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Semistable Quotients

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Let G be a complex reductive group, and let X be a (reduced) complex space with a holomorphic action of G. A complex space Y together with a holomorphic map $\pi : X \to Y$ is said to be a semistable quotient of X with respect to the G-action if:

(i) π is a G-invariant locally Stein map, and

(ii) $\mathcal{O}_Y = \pi_* \mathcal{O}_X^G$.

If a semistable quotient exists, then it is unique up to biholomorphism, and will be denoted by $X//G$.

In the algebraic category quotients of this type are often called good quotients and have been studied intensively. They also arise naturally in the context of Hamiltonian group actions; more precisely, let Z be a complex space with a holomorphic G-action, let K be a maximal compact subgroup of G, and assume that there is a moment map $\mu : Z \to (\mathcal{L}ie\, K)^*$ with respect to a K-invariant Kählerian structure ω on Z. Then the set $X := \{z \in Z;\ \overline{G \cdot z} \cap \mu^{-1}(0) \neq \emptyset\}$ of semistable points of Z with respect to μ is an open G-stable subset of Z, and the quotient $X//G$ exists ([H-L], [S]). Moreover, in the case where Z is a projective manifold, it can be shown that X coincides with a subset of semistable points in the sense of geometric invariant theory, i.e., there is an ample G-line bundle L on Z such that X is the set of semistable points with respect to the linearization induced by L (see [H-M]).

The goal of this paper is to reduce the question of existence of a semistable quotient to the case where G is an abelian connected Lie group; more precisely we prove the following:

THEOREM. *The semistable quotient $X//G$ exists if and only if $X//T$ exists for some maximal algebraic torus T in G.*

The theorem solves a problem of Białynicki-Birula which he posed during his stay at the Ruhr–Universität Bochum and the University of Florence.

The result is well known in the algebraic category (see [BB-S1], [BB-S2]), under the weaker assumption that the semistable quotient $X//T_0$ exists for all

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one-dimensional algebraic subgroups T_0 of T. Since it is easy to construct counterexamples in the holomorphic setting to this more general statement (see Section 6), the above theorem is the best possible in the holomorphic framework.

One of the facts which is used in the proof of the above mentioned result for algebraic actions is that the closure of a G-orbit in X contains a closed orbit. The main problem which arise in the analytic category is to show that, if the existence of $X//T$ is assumed, then the G-orbits do not behave too wildly, e.g. that a G-orbit is open in its analytic Zariski-closure; we show this by using properties of subanalytic sets.

1. – Generalities on semistable quotients

Let K be a Lie group and X a complex K-space, i.e., X is a reduced Hausdorff complex space with countable topology and K acts on X by holomorphic transformations such that the action $K \times X \to X$, $(k, x) \to k \cdot x$, is real analytic. Let $\mathcal{O}(X)^K$ denote the algebra of K-invariant holomorphic functions on X; associated with $\mathcal{O}(X)^K$ is the equivalence relation $\sim := \{(x_1, x_2) \in X \times X;\ f(x_1) = f(x_2)$ for all $f \in \mathcal{O}(X)^K\}$. Let $\pi : X \to X/\sim$ be the quotient map. In the case where X is assumed to be a Stein space and K is a compact Lie group one has the following result ([H]).

The quotient $X/\sim$ is a Stein space such that $\mathcal{O}(Q) = \mathcal{O}(\pi^{-1}(Q))^K$ for any open subset Q of $X/\sim$.

Moreover, $X/\sim$ is the categorical quotient of X in the category of complex spaces, and will be denoted by $X//K$.

In order to define a natural extension of this concept, let K be a compact Lie group and X a complex K-space. We say that a complex space Y together with a surjective holomorphic map $\pi : X \to Y$ is a semistable quotient of X if every $y \in Y$ has an open Stein neighborhood Q such that $\pi^{-1}(Q)$ is an open Stein subset of X, and the restriction $\pi : \pi^{-1}(Q) \to Q$ induces an isomorphism $\pi^{-1}(Q)//K \cong Q$.

A semistable quotient Y of X is unique up to isomorphism and will be denoted by $X//K$.

Let K be a compact Lie group, and X a complex K-space such that $X//K$ exists. The following properties of the semistable quotient $\pi : X \to X//K$ follow from the corresponding properties in the Stein setting.

(i) *If A_j, $j = 1, 2$, are closed K-stable complex subspaces of X, then $\pi(A_1) \cap \pi(A_2) = \pi(A_1 \cap A_2)$.*

(ii) *For a closed K-stable complex subspace Z of X, the image $\pi(Z)$ is a closed analytic subspace of $X//K$, the semistable quotient $Z//K$ exists and the embedding $Z \hookrightarrow X$ induces an isomorphism $Z//K \cong \pi(Z)$.*

(iii) *If Y is a locally closed analytic subspace of $X//K$, then the embedding $\pi^{-1}(Y) \hookrightarrow X$ induces an isomorphism $Y \cong \pi^{-1}(Y)//K$.*

Let K be a compact Lie group, and let $G := K^{\mathbb{C}}$ be the complexification of K, i.e., G is a complex reductive group with maximal compact subgroup K. Since the Lie algebra $\mathcal{L}ie\, G$ of G is the complexification of the Lie algebra $\mathcal{L}ie\, K$ of K, and since K intersects every connected component of G, we have $\mathcal{O}(X)^G = \mathcal{O}(X)^K$ for every holomorphic G-space X, i.e., for every complex G-space X such that the action $G \times X \to X$ is holomorphic. Moreover, for the same reasoning, every K-stable closed analytic subset of X is G-stable.

A holomorphic G-space X may be considered as a complex K-space. If in this case a semistable quotient $X//K$ exists, then we set $X//G := X//K$; this makes sense since $X//G$ does not depend on the choice of a maximal compact subgroup K of G and $\pi : X \to X//G$ satisfies the conditions (i) and (ii) in the definition of a semistable quotient.

REMARK 1. If $X//G$ exists, then we claim that $\pi(A)$ is closed in $X//G$ for any G-stable closed subset A of X. Here A is not assumed to be analytic as in (ii). In order to see this, one may assume that X is a Stein space; furthermore there is a moment map μ on X such that the embedding $\mu^{-1}(0) \hookrightarrow X$ induces a homeomorphism $\mu^{-1}(0)/K \sim X//G$. Since $\pi(A) = \pi(A \cap \mu^{-1}(0))$, the claim follows. Moreover, this also implies that $X//G$ is the categorical quotient of X with respect to G in the category of topological Hausdorff spaces. For more details see e.g. [H-H-K].

