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Global solutions of the Cauchy problem for a viscous polytropic ideal gas

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Abstract. We consider the Cauchy problem for a viscous polytropic ideal gas in $\mathbb{R}^n$ ($n = 2$ or 3). First we derive an a priori estimate for (smooth) solutions for small e_0 which may be used to show the existence of weak solutions, then we prove the existence and uniqueness of global (smooth) solutions for small E_0, where e_0 and E_0, depending on dimensions, are bounded from above by the Sobolev norms of the initial data. (In two dimensions e_0 and E_0 are bounded from above by the $L^2 \cap L^\infty \times H^1 \times H^1$- and $H^1 \cap W^{1,\alpha} \times H^1 \times H^1$-norms of $(\rho_0 - \bar\rho, v_0, \theta_0 - \bar\theta)$ respectively, where ρ_0, v_0 and θ_0 are the initial density, the initial velocity, and the initial temperature respectively, $\alpha \in (2, \infty)$, $\bar\rho, \bar\theta > 0$ are constants.)

1. – Introduction

The motion of a viscous polytropic ideal gas in $\mathbb{R}^n$ ($n = 2$ or 3) is described by the following equations in Eulerian coordinates (cf. [3], [26], [2])

$$(1.1) \qquad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0,$$

$$(1.2) \qquad \rho \left[\frac{\partial v}{\partial t} + (v \cdot \nabla)v \right] = \mu \Delta v + (\lambda + \mu)\nabla(\operatorname{div} v) - R\nabla(\rho\theta),$$

$$(1.3) \quad c_V \rho \left[\frac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta \right] = \kappa \Delta \theta - R\rho\theta(\operatorname{div} v) + \lambda(\operatorname{div} v)^2 + 2\mu D \cdot D.$$

Here ρ, θ, and $v = (v_1, \cdots, v_n)$ are the density, the absolute temperature and the velocity respectively, R, c_V and κ are positive constants; λ and μ are the constant viscosity coefficients, $\mu > 0$, $\lambda + 2\mu/n \geq 0$; $D = D(v)$ is the deformation tensor

$$D_{ij} := \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad \text{and} \quad D \cdot D := \sum_{i,j=1}^{n} D_{ij}^2.$$

We shall consider the Cauchy problem for (1.1)-(1.3) with the initial data

(1.4) $\rho(x, 0) = \rho_0(x), \quad v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \mathbb{R}^n$.

The Cauchy problem and initial boundary value problems for (1.1)-(1.3) have been studied by many authors. The global existence and asymptotic stability of (smooth) solutions have been established for the initial data close to a constant state in $H^3(\mathbb{R}^n)$; see [17], [18] for the Cauchy problem, [19], [20], [34], [21], [4], [5] for initial boundary value problems, [33] (among others) for a survey. For large data we have the global existence and asymptotic stability only in the spherically symmetric case (see [23], [6], [7], [11], [12]). We also mention that for barotropic (either isothermal or isentropic) fluids a lot of works have been done on the global existence (see e.g. [15], [16], [27], [9], [10], [14], [30], [31], [32] and the references therein, also see the survey article [33]).

In this paper we first establish the *a priori estimates* (Theorem 1.1) for solutions of the Cauchy problem (1.1)-(1.4) for small e_0, then we prove the existence and stability of global (smooth) solutions for E_0 small enough, where e_0 and E_0 (depending on n and defined by (1.6)-(1.7) below) are bounded from above by the Sobolev norms of the initial data. (For example, in two dimensions $e_0 = \|\rho_0 - \bar{\rho}\|_{L^2 \cap L^\infty} + \|(v_0, \theta_0 - \bar{\theta})\|_{H^1}$, $E_0 = e_0 + \|\nabla \rho_0\|_{L^2 \cap L^\alpha}$, where $\alpha > 2$ is an arbitray but fixed constant.) Thus we improve Matsumura-Nishida's results [17], [18] on the Cauchy problem in the case of a viscous polytropic ideal gas. Moreover, from the established a priori estimates a great deal of qualitative information about solutions can be obtained. For example, we see that v, θ, the vorticity, and the "effective viscous flux" F (defined below) are relatively smooth in positive time. We may use the a priori estimates to obtain the existence and uniqueness of global weak solutions (cf. Remark 1.1).

