ν_1, and $\partial C_{i_1}^{\nu_0}$ contains a set Γ as in Proposition 3.1 (with respect to the direction ν_1).

In what follows we implicity assume that $\epsilon > 0$ means $0 < \epsilon \leq \epsilon_0$, $\delta > 0$ means $0 < \delta \leq \delta_0$.

Step 1. Here we show that the function $\lambda_0(\nu)$ is continuous, i.e. for each $\epsilon > 0$ there exists $\delta > 0$ such that if $\nu \in I_\delta(\nu_0)$ then

$$\lambda_0(\nu_0) - \epsilon < \lambda_0(\nu) < \lambda_0(\nu_0) + \epsilon\,. \tag{i}$$

Moreover for each $\nu \in I_{\delta(\epsilon_0)}(\nu_0)$ we have

$$\exists\, i \in \{1, \dots, n_0\} \text{ such that } \widetilde{C}_i^{\nu_0} \in \widetilde{\mathcal{F}}_\nu\,. \tag{ii}$$

Proof of Step 1. Let $\epsilon > 0$ ($\epsilon \leq \epsilon_0$) be fixed. By the definition of $\lambda_0(\nu_0)$ there exist $\lambda \in (\lambda_0(\nu_0), \lambda_0(\nu_0) + \epsilon)$ and $x \in \Omega_\lambda^{\nu_0}$ such that $u(x) > u_\lambda^{\nu_0}(x)$. By continuity there exists $\delta_1 > 0$ such that for every $\nu \in I_{\delta_1}(\nu_0)$ x belongs to Ω_λ^ν and $u(x) > u_\lambda^\nu(x)$. This implies that for all $\nu \in I_{\delta_1}(\nu_0)$ we have

$$\lambda_0(\nu) < \lambda < \lambda_0(\nu_0) + \epsilon\,.$$

Next we show that there exists $\delta_2 > 0$ such that $\lambda_0(\nu) > \lambda_0(\nu_0) - \epsilon$ for any $\nu \in I_{\delta_2}(\nu_0)$. Suppose the contrary, then there exists a sequence $\{\nu_n\}$ of directions such that $\nu_n \to \nu_0$ and $\lambda_0(\nu_n) \leq \lambda_0(\nu_0) - \epsilon$. Up to a subsequence we have that $\lambda_0(\nu_n)$ converges to a number $\overline{\lambda} \leq \lambda_0(\nu_0) - \epsilon$. Moreover, since $\lambda_0(\nu_n) > a(\nu_n)$ and $a(\nu_n) \to a(\nu_0)$, we also have that $a(\nu_0) \leq \overline{\lambda}$.

Actually this inequality is strict because the caps $\Omega^{\nu_n}_{\lambda_0(\nu_n)}$ have measure greater than or equal to the number α of Theorem 2.2 (otherwise $\lambda_0(\nu_n)$ would not be maximal with respect to the inequality $u \leq u^{\nu_n}_{\lambda}$, see the proof of Step 1 in Theorem 3.1); then also $|\Omega^{\nu_0}_{\overline{\lambda}}| \geq \alpha$, which implies that $\overline{\lambda} > a(\nu_0)$.

Now, since $\overline{\lambda} < \lambda_0(\nu_0)$, by (3.8) of Theorem 3.1, we have

$$u < u^{\nu_0}_{\overline{\lambda}} \qquad \text{in } \Omega^{\nu_0}_{\overline{\lambda}} \setminus Z^{\nu_0}_{\overline{\lambda}}\,. \tag{4.6}$$

Arguing as in Step 2 of Theorem 3.1 we can construct an open set $A \subset \Omega^{\nu_0}_{\overline{\lambda}}$ and a compact set $K \subset \Omega^{\nu_0}_{\overline{\lambda}}$ such that

$$Z^{\nu_0}_{\overline{\lambda}} \subset A\,, \quad \sup_A \left(|Du| + |Du^{\nu_0}_{\overline{\lambda}}|\right) < \frac{M}{2}\,, \quad \left|\Omega^{\nu_0}_{\overline{\lambda}} \setminus K\right| < \frac{\alpha}{2}$$

and, if $K \setminus A \neq \emptyset$,

$$u^{\nu_0}_{\overline{\lambda}} - u \geq m > 0 \qquad \text{in } K \setminus A$$

(α and M being the usual numbers which come from Theorem 2.2).