Now let G be a complex reductive group, and X a holomorphic G-space such that the semistable quotient exists. In the definition of a semistable quotient the map $\pi : X \to X//G$ is only required to be a locally Stein map; the following result shows that this is also globally the case.

THEOREM. *If $X//G$ is a Stein space, then X is a Stein space.*

For the proof of the Theorem we need the following:

LEMMA. *Let $\pi : X \to X//G$ be a semistable quotient. For every $q \in X//G$ there exist an open neighborhood Q of q, a G-representation W, and a G-equivariant holomorphic map $\phi : X \to W$ such that ϕ embeds $\pi^{-1}(Q)$ as a closed analytic subset of $\pi_W^{-1}(P)$, where P is a suitable open subset of $W//G$ and $\pi_W : W \to W//G$ denotes the quotient map.*

PROOF. If X is a Stein space, then the lemma is proved in [H] Section 6 (see also [Sn]). Thus, in general, there exist an open Stein neighborhood Q of q and a G-equivariant holomorphic map ϕ_U from $U := \pi^{-1}(Q)$ into a G-representation space W such that $\phi_U : U \to \pi_W^{-1}(P)$ is a closed embedding, where P is an open subset of $W//G$.

On $Y := X//G$ we have the sheaf $\mathcal{H}$ of germs of G-equivariant holomorphic maps into W, i.e., $\mathcal{H}(\tilde{Q}) = \{f;\ f : \pi^{-1}(\tilde{Q}) \to W$ is a G-equivariant holomorphic map$\}$ for $\tilde{Q} \subset Y$ open; it is a coherent sheaf of $\mathcal{O}_Y$-modules ([R]), and $\phi_U \in \mathcal{H}(Q)$. Let $\mathcal{I}_q$ denote the ideal sheaf of the point q; then ϕ_U defines a global section ϕ_q of the quotient sheaf $\mathcal{H}/\mathcal{I}_q^2\mathcal{H}$. Since Y is assumed to be a Stein space, the natural map $r : \mathcal{H}(Y) \to (\mathcal{H}/\mathcal{I}_q^2\mathcal{H})(Y)$ is surjective. Therefore there exists a G-equivariant holomorphic map $\phi : X \to W$ such that $r(\phi) = \phi_q$. The map $\phi - \phi_U$ is a holomorphic map from U into W with vanishing order two on $\pi^{-1}(q)$, thus $\phi : X \to W$ is a G-equivariant holomorphic map such that:

(i) $\phi|\pi^{-1}(q)$ is a closed embedding, and

(ii) ϕ is an immersion at every point in $\pi^{-1}(q)$.

This implies that, after shrinking Q, the map ϕ has the desired properties ([H], Section 6, see also Section 5, Proposition 1 for a more general statement). □

Proof of the Theorem. It follows directly from the lemma that X is holomorphically separable. We have to prove that X is holomorphically convex, i.e., for a sequence (x_n) in X such that $\lim x_n = \infty$, we have to show that $\lim f(x_n) = \infty$ for a subsequence of (x_n) and some holomorphic function $f : X \to \mathbb{C}$. Since $X//G$ is a Stein space, we may assume that $\pi(x_n)$ converges to $q \in X//G$; therefore we may assume that $x_n \in \pi^{-1}(Q)$ for all n, where Q is an open neighborhood of q with the properties as stated in the lemma: it follows that $(\phi(x_n))$ is a discrete sequence in W. If $g : W \to \mathbb{C}$ is a holomorphic function on W such that $\lim g(\phi(x_n)) = \infty$, then $f := g \circ \phi$ has the desired properties. □

Remark 2. If X is a holomorphic principal bundle with complex structure group G over a Stein space $Y = X/G$, then X is a Stein space if and only if the complex manifold G is Stein ([M-M]).

Corollary. *Let X be a holomorphic G-space such that $X//G$ exists. Then the quotient map $\pi : X \to X//G$ is a Stein map,* i.e., *the inverse image of a Stein subspace of $X//K$ is Stein.*

Proof. If Y is a Stein subspace of $X//G$ then the restriction $\pi|\pi^{-1}(Y)$: $\pi^{-1}(Y) \to Y$ is the quotient map. Thus $\pi^{-1}(Y)$ is a Stein space. □

2. – Saturation

Let G be a complex reductive group and X a holomorphic G-space.

For a subset A of X let $\mathcal{S}_G(A) = \{x \in X;\ \overline{G \cdot x} \cap A \neq \emptyset\}$ be the saturation of A with respect to G. If the ambient space X is relevant for our considerations, we use the notation $\mathcal{S}_G^X(A)$. A subset A is said to be saturated if $\mathcal{S}_G(A) = A$. A G-stable subset U of X is said to be G-complete if $\overline{G \cdot x} \cap U = \overline{G \cdot x}$ holds for all $x \in U$, i.e., if the closure of a G-orbit in U coincides with its closure in X.

Note that an open G-stable subset of X is saturated, and that a closed G-subset is G-complete. Moreover, a G-stable subset A of X is saturated if and only if $U := X \setminus A$ is G-complete.

Remark. If the semistable quotient $\pi : X \to X//G$ exists, then a G-stable open subset U of X is G-complete if and only if $U = \pi^{-1}(\pi(U))$, i.e., if and only if it is saturated with respect to π (Section 1 Remark 1). More generally, it follows from Remark 1 in Section 1 that $\mathcal{S}_G(A) = \pi^{-1}(\pi(A))$ for a closed G-stable subset A of X.

If a holomorphic G-space X can be covered with G-complete open G-stable subsets $\{U_\alpha\}$ such that $U_\alpha//G$ exist, then the semistable quotients can be glued together (Remark) to a possibly not Hausdorff complex space $X//G$, which has all properties of a semistable quotient except that the Hausdorff property may fail. This observation is sufficient to show the following

PROPOSITION 1. *If there exists a G-invariant locally Stein map ϕ from X into a complex space Y, then a semistable quotient $X//G$ exists.*

PROOF. There exist a covering of Y by open Stein subsets which gives us a covering of X by G-complete open Stein subsets, because of the G-invariance of the map; we have to prove that $X//G$ is Hausdorff, i.e., that two disjoint closed orbits $G \cdot x$ and $G \cdot y$ have disjoint G-complete open neighborhoods. It suffices to consider the case where $\phi(x) = \phi(y)$. Let Q be an open Stein neighborhood of $\phi(x)$ such that $U := \phi^{-1}(Q)$ is Stein; since a G-complete subset of U is already G-complete in X and $U//G$ is Hausdorff, the existence of disjoint open G-complete neighborhoods follows from the same result in the Stein setting. □

COROLLARY. *Let G be a complex reductive group, and let X be a holomorphic G-space. If a semistable quotient $X//G$ exists, then $X//H$ exists for any complex reductive subgroup H of G.* □

For an algebraic analog of this corollary see Proposition 2.1 in [BB-S1].