Before stating the main results, we explain the notation used throughout the paper. We denote $\int f \, dx = \int_{\mathbb{R}^n} f \, dx$. Let $m \geq 0$ be an non-negative integer and let $1 \leq p \leq \infty$. By $W^{m,p}$ we denote the usual Sobolev space defined on $\mathbb{R}^n$ with norms $\|\cdot\|_{W^{m,p}}$ (see e.g. [1]); $H^m \equiv W^{m,2}$ with norm $\|\cdot\|_{H^m}$, $L^p \equiv W^{0,p}$ with norm $\|\cdot\|_{L^p}$. $\|\cdot\|$ stands for the norm in $L^2(\mathbb{R}^n)$. $L^p(I, B)$ respectively $C^0(I, B)$ denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p = \infty$) respectively continuous functions from I to B, $I \subset \mathbb{R}$ an interval, B a Banach space. For a vector valued function $f = (f_1, \cdots, f_m)$ and a normed space X with the norm $\||\cdot\||$, $f \in X$ means that each component of f is in X; we put $\||f\|| := \||f_1\|| + \cdots + \||f_m\||$. Given a velocity field $v(x, t)$ we denote the corresponding vorticity matrix $\omega(x, t)$ by

(1.5) $\omega^{j,k} := \partial_k v_j - \partial_j v_k$,

and the material derivative $d/dt = \dot{}$, given by

$$\dot{f} = \frac{d}{dt} f = \partial_t f + v \cdot \nabla f = \partial_t f + v_j \partial_j f .$$

Here and throughout the paper repeated indices mean summation from 1 to n.

For a given constant state $(\bar{\rho}, 0, \bar{\theta})$, $\bar{\rho}, \bar{\theta} > 0$ we denote

$$(1.6) \quad e_0 := \|\rho_0 - \bar{\rho}\| + \|\rho_0 - \bar{\rho}\|_{L^\infty} + \|(v_0, \theta_0 - \bar{\theta})\|_{H^1}$$
$$+ (n - 2)(\|v^1\| + \|\omega_0\|_{L^4}^2),$$

$$(1.7) \quad E_0 := e_0 + \|\nabla \rho_0\| + \|\nabla \rho_0\|_{L^\alpha} + (n - 2)\|\nabla v^1\|, \qquad n = 2 \text{ or } 3,$$

where α is an arbitrary but fixed constant satisfying $n < \alpha < 2n/(n - 2)$, v^1 and the initial vorticity ω_0 are defined by

$$v^1 := \rho_0^{-1}\{\mu \Delta v_0 + (\lambda + \mu)\nabla \operatorname{div} v_0 - R\nabla(\rho_0 \theta_0)\}, \qquad \omega_0 := \omega|_{t=0}.$$

In fact $v^1 = \dot{v}\big|_{t=0}$ if (ρ, v, θ) is a solutiuon of (1.1)-(1.4).

To facilitate the discussion we introduce the notation for $n = 2$ or 3,

$$A(T) := \sup_{0 \leq t \leq T} \left\{ \|(\rho - \bar{\rho}, v, \theta - \bar{\theta}, \nabla v, \nabla \theta)\|^2 + \phi^{3-n}(\|\dot{v}\|^2 + \|\omega\|_{L^4}^4) \right.$$

$$(1.8) \qquad \left. + \phi^2\|\Delta \theta\|^2 + \phi\|\dot{\theta}\|^2 \right\}(t) + \int_0^T \left\{ \|(\nabla v, \nabla \theta, \dot{v}, \dot{\theta}, \nabla \omega, \Delta \theta)\|^2 \right.$$

$$\left. + \|\nabla v\|_{L^4}^4 + \phi^{3-n}\|\nabla \dot{v}\|^2 + \phi\|\nabla \dot{\theta}\|^2 \right\}(t)dt,$$

where $\phi \equiv \phi(t) := \min\{t, 1\}$.

The main results of this paper are the following two theorems.

