Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 26, nº 1 (1998), p. 75-95
<http://www.numdam.org/item?id=ASNSP_1998_4_26_1_75_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVI (1998), pp. 75-95

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FUMIHIKO HIROSAWA

1. - Introduction

We shall consider the Cauchy problem

$$(1)\qquad \begin{cases} u_{tt}(t,x) + \Phi\left((Au(t,\cdot), u(t,\cdot))\right) Au(t,x) = f(t,x) & (t > 0, x \in \mathbb{R}^n), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \end{cases}$$

where $Au(t,x) = \sum_{hk} D_{x_h}(a_{hk}(x) D_{x_k} u(t,x))$, $D_{x_h} = \frac{\partial}{i\partial x_h}$ $(h = 1, \cdots, n)$, $(Au(t,\cdot), u(t,\cdot))$ is the inner product of $Au(x)$ and $u(x)$ in $L^2(\mathbb{R}^n_x)$, and $\Phi(\eta)$ is a nonnegative function in $C^1([0,\infty))$. When $A = \sum_{h=1}^n D_{x_h}^2 = -\Delta$ (i.e. $a_{hk}(x) = \delta_{hk}$, Kronecker's δ) equation (1) is called the Kirchhoff equation, which has been studied by many authors (cf. [2], [3], [5], [12], [13], [14], [16], etc.). The problem which we shall treat in this paper is a generalization of $-\Delta$ to a degenerate elliptic operator A, where $[a_{hk}(x)]_{hk}$ is a real symmetric matrix which satisfies

$$(2)\qquad \sum_{hk} a_{hk}(x)\xi_h\xi_k \geq 0 \quad (\forall x \in \mathbb{R}^n_x,\ \forall \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n_\xi).$$

We know some results for the problem (1), that are the following. When the coefficients a_{hk} and the Cauchy data u_0 and u_1 belong to real analytic class, Kajitani-Yamaguti [9] proved global existence and uniqueness for the solution of (1) in case that $\Phi(\eta) \in C^1([0,\infty))$ and $\Phi(\eta) \geq 0$, and later Hirosawa [7] relaxed the assumption on Φ to $\Phi(\eta) \in C^0([0,\infty))$. In case of quasi-analytic class data, Yamaguti [17] proved the global solvability for (1) in case that u_0 and u_1 belong to a certain subclass of quasi-analytic functions (for the definition of quasi-analytic class, see [10]) under the assumptions that $\Phi(\eta) \in C^1([0,\infty))$, $\Phi(\eta) \geq \exists\, \Phi_0 > 0$ and that $[a_{hk}(x)]$ is real analytic. Actually in case $A = -\Delta$, Nishihara [13] investigated the global solvability for data in the quasi-analytic class. The class of functions which was introduced in [13] is more general than

Pervenuto alla Redazione il 25 ottobre 1996 e in forma definitiva il 29 aprile 1997.

Yamaguti's one in [17]. This paper enhances Nishihara's work to a general degenerate elliptic operator A whose coefficients satisfy a particular condition, specified below, with respect to the space variable x. We remark that the problem which was treated in [13] is strictly hyperbolic, and the author showed a sufficient condition for the existence of a local solution in a more general class. But our problem is weakly hyperbolic, so we can't get the same result for local existence in general.

At first we define some function spaces to state our main theorem.

DEFINITION. Let $M_0, M_1, \cdots$ be a sequence of positive real numbers and ρ a positive constant.

(i) A $C^\infty(\mathbb{R}^n)$ function f said to belong to the class $C(\{M_j\})_\rho$ if there exists a $C > 0$ independent of α, ρ and x such that

$$|D_x^\alpha f(x)| \le C\rho^{-|\alpha|} M_{|\alpha|} ,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index of nonnegative integers, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $f(x) \in C(\{M_j\})_\rho$ we define the norm $|f|_{C(\{M_j\})_\rho}$ by

$$|f|_{C(\{M_j\})_\rho} \equiv \sup_{\alpha\in\mathbb{N}^n, x\in\mathbb{R}^n} \left\{ \frac{\rho^{|\alpha|}}{M_{|\alpha|}} |D^\alpha f(x)| \right\} .$$

(ii) Let $\{M_j\}$ be a monotone increasing, logarithmically convex sequence, that is, $\frac{M_j}{jM_{j-1}} \le \frac{M_k}{kM_{k-1}}$ for any $j \le k$. A $C^\infty(\mathbb{R}^n)$ function f said to belong to the class $C^\sharp(\{M_j\})_\rho$ if there is a $C > 0$ independent of α and ρ such that

$$\left\|D^\alpha f\right\| \le C\rho^{-|\alpha|} M_{|\alpha|} ,$$

where $\|\cdot\|$ is the usual norm in $L^2(\mathbb{R}_x^n)$. We set $C^\sharp(\{M_j\}) \equiv \bigcup_{\rho>0} C^\sharp(\{M_j\})_\rho$. Moreover if the summation

$$\sum_{j=0}^{\infty} \frac{\rho^j}{M_j} \left\{ \sum_{|\alpha|=j} \left\|D^\alpha f\right\|^2 \right\}^{\frac{1}{2}}$$

is finite, then we say that f belongs to the class $\mathcal{C}^\sharp(\{M_j\})_\rho$ and define the norm $\|f\|_{\mathcal{C}^\sharp(\{M_j\})_\rho}$ by

$$\|f\|_{\mathcal{C}^\sharp(\{M_j\})_\rho} \equiv \sum_{j=0}^{\infty} \frac{\rho^j}{M_j} \left\{ \sum_{|\alpha|=j} \left\|D^\alpha f\right\|^2 \right\}^{\frac{1}{2}} .$$

We see that the classes $C^\sharp(\{M_j\})_\rho$ and $\mathcal{C}^\sharp(\{M_j\})_\rho$ satisfy the inclusion $C^\sharp(\{M_j\})_{\rho'} \dot{\subset} \mathcal{C}^\sharp(\{M_j\})_\rho$ for $0 < \forall\rho < \rho'$ (see Lemma D in Appendix) and the product of $a(x) \in C(\{M_j\})_{\rho_0}$ and $v(x) \in \mathcal{C}^\sharp(\{M_j\})_{\rho_0}$ belongs to $\mathcal{C}^\sharp(\{M_j\})_\rho$ for $0 < \forall\rho < \rho_0$ (see Lemma C).

MAIN THEOREM. *Let ρ_0 and ρ_1 be positive numbers satisfying $\rho_0 < \rho_1$, $\{M_j\}$ be a positive, logarithmically convex sequence and* $\sup_j Q(j) \equiv \sup_j\{M_j/j^{\frac{3}{2}}M_{j-1}\} < \infty$. *Assume that* $[a_{hk}(x)] \in [C(\{M_j\})_{\rho_1}]$ *satisfies* (2), $u_0(x), u_1(x) \in C^\sharp(\{M_j\})_{\rho_0}$, $f(t,x) \in C^0([0,T]; C^\sharp(\{M_j\})_{\rho_0})$ *and that* $\Phi(\eta) \in C^1([0,\infty))$ *satisfies* $\Phi(\eta) \geq \exists\, \Phi_0 > 0$ *for all* $\eta \in [0,\infty)$. *Then there are a positive number* $T_0(\{M_j\}) = T_0 \leq T$, *a monotone decreasing positive function* $\rho(t) \in C^1([0,T_0])$ *and the unique solution* $u(t,x)$ *of* (1) *such that*

$$u(t,x) \in C^2\left([0,T_0]; C^\sharp(\{M_j\})_{\rho(t)}\right).$$

Moreover, if $\{M_j\}$ *satisfies a quasi-analytic condition, that is,* $\sum_{j=1} M_j/M_{j+1} = \infty$ (*see Theorem* A) *and* $Q(j) \to 0$ *as* $j \to \infty$, *then we can take* $T_0 = T$.

REMARK 1. Quasi-analytic condition holds for $\{M_j\} = \left\{1\ (j=1),\ \prod_{k=2}^{j} k \log k\ (j \geq 2)\right\}$, $\{M_j\} = \{j!^s\}$ $(s \leq 1)$ and so on. The conditon $\sup_j Q(j) < \infty$ holds for $\{M_j\} = \{j!^s\}$ $s \leq 3/2$, and so on. In the linear weakly hyperbolic case, that is for $\Phi(\eta) \equiv$ constant, the condition $\sup_j Q(j) < \infty$ is optimal (see [4]).

REMARK 2. Roughly speaking, the class of the solution $\mathcal{H}(M)$ introduced in [13] is defined as follows. For a strictly increasing nonnegative continuous function $M(r)$, $\mathcal{H}(M)$ is the set of functions $f(x)$ satisfy $\|M(|\cdot|)\hat{f}(\cdot)\|_{L^2} < \infty$, where $|\xi| = \sqrt{\xi_1^2 + \ldots + \xi_n^2}$ and $\hat{f}(\cdot)$ is the Fourier image of $f(x)$. Then the condition for $M(r)$ which ensures the existence of a global solution is the following:

$$\int_c^\infty \frac{ds}{s(d_0 + M^{-1}(s))} = \infty, \tag{3}$$

for some positive constants c and d_0, where M^{-1} is the inverse function of M. On the other hand, by the Denjoy-Carleman theorem (Theorem A), we can easily see that the quasi-analytic condition $\sum M_j/M_{j+1} = \infty$ is equivalent to (3) by considering $M(r) \equiv \sup_{j\geq 1}\{r^j/M_j\}$, here $M(r)$ is the associated function of $\{M_j\}$.