Now let T be a maximal algebraic torus in G, and assume that $X//T$ exists. The following result, which underlines the close connection between the G-action and the action of T, will be used later on. It is based on an observation of Richardson. For an algebraic analog see also [BB-S2].

PROPOSITION 2. *If A is a closed G-stable subset of X, then*

$$\mathcal{S}_G(A) = K \cdot \mathcal{S}_T(A).$$

PROOF. Given $x \in X$, since $G = KTK$, we have to show that

$$\overline{T \cdot K \cdot x} \cap A \neq \emptyset$$

implies

$$\overline{T \cdot k \cdot x} \cap A \neq \emptyset$$

for some $k \in K$.

Assume that $\overline{T \cdot k \cdot x} \cap A = \emptyset$ for all $k \in K$. Then to every $y \in K \cdot x$ there exist an open neighborhood U_y and a T-invariant continuous function f_y on X such that $f_y > 0$ on U_y and $f_y = 0$ on A (Section 1, Remark 1); but $K \cdot x$ is compact, and therefore the sum of finitely many f_y gives a continuous T-invariant function f on X such that $f|K \cdot x \geq m > 0$ and $f|A = 0$. Thus $\overline{TK \cdot x} \cap A = \emptyset$. □

3. – Closures of orbits

If a semistable quotient $X//G$ exists, then the closure of a G-orbit contains a closed G-orbit in its closure; in this section we show that this remains true if we only assume that $X//T$ exists, where T is a maximal algebraic torus in G. We show this by considering special subanalytic sets in X.

Let X be a (reduced) real analytic space. A subset A of X is said to be subanalytic in X if for any point $x \in X$ there are an open neighborhood U, finitely many real analytic spaces Y_i, Z_i, and proper analytic maps $f_i : Y_i \to U$, $g_i : Z_i \to U$ such that $A \cap U = \cup(f_i(Y_i) \setminus g_i(Z_i))$.

Note that a closed analytic subset of X is a subanalytic set in X, and that the set of subanalytic sets in X is closed with respect to the finite set theoretical operations of taking unions, intersections and complements. Moreover, the inverse image of a subanalytic set with respect to an analytic map is subanalytic. We also need the following property of subanalytic sets ([Hi], 3.8.2; [B-M], Theorem 0.1).

$*$ *Let $\phi : X \to Y$ be an analytic map and A a subanalytic set in X. If $\phi|\bar{A} : \bar{A} \to Y$ is proper, then $\phi(A)$ is subanalytic in Y.*

Here $\bar{A}$ denotes the topological closure of A in X. In the proof of $*$ Hironaka uses the Desingularization Theorem; in our application we use $*$ only in the case where X and Y are closed analytic subsets in a real analytic manifold: in this case the Desingularization Theorem can be replaced by the more elementary Theorem 0.1 of Bierstone and Millman in [B-M].

LEMMA. *Let X be an irreducible complex space, and Z a proper closed analytic subspace in X. Let ϕ be a meromorphic map from X into a complex space Y such that $\phi_\Omega := \phi|\Omega$, where $\Omega := X \setminus Z$, is holomorphic. If D is a relatively compact subanalytic set in X, then $\phi_\Omega(D \cap \Omega)$ is subanalytic in Y and $\phi_\Omega^{-1}(\phi_\Omega(D \cap \Omega))$ is subanalytic in X.*

PROOF. Since ϕ is meromorphic, there exist a proper modification $\Gamma \xrightarrow{p} X$ such that the restriction $p_\Omega : p^{-1}(\Omega) \to \Omega$, $p_\Omega := p|p^{-1}(\Omega)$ is biholomorphic and a holomorphic map $\Phi : \Gamma \to Y$ such that $\phi_\Omega = \Phi \circ p_\Omega^{-1}$; since $D \cap \Omega$ is subanalytic in X, the inverse image $S := p^{-1}(D \cap \Omega)$ is subanalytic in Γ. Now the topological closure $\bar{S} \subset p^{-1}(\bar{D})$ is compact and therefore $\Phi|\bar{S} : \bar{S} \to Y$ is proper; from $*$ it follows that $\Phi(S) = \phi_\Omega(D \cap \Omega)$ is subanalytic in Y.

In order to show that $\phi_\Omega^{-1}(\phi_\Omega(D \cap \Omega))$ is subanalytic in X, set $E := p^{-1}(Z)$. Since p is proper and $\Phi^{-1}(\Phi(S))$ is subanalytic in Γ, it follows that $p(\Phi^{-1}(\Phi(S)) \setminus E) = \phi_\Omega^{-1}(\phi_\Omega(D \cap \Omega))$ is subanalytic in X. □

We will now apply this lemma in the setting where an algebraic torus T is acting holomorphically on X.

PROPOSITION. *Let X be a holomorphic T-space such that the semistable quotient $\pi : X \to X//T$ exists, and let A be a subanalytic set in X such that $\pi|A : A \to X//T$ is proper. Then $T \cdot A$ is subanalytic in X.*

PROOF. There are coverings $\{V_\alpha\}$ and $\{C_\beta\}$ of $X//T$ which have the following properties.

(i) V_α is open in $X//T$, and $\pi^{-1}(V_\alpha)$ is T-equivariantly biholomorphic to a closed analytic subset of an open semi-stable set in a T-representation W_α for each α,

(ii) for every β there is some α such that $C_\beta \subset V_\alpha$,

(iii) C_β is a compact subanalytic set in $X//T$ for every β, and

(iv) $\{C_\beta\}$ is a locally finite covering of $X//T$.

For $B_\beta := \pi^{-1}(C_\beta)$ and $A_\beta := A \cap B_\beta$ we have $A = \cup A_\beta$ and $T \cdot A = \cup T \cdot A_\beta$. Since $\pi | A : A \to X//T$ is proper and $\{B_\beta\}$ is locally finite, A_β is a compact subanalytic set in X and $\{T \cdot A_\beta\}$ is locally finite; thus it is sufficient to show that $T \cdot A_\beta$ is subanalytic in X. This follows from the following

CLAIM. Let X be an affine T-variety and let D be a compact subanalytic set in X. Then $T \cdot D$ is subanalytic in X.