By continuity there exist $r, \delta > 0$ such that

$$\sup_A \left(|Du| + |Du^{\nu}_{\lambda}|\right) < M\,, \quad \left|\Omega^{\nu}_{\lambda} \setminus K\right| < \alpha\,, \quad u^{\nu}_{\lambda} - u > 0 \quad \text{in } K \setminus A$$

if $\nu \in I_r(\nu_0)$ and $\lambda \in \left(\overline{\lambda} - \delta, \overline{\lambda} + \delta\right)$.

For such values of ν and λ, applying Theorem 2.2 exactly as in Step 2 of Theorem 3.1, we get $u \leq u^{\nu}_{\lambda}$ in Ω^{ν}_{λ}. This in particular holds for $\nu = \nu_n$, $\lambda = \lambda_0(\nu_n) + \eta$ for any n sufficiently large and η sufficiently small, contradicting the definition of $\lambda_0(\nu_n)$. Hence (i) is proved.

Observe that, since we implicity assume that $\epsilon \leq \epsilon_0$ and $\delta \leq \delta_0$, by (i) and (4.1) we have that

$$\lambda_1(\nu) > \lambda_0(\nu) \qquad \forall\, \nu \in I_{\delta(\epsilon_0)}(\nu_0) \tag{4.7}$$

and (4-2)-(4-5) hold for $\nu \in I_{\delta(\epsilon_0)}(\nu_0)$, $\lambda = \lambda_0(\nu)$.

Let us now prove (ii) and fix a direction $\nu \in I_{\delta(\epsilon_0)}(\nu_0)$. Suppose that there exist $i \in \{1, \dots, n_0\}$ and a point $x_i \in K_i$ such that $u(x_i) = u^{\nu}_{\lambda_0(\nu)}(x_i)$. Since $Du(x_i) \neq 0$, by Theorem 2.3 we get $u \equiv u^{\nu}_{\lambda_0(\nu)}$ in the component $C^{\nu} \in \mathcal{F}_{\nu}$ to which x_i belongs. We also have that $x_i \in \widetilde{C}^{\nu} \cap \widetilde{C}^{\nu_0}_i$ because $K_i \subset C^{\nu_0}_i \subseteq \widetilde{C}^{\nu_0}_i$ and hence, by Remark 4.1, $\widetilde{C}^{\nu} \equiv \widetilde{C}^{\nu_0}_i$ and (ii) is proved.

The other possibility is that for each $i \in \{1, \dots, n_0\}$ and every $x \in K_i$ we have $u(x) < u^{\nu}_{\lambda_0(\nu)}(x)$. If this is the case, by (4.2)-(4.5) the inequality $u < u^{\nu}_{\lambda_0(\nu)}$ holds in $\Omega^{\nu}_{\lambda_0(\nu)}$, except for a set with measure less than α and for the set $Z^{\nu}_{\lambda_0(\nu)}$. Then, arguing as in the proof of (i) (i.e. repeating the proof of Step 2 of Theorem 3.1), we get the inequality $u \leq u^{\nu}_{\lambda}$ in Ω^{ν}_{λ} for $\lambda > \lambda_0(\nu)$, $\lambda - \lambda_0(\nu)$ small, contradicting the definition of $\lambda_0(\nu)$. So (ii) is completely proved. □

STEP 2. Here we show that there exist a direction $\nu_1 \in I_{\delta(\epsilon_0)}(\nu_0)$, a neighbourhood $I_{\delta_1}(\nu_1)$ and an index $i_1 \in \{1, \dots, n_0\}$ such that for any $\nu \in I_{\delta_1}(\nu_1)$ the set $\widetilde{C}_{i_1}^{\nu_0}$ belongs to the collection $\widetilde{\mathcal{F}}_\nu$.

PROOF OF STEP 2. Before starting the proof let us recall that the statement (ii) of Step 1 asserts that for any direction ν in the neighborhood $I_{\delta(\epsilon_0)}(\nu_0)$ of the direction ν_0 there exists a set $\widetilde{C}_i^{\nu_0}$ in the collection $\widetilde{\mathcal{F}}_{\nu_0}$ which also belongs to $\widetilde{\mathcal{F}}_\nu$.