THEOREM 1.1. *Let (ρ, v, θ) be a (smooth) solution of (1.1)-(1.4) defined up to a positive time T. Then there exists positive constants $\epsilon \leq 1$ and Γ, depending only on $\mu, \lambda, c_V, \kappa, \bar{\rho}$, and $\bar{\theta}$, such that if $e_0 \leq \epsilon$, then*

$$A(T) \leq \Gamma e_0^2, \qquad \sup_{0 \leq t \leq T} \left\{ \|\rho(t) - \bar{\rho}\|_{L^\infty} + \phi(t)\|\theta(t) - \bar{\theta}\|_{L^\infty} \right\} \leq \Gamma e_0,$$

and for all $q \in (2, \infty)$, $\|\rho(T) - \bar{\rho}\|_{L^\infty} + \|(v(T), \theta(T) - \bar{\theta})\|_{L^q} \to 0$ as $T \to \infty$.

As a consequence of Theorem 1.1 we can bound higher order derivatives of (ρ, v, θ) by means of the energy method to obtain

THEOREM 1.2. *Let $(\rho_0 - \bar{\rho}, v_0, \theta - \bar{\theta}) \in H^3$ and $\inf \theta_0 > 0$. Then there exists a positive constant $\epsilon_1 \leq 1$, depending only on $\mu, \lambda, c_V, \kappa, \bar{\rho}, \bar{\theta}$, and α, such that if $E_0 \leq \epsilon_1$, then the Cauchy problem (1.1)-(1.4) has a unique solution (ρ, v, θ) on $\mathbb{R}^n \times [0, \infty)$ satisfying*

$$(\rho - \bar{\rho}, v, \theta - \bar{\theta}) \in C^0([0, \infty), H^3), \quad \partial_t \rho \in C^0([0, \infty), H^2),$$

$$(1.9) \qquad (\partial_t v, \partial_t \theta) \in C^0([0, \infty), H^1), \quad (\partial_t \rho, \partial_t v, \partial_t \theta) \in L^2([0, \infty), H^2),$$

$$\sup_{t \geq 0} \|\rho(t) - \bar{\rho}\|_{L^\infty} \leq \bar{\rho}/2, \quad \inf_{x \in \mathbb{R}^n, t \geq 0} \theta(x, t) > 0.$$

Moreover,

(1.10) $$\|(\rho - \bar{\rho}, v, \theta - \bar{\theta})(t)\|_{L^\infty} \to 0 \qquad as \ t \to \infty.$$

REMARK 1.1. We might mollify the initial data and apply Theorem 1.1 to obtain the existence and uniqueness of global weak solutions of (1.1)-(1.4) by a limit procedure (see [9, Section 4], [10]). From the proof we see that to show Theorem 1.1 the regularity $\rho \in C^0([0, T], H^1) \cap C^0([0, T], L^\infty)$, $v, \theta \in C^0([0, T], H^2)$, $v_t, \theta_t \in C^0([0, T], L^2)$ is sufficient.

REMARK 1.2. We can show in the same way as in [18, pp. 101-103] that the solution (ρ, v, θ) established in Theorem 1.2 is a classical one for $t > 0$. Since to obtain a local solution we do not need $\inf \theta_0 > 0$ (see Theorem 4.7 (the local existence) in Section 4), the condition $\inf \theta_0 > 0$ in Theorem 1.2 is not necessary.