We prove the claim by induction over the dimension of X. We may assume that X is irreducible. By a theorem of Rosenlicht ([Ro]), there exist a T-stable Zariski-open subset Ω of X, a projective variety Y and a rational map ϕ from X into Y such that $\phi_\Omega := \phi|\Omega$ is regular, $U := \phi_\Omega(\Omega)$ is Zariski-open in Y and $\phi_\Omega : \Omega \to U$ is the geometric quotient of Ω with respect to the T-action. Thus $T \cdot (D \cap \Omega) = \phi_\Omega^{-1}(\phi_\Omega(D \cap \Omega))$ is subanalytic in X (lemma). By induction, $T \cdot (D \cap (X \setminus \Omega))$ is subanalytic in $Z := X \setminus \Omega$. Since Z is closed, the claim follows. $\square$

REMARK. In general $T \cdot A$ is not subanalytic for a subanalytic set A. For example, let $\mathbb{C}^*$ act on $\mathbb{C} \times \mathbb{C}^*$ by multiplication on the second factor; then $A := \{(\frac{1}{n}, n);\ n \in \mathbb{N}\}$ is analytic in $\mathbb{C} \times \mathbb{C}^*$, but $\mathbb{C}^* \cdot A$ is not subanalytic in $\mathbb{C} \times \mathbb{C}^*$.

Let G be a complex reductive group and T an maximal algebraic torus in G.

COROLLARY 1. *Let X be a holomorphic G-space such that the semistable quotient $X//T$ exists. Then $G \cdot A$ is subanalytic in X for every compact subanalytic set A in X.*

PROOF. . Let K be a maximal compact subgroup of G such that $G = KTK$. Then, since $K \cdot A$ is a compact subanalytic set, $T \cdot K \cdot A$ is a subanalytic set in X; thus $G \cdot A = K \cdot T \cdot K \cdot A$ is subanalytic in X. $\square$

COROLLARY 2. *Let X be a holomorphic G-space such that $X//T$ exists. Then*

$$\dim G \cdot y < \dim G \cdot x$$

for every $x \in X$ and $y \in \overline{G \cdot x} \setminus G \cdot x$.

PROOF. Since $G \cdot x$ is subanalytic, $\dim_y(\overline{G \cdot x} \setminus G \cdot x) < \dim_x G \cdot x$ for every $y \in \overline{G \cdot x} \setminus G \cdot x$ ([Hi], 4.8.1). Thus $\dim G \cdot y < \dim G \cdot x$ follows. $\square$

COROLLARY 3.. *Let X be a holomorphic G-space such that $X//T$ exists. Then every G-orbit contains a closed G-orbit in its closure.*

PROOF. An orbit of minimal dimension in $\overline{G \cdot x}$ is closed. $\square$

4. – Proper actions

If X is a holomorphic G-space such that the semistable quotient exists, then every closed G-orbit is affine, i.e., the isotropy group is reductive. In this section we show that this remains true if one only assumes that the semistable quotient with respect to a maximal algebraic torus in G exists.

Let G be a Lie group and X a complex G-space. The G-action on X is said to be proper if the map $G \times X \to X \times X$, $(g, x) \to (g \cdot x, x)$, is proper. This is the case if and only if ([P])

(i) *the orbit space X/G is Hausdorff,*

(ii) *every $x \in X$ has a compact isotropy group G_x, and*

(iii) *every $x \in X$ has a slice neighborhood U,* i.e., *a G-stable open neighborhood U such that, for some closed G_x-stable subset S of U, the natural map $G \times_{G_x} S \to U$, $[g, s] \to g \cdot s$, is a homeomorphism.*

Here $G \times_{G_x} S$ denotes the fiber bundle associated with the G_x-principal bundle $G \to G/G_x$.

In the holomorphic framework the correct analog of a compact Lie group is a complex reductive group. In order to construct a holomorphic slice at some orbit, it is often useful to first consider orbits with a reductive isotropy group.

Let G be a complex Lie group, and let X be a holomorphic G-space. Let $x_0 \in X$ be a point with a reductive isotropy group $H := G_{x_0}$ and let L be a maximal compact subgroup of H.

Lemma 1. *There exist a holomorphic Stein H-space S, $s_0 \in S$ and an H-equivariant holomorphic map $\iota_S : S \to X$ such that $\iota_S(s_0) = x_0$ and the induced G-equivariant holomorphic map $\iota : G \times_H S \to X$, $\iota[g, v] = g \cdot \iota_S(v)$, maps $G \times_H S$ locally biholomorphically onto an open neighborhood of x_0.*

Proof. Since H fixes x_0, the tangent space $T_{x_0}X$ is an H-representation. Let $T_{x_0}X = T_{x_0}(G \cdot x_0) \oplus V$ be an H-equivariant splitting. Using Cartan's Linearization lemma, one sees that an L-stable open neighborhood U of x_0 can be L-equivariantly identified with an L-stable closed analytic subset A in an L-stable ball $B \subset T_{x_0}X$. Let $\iota_A : A \to U$ denote such an isomorphism with $\iota_A(0) = x_0$. Then $D := A \cap V$ is an L-stable analytic subset of the ball $B_V := B \cap V$. It follows that $S := H \cdot D$ is a closed analytic subset of the open Stein subset $H \cdot B_V$ of V and that the map $\iota_D := \iota_A | D$ extends to an H-equivariant holomorphic map $\iota_S := S \to X$ ([H], Section 1.5 Extension lemma and Section 6.6 Complexification theorem). Thus there is an induced G-equivariant holomorphic map $\iota : G \times_H S \to X$, $[g, v] \to g \cdot \iota_S(v)$.

Claim. If U is sufficiently small, then ι is locally biholomorphic.

Note that the claim is obvious if x_0 is a smooth point. In the singular case one can argue as follows.

We fix an L-equivariant isomorphism ι_{A_1} from an L-stable locally analytic subset A_1 in $T_{x_0}X$ onto an open neighborhood U_1 of x_0. Then there are an open L-stable neighborhood N of $1 \in G$, where L-acts on G by conjugation, and an open L-stable neighborhood U of x_0, such that $N \cdot U \subset U_1$ and $A := \iota_{A_1}^{-1}(U)$ is closed in an L-stable ball $B \subset T_{x_0}X$. The G-action on X induces a local action on U, which gives a local action on A. In a neighborhood of $0 \in A$ the local action is determined by a holomorphic map $\phi : N \times A \to A_1$, $(g, v) \to \phi(g, v)$.