Here instead we want to prove that there exists a set $\widetilde{C}_{i_1}^{\nu_0}$ in $\widetilde{\mathcal{F}}_{\nu_0}$ which belongs to $\widetilde{\mathcal{F}}_\nu$ for any direction ν in a suitable neighborhood $I_{\delta_1}(\nu_1)$ of a certain direction $\nu_1 \in I_{\delta(\epsilon_0)}(\nu_0)$.

Now let us observe that in the proof of (ii) of Step 1 we have seen that if $\nu \in I_{\delta(\epsilon_0)}(\nu_0)$ and if there exists $x_i \in K_i$, for some $i \in \{1, \dots, n_0\}$, such that $u(x_i) = u^\nu_{\lambda_0(\nu)}(x_i)$, then $u \equiv u^{\nu_0}_{\lambda_0(\nu_0)} \equiv u^\nu_{\lambda_0(\nu)}$ in K_i and $\widetilde{C}_i^{\nu_0} \in \widetilde{\mathcal{F}}_\nu$.

So for any $\nu \in I_{\delta(\epsilon_0)}(\nu_0)$ and each $i \in \{1, \dots, n_0\}$ we have two alternatives: either $u \equiv u^\nu_{\lambda_0(\nu)}$ in K_i (and hence in $\widetilde{C}_i^{\nu_0}$ so that $\widetilde{C}_i^{\nu_0} \in \widetilde{\mathcal{F}}_\nu$) or $u < u^\nu_{\lambda_0(\nu)}$ in K_i. In the latter case, since $u^\nu_{\lambda_0(\nu)} - u \geq m > 0$ in K_i, and the function $\lambda_0(\nu)$ is continuous with respect to ν, we get that the inequality $u < u^\mu_{\lambda_0(\mu)}$ holds in K_i for any μ in a suitable neighborhood $I(\nu)$ of ν; this implies that $\widetilde{C}_i^{\nu_0} \notin \widetilde{\mathcal{F}}_\mu$ for any $\mu \in I(\nu)$.

To prove the statement of Step 2 let us note that if $n_0 = 1$ the assertion is proved by (ii) of Step 1, otherwise we start by taking a set in $\widetilde{\mathcal{F}}_{\nu_0}$, say $\widetilde{C}_1^{\nu_0}$ and argue as follows. If $\widetilde{C}_1^{\nu_0}$ belongs to $\widetilde{\mathcal{F}}_\nu$ for any ν in $I_{\delta(\epsilon_0)}(\nu_0)$ then the assertion is proved. Otherwise, for what we explained above, there exists a direction $\mu_1 \in I_{\delta(\epsilon_0)}(\nu_0)$ such that $\widetilde{C}_1^{\nu_0} \notin \widetilde{\mathcal{F}}_\mu$ for any μ in a suitable neighborhood $I_{\delta_1}(\mu_1) \subset I_{\delta(\epsilon_0)}(\nu_0)$. Now, by (ii) of Step 1 there exists a set in $\widetilde{\mathcal{F}}_{\nu_0}$, say $\widetilde{C}_2^{\nu_0}$, such that $\widetilde{C}_2^{\nu_0} \in \widetilde{\mathcal{F}}_{\mu_1}$. Thus either $\widetilde{C}_2^{\nu_0}$ belongs to $\widetilde{\mathcal{F}}_\mu$ for any $\mu \in I_{\delta_1}(\mu_1)$ and the assertion is proved, or there exists a direction $\mu_2 \in I_{\delta_1}(\mu_1)$ such that $\widetilde{C}_2^{\nu_0} \notin \widetilde{\mathcal{F}}_\mu$ for any μ in a suitable neighborhood $I_{\delta_2}(\mu_2) \subset I_{\delta_1}(\mu_1)$. Hence for all directions μ in $I_{\delta_2}(\mu_2)$ (in particular for μ_2) we have that neither $\widetilde{C}_1^{\nu_0}$ nor $\widetilde{C}_2^{\nu_0}$ belongs to $\widetilde{\mathcal{F}}_\mu$. Thus if $n_0 = 2$ we reach a contradiction with (ii) of Step 1 and the assertion is proved, while if $n_0 > 2$ we proceed as before taking a set $\widetilde{C}_3^{\nu_0}$ in $\widetilde{\mathcal{F}}_{\nu_0}$ such that $\widetilde{C}_3^{\nu_0} \in \widetilde{\mathcal{F}}_{\nu_2}$. Arguing as we did for $\widetilde{C}_2^{\nu_0}$ and $\widetilde{C}_1^{\nu_0}$ either after $k < n_0$ steps we reach a set $\widetilde{C}_k^{\nu_0}$ which belongs to $\widetilde{\mathcal{F}}_\mu$ for any $\mu \in I_{\delta_{k-1}}(\mu_{k-1})$, proving the assertion, or after n_0 steps we get a direction $\mu_{n_0} \in I_{\delta(\epsilon_0)}(\nu_0)$ such that $\widetilde{C}_i^{\nu_0} \notin \widetilde{\mathcal{F}}_{\mu_{n_0}}$ for any $i \in \{1, \dots, n_0\}$. This contradicts (ii) of Step 1 and proves again the assertion. □