REMARK 3. In strictly hyperbolic case (i.e., when the matrix $[a_{hk}]$ is strictly positive definite), we can omit the assumption $\sup_j Q(j) < \infty$ to prove the local existence of the solution.

ACKNOWLEDGMENT. I would like to thank Prof. K. Kajitani and Dr. T. Kinoshita for their encourgement and many helpful conversations.

2. – Proof of Main Theorem

In case that the coefficients $a_{hk}(x)$ $(h,k=1,\cdots,n)$ and the Cauchy data are real analytic, we know the following result:

THEOREM 1 (K. Kajitani-K. Yamaguti [9]). *Let ρ_0 and ρ_1 be positive constants satisfying $\rho_0 < \rho_1$. Assume that $\Phi(\eta) \in C^1([0,\infty))$, $\Phi(\eta) \geq 0$ and $[a_{hk}(x)] \in [C(\{j!\})_{\rho_1}]$. Then if $u_0(x), u_1(x) \in \mathcal{C}^\sharp(\{j!\})_{\rho_0}$ and $f(t,x) \in C^0([0,T];\mathcal{C}^\sharp(\{j!\})_{\rho_0})$ for some $T>0$, there is a nonnegative function $\rho(t) \in C^1([0,T])$ such that the Cauchy problem* (1) *has the unique solution $u(t,x) \in C^2\left([0,T];\mathcal{C}^\sharp(\{j!\})_{\rho(t)}\right)$.*

We shall relax the assumption of Theorem 1 from $\{j!\}$ to general $\{M_j\}$.

For given u_0, u_1, a_{hk} and f satisfying the assumptions of Main Theorem, we define the real analytic functions $u_0^{(\nu)}$, $u_1^{(\nu)}$, $a_{hk}^{(\nu)}$ and $f^{(\nu)}$ as

$$u_0^{(\nu)}(x) \equiv \chi_{\frac{1}{\nu}} * u_0(x),\ u_1^{(\nu)}(x) \equiv \chi_{\frac{1}{\nu}} * u_1(x),$$

$$a_{hk}^{(\nu)}(x) \equiv \chi_{\frac{1}{\nu}} * a_{hk}(x)\ (h,k=1,\cdots,n),\ f^{(\nu)}(x) \equiv \chi_{\frac{1}{\nu}} * f(x),\ (\nu=1,2,\cdots),$$

where $\chi_{\frac{1}{\nu}}*$ is Friedrichs' mollifier and $\chi(y)$ satisfies $\chi(y)\geq 0$ and $\int |D_y^\alpha \chi(y)|dy \leq \rho_0^{-|\alpha|}|\alpha|!$

Note that, $\chi_{\frac{1}{\nu}}*$ is an analytic convolution, so $u_0^{(\nu)}(x)$, $u_1^{(\nu)}(x)$, $a_{hk}^{(\nu)}(x)$ and $f^{(\nu)}(x)$ are real analytic functions, namely, $u_0^{(\nu)}(x), u_1^{(\nu)}(x) \in \mathcal{C}^\sharp(\{j!\})_{\rho_0}$, $[a_{hk}^{(\nu)}(x)] \in [C(\{j!\})_{\rho_1}]$ and $f^{(\nu)}(t,x) \in C^0([0,T];\mathcal{C}^\sharp(\{j!\})_{\rho_0})$ for any fixed ν.

These regularized functions satisfy the following properties:

LEMMA 1.
(i) *$D_x^\alpha a_{hk}^{(\nu)}(x)$ converges to $D_x^\alpha a_{hk}(x)$ in $C(\{M_j\})_{\rho_1}$ $(h,k=1,\cdots,n)$.*
(ii) *$u_0^{(\nu)}(x)$ and $u_1^{(\nu)}(x)$ converge to $u_0(x)$ and $u_1(x)$ respectively in $\mathcal{C}^\sharp(\{M_j\})_{\rho_0}$ as $\nu \to \infty$.*
(iii) *$f^{(\nu)}(t,x)$ converges to $f(t,x)$ uniformly with respect to $t \in [0,T]$ in $\mathcal{C}^\sharp(\{M_j\})_{\rho_0}$ as $\nu \to \infty$.*

PROOF. (i) can be easily seen by using the Lebesgue convergence theorem.

We can prove (ii) by using a property of Fourier transformation. Indeed, the difference of the α-derivative of $u_0^{(\nu)}$ and that of u_0 can be estimated as

$$\begin{aligned}\left\|D^\alpha u_0^{(\nu)}(\cdot) - D^\alpha u_0(\cdot)\right\| &= \left\|\chi_{\frac{1}{\nu}} * D^\alpha u_0(\cdot) - D^\alpha u_0(\cdot)\right\| \\ &= \left\{\int \left|\xi^\alpha \hat{u}_0(\xi)\left\{(2\pi)^{\frac{n}{2}}\hat{\chi}_{\frac{1}{\nu}}(\xi) - 1\right\}\right|^2 d\xi\right\}^{\frac{1}{2}} \\ &\leq \sup_{\xi\in\mathbb{R}^n}\left|(2\pi)^{\frac{n}{2}}\hat{\chi}\,(\xi/\nu) - 1\right| \left\|D^\alpha u_0(\cdot)\right\|.\end{aligned}$$

Using the property

$$(2\pi)^{\frac{n}{2}}\hat{\chi}(\xi/\nu) = \int e^{-ix\cdot\xi/\nu}\chi(x)dx \longrightarrow \int \chi(x)dx = 1 \quad (\nu \to \infty),$$

we have

$$\sum_{j=0}^{\infty}\frac{\rho_0^j}{M_j}\left\{\sum_{|\alpha|=j}\left\|D^\alpha u_0^{(\nu)}(\cdot) - D^\alpha u_0(\cdot)\right\|^2\right\}^{\frac{1}{2}}$$
$$\leq C_{u_0}\sup_{\xi\in\mathbb{R}^n}\left|(2\pi)^{\frac{n}{2}}\hat{\chi}(\xi/\nu) - 1\right| \longrightarrow 0 \quad (\nu\to\infty).$$

Applying the same argument to (ii), we have (iii). □

For the real analytic functions $u_0^{(\nu)}$, $u_1^{(\nu)}$, $a_{hk}^{(\nu)}$ $(h, k = 1, \cdots, n)$ and $f^{(\nu)}$ constructed above, we shall consider the following Cauchy problem

$$(4)\quad \begin{cases} u_{tt}^{(\nu)}(t,x)+\Phi\left((A_\nu u^{(\nu)}(t,\cdot), u^{(\nu)}(t,\cdot)\right) A_\nu u^{(\nu)}(t,x)=f^{(\nu)}(t,x) \quad (t>0), \\ u^{(\nu)}(0,x)=u_0^{(\nu)}(x), \quad u_t^{(\nu)}(0,x)=u_1^{(\nu)}(x), \end{cases}$$

where $A_\nu u(t,x) = \sum_{hk} D_{x_h}(a_{hk}^{(\nu)}(x)D_{x_k}u(t,x))$. Here we remark that condition (2) is satisfied by the coefficients $[a_{hk}^{(\nu)}(x)]$.

Suppose now that $u^{(\nu)}(t,x)$ is a solution of (4). We define the infinite order energy $E_\nu(t)$ and its j-th order element $e_j^{(\nu)}(t)$ as

$$(5)\quad E_\nu(t) \equiv \sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}e_j^{(\nu)}(t),$$

$$e_j^{(\nu)}(t)^2 \equiv \sum_{|\alpha|=j-1}\Big\{\Psi_\nu(t)(A_\nu D^\alpha u^{(\nu)}(t,\cdot), D^\alpha u^{(\nu)}(t,\cdot)) + j^2\|D^\alpha u^{(\nu)}(t,\cdot)\|^2$$
$$(6)\qquad +\|D^\alpha u_t^{(\nu)}(t,\cdot)\|^2\Big\} + j^{-1}\sum_{|\alpha|=j}\|D^\alpha u^{(\nu)}(t,\cdot)\|^2,$$

where $\Psi_\nu(t) \equiv \Phi\left((A_\nu u^{(\nu)}(t,\cdot), u^{(\nu)}(t,\cdot))\right)$. (From now on we write $u^{(\nu)}(t,\cdot) = u$ in the norm and inner product.)

We prove here some estimates to the solution of (4), but, in order to obtain them, we shall introduce a lemma for the Kirchhoff type problem.