Now let $\hat{N}$ be an open L-stable Stein neighborhood of $e_0 := 1 \cdot H \in G/H$, and $\tau : \hat{N} \to G$ a holomorphic section such that:

(i) $\tau(e_0) = 1$, $\tau(\hat{N}) \subset N$, and

(ii) τ is L-equivariant, i.e., $\tau(h \cdot t) = h\tau(t)h^{-1}$ for all $h \in L$ and $t \in \hat{N}$.

Then $\hat{\phi} : \hat{N} \times A \to T_{x_0}X$, $\hat{\phi}(t, v) = \phi(\tau(t), v)$ is an L-equivariant holomorphic map; since A is a closed analytic subset of B, the map $\hat{\phi}$ extends to an L-equivariant holomorphic map $\hat{\Phi} : \hat{N} \times B \to T_{x_0}X$ such that $\hat{\Phi}(e_0, v) = v$ for all $v \in B$. Now the map $G/H \to G \cdot x_0 \subset X$, $g \cdot e_0 \to g \cdot x_0$, is an injective immersion and V is transversal to $T_{x_0}(G \cdot x_0)$; therefore, after shrinking $\hat{N}$ and B, the map $\hat{\Phi}|\hat{N} \times B_V$ is biholomorphic onto its image Ω. Thus $\hat{\phi}_D : \hat{N} \times D \to \Omega$, $\hat{\phi}_D = \hat{\Phi}|\hat{N} \times D$, is biholomorphic onto its image $\hat{A} := \hat{\phi}_D(\hat{N} \times D) \subset \Omega \cap A_1$. In order to show that $\hat{A} = \Omega \cap A_1$, we have to assume that $\hat{N}$ is connected and that every irreducible component A_α of $\Omega \cap A_1$ contains $0 \in T_{x_0}X$; then $D_\alpha := A_\alpha \cap V$ is not empty and $\hat{\phi}_D(\hat{N} \times D_\alpha) \subset A_\alpha$. From dim $\hat{N} \times D_\alpha =$ codim $V +$ dim $D_\alpha \leq$ dim $A_\alpha \leq$ dim $(A_\alpha \cap V) +$codim $V =$ dim $D_\alpha +$codim V follows that $\hat{\phi}_D(\hat{N} \times D_\alpha) = A_\alpha$. This implies that $\hat{A} = \Omega \cap A_1$.

Finally note that $\hat{N} \times D$ can be viewed as an open subset of $G \times_H S$, where $\hat{N} \times D \hookrightarrow G \times_H S$, $(t, v) \to [\sigma(t), v]$; thus $\iota : G \times_H S \to X$, $\iota[g, v] = g \cdot \iota_S(v)$ is biholomorphic on $\hat{N} \times D$. Equivariance implies that ι is localy biholomorphic. □

Remark 1. For the proof of the lemma, one needs that the image of $H = G_{x_0}$ in $\mathrm{GL}(T_{x_0}X)$ is reductive; in particular, the statement of the lemma also holds if H is a compact complex Lie group, since in this case the image of H in $\mathrm{GL}(T_{x_0}X)$ is compact and therefore finite. In this context the proof of Lemma 1 is essentially due to Holmann ([Hol]).

Corollary 1. *If the G-action on X is proper, then every $x \in X$ has a G-stable slice neighborhood,* i.e., *there exists a locally closed G_x-stable subspace S of X with $x \in S$ such that $G \cdot S$ is open in X and*

$$G \times_{G_x} S \to G \cdot S, \ [g, s] \to g \cdot s,$$

is biholomorphic.

Proof. Since G_x acts as a finite group on S, after shrinking S, there exists an open neighborhood N of $1 \in G$, stable by G_x with respect to right multiplication, such that

$$N \times_{G_x} S \to X$$

is an open embedding. Properness of the G-action implies the existence of an open G_x-stable neighborhood V of x in X such that $\{g \in G;\ g \cdot V \cap V \neq \emptyset\} \subset N$;

after replacing S with $S \cap V$ we claim that $\iota : G \times_{G_x} S \to X$ is an open embedding.

We have to show injectivity: for this it is sufficient to prove that $g \cdot s_1 = s_2$ implies $g \in G_x$ for $g \in G$, $s_1, s_2 \in S$. Now, from $g \cdot s_1 = s_2$ follows that $g \cdot V \cap V \neq \emptyset$ and therefore $g \in N$. Thus $g = \tau(u) \cdot h$ for some $u \in \hat{N}$ and $h \in G_x$ (we use the notation of the proof of Lemma 1). Since $h \in G_x$, we have $h \cdot s_1 \in S$ and $\tau(u)h \cdot s_1 = s_2 = \tau(1 \cdot G_x) \cdot s_2$. The injectivity of $\hat{\Psi}$ implies $\tau(u) = 1$ and therefore $g \in G_x$ follows. □

For the following consequence see also [Hol].

COROLLARY 2. *If G acts properly on X, then X/G is a complex space. In particular, if G is assumed to be a complex reductive group which acts properly on X, then $X//G = X/G$ exists.* □

REMARK 2. Let G be a complex reductive group, and let X be a holomorphic G-space such that the semistable quotient exists. Then the complex analytic version of Luna's slice theorem (see [H] or [Sn]) implies that the G-action on X is proper if and only if $\dim G \cdot x = \dim G$ for all $x \in X$.

Now let G be a Lie group, T a Lie subgroup of G, and assume that there exist compact subgroups K_1, K_2 of G such that $G = K_1 T K_2$. Let G act topologically on X and assume that the T-action on X is proper; then we have the following:

LEMMA 2. *If T acts properly on X, then the G-action on X is proper.*

PROOF. We have to show that any sequence (g_n, x_n) in $G \times X$ such that $(g_n \cdot x_n, x_n)$ converges to $(y_0, x_0) \in X \times X$ has a convergent subsequence with $(g_n, x_n) \to (g_0, x_0)$.

For this, we write $g_n = k_n t_n h_n$ with $k_n \in K_1$, $t_n \in T$, $h_n \in K_2$. We may assume that $k_0 = \lim k_n$ and $h_0 = \lim h_n$ exist; since $\lim (k_n t_n h_n) \cdot x_n = y_0$, it follows that $\lim h_n \cdot x_n = h_0 \cdot x_0$ and $\lim t_n \cdot (h_n \cdot x_n) = k_0^{-1} \cdot y_0$. The properness of the T-action implies that a subsequence of (t_n) converges to $t_0 \in T$; thus (g_n) converges to $k_0 t_0 h_0$. □

REMARK. If G is a reductive group and T is its maximal torus, the same statement is proved in [Mu], chap.II, Proposition 2.4. for algebraic actions. There the use of the decomposition $G = K_1 T K_2$ is replaced by the use of a theorem of Iwahori.