STEP 3. Let ν_1, i_1, δ_1 be as in Step 2 and set $C = C_{i_1}^{\nu_0}$. Here we show that $\partial C \cap \Omega^{\nu_1}_{\lambda_0(\nu_1)}$ contains a subset Γ on which u is constant and whose projection on the hyperplane $T^{\nu_1}_{\lambda_0(\nu_1)}$ contains an open subset of the hyperplane.

Since $Du = 0$ on $\partial C \cap \Omega^{\nu_1}_{\lambda_0(\nu_1)}$ this gives a contradiction with Proposition 3.1 and ends the proof of Theorem 1.1.

PROOF OF STEP 3. For sake of simplicity assume that $\nu_1 = e_1 = (1, 0, \ldots, 0)$ and set $\lambda_0 = \lambda_0(e_1)$, $\Omega_{\lambda_0} = \Omega^{\nu_1}_{\lambda_0(\nu_1)}$, etc.

We have by Step 2 that for each $\nu \in I_{\delta_1}(e_1)$

$$u \equiv u^{\nu}_{\lambda_0(\nu)} \quad \text{in } \widetilde{C} \qquad \text{and} \qquad \widetilde{C} \in \widetilde{\mathcal{F}}_{\nu}. \tag{4.8}$$

Now we remark that, since $\widetilde{C}$ is open, the sets $\widetilde{C}^{\nu} = R^{\nu}_{\lambda_0(\nu)}(\widetilde{C})$ cannot be disjoints from $\widetilde{C}^{e_1}$, for ν sufficiently close to e_1. Moreover, since $Du = 0$ on $\partial\widetilde{C}$, by (4.8) we also have that $Du = 0$ on $\partial\widetilde{C}^{\nu}$ and then, arguing as in Remark 4.1, we get that $\widetilde{C}^{e_1} \equiv \widetilde{C}^{\nu}$ for ν sufficiently close to e_1. Then we take a point $\overline{x} = (\overline{y}, \overline{z})$ in $\partial C \cap \Omega_{\lambda_0}$ and consider the point $\overline{x}' = (2\lambda_0 - \overline{y}, \overline{z})$ symmetric with respect to the hyperplane T_{λ_0}. By reflecting $\overline{x}'$ through the hyperplanes $T^{\nu}_{\lambda_0(\nu)}$ for $\nu \in I_{\delta_1}(e_1)$ we obtain the points

$$A(\nu) = (y(\nu), z(\nu)) = \overline{x}' + 2\,\left(\lambda_0(\nu) - \overline{x}' \cdot \nu\right)\,\nu$$

which belong to ∂C for what we remarked before. Since $\overline{x}' \notin T_{\lambda_0}$ we can suppose, taking δ_1 smaller if necessary, that for each $\nu \in I_{\delta_1}(e_1)$ the point $A(\nu)$ belongs to $\Omega_{\lambda_0} = \Omega^{e_1}_{\lambda_0(e_1)}$ and that $\lambda_0(\nu) - \overline{x}' \cdot \nu < 0$. Observe that, since the function $\nu \to \lambda_0(\nu)$ is continuous, also the function $\nu \to A(\nu)$ is continuous (and it is injective as it is easy to see).

By (4.8) the function u is constant on the set $\Gamma = \{A(\nu) : \nu \in I_{\delta_1}(e_1)\}$, and the gradient of u vanishes on Γ. We will prove that the projection of Γ on the hyperplane T_{λ_0} contains an open subset of T_{λ_0}. In this way we will obtain a contradiction with Proposition 3.1 and the proof of Theorem 1.1 will be concluded.