LEMMA 2 (Proposition 6.1 [9]). *If $u_0 \in H^1$, $u_1 \in L^2$ and $f(t,x) \in C^0([0,T]; L^2)$ for $T > 0$, then there is a constant C_T independent of $t \in [0,T]$ such that the solution of the Cauchy problem* (1) *satisfies*

$$\|u_t(t,\cdot)\| \leq C_T, \quad \|u(t,\cdot)\| \leq C_T, \quad (Au(t,\cdot), u(t,\cdot)) \leq C_T,$$

where H^1 is the Sobolev space defined by $H^1 = \left\{f\in L^2(\mathbb{R}^n); \sum_{|\alpha|\leq 1}\|D^\alpha f\|^2 < \infty\right\}$.

PROOF. Let $F(\eta) = \int_0^\eta \Phi(s)ds$. Define $e(t)$ for the solution of (1) as follows:

$$e(t)^2 \equiv \frac{1}{2}\left\{\|u_t(t,\cdot)\|^2 + F\left((Au(t,\cdot), u(t,\cdot))\right)\right\}.$$

Then computing the first order derivative of $e(t)^2$ we have

$$\begin{aligned}2e'(t)e(t) &= \Re(u_{tt}(t,\cdot), u_t(t,\cdot)) + \Phi\left((Au(t,\cdot), u(t,\cdot))\right)(Au(t,\cdot), u_t(t,\cdot)) \\ &= \Re(f(t,\cdot), u_t(t,\cdot)) \\ &\le \sqrt{2}e(t)\|f(t,\cdot)\|.\end{aligned}$$

Applying Gronwall's lemma, we obtain $e(t) \le e(0) + \frac{1}{\sqrt{2}}\int_0^t \|f(\tau,\cdot)\|d\tau \le C_T$ for $\forall t \in [0,T]$, that is, $\|u_t\| \le C_T$. Since $\Phi(\eta)$ is positive, we have

$$F((Au,u)) = \int_0^{(Au,u)} \Phi(s)ds \ge \Phi_0(Au,u),$$

hence we get $(Au,u) \le F((Au,u))/\Phi_0 \le C_T$. □

Now we shall estimate the first order derivative of $e_j^{(\nu)}(t)^2$. Adopting equation (4), we have

$$\begin{aligned}\frac{d}{dt}e_j^{(\nu)}(t)^2 \le{}& \frac{|\Psi_\nu{}'(t)|}{\Psi_\nu(t)}e_j^{(\nu)}(t)^2 + 2\Psi_\nu(t)\sum_{|\alpha|=j-1}\Re([A_\nu, D^\alpha]u, D^\alpha u_t) \\ &+ 2\sum_{|\alpha|=j-1}\Re(D^\alpha f^{(\nu)}, D^\alpha u_t) + 2j^2\left(\sum_{|\alpha|=j-1}\|D^\alpha u\|^2\right)^{\frac{1}{2}} \\ &\times\left(\sum_{|\alpha|=j-1}\|D^\alpha u_t\|^2\right)^{\frac{1}{2}} + 2j^{-1}\sum_{|\alpha|=j}\Re(D^\alpha u, D^\alpha u_t) \\ \le{}& \frac{|\Psi_\nu{}'(t)|}{\Psi_\nu(t)}e_j^{(\nu)}(t)^2 + 2\Psi_\nu(t)\left(\sum_{|\alpha|=j-1}\|[A_\nu, D^\alpha]u\|^2\right)^{\frac{1}{2}} e_j^{(\nu)}(t) \\ &+2\left(\sum_{|\alpha|=j-1}\|D^\alpha f^{(\nu)}\|^2\right)^{\frac{1}{2}} e_j^{(\nu)}(t) + 2je_j^{(\nu)}(t)^2 + 2j^{-\frac{1}{2}}e_j^{(\nu)}(t)e_{j+1}^{(\nu)}(t).\end{aligned}$$

Dividing by $2e_j^{(\nu)}(t)$, we get

$$\begin{aligned}e_j^{(\nu)\prime}(t) \le{}& \frac{|\Psi_\nu{}'(t)|}{2\Psi_\nu(t)}e_j^{(\nu)}(t) + je_j^{(\nu)}(t) + \Psi_\nu(t)\left(\sum_{|\alpha|=j-1}\|[A_\nu, D^\alpha]u\|^2\right)^{\frac{1}{2}} \\ &+ j^{-\frac{1}{2}}e_{j+1}^{(\nu)}(t) + \left(\sum_{|\alpha|=j-1}\|D^\alpha f^{(\nu)}\|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{7}$$

Now, differentiating $E_\nu(t)$, and applying the above estimate of $e_j^{(\nu)\prime}(t)$, we obtain

$$
\begin{aligned}
\frac{d}{dt}E_\nu(t) &= \sum_{j=1}^{\infty}\left\{\frac{j\rho'(t)\rho(t)^{j-1}}{M_j}e_j^{(\nu)}(t)+\frac{\rho(t)^j}{M_j}e_j^{(\nu)\prime}(t)\right\} \\
&\le \sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}\left(\frac{j\rho'(t)}{\rho(t)}+j\right)e_j^{(\nu)}(t)+\frac{|\Psi_\nu{}'(t)|}{2\Psi_\nu(t)}\sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}e_j^{(\nu)}(t) \\
&\quad + \Psi_\nu(t)\sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}\left(\sum_{|\alpha|=j-1}\|[A_\nu, D^\alpha]u\|^2\right)^{\frac{1}{2}} \\
&\quad + \sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}j^{-\frac{1}{2}}e_{j+1}^{(\nu)}(t)+\sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}\left(\sum_{|\alpha|=j-1}\|D^\alpha f^{(\nu)}\|^2\right)^{\frac{1}{2}}.
\end{aligned}
\tag{8}
$$

Now we shall calculate the commutator $[A_\nu, D^\alpha]$ in order to estimate the third term in (8). Applying Leibniz' rule, $[A_\nu, D^\alpha]u$ can be rewrited as follows:

$$
\begin{aligned}
[A_\nu, D^\alpha]u = \; & A_\nu D^\alpha u - D^\alpha A_\nu u \\
= \; & \sum_{hk}\left(D^{\mathbf{e}_h}a_{hk}^{(\nu)}D^{\alpha+\mathbf{e}_k}u - D^{\alpha+\mathbf{e}_h}a_{hk}^{(\nu)}D^{\mathbf{e}_k}u\right) \\
= & -\sum_{hk}\left\{\sum_{\beta<\alpha}\binom{\alpha}{\beta}(D^{\alpha-\beta+\mathbf{e}_h}a_{hk}^{(\nu)})D^{\beta+\mathbf{e}_k}u\right. \\
& \left. +\sum_{\beta<\alpha}\binom{\alpha}{\beta}(D^{\alpha-\beta}a_{hk}^{(\nu)})D^{\beta+\mathbf{e}_h+\mathbf{e}_k}u\right\} \\
\equiv \; & I_\alpha + II_\alpha + III_\alpha\,,
\end{aligned}
\tag{9}
$$

where

$$
\begin{cases}
I_\alpha &= -\displaystyle\sum_{hk}\sum_{\beta<\alpha}\binom{\alpha}{\beta}(D^{\alpha-\beta+\mathbf{e}_h}a_{hk}^{(\nu)})D^{\beta+\mathbf{e}_k}u\,, \\
II_\alpha &= -\displaystyle\sum_{hk}\sum_{\substack{\beta<\alpha \\ |\beta|\le|\alpha|-2}}\binom{\alpha}{\beta}(D^{\alpha-\beta}a_{hk}^{(\nu)})D^{\beta+\mathbf{e}_h+\mathbf{e}_k}u, \\
III_\alpha &= -\displaystyle\sum_{hk}\sum_{\substack{\beta<\alpha \\ |\beta|=|\alpha|-1}}\binom{\alpha}{\beta}(D^{\alpha-\beta}a_{hk}^{(\nu)})D^{\beta+\mathbf{e}_h+\mathbf{e}_k}u\,,
\end{cases}
$$

and $\mathbf{e}_h = (0, \cdots, \overset{h}{\check{1}}, \cdots, 0)$.

Using the equality $|\alpha-\beta+\mathbf{e}_h| = j-|\beta|$ and the inequality $\binom{\alpha}{\beta} \le \binom{|\alpha|}{|\beta|}$, we have

$$(10)\quad \left(\sum_{|\alpha|=j-1} \|I_\alpha\|^2\right)^{\frac{1}{2}} \le C_1 \left\{ \sum_{|\alpha|=j-1} \left(\sum_{hk}\sum_{\beta<\alpha} \binom{j-1}{|\beta|} M_{j-|\beta|} \rho_1^{-(j-|\beta|)} \|D^{\beta+\mathbf{e}_k}u\| \right)^2 \right\}^{\frac{1}{2}},$$

where C_1 is a constant independent of j.

We now introduce the following lemma to estimate (10).