For a complex reductive group G with a maximal algebraic torus T, this implies the following (see [BB-S1] for a similar statement for algebraic actions):

COROLLARY 3. *If X is a holomorphic G-space such that the geometric quotient X/T exists, then the geometric quotient X/G exists.* □

Let H be a closed complex subgroup of the reductive group G. If for $X := G/H$ the semistable quotient $X//T$ exists, then our main result in this special case simply states that H is reductive. For this we need the following decomposition theorem for complex linear groups.

PROPOSITION 1. *Let G be a connected complex Lie subgroup of a complex linear group GL(W). Then G is a semidirect product $G = H \cdot U$, where H is a complex reductive group and U is a normal solvable simply connected complex subgroup of G.*

This result is well known. It is the complex analytic analog of the same statement for real Lie groups [Ho] (p. 223); for the convenience of the reader we give here the proof in the complex setting.

We need the following remark.

LEMMA 3. *Let G be a connected complex Lie group and A a connected closed normal complex subgroup of G, which does not contain a non-trivial compact subgroup. If G/A is reductive, then A is a semidirect factor of G.*

PROOF. Let K be a maximal compact subgroup of G. Then, since A is connected, $\pi(K)$ is a maximal compact subgroup of G/A ([I]), where $\pi : G \to G/A$ denotes the quotient map; thus, by the assumption on A, the restriction $\pi|K : \pi(K) \to K$ is an isomorphism. Since G/A is the universal complexification of the compact group $\pi(K)$, the inverse homomorphism $\sigma : \pi(K) \to G$, $\sigma := (\pi|K)^{-1}$ extends to a holomorphic homomorphism $\sigma^c : G/A \to G$. □

PROOF OF PROPOSITION 1. Assume that the commutator subgroup G' of G is reductive. Then the radical R of G is abelian, and therefore $R = L^{\mathbb{C}} \times V$, where $L \cong (S^1)^l$ is the maximal compact subgroup of R, and V is a vector group. Note that the automorphism group of $L^{\mathbb{C}}$ is finite and therefore $L^{\mathbb{C}}$ is central; thus, if we write $G = SR$, where S is a semisimple subgroup of G, the group $H = SL^{\mathbb{C}}$ is reductive and $G = H \cdot V$ is a semidirect product.

Note that $G' = \bar{G}'$, where $\bar{G}$ denotes the Zariski-closure of G in $\mathrm{GL}(W)$. If G' is not reductive, it has a non-trivial unipotent radical R_U, which is closed and normal in G and in G'; the group $\tilde{G} := G/R_U$ is a linear group, because it is contained in the linear algebraic group $\bar{G}/R_U$. Apply induction to the quotient, i.e. $\tilde{G} = \tilde{H} \cdot \tilde{U}$ is a semidirect product, where $\tilde{H}$ is reductive and $\tilde{U}$ is a simply connected solvable normal subgroup of $\tilde{G}$. If $\pi : G \to \tilde{G}$ is the natural map, define $U := \pi^{-1}(\tilde{U})$; then G/U is reductive and an application of Lemma 3 shows that $G = H \cdot U$ is a semidirect product. □

PROPOSITION 2. *If for $X := G/H$ the semistable quotient $X//T$ exists, then H is a complex reductive group. In particular, X is affine.*

PROOF. Let H^0 denote the connected component of the identity of H. Since H^0 is a connected complex linear group, there is a closed complex normal simply connected solvable subgroup U of H^0, and a complex reductive subgroup L of H^0 such that $H^0 = L \cdot U$ is a semidirect product. Thus the fibration $G/U \to G/H^0$ has $L = H^0/U$ as typical fiber and therefore it is a Stein map; moreover, since the fibration $G/H^0 \to G/H$ is a covering, it is also a Stein map: thus $G/U \to G/H$ is a Stein map. Hence, for $Y := G/U$, the semistable quotient $Y//T$ exists; but, since U is solvable and simply connected and the T-isotropy groups are reductive, the T-action on Y is free. Therefore G acts properly on G/U, and this implies $U = \{e\}$, i.e., H^0 is a reductive group.

We have to show that H/H^0 is finite: for this, set $N_0 := N/H^0$, where $N := N_G(H^0)$ denotes the normalizer of H^0 in G. The groups N and N_0 are reductive and we may assume that the maximal torus T_N of N is contained in T and that $T_{N_0} := T_N/T_N \cap H^0$ is a maximal algebraic torus of N_0; note that the fiber $X_N := N/H$ of the fibering $G/H \to G/N$ is closed in $X = G/H$ and therefore $X_N//T_N$ exists. Since $X_N = N/H = (N/H^0)/(H/H^0) = N_0/\Gamma$, where $\Gamma := H/H^0$ is a discrete subgroup of N_0, the semistable quotient $X_N//T_{N_0}$ exists. But T_{N_0} is a maximal torus of the reductive group N_0 and Γ is a discrete subgroup, thus T_{N_0} acts properly on $X_N = N_0/\Gamma$ (Remark 2), and therefore N_0 also acts properly: this implies that $\Gamma = H/H^0$ is finite. □

REMARK. For an algebraic proof see Proposition 2.3 in [BB-S1].

COROLLARY 4. *If X is a holomorphic G-space such that $X//T$ exists, then every closed G-orbit in X is affine.* □

5. – Stein neighborhoods

For the existence of a semistable quotient, it is necessary that every closed G-orbit has a G-stable open Stein neighborhood. In this section we show that this already follows if one assumes the existence of a semistable quotient with respect to a maximal algebraic torus in G.

Let K be a compact Lie group, and let X_j be complex K-spaces such that the semistable quotients $X_j//K$ exist, $j = 1, 2$. Let $\pi_j : X_j \to X_j//K$ denote the quotient map. The following is a consequence of the holomorphic analog of Luna's Slice Theorem ([H], Section 6.3).

PROPOSITION 1. *If $\phi : X_1 \to X_2$ is a locally biholomorphic K-equivariant map which maps a K-stable closed analytic subset A_1 biholomorphically onto a closed analytic subset A_2 of X_2, then ϕ maps a π_1-saturated open neighborhood of A_1 biholomorphically onto a π_2-saturated open neighborhood of A_2.*

PROOF. The map ϕ induces a holomorphic map $\bar{\phi} : X_1//K \to X_2//K$ such that:

(i) $\bar{\phi}$ maps $\pi_1(A_1)$ biholomorphically onto $\pi_2(A_2)$, and

(ii) $\bar{\phi}$ is locally biholomorphic along $\pi_1(A_1)$.