Let us now write the generic direction $\nu \in S^{N-1}$ as $\nu = (\nu_y, \nu_z)$, with $\nu_y \in \mathbb{R}$, $\nu_z \in \mathbb{R}^{N-1}$. If ν is close to e_1, then $\nu_y = \sqrt{1 - |\nu_z|^2}$.

We take now $\beta > 0$ small and consider the set

$$K = \left\{\nu = (\nu_y, \nu_z) : \nu_z \in \overline{B}_{\beta},\, \nu_y = \sqrt{1 - |\nu_z|^2}\right\}$$

where $\overline{B}_{\beta} = \{z \in \mathbb{R}^{N-1} : |z| \leq \beta\}$ is the closed ball in $\mathbb{R}^{N-1}$ centered at the origin with radius β.

K is a compact neighbourhood of e_1 in the metric space S^{N-1}, and if β is small then K is contained in $I_{\delta_1}(e_1)$. We will show that if $A(\nu) = (y(\nu), z(\nu))$ then the set $\{z(\nu) : \nu \in K\}$ contains an open set in $\mathbb{R}^{N-1}$.

Now $z(\nu) = \overline{z} + 2\,(\lambda_0(\nu) - \overline{x}' \cdot \nu)\,\nu_z$, where $\nu = (\sqrt{1 - |\nu_z|^2}, \nu_z)$, $\nu_z \in \overline{B}_{\beta}$.

So we have to prove that the image of the function

$$F(\nu_z) = 2\,\left(\lambda_0(\nu) - \overline{x}' \cdot \nu\right)\,\nu_z\,, \qquad \nu_z \in \overline{B}_{\beta} \quad \nu = \left(\sqrt{1 - |\nu_z|^2}, \nu_z\right)$$

contains a $(N-1)$-dimensional ball centered at the origin.

Let us now consider a point $l \in S^{N-2} = \{z \in \mathbb{R}^{N-1} : |z| = 1\}$ and the segment $S_l = \{t\,l : |t| \leq \beta\}$ contained in $\overline{B}_\beta$. The image $F(S_l)$ is a segment contained in the line passing through the origin with direction l in $\mathbb{R}^{N-1}$, because F is continuous. Moreover, since $\lambda_0(\nu) - \overline{x}' \cdot \nu < 0$ and S_l contains points $\nu_z = t\,l$ with t both positive and negative, we have that the origin is an interior point of $F(S_l)$. Hence we can write

$$F(S_l) = \{tl : t \in [d_1(l), d_2(l)]\} \qquad d_1(l) < 0 < d_2(l)\,.$$

Since d_1 and d_2 are continuous in S^{N-2}, which is compact, they have respectively a negative maximum $\underline{d}$ and a positive minimum $\overline{d}$. If $d = \min\{-\underline{d}, \overline{d}\}$ we obtain that

$$\{z \in \mathbb{R}^{N-1} : |z| \leq d\} = \overline{B}_d \subseteq F(\overline{B}_\beta)$$

which ends the proof. □

Now we prove Corollary 1.2 (Corollary 1.1 being an immediate consequence of Theorem 1.1).

Proof of Corollary 1.2. If $\Omega = B_R(0)$ by Theorem 1.1 we immediately deduce that u is radially symmetric, $u(x) = U(|x|)$, with $U'(r) \leq 0$ for all $r \in [0, R]$. If $0 < r < R$ and $G = B_R \setminus \overline{B}_r$, then $m = U(r)$ is the maximum of u in $\overline{G}$ and the minimum of u in $\overline{B}_r$.

In case $f(m) \leq 0$, as in the proof of Proposition 3.1 we observe that by the lipschitzianity of f there exists $\Lambda \geq 0$ such that

$$-\Delta_p(u-m) + \Lambda(u-m) \leq 0 \qquad \text{in } G\,.$$

Moreover since $m > 0$ and $u = 0$ on ∂B_R we have that u is not constant in G. By the Hopf's lemma (Theorem 2.1) we get $U'(r) < 0$.

If instead $f(m) > 0$ then u is not constant in B_r (otherwise $f(m)$ should be zero) and

$$-\Delta_p(u-m) + \Lambda(u-m) > 0 \qquad \text{in } B_r$$

for some $\Lambda \geq 0$. Again by Hopf's lemma we obtain that $U'(r) < 0$. □

Remark 4.2. If $f(0) \geq 0$ then we also get $U'(R) < 0$ with the same proof.

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