Lemma 3 (Lemma 2.1 [4]). *Let $\{X_\alpha\}$ ($\alpha \in \mathbb{N}^n$) be a sequence of non-negative real numbers. Then, for every integer l and $l'(\le l)$, and every real number $0<\kappa<1$, there is a constant $C(n,\kappa)$ such that*

$$\left\{ \sum_{|\alpha|=l} \left(\sum_{\substack{\beta\le\alpha \\ |\beta|\le l'}} X_\beta \right)^2 \right\}^{\frac{1}{2}} \le C(n,\kappa) \sum_{r=0}^{l'} \kappa^{-(l-r)} \left(\sum_{|\beta|=r} X_\beta^2 \right)^{\frac{1}{2}}.$$

Applying this lemma, (10) can be estimated by

$$\begin{aligned}
(10) &\le C_2(n,\kappa) \sum_{r=0}^{j-2} \kappa^{-(j-1-r)} \left\{ \sum_{|\beta|=r} \left(\sum_{hk} \binom{j-1}{r} M_{j-r} \rho_1^{-(j-r)} \|D^{\beta+\mathbf{e}_k}u\| \right)^2 \right\}^{\frac{1}{2}} \\
&= C_2 \sum_{r=1}^{j-1} \kappa^{-(j-r)} \left\{ \sum_{|\beta|=r-1} \left(\sum_{hk} \binom{j-1}{r-1} M_{j-r+1} \rho_1^{-(j-r+1)} \|D^{\beta+\mathbf{e}_k}u\| \right)^2 \right\}^{\frac{1}{2}} \\
&\le C_3 \sum_{r=1}^{j-1} (\kappa\rho_1)^{-(j-r+1)} \binom{j-1}{r-1} M_{j-r+1} \left(\sum_{|\beta|=r} \|D^\beta u\|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

where C_2 and C_3 are constants independent of j.

For the term II_α, using an analogous inequality with $\binom{j-1}{r-2}$ instead of $\binom{j-1}{r-1}$, we get

$$(11)\quad \left(\sum_{|\alpha|=j-1} \|II_\alpha\|^2\right)^{\frac{1}{2}} \le C_4 \sum_{r=2}^{j-1} (\kappa\rho_1)^{-(j-r+1)} \binom{j-1}{r-2} M_{j-r+1} \left(\sum_{|\beta|=r} \|D^\beta u\|^2 \right)^{\frac{1}{2}}.$$

Hence we have

$$\begin{aligned}
&\left(\sum_{|\alpha|=j-1} \|I_\alpha\|^2\right)^{\frac{1}{2}} + \left(\sum_{|\alpha|=j-1} \|II_\alpha\|^2\right)^{\frac{1}{2}} \\
&\qquad\qquad \le C_5 \sum_{r=1}^{j-1} (\kappa\rho_1)^{-(j-r+1)} \binom{j}{r-1} M_{j-r+1} \left(\sum_{|\beta|=r} \|D^\beta u\|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

for some constant C_5 independent of j, where we have used the identity $\binom{j-1}{r-2}+\binom{j-1}{r-1}=\binom{j}{r-1}$.

Finally, to estimate the term III_α, we apply the following lemma due to O. A. Oleinik.

LEMMA 4 (Lemma 4 [15]). *Let $[a_{hk}^{(\nu)}(x)]$ be a Hermitian non-negative matrix of functions in $\mathcal{B}^2(\mathbb{R}^n)$. Then, for every $n\times n$ symmetric matrix $[\xi_{hk}]$, for $\mu=1,\cdots,n$, there is a constant C such that*

$$\left|\sum_{hk} D_{x_\mu} a_{hk}^{(\nu)}(x)\xi_{hk}\right|^2 \le C\sum_{hkq} a_{hk}^{(\nu)}(x)\xi_{hq}\xi_{kq}\,.$$

Applying this lemma, we have

$$\left(\sum_{|\alpha|=j-1}\|III_\alpha\|^2\right)^{\frac12} \le \left(\sum_{|\alpha|=j-1}\sum_{\substack{\beta<\alpha\\|\beta|=|\alpha|-1}}(j-1)^2\left\|\sum_{hk}(D^{\alpha-\beta}a_{hk}^{(\nu)})D^{\beta+\mathbf{e}_h+\mathbf{e}_k}u\right\|^2\right)^{\frac12}$$

$$\le C_6(j-1)\left\{\sum_{|\alpha|=j-1}\sum_{\substack{\beta<\alpha\\|\beta|=|\alpha|-1}}\sum_{q=1}^{n}\left(\sum_{hk}(D^{\mathbf{e}_k}a_{hk}^{(\nu)}D^{\mathbf{e}_h}D^{\beta+\mathbf{e}_q}u,\, D^{\beta+\mathbf{e}_q}u\right)\right\}^{\frac12}$$

$$= C_6(j-1)\left(\sum_{|\alpha|=j-1}(A_\nu D^\alpha u, D^\alpha u)\right)^{\frac12} \le \frac{C_6(j-1)}{\sqrt{\Psi_\nu(t)}}e_j^{(\nu)}(t)\,.$$

Hence the third term of (8) can be estimated as follows

$$\begin{aligned}
&\Psi_\nu(t)\sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}\left(\sum_{|\alpha|=j-1}\|[A_\nu, D^\alpha]u\|^2\right)^{\frac12}\\
&\le C_7\Psi_\nu(t)\sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}\left\{\sum_{r=1}^{j-1}(\kappa\rho_1)^{-(j-r+1)}\binom{j}{r-1}M_{j-r+1}\left(\sum_{|\beta|=r}\|D^\beta u\|^2\right)^{\frac12}\right\}\\
&\quad + C_8\sqrt{\Psi_\nu(t)}\sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}(j-1)e_j^{(\nu)}(t)
\end{aligned}\tag{12}$$

$$= C_7\Psi_\nu(t)\rho(t)^{-2}\sum_{r=1}^{\infty}\sum_{j=r+1}^{\infty}\left(\frac{\rho(t)}{\kappa\rho_1}\right)^{j-r+1}\frac{M_{j-r+1}}{M_j}\binom{j}{r-1}\rho(t)^{r+1}\left(\sum_{|\alpha|=r}\|D^\alpha u\|^2\right)^{\frac12}$$

$$+C_8\sqrt{\Psi_\nu(t)}\sum_{j=1}^{\infty}\frac{\rho(t)^j}{M_j}(j-1)e_j^{(\nu)}(t)\,.$$

Let us consider $\rho(t) \le \rho_0$. By using Lemma B (ii), the first term of (12) can be estimated as

$$\text{first term of (12)} \le \frac{C_7 \Psi_\nu(t)}{\kappa\rho_1(\kappa\rho_1 - \rho_0)} \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j}(j-1)e_j^{(\nu)}(t)$$
$$\le C_9 \Psi_\nu(t) \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j}(j-1)e_j^{(\nu)}(t)\,,$$

where C_9 is a constant independent of t.

Applying the assumption $\sup_j Q(j) = \sup_j\{M_j / j^{\frac{3}{2}} M_{j-1}\} < \infty$, we obtain

$$\text{the fourth term of (8)} \le C_{10}\, \rho(t)^{-1} \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} j\, e_j^{(\nu)}(t) Q(j)\,,$$

where C_{10} is some constant independent of t.

Moreover we have

$$\text{the fifth term of (8)} \le C_{11}$$

for some constant C_{11} independent of ν (by the definition of $f^{(\nu)}$).

Then, taking together the preceeding estimates for the derivative of $E_\nu(t)$, we get the following bound

$$\begin{aligned} \frac{d}{dt}E_\nu(t) \le \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} \Bigg\{ & \frac{j\rho'(t)}{\rho(t)} + j + C_8\sqrt{\Psi_\nu(t)}(j-1) \\ & + C_9\Psi_\nu(t)(j-1) + C_{10}\frac{jQ(j)}{\rho(t)} \Bigg\} e_j^{(\nu)}(t) + \frac{|\Psi_\nu{}'(t)|}{2\Psi_\nu(t)}E_\nu(t) + C_{11}\,. \end{aligned} \tag{13}$$

We now proceed to a choice of $\rho(t)$. Applying Lemma 2, in the parenthesis of the first term of (13) there is a constant C_{12} independent of j and t such that

$$\text{the first term in (13)} \le j\rho(t)^{-1}\left(\rho'(t) + C_{12}\rho(t) + C_{10}Q_0\right)\,,$$

where $Q_0 = \sup_j Q(j)$. Let us assume that $\rho(0) > 0$, then we can easily see that there is $T_1 > 0$ such that the ordinary differential inequality

$$\rho'(t) + C_{12}\rho(t) + C_{10}Q_0 \le 0\,, \tag{14}$$

has a positive decreasing solution on $[0, T_1]$. Moreover if $\lim_{j\to\infty} Q(j) = 0$, let $j_0(T)$ large enough such that the ordinary differential inequality

$$\rho'(t) + C_{12}\rho(t) + C_{10}Q(j) \le 0$$

has a positive decreasing solution on $[0, T]$ for any $j \geq j_0$. On the other hand, if $j < j_0$, we obtain

$$j\rho(t)^{-1}(\rho'(t) + C_{12}\rho(t) + C_{10}Q(j)) \leq \rho(T)^{-1}(C_{12}\rho(0) + C_{10}Q_0) \equiv C_{13}$$

for some constant C_{13} independent of j and t.