Since $\pi_j(A_j)$ are closed analytic subsets of $X_j//K$, this implies that $\bar{\phi}$ maps an open neighborhood Q_1 of $\pi_1(A_1)$ biholomorphically onto an open neighborhood Q_2 of $\pi_2(A_2)$. Thus we may assume that $\bar{\phi} : X_1//K \to X_2//K$ is biholomorphic; by applying Luna's Slice Theorem again ([H], Section 6.3), it follows that $\phi : X_1 \to X_2$ is injective and therefore an isomorphism. □

Now let G be a complex reductive group, T a maximal algebraic torus in G, and let X be a holomorphic G-space such that $X//T$ exists.

PROPOSITION 2. *Every closed G-orbit $G \cdot x_0$ in X has a G-stable open Stein neighborhood.*

The proof of Proposition 2 requires some preparation.

Let K be a maximal compact subgroup of G, and U a K-stable subset of X; we say that U is orbit connected if for every $x \in U$ the set $\{g \in G;\ g \cdot x \in U\}/K$ is connected. Here K acts on G by multiplication from the left. A K-stable subset U is said to be orbit convex if for all $x \in U$ and $\xi \in \mathcal{L}ie\, K$ the set $\{t \in \mathbb{R};\ \exp it\xi \cdot x \in U\}$ is connected.

In the following we fix a maximal compact subgroup K of G such that $K \cap T$ is a maximal torus in K. The following was observed in a slightly different form by Koras ([K]).

LEMMA. *A K-invariant open subset U of X, which is orbit connected with respect to the T-action, is also orbit connected with respect to the G-action.*

PROOF. We may assume that G, and therefore also K, are connected; since for every $x \in U$ and $k \in K$ we have $\{t \in T;\ ktk^{-1} \cdot x \in U\} = \{t \in T;\ t \cdot (k^{-1} \cdot x) \in U\}$, the set $\{t \in T;\ ktk^{-1} \cdot x \in U\}$ is connected for every $k \in K$ and $x \in U$.

Now assume that for $g \in G$ and $x \in U$ we have $g \cdot x \in U$. Using the decomposition $G = KTK$ we can write $g = k_1 k_0 t_1 k_0^{-1}$ where $k_1, k_0 \in K$ and $t_1 \in T$; from $k_0 t_1 k_0^{-1} \cdot x \in U$ it follows that there is a path $\alpha : [0,1] \to T$ with $\alpha(0) = 1$, $\alpha(1) = t_1$ and $\alpha(s) \cdot x \in U$ for all $s \in [0,1]$. Let $\beta : [0,1] \to K$ be a path with $\beta(0) = 1$ and $\beta(1) = k_1$; then $\gamma : [0,1] \to G$, $\gamma(s) = \beta(s) k_0 \alpha(s) k_0^{-1}$ satisfies $\gamma(0) = 1, \gamma(1) = g$ and $\gamma(s) \cdot x \in U$ for all $s \in [0,1]$. □

Let H be a complex reductive subgroup of G such that $L := H \cap G$ is a maximal compact subgroup of H. Let V be an H-representation space, and identify $K/L \hookrightarrow G/H \hookrightarrow G \times_H V$ with the corresponding subsets of the zero section in $G \times_H V$. The following is proved in [H].

Every open neighborhood U of K/L in $G \times_H V$ contains a K-invariant open Stein neighborhood Ω of K/L in $G \times_H V$ which is orbit convex with respect to G.

PROOF OF PROPOSITION 2. Since $G \cdot x_0$ is a closed orbit in X, the isotropy group $H := G_{x_0}$ is reductive; further we may assume that $L := K_{x_0} = K \cap G_{x_0}$ is a maximal compact subgroup of H. By the Lemma 1 in Section 4, there exists a locally biholomorphic G-equivariant map $\phi : G \times_H S \to X$, where S is an open H-stable neighborhood of zero in an H-representation space, which maps G/H biholomorphically onto its image; thus ϕ maps a T-stable open Stein neighborhood U of G/H in $G \times_H S$ biholomorphically onto its image. There exists a K-invariant open Stein neighborhood $\Omega \subset U$ of K/L in $G \times_H S$ which is orbit convex with respect to G. Hence, after identifying Ω with $\phi(\Omega)$, we see that $K \cdot x_0$ has a K-stable open Stein neighborhood Ω which is orbit convex with respect to T. This implies that Ω is orbit connected with respect to G as a subset of X, and therefore $G \cdot \Omega$ coincides with its universal complexification $\Omega^{\mathbb{C}}$ ([H]) which is a Stein space. □

6. – Existence of semistable quotients

Let G be a complex reductive group, K a maximal compact subgroup, and T a maximal algebraic torus such that $G = KTK$; let X be a holomorphic G-space. The following are used in the proof of the main result of this section.

LEMMA 1. *Let $\tilde{U}_j$ be T-invariant subsets of X such that $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$. Then, for any K-invariant subsets $U_j \subset \tilde{U}_j$, we have $G \cdot U_1 \cap G \cdot U_2 = \emptyset$.*

PROOF. Since $G = KTK$, we have that

$$G \cdot U_1 \cap G \cdot U_2 = G \cdot (U_1 \cap G \cdot U_2) = G \cdot (U_1 \cap KTU_2) = G \cdot (U_1 \cap T \cdot U_2).$$

Thus the lemma follows. □

LEMMA 2. *If $X//T$ exists, then two different closed G-orbits in X have disjoint G-stable open Stein neighborhoods which are G-complete with respect to G.*

PROOF. Let Y_j be closed G-orbits, and let $\tilde{U}_j$, $j = 1, 2$, be T-stable open neighborhoods of Y_j such that $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$. Since Y_j is K stable, there exists an open K-stable neighborhood of Y_j which is contained in $\tilde{U}_j$. By Lemma 1, there exist disjoint G-stable open neighborhoods U_j of Y_j; moreover, we may assume that U_j are open Stein subspaces of X (see Section 5). Since $A_j := X \setminus U_j$ is closed, it follows that $\tilde{A}_j := \mathcal{S}_G(A_j) = K \cdot \mathcal{S}_T(A_j)$ is closed (Section 2); thus $\tilde{V}_j := X \setminus \tilde{A}_j \subset U_j$ are G-complete with respect to G in X. Let $\pi_j : U_j \to U_j//G$ denote the quotient map; the semistable quotient $U_j//G$ exists, since U_j is Stein. Now, $\pi_j(\tilde{V}_j)$ is an open neighborhood of $q_j := \pi_j(Y_j)$; thus, there is an open Stein neighborhood Q_j of q_j in $U_j//G$ such that $V_j := \pi_j^{-1}(Q_j) \subset \tilde{V}_j$. Since V_j is G-complete in U_j and $\tilde{V}_j$ are G-complete in X, this implies that V_j is G-complete in X. □

THEOREM. *A semistable quotient $X//G$ exists if and only if $X//T$ exists.*

PROOF. We already proved that the existence of $X//G$ implies that $X//H$ exists for any reductive subgroup H of G (Section 2).