Thus we get the following estimate

$$\frac{d}{dt}E_\nu(t) \leq \left(\frac{|\Psi_\nu{}'(t)|}{2\Psi_\nu(t)} + C_{13}\right) E_\nu(t) + C_{11} \quad \forall t \in [0, T_1]. \tag{15}$$

Note that $Q(j) \to 0$ as $j \to \infty$ and then we can take $T_1 = T$.

Next we estimate the nonlinear part of (15), $|\Psi'{}_\nu(t)| = \left|\frac{d}{dt}\Phi\big((A_\nu u(t,\cdot), u(t,\cdot))\big)\right|$.

We now introduce a lemma, which is classical in convex analysis.

LEMMA 5 (Theorem 243 [6]). *Let μ and σ be continuous and strictly increasing. We define $\mathfrak{M}_\mu(f)$ by*

$$\mathfrak{M}_\mu(f) = \mu^{-1}\left(\int \mu\left(f(x)\right) p(x)dx\right),$$

where f and p are nonnegative functions such that $\int p(x)dx = 1$ and $\int \mu(f(x))p(x)dx$ exists. Then, in order that $\mathfrak{M}_\mu(f) \leq \mathfrak{M}_\sigma(f)$ for all f, it is necessary and sufficient that $\sigma \circ \mu^{-1}$ should be convex.

Using the inequality $|(A_\nu u, u_t)| \leq (A_\nu u, u)^{\frac{1}{2}}(A_\nu u_t, u_t)^{\frac{1}{2}}$, the Plancherel theorem and Lemma 2, we have

$$\begin{aligned} |\Psi_\nu{}'(t)| &= 2\left|\Re(A_\nu u, u_t)\Phi'\left((A_\nu u, u)\right)\right| \\ &\leq 2(A_\nu u, u)^{\frac{1}{2}}(A_\nu u_t, u_t)^{\frac{1}{2}}\left|\Phi'\left((A_\nu u, u)\right)\right| \\ &\leq C_{T,A,\Phi}\left\||\cdot|\hat{u}_t(t,\cdot)\right\|, \end{aligned} \tag{16}$$

where $C_{T,A,\Phi}$ is a constant independent of u.

When $\left\|\hat{u}_t(t,\cdot)\right\| \geq 1$, we see $1/\left\|\hat{u}_t(t,\cdot)\right\| \leq 1$. So

$$\begin{aligned} \left\||\cdot|\hat{u}_t(t,\cdot)\right\| &= \left\|\hat{u}_t(t,\cdot)\right\|\left(\int_{\mathbb{R}^n_\xi}\frac{|\hat{u}_t(t,\xi)|^2}{\left\|\hat{u}_t(t,\cdot)\right\|^2}|\xi|^2 d\xi\right)^{\frac{1}{2}} \\ &\leq C_T N^{-1}\left(\int_{\mathbb{R}^n_\xi}\frac{|\hat{u}_t(t,\xi)|^2}{\left\|\hat{u}_t(t,\cdot)\right\|^2}N(|\xi|)d\xi\right), \end{aligned}$$

where N is a nonnegative function such that $\lambda \to N(\lambda^{\frac{1}{2}})$ is convex, having used Lemma 5 for $\mu = s^2$, $\sigma = N(s)$ and $p(t,\xi) = |\hat{u}_t(t,\xi)|^2/\left\|\hat{u}_t(t,\cdot)\right\|^2$. When

$\|\hat{u}_t(t,\cdot)\| < 1$, let us take $p_\theta(t,\xi) \equiv (1 - \|\hat{u}_t(t,\cdot)\|^2)\chi_\theta(\xi)$, where $\chi_\theta(\xi) \equiv \theta^{-n}\chi(\theta\xi)$, with $\int \chi(\xi)d\xi = 1$ and $0 < \theta < 1$, is Friedrichs' mollifier. It is easy to see that $p_\theta(t,\xi)$ satisfies $0 < \int p_\theta(t,\xi)d\xi$ and $\int \{|\hat{u}_t(t,\xi)|^2 + p_\theta(t,\xi)\} = 1$. Now, by using Lemma 5 with $p(t,\xi) = |\hat{u}_t(t,\xi)|^2 + p_\theta(t,\xi)$, we have

$$\||\cdot|\hat{u}_t(t,\cdot)\| \le \left\{\int_{\mathbb{R}^n_\xi} |\xi|^2 \left(|\hat{u}_t(t,\xi)|^2 + p_\theta(t,\xi)\right) d\xi\right\}^{\frac{1}{2}}$$
$$\le N^{-1}\left(\int_{\mathbb{R}^n_\xi} N(|\xi|)\left(|\hat{u}_t(\xi)|^2 + p_\theta(t,\xi)\right) d\xi\right).$$

In addition, we choose $\chi_\theta(\xi)$ satisfies that $\operatorname{supp} p_\theta(t,\xi) \subset \mathbb{R}_+ \times \{\xi \in \mathbb{R}^n;\ |\xi| \le \theta\}$ and so we have

$$\int_{\mathbb{R}^n_\xi} N(|\xi|)p_\theta(t,\xi)d\xi = \int_{|\xi|\le\theta} N(|\xi|)p_\theta(t,\xi)d\xi$$
$$\le \left(\int_{|\xi|\le\theta} N(|\xi|)^2 d\xi\right)^{\frac{1}{2}}\left(\int_{|\xi|\le\theta} p_\theta(t,\xi)^2 d\xi\right)^{\frac{1}{2}} \longrightarrow 0,\quad (\theta\to 0).$$

Hence we obtain

$$\||\cdot|\hat{u}_t(t,\cdot)\| \le C_T N^{-1}\left(\left\|N(|\cdot|)^{\frac{1}{2}}\hat{u}_t(t,\cdot)\right\|^2\right). \tag{17}$$

Now we introduce the following lemma.

LEMMA 6. *Let $v(x) \in C^\sharp(\{M_j\})_\rho$ and let $M(r)$ be the associated function of $\{M_j\}$. If $\sup_j Q(j) < \infty$, then for any $\varepsilon > 0$, there is a constant $C_{\varepsilon,n}$ such that*

$$\|M(\tilde{\rho}|\cdot|)\hat{v}(\cdot)\| \le C_{\varepsilon,n}\sum_{j=1}^\infty \frac{\rho^j}{M_j}\left(\sum_{|\alpha|=j-1}\|D^\alpha v\|^2\right)^{\frac{1}{2}}, \tag{18}$$

where $\tilde{\rho} = \rho/(1+\varepsilon)$.

PROOF. The right hand side of (18) can be estimated as the following:

$$\sum_{j=1}^\infty \frac{\rho^j}{M_j}\left(\sum_{|\alpha|=j-1}\|D^\alpha v\|^2\right)^{\frac{1}{2}} \ge \sum_{j=1}^\infty \frac{\rho^j}{M_j}\binom{j-2+n}{n-1}^{-\frac{1}{2}} \sum_{|\alpha|=j-1}\|D^\alpha v\|$$
$$\ge C'_{\varepsilon,n}\sum_{j=1}^\infty \frac{\rho^j(1+\varepsilon)^{-\frac{j}{2}}}{M_j}\left\{\int_{\mathbb{R}^n}\left(\sum_{|\alpha|=j-1}|\xi^\alpha|\right)^2|\hat{v}(\xi)|^2 d\xi\right\}^{\frac{1}{2}}$$
$$\ge C''_{\varepsilon,n}\sum_{j=1}^\infty\left\{(1+\varepsilon)^{\frac{j}{2}} j^{-\frac{1}{2}}\frac{j^{\frac{1}{2}}M_j}{M_{j+1}}\right\}\frac{\{\rho/(1+\varepsilon)\}^j}{M_j}\left\||\cdot|^j\hat{v}(\cdot)\right\|$$
$$\ge C_{\varepsilon,n}\sum_{j=1}^\infty \frac{\tilde{\rho}^j}{M_j}\left\||\cdot|^j\hat{v}(\cdot)\right\|,$$

for any $\varepsilon > 0$, where we have applied the boundedness of $Q(j)$ and the following inequalities

$$\sum_{|\alpha|=N} 1 = \binom{N+n-1}{n-1}, \quad \left(\sum_{i=1}^{M} a_i^2\right)^{\frac{1}{2}} \geq M^{-\frac{1}{2}} \sum_{i=1}^{M} a_i, \quad (a_i \geq 0),$$

$$\binom{N+n}{n} \leq \frac{(1+\varepsilon)^{N+n}}{\varepsilon^n} \quad (\forall \varepsilon > 0), \quad \sum_{|\alpha|=N} |x^\alpha| \geq |x|^N, \quad \frac{(1+\varepsilon)^j}{j} \geq \exists C_\varepsilon > 0.$$

Now let R_k $(k = 0, 1, \cdots)$ and Ω_k $(k = 1, 2, \cdots)$ be chosen as follows:

$$R_k \equiv \max\left\{ r \geq 0; \sup_{j\geq 1}\left\{\frac{(\tilde{\rho}r)^j}{M_j}\right\} = \frac{(\tilde{\rho}r)^k}{M_k}\right\}, \quad (k \geq 1), \qquad R_0 \equiv 0,$$

$$\Omega_k \equiv \left\{\xi \in \mathbb{R}^n; R_{k-1} \leq |\xi| < R_k\right\}),$$

where we note that $\bigcup_{k=1}^{\infty} \Omega_k = \mathbb{R}^n$. Hence we have

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{\tilde{\rho}^j}{M_j} \left\| |\cdot|^j \hat{v}(\cdot) \right\| &\geq \left\{ \sum_{k=1}^{\infty} \int_{\Omega_k} \left| \frac{(\tilde{\rho}|\xi|)^k}{M_k} \right|^2 |\hat{v}(\xi)|^2 \, d\xi \right\}^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^{\infty} \int_{\Omega_k} \left| \sup_{j\geq 1} \left\{ \frac{(\tilde{\rho}|\xi|)^j}{M_j} \right\} \right|^2 |\hat{v}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \\
&= \left\| M(\tilde{\rho}|\cdot|)\hat{v}(\cdot) \right\|. \qquad \square
\end{aligned}$$

Here we remark that the estimate (17) remains valid for any continuous strictly increasing function N such that $\lambda \to N(\lambda^{\frac{1}{2}})$ is convex. Now, considering that $N(r) = \left\{ M\left(\frac{\rho(t)r}{1+\varepsilon}\right) / C_{\varepsilon,n} \right\}^2$ for any fixed $t \in [0, T]$ and $\varepsilon > 0$, then we see that $N(r)$ is a continuous strictly increasing function such that $\lambda \to N(\lambda^{\frac{1}{2}})$ is convex by definition of $M(r)$, where $C_{\varepsilon,n}$ is the constant in Lemma 6. Finally, applying Lemma 6, we obtain

$$\text{(19)} \qquad \left\| N(|\cdot|)^{\frac{1}{2}} \hat{u}_t(t,\cdot) \right\| \leq \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} \left(\sum_{|\alpha|=j-1} \left\| D^\alpha u_t(t,\cdot) \right\|^2 \right)^{\frac{1}{2}} \leq E_\nu(t).$$

Therefore we have the estimate

$$\frac{d}{dt} E_\nu(t)^2 \leq C N^{-1}\left(E_\nu(t)^2\right) E_\nu(t)^2 + C' E_\nu(t).$$

Now, without loss of generality let $E_\nu(t) \geq 1$, and consider the inequality

$$\frac{d}{dt} E_\nu(t)^2 \leq C N^{-1}\left(E_\nu(t)^2\right) E_\nu(t)^2 + C' E_\nu(t)^2 . \tag{20}$$

Next we shall investigate the condition which must be satisfied by N in order that some estimates on the solution of (4) hold. For this purpose we introduce the following lemmas.

LEMMA 7. *Let $f : [0, \infty) \to [0, \infty)$ be continuous, $g : (0, \infty) \to (0, \infty)$ continuous and nondecreasing, and c a positive constant. Then the inequality*

$$f(t) \leq c + \int_0^t g(f(s))ds \quad (0 \leq t < \infty),$$

implies that

$$f(t) \leq G^{-1}(G_0) < +\infty \quad (0 \leq t \leq G_0),$$

for any fixed number G_0 less than $G(\infty)$, where $G(t) = \int_c^t 1/g(s)ds$ for $t \geq c$. Moreover, if $G(\infty) = \infty$, then the inequality

$$f(t) \leq G^{-1}(t),$$

is valid for all $t \geq 0$.

The proof of Lemma 3 is given in [13].

LEMMA 8. *Let $M(r)$ be a continuous nondecreasing function such that*

$$\int_c^\infty \frac{ds}{s(d_0 + M^{-1}(s))} = \infty,$$

for some positive constant c and d_0, then for any positive constant ρ, the function $N(r)$ defined by $N(r) \equiv M(\rho r)^2$ satisfies

$$\int_{c'}^\infty \frac{ds}{s(\tilde{\rho}^{-1} d_0 + N^{-1}(s))} = \infty,$$

for some positive constant c' and for any $0 < \tilde{\rho} < \rho$.

Now we introduce a lemma to prove the Lemma 8.

Let $M(r)$ be the associated function of $\{M_j\}$. We define the regularized associated function $M_\varepsilon(r)$ of $M(r)$, written in the form $M_\varepsilon(r) \equiv \chi_\varepsilon * M(r)$ for $\chi(r) \in C_0^\infty(\mathbb{R})$, such that $\operatorname{supp} \chi \subset [-\delta, \delta]$, $\int \chi(r)dr = 1$ and $\chi_\varepsilon(r) \equiv \varepsilon^{-1}\chi(\varepsilon^{-1}r)$ for $\varepsilon > 0$ and $\delta > 0$. Then we have the following lemma.

LEMMA 9 (W. Matsumoto [11]). *For $0 < \forall\varepsilon < 1$ and $\forall\delta > 0$, a regularized associated function $M_\varepsilon(r) \in C^\infty(\mathbb{R}_+)$ satisfies the inequality*

$$M((1-\varepsilon)r) \leq M_\varepsilon(r) \leq M((1+\varepsilon)r)$$

for any $r \geq \delta$.

PROOF. For $0 < \forall \varepsilon < 1$,

$$\frac{M_\varepsilon(r)}{M((1 \mp \varepsilon)r)} = \int_{-\delta}^{\delta} \chi(s) \frac{M(r-\varepsilon s)}{M((1 \mp \varepsilon)r)} ds \gtreqless \int_{-\delta}^{\delta} \chi(s) ds = 1,$$

where we used $M(r - \varepsilon s) \gtreqless M((1 \mp \varepsilon)r)$ for $-\delta \leq \forall s \leq \delta$. □

PROOF OF LEMMA 8. Let $N(r) = M(\rho r)^2$ and $N_\varepsilon(r) = M_\varepsilon(\rho(1-\varepsilon)r)^2$, where M_ε is a regularized associated function of M defined above. Then we see that $N^{-1}(s) \leq N_\varepsilon^{-1}(s)$, and hence we have

$$\begin{aligned}
\int_c^\infty \frac{ds}{s(\tilde{\rho}^{-1}d_0 + N^{-1}(s))} &\geq \int_c^\infty \frac{ds}{s(\tilde{\rho}^{-1}d_0 + N_\varepsilon^{-1}((s))} \\
&= 2\rho(1-\varepsilon) \int_{N_\varepsilon^{-1}(c)}^\infty \frac{M_\varepsilon{}'(\rho(1-\varepsilon)r)}{M_\varepsilon(\rho(1-\varepsilon)r)(\tilde{\rho}^{-1}d_0+r)} dr \\
&\geq 2\rho(1-\varepsilon) \int_{\frac{M_\varepsilon(\rho(1-\varepsilon)N_\varepsilon^{-1}(c))}{(1-\varepsilon)}}^\infty \frac{ds}{s\{\rho(1-\varepsilon)\tilde{\rho}^{-1}d_0 + M^{-1}(s)\}},
\end{aligned}$$

where we used the equality $\frac{ds}{dr} = 2\rho(1-\varepsilon)M_\varepsilon{}'(\rho(1-\varepsilon)r)M_\varepsilon(\rho(1-\varepsilon)r)$ and the inequality $M_\varepsilon^{-1}(s) \leq M^{-1}(s/(1-\varepsilon))$. Putting $\varepsilon = 1 - \tilde{\rho}/\rho$, we get Lemma 8. □

Let $C = C'\tilde{\rho}/d_0$ in (20) and then (20) can be rewritten as

$$\frac{d}{dt} E_\nu(t)^2 \leq C' E_\nu(t)^2 \left(N^{-1}\left(E_\nu(t)^2\right) + \tilde{\rho}^{-1} d_0\right). \tag{21}$$

Then, regarding $g(f)$ in Lemma 7 as $g(f) \equiv C_1 f(N^{-1}(f) + \tilde{\rho}^{-1}d_0)$ and applying Lemma 8, we see that $\int^\infty \frac{ds}{g(s)} = \infty$, where $N(r) \equiv M(\rho(t)r)^2$ and $\tilde{\rho}$ is a constant such that $0 < \tilde{\rho} < \rho(T)$. Hence, if $\{M_j\}$ satisfies a quasi-analytic condition, by applying Lemma 7, we have the energy estimate

$$E_\nu(t) \leq C_T \quad (0 \leq \forall t \leq T), \tag{22}$$

where C_T is a constant independenet of ν.

By the foregoing arguments, finally we have the following proposition.