Thus assume that $X//T$ exists. Then, since for every $x \in X$ the closure of $G \cdot x$ contains a closed orbit, and closed orbits have open G-stable Stein neighborhoods which are G-complete with respect to G, there is an open covering $\{U_\alpha\}$ of X such that U_α is G-complete with respect to G and Stein; thus the semistable quotients $U_\alpha//G$ exists and can be glued together. Since closed G-orbits can be separated by G-complete open G-subsets, the resulting space $X//G$ is Hausdorff, and $X//G$ is a semistable quotient of X. □

In contrast to the algebraic case, the existence of semistable quotients for all one-dimensional algebraic subtori is not sufficient in the above Theorem. In order to give a concrete example, we consider a lattice Γ of rank $2n-1$ in $\mathbb{C}^n$, and denote by V the $2n-1$-dimensional real subspace of $\mathbb{C}^n$ spanned by Γ; moreover we choose Γ such that:

(i) $\mathbb{Z}^n$ is a direct factor of Γ, i.e., $\Gamma = \mathbb{Z}^n \oplus \Lambda$ for some sublattice Λ, and
(ii) $i\mathbb{Z}^n \cap V = \{0\}$.

Thus $T := (\mathbb{C}^*)^n = \mathbb{C}^n/\mathbb{Z}^n$ acts holomorphically and transitively on $X := \mathbb{C}^n/\Gamma = (\mathbb{C}^n/\mathbb{Z}^n)/(\Gamma/\mathbb{Z}^n)$. Since $\Gamma/\mathbb{Z}^n$ is not finite, X is not a Stein manifold and therefore a semistable quotient of X with respect to T does not exist.

We claim that every one-dimensional algebraic subtorus A of T acts properly on X; in particular, X/A is the semistable quotient of X with respect to A. In order to see this, it is sufficient to show that the image $\tilde{A}$ of A in $\mathbb{C}^n/\Gamma = (\mathbb{C}^n/\mathbb{Z}^n)/(\Gamma/\mathbb{Z}^n)$ is not compact and closed. We may assume that $\tilde{A}$ is the image of the line $\mathbb{C}\cdot a \subset \mathbb{C}^n$ with respect to the quotient $q : \mathbb{C}^n \to \mathbb{C}^n/\Gamma$, where $a \in \mathbb{Z}^n \setminus \{0\}$; the condition (ii) implies that q maps $i\mathbb{R}\cdot a$ isomorphically onto its image R. Moreover $p : \mathbb{C}^n/\Gamma \to \mathbb{C}^n/V \cong \mathbb{R}$ maps R isomorphically onto $\mathbb{C}^n/V \cong \mathbb{R}$; but p is a trivial fibration and therefore R is a closed subgroup of $X = \mathbb{C}^n/\Gamma$. Finally, $\tilde{A} = S \cdot R$, where $S = q(\mathbb{R}\cdot a)$ is the maximal compact subgroup of $\tilde{A}$, implies that $\tilde{A}$ is closed in X.

REFERENCES

[BB-S1] A. BIAŁYNICKI-BIRULA – J. A. ŚWIĘCICKA, *A reduction theorem for existence of good quotients*, Amer. J. Math. **113** (1990), 189-201.

[BB-S2] A. BIAŁYNICKI-BIRULA – J. A. ŚWIĘCICKA, *Three theorems on existence of good quotients*, preprint 1995,1-11.

[B-M] E. BIERSTONE – P. D. MILLMAN, *Semianalytic and subanalytic sets*, Publ. Math. IHES **67** (1988), 5-42.

[H] P. HEINZNER, *Geometric invariant theory on Stein spaces*, Math. Ann. **289** (1991), 631-662.

[H-H-K] P. HEINZNER – A. T. HUCKLEBERRY – F. KUTZSCHEBAUCH, Abels' Theorem in the real analytic case and applications to complexifications, In: Complex Analysis and Geometry, Lecture Notes in Pure and Applied Mathematics, Marcel Decker, 1995, 229-273.

[H-L] P. HEINZNER – F. LOOSE, *Reduction of complex Hamiltonian G-spaces*, Geom. Funct. Anal. **4** (1994), 288-297.

[H-M] P. HEINZNER – L. MIGLIORINI, *Projectivity of moment map quotients*, preprint 1996.

[Hi] H. HIRONAKA, Subanalytic sets, In: Number Theory, Algebraic Geometry and Commutative Algebra in honor of Y. Akizuki, Kinokunuya, Tokyo, 1973, p. 453-493.

[Ho] G. HOCHSCHILD, The Structure of Lie groups, San Francisco, London, Amsterdam: Holden-Day, 1965.

[Hol] H. HOLMANN, *Quotienten komplexer Räume*, Math. Ann. **142** (1961), 407-440.

[I] K. IWASAWA, *On some types of topological groups*, Ann. Math. **50** (1949), 507-558.

[K] M. KORAS, Linearization of reductive group actions, In: Group actions and vector fields. Lecture Notes in Mathematics 956. Springer-Verlag, Berlin, Heidelberg, New York, 1982, p. 92-98.

[M-M] Y. MATSUSHIMA – A. MORIMOTO, *Sur certains espaces fibrés holomorphes sur une variété de Stein*, Bull. Soc. Math. France **88** (1960), 137-155.

[Mu] D. MUMFORD – J. FOGARTY, Geometric Invariant Theory, Ergeb. Math. 34, Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[P] R. S. PALAIS, *On the existence of slices for actions of non-compact Lie groups*, Ann. Math. **73** (1961), 295-323.

[R] M. ROBERTS, *A note on coherent G-sheaves*, Math. Ann. **275** (1986), 573-582.

[Ro] M. ROSENLICHT, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401-443.

[S] R. SJAMAAR, *Holomorphic slices, symplectic reduction and multiplicities of representations*, Ann. Math. **141** (1995), 87-129.

[Sn] D. M. SNOW, *Reductive group actions on Stein Spaces*, Math. Ann. **259** (1982), 79-97.

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