PROPOSITION 1. *Let ρ_0 and ρ_1 be positive numbers such that $\rho_0 < \rho_1$, $\{M_j\}$ be positive, logarithmically convex and* $\sup_j Q(j) \equiv \sup_j\{M_j/j^{\frac{3}{2}}M_{j-1}\} < \infty$. *Assume that $[a_{hk}(x)] \in [C(\{M_j\})_{\rho_1}]$ satisfies* (2), *u_0 and $u_1 \in C^\sharp(\{M_j\})_{\rho_0}$, $f(t,x) \in C^0([0,T]; C^\sharp(\{M_j\})_{\rho_0})$ and that $\Phi(\eta) \in C^1([0,\infty))]$ satisfies the inequality $\Phi \geq \exists\, \Phi_0 > 0$. Then we get the following estimates of $E_\nu(t)$ to the solution of* (4).

i) *There exist positive constants $T_0(\leq T)$ and C independent of ν, and a positive function $\rho(t)$ on $[0, T_0]$ such that*

$$E_\nu(t) \leq C_{T_0} \quad (0 \leq \forall t \leq T_0).$$

ii) *If* $\{M_j\}$ *satisfies a quasi-analytic condition and* $Q(j) \to 0$ *as* $j \to \infty$, *there exists a constant* C_T *independent of* ν *such that*

$$E_\nu(t) \le C_T \quad (0 \le \forall t \le T).$$

Now we shall show the existence of the solution for the nonlinear Cauchy problem (1) in case that the data and the coefficients satisfies the assumption of the Main Theorem.

Let $u^{(\nu)}(t,x)$ and $u^{(\nu')}(t,x)$ be solutions of (4). Define $v^{(\nu,\nu')}(t,x) \equiv u^{(\nu)}(t,x) - u^{(\nu')}(t,x)$. We shall consider the following Cauchy problem for $v^{(\nu,\nu')}(t,x)$

$$\tag{23} \begin{cases} v_{tt}^{(\nu,\nu')}(t,x) + \Psi_\nu(t) A_\nu v^{(\nu,\nu')}(t,x) = f^{(\nu)}(t,x) - f^{(\nu')}(t,x) \\ \qquad\qquad + (\Psi_{\nu'}(t)A_{\nu'} - \Psi_\nu(t)A_\nu) u^{(\nu')}(t,x) \quad (t > 0, x \in \mathbb{R}^n), \\ v^{(\nu,\nu')}(0,x) = u_0^{(\nu)}(x) - u_0^{(\nu')}(x), \quad v_t^{(\nu,\nu')}(0,x) = u_1^{(\nu)}(x) - u_1^{(\nu')}(x). \end{cases}$$

From now on we will show that the solution of the Cauchy problem (23) $v^{(\nu,\nu')}(t,x)$ converges to 0 in $C^2([0,T_0]; C^\sharp(\{M_j\})_{\tilde\rho(t)}$ for some $\tilde\rho(t) > 0$.

We define the infinite order energy $E_{\nu,\nu'}(t)$ and its j-th order element $e_j^{(\nu,\nu')}(t)$ as

$$\tag{24} E_{\nu,\nu'}(t) \equiv \sum_{j=1}^{\infty} \frac{\tilde\rho(t)^j}{M_j} e_j^{(\nu,\nu')}(t)$$

with

$$\tag{25} \begin{aligned} e_j^{(\nu,\nu')}(t)^2 \equiv \sum_{|\alpha|=j-1} &\Big\{ \Psi_\nu(t)(A_\nu D^\alpha v^{(\nu,\nu')}(t,\cdot), D^\alpha v^{(\nu,\nu')}(t,\cdot)) + j^2 \| D^\alpha v^{(\nu,\nu')}(t,\cdot)\|^2 \\ &+ \| D^\alpha v_t^{(\nu,\nu')}(t,\cdot)\|^2 \Big\} + j^{-1} \sum_{|\alpha|=j} \| D^\alpha v^{(\nu,\nu')}(t,\cdot)\|^2 . \end{aligned}$$

Then, applying Proposition 1 and the estimate obtained for $E_\nu(t)$, we conclude that there is a constant C_1 independent of ν and ν' such that

$$\tag{26} \begin{aligned} \frac{d}{dt} E_{\nu,\nu'}(t) \le &\sum_{j=1}^{\infty} \frac{\tilde\rho(t)^j}{M_j} \left\{ \sum_{|\alpha|=j-1} \left\| D^\alpha \left\{ (\Psi_{\nu'}(t)A_{\nu'} - \Psi_\nu(t)A_\nu) u^{(\nu')} + f^{(\nu)} - f^{(\nu')} \right\} \right\|^2 \right\}^{\frac{1}{2}} \\ &+ C_1 E_{\nu,\nu'}(t) \end{aligned}$$

on $[0, T_0]$.

As to the first term of (26), we see that

$$\text{(27) first term of (26)} \leq \sqrt{3}|\Psi_{\nu'}(t)-\Psi_\nu(t)|\sum_{j=1}^{\infty}\frac{\tilde{\rho}(t)^j}{M_j}\left\{\sum_{|\alpha|=j-1}\|D^\alpha A_{\nu'}u^{(\nu')}\|^2\right\}^{\frac{1}{2}}$$

$$\text{(28)} \qquad +\sqrt{3}\Psi_\nu(t)\sum_{j=1}^{\infty}\frac{\tilde{\rho}(t)^j}{M_j}\left\{\sum_{|\alpha|=j-1}\|D^\alpha(A_{\nu'}-A_\nu)u^{(\nu)}\|^2\right\}^{\frac{1}{2}}$$

$$\text{(29)} \qquad +\sqrt{3}\Psi_\nu(t)\sum_{j=1}\infty\frac{\tilde{\rho}(t)^j}{M_j}\left\{\sum_{|\alpha|=j-1}\|D^\alpha(f^{(\nu)}-f^{(\nu')})\|^2\right\}^{\frac{1}{2}}.$$

Applying Lemma 1 and Lemma 2 to (27) and (29), we find a constant C_2, independent of ν and ν', such that

$$(27)+(29) \leq C_2 E_{\nu,\nu'}(t)+\varepsilon_1(\nu,\nu'),$$

where $\varepsilon_1(\nu,\nu') \to 0$ as $j \to \infty$ for any $t \in [0, T_0]$.

Next we shall estimate (28) as follows:

$$\text{(30)} \quad \begin{aligned}(28) \leq \sqrt{6}\,\Psi_\nu(t)\sum_{j=1}^{\infty}\frac{\tilde{\rho}(t)^j}{M_j}&\left\{\sum_{|\alpha|=j-1}\|D^\alpha(A_{\nu'}-A)u^{(\nu)}\|^2\right\}^{\frac{1}{2}}\\ +\sqrt{6}\,\Psi_\nu(t)\sum_{j=1}^{\infty}\frac{\tilde{\rho}(t)^j}{M_j}&\left\{\sum_{|\alpha|=j-1}\|D^\alpha(A_\nu-A)u^{(\nu)}\|^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Now, applying Lemma 1, Lemma C and the fact $C(\{M_j\})_\rho \subset C(\{M_{j+1}\})_\rho$, there are constants C_3 and $\varepsilon_2(\nu)$ such that $\|D^\alpha(A_\nu - A)u^{(\nu)}\|$ can be estimated by

$$\|D^\alpha(A_\nu-A)u^{(\nu)}\| \leq C_3\,\varepsilon_2(\nu)(\kappa\rho(t))^{-(|\alpha|+2)}M_{|\alpha|+2}$$

for $0 < \forall\kappa < 1$, where $\varepsilon_2(\nu) \to 0$ as $\nu \to \infty$ and C_3 is a constant independent of ν. Similarly, we have

$$\|D^\alpha(A_{\nu'}-A)u^{(\nu)}\| \leq C_3\,\varepsilon_2(\nu')(\kappa\rho(t))^{-(|\alpha|+2)}M_{|\alpha|+2}.$$

Hence we get the following estimate

$$(28) \leq C_4(\varepsilon_2(\nu)+\varepsilon_2(\nu'))\sum_{j=1}^{\infty}\left(\frac{\tilde{\rho}(t)}{\kappa'\rho(t)}\right)^j$$

for $0 < \forall\kappa' < 1$, where we have applied the same arguments as the proof of Lemma 6. Here let us consider that $\tilde{\rho(t)} < \kappa'\rho(t)$, so that we have $\sum_{j=1}^{\infty}(\tilde{\rho}(t)/\kappa'\rho(t))^j < \infty$.

Therefore by taking together the previous estimates, there are constants C_5 and C_6 such that we get the following estimate of $E_{\nu,\nu'}(t)$:

$$\frac{d}{dt}E_{\nu,\nu'}(t) \leq C_5\left(\varepsilon_1(\nu,\nu') + \varepsilon_2(\nu) + \varepsilon_2(\nu')\right) + C_6 E_{\nu,\nu'}(t) \quad \forall t \in [0, T_0]\,. \tag{31}$$

Hence by applying Gronwall's lemma, Lemma 1 and the definition of ε_1 and ε_2, there are C and C' such that

$$E_{\nu,\nu'}(t) \leq CE_{\nu,\nu'}(0) + C'\left(\varepsilon_1(\nu,\nu') + \varepsilon_2(\nu) + \varepsilon_2(\nu')\right) \to 0$$

as $\nu, \nu' \to \infty$ on $[0, T_0]$.

3. – Appendix

THEOREM A (Denjoy-Carleman). *The following three statements are equivalent:*

(i) $C^{\sharp}(\{M_j\})$ *is a quasi-analytic class,*

(ii) $\displaystyle\int^{\infty}\frac{\log M(r)}{r^2}dr = \infty,$

(iii) $\displaystyle\sum_{j=0}^{\infty} M_j/M_{j+1} = \infty\,,$

where $M(r) \equiv \sup_j\{r^j/M_j\}$.

The proof is given in [10].

LEMMA B (Kinoshita [8]). *Let* $\{M_j\}$ *be a positive sequence of real numbers. Assume that* $\{M_j\}$ *is logalithmically convex. Then the the following inequalities are established.*

(i) $\binom{j}{\nu}M_{j-\nu}M_\nu \leq M_j$.

(ii) $\binom{j}{\nu-1}M_{j-\nu+1}M_{\nu+1} \leq \exists CM_j\nu(\nu+1)$ *for* $\forall j \geq 2\nu$, *where* C *is a constant independent of* j.

PROOF. (i) can be proved by the same argument to (ii). For (ii), we shall prove that there is a constant C independent of j and ν such that $\binom{j}{\nu-1}M_{j-\nu+1}M_{\nu+1}/M_j \le C\nu(\nu+1)$.

$$\binom{j}{\nu-1}\frac{M_{j-\nu+1}M_{\nu+1}}{M_j}$$

$$=\frac{\dfrac{M_{j-\nu+1}}{(j-\nu+1)M_{j-\nu}}\dfrac{M_{j-\nu}}{(j-\nu)M_{j-\nu-1}}\cdots\dfrac{M_2}{2M_1}\dfrac{M_1}{1}\times\dfrac{M_{\nu+1}}{(\nu+1)M_\nu}\cdots\dfrac{M_2}{2M_1}\dfrac{M_1}{1}}{\dfrac{M_j}{jM_{j-1}}\dfrac{M_{j-1}}{(j-1)M_{j-2}}\cdots\dfrac{M_2}{2M_1}\dfrac{M_1}{1}}.$$

In case that $j-\nu+1 \ge \nu+1$, we see that the following inequality holds:

$$=\frac{\dfrac{M_{j-\nu+1}}{(j-\nu+1)M_{j-\nu}}\cdots\dfrac{M_3}{3M_2}}{\dfrac{M_j}{jM_{j-1}}\cdots\dfrac{M_{\nu+2}}{(\nu+2)M_{\nu+1}}}\times\frac{\dfrac{M_2}{2M_1}\dfrac{M_1}{1}}{1}\le C_{M_1,M_2}.$$

In case that $j-1-\nu \ge \nu$, we have the inequality:

$$=\frac{\dfrac{M_{j-1-\nu}}{(j-1-\nu)M_{j-2-\nu}}\dfrac{M_{j-2-\nu}}{(j-2-\nu)M_{j-3-\nu}}\cdots\dfrac{M_1}{1}}{\dfrac{M_{j-1}}{(j-1)M_{j-2}}\dfrac{M_{j-2}}{(j-2)M_{j-3}}\cdots\dfrac{M_{\nu+1}}{(\nu+1)M_\nu}}\times\frac{\dfrac{M_\nu}{\nu M_{\nu-1}}\dfrac{M_{\nu-1}}{(\nu-1)M_{\nu-2}}\cdots\dfrac{M_1}{1}}{\dfrac{M_\nu}{\nu M_{\nu-1}}\dfrac{M_{\nu-1}}{(\nu-1)M_{\nu-2}}\cdots\dfrac{M_1}{1}}\le C.$$

Where we used the logalithmically convexity of $\{M_j\}$, that is, $\frac{M_j}{jM_{j-1}} \le \frac{M_k}{kM_{k-1}}$ for any $j \le k$. □

LEMMA C. *The product of $a(x) \in C(\{M_j\})_{\rho_0}$ and $v(x) \in C^\sharp(\{M_j\})_{\rho_0}$ belongs to the class $C^\sharp(\{M_j\})_\rho$ for $0 < \forall\,\rho < \rho_0$.*

PROOF. By applying Lemma B (i), $D^\alpha(a(x)v(x))$ can be estimated as follows:

$$\|D^\alpha(a(x)v(x))\| \le C(\kappa\rho_0)^{-|\alpha|}M_{|\alpha|}$$

for any $\kappa > 1$ and $\alpha \in \mathbb{N}^n$, where C is a constant independent of α. Then using the same method as the proof of Lemma 6, we have

$$\sum_{j=0}^{\infty}\frac{\rho^j}{M_j}\left\{\sum_{|\alpha|=j}\|D^\alpha(a(x)v(x))\|^2\right\}^{\frac{1}{2}} \le C\sum_{j=0}^{\infty}(\rho\kappa^2/\rho_0)^j.$$

Hence, by choosing $\kappa < \sqrt{\rho_0/\rho}$, we have the lemma. □

LEMMA D. *$C^\sharp(\{M_j\})_{\rho'} \subset C^\sharp(\{M_j\})_\rho$ for any $0 < \rho < \rho'$.*

Proof. Let $f(x)$ be a function belonging the class $C^{\sharp}(\{M_j\})_{\rho'}$. Then we can get the following estimates:

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{\rho^j}{M_j}\left(\sum_{|\alpha|=j}\|D^\alpha f\|^2\right)^{\frac{1}{2}} &\le C\sum_{j=0}^{\infty}\frac{\rho^j}{M_j}\left\{\sum_{|\alpha|=j}\left(\rho'^{-|\alpha|}M_{|\alpha|}\right)^2\right\}^{\frac{1}{2}}\\ &= C\sum_{j=0}^{\infty}\left(\frac{\rho}{\rho'}\right)^j\binom{j+n-1}{n-1}^{\frac{1}{2}}\\ &\le C'_{\varepsilon,n}\sum_{j=0}^{\infty}\left(\frac{\rho}{\rho'}\right)^j(1+\varepsilon)^{\frac{j}{2}},\end{aligned}$$

where we used the relations $\sum_{|\alpha|=j}1=\binom{j+n-1}{n-1}$ and $\binom{j+n-1}{n-1}\le(1+\varepsilon)^{j+n-1}/\varepsilon^{n-1}$ $\le C_{\varepsilon,n}(1+\varepsilon)^j$. Hence, by choosing $0<\varepsilon<(\rho'/\rho)^2-1$, we get the lemma. □

REFERENCES

[1] A. Arosio – S. Spagnolo, *Global existence for abstract evolution equations of weakly hyperbolic type*, J. Math. Pures Appl. **65** (1986), 263-305.

[2] A. Arosio – S. Spagnolo, *Global solution to the Cauchy problem for a nonlinear equation*, Pitman Res. Notes Math. **109** (1984), 1-26.

[3] S. Bernstein, *Sur une class d'équations fonctionnelles aux dérivées partielles*, Izv. Akad. Nauk SSSR, Ser. Mat. **4** (1940), 17-26.

[4] P. D'Ancona, *Gevrey well posedness of an abstract Cauchy problem of weakly hyperbolic type*, Publ. Res. Inst. Math. Sci, Kyoto Univ. **24** (1988), 433-449.

[5] P. D'Ancona – S. Spagnolo, *Global solvability for the degenerate Kirchhoff equation with real analytic data*, Invent. Math. **108** (1992), 247-262.

[6] G. H. Hardy – J. E. Littlewood – G. Polya, "Inequalities", Cambridge UP, Cambridge,1952.

[7] F. Hirosawa, *Global solvability for the generalized degenerate Kirchhoff equation with real-analytic data in* $\mathbb{R}^n$, Tsukuba J. Math. **21** (1997), 483-503.

[8] T. Kinoshita, *On the wellposedness in the ultradifferentiable classes of the Cauchy ploblem for a weakly hyperbolic equation of second order*, Tsukuba J. Math. (to appear).

[9] K. Kajitani – K. Yamaguti, *On global real analytic solutions of the degenerate Kirchhoff equation*, Ann. Scuola Norm. Sup. Pisa. Cl. Sci. (4) **21** (1994), 279-297.

[10] G. Krantz – H. R. Parks, "A Primer of Real Analytic Functions", Birkhäuser Verlag, Basel - Boston - Berlin, 1992.

[11] W. Matsumoto, *Theory of pseudo-differential operators of ultradifferentiable class*, J. Math. Kyoto Univ. **27** (1987), 453-500.

[12] T. Nishida, *A note on the nonlinear vibrations of the elastic string*, Mem. Fac. Engrg. Kyoto Univ. **33** (1971), 329-341.

[13] K. Nishihara, *On a global solution of some quasilinear hyperbolic equation*, Tokyo J. Math. **7** (1984), 437-459.

[14] G. Perla, *On classical solutions of a quasilinear hyperbolic equation*, Nonlinear Anal. **3** (1979), 613-627.

[15] O. A. Oleinik, *On linear equations of second order with non-negative characteristic form*, Mat. Sb. N.S. **69** (1966), 111-140 (English translation: Transl. Amer. Mat. Soc. **65**, 167-199).

[16] S. I. Pohozaev, *On a class of quasilinear hyperbolic equations*, Math. USSR-Sb. **96** (1975), 152-166.

[17] K. Yamaguti, *On global quasi-analytic solution of the degenerate Kirchhoff equation*, Tsukuba J. Math. (to appear).

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