The main ingredients in the proof of Theorems 1.1 and 1.2 are smoothing properties of parabolic parts in (1.2)-(1.3), the second law of thermodynamics, the careful (weighted) energy estimates, and a certain exponential decay of $\rho - \bar{\rho}$ (cf. equation (2.5)) that is crucial in deriving pointwise bounds for ρ. For compressible viscous barotropic fluids Hoff [9], [10] recently proved the existence of global weak solutions when ρ_0 is close to a constant in $L^\infty \cap L^2$, v_0 is small in L^2 and bounded in L^{2^n}, and the ratio λ/μ is small. The method of proof here is based on that used by Hoff [9], [10]. For our system (1.1)-(1.3), since the temperature and equation (1.3) appear, the difficulties here arise from the coupling of ρ and θ and the nonlinear terms of high order in equation (1.3) which have to be appropriatly controlled, when we derive bounds for the L^2- and H^1-norms of $\theta - \bar{\theta}$ and v, pointwise bounds of $(\theta - \bar{\theta})\phi$ and $\rho - \bar{\rho}$ without assuming the smallness of $\theta_0 - \bar{\theta}$ in L^∞. To overcome such difficulties new arguments are used. In particular, we derive a useful identity which is motivated by the second law of thermodynamics and embodies the dissipative effects of viscosity and thermal diffusion to estimate $h(T) := \|(\rho - \bar{\rho}, v, \theta - \bar{\theta})(T)\|_{L^2}^2 + \int_0^T \|(\nabla v, \nabla \theta)(s)\|_{L^2}^2 ds$ for $T > 1$, while the estimate of $h(T)$ for $T \leq 1$ is obtained using the L^2-energy method. This avoids the smallness assumption of $\|\theta_0 - \bar{\theta}\|_{L^\infty}$, but requires the control of $\int_0^T \|\nabla v\|_{L^4}^4 ds$ that can be bounded by $\int_0^T (\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4) ds$ using the Marcinkiewicz multiplier theorem and a useful equation for v. With the help of Gagliado-Nirenberg's inequality, the Marcinkiewicz multiplier theorem and a fundamental partial differential equation for F, the norm of F in $L^4((0, T) \times \mathbb{R}^n))$ is bounded by the term $\int_0^T \|\nabla v\|_{L^2}^{4-n} \|\dot{v}\|_{L^2}^n ds$, and in order to control this term we need the smallness of $\|(\nabla v_0, \nabla \theta_0)\|_{L^2}$ but no condition on the ratio λ/μ. Moreover, in the case of $n = 3$ we have to bound $\sup \|\dot{v}(t)\|^2$, this requires the smallness of $\dot{v}|_{t=0}$ in L^2 (cf. the proof of (2.7) and (3.1)). On the other hand we utilize the careful (weighted) energy estimates and the integrability of $\phi^\delta(t)$ on $(0, 1)$ for $0 < \delta < 1$ to estimate $\int_0^T \|\omega\|_{L^4}^4 ds$ as well as $\int_0^1 \|(F, \theta - \bar{\theta})\|_{L^\infty} ds$ which are used in the derivation of pointwise bounds for $(\theta - \bar{\theta})\phi$ and $\rho - \bar{\rho}$ (cf. Lemma 3.1 and (2.9)).

Now we define the effective viscous flux F, similarly as in [9], by

$$(1.11) \qquad F := (\lambda + 2\mu) \operatorname{div} v - R(\rho\theta - \bar{\rho}\bar{\theta}).$$

F will play an important role in the proof of Theorem 1.1. The quantities ρ and $\rho\theta$ evidently may be as rough as ρ_0, and ∇v no smoother than an L^2 function. But the jumps in $R\rho\theta$ and in $(\lambda + 2\mu) \operatorname{div} v$ may exactly cancel, so that F becomes somewhat smoother than either $\rho\theta$ or $\operatorname{div} v$. (In fact from the derivation of (2.11) we have $F(\cdot, t) \in W^{1,4}$ for a.e. $t > 0$, which yields $F(\cdot, t) \in C^\beta(\mathbb{R}^n)$ for some $\beta \in (0, 1)$ and a.e. $t > 0$). See the introduction of [9] for an extensive discussion at the heuristic level concerning jump discontinuities and smoothing properties in solutions for the barotropic case. The conjecture concerning the regularity of F, v, and ω will be made using the fundamental partial differential equation for F,

$$(1.12) \qquad \Delta F = \operatorname{div}(\rho\dot{v}),$$

which is obtained as follows. We use (1.5) to obtain an equivalent form of (1.2): $\rho\dot{v}_j = \partial_j F + \mu\partial_k\omega^{j,k}$, which by taking ∂_j and summing over j gives (1.12).

From the definition of F and ω we get after a straightforward calculation the following equation:

$$(1.13) \quad (\lambda + 2\mu)\Delta v_j = \partial_j F + (\lambda + 2\mu)\partial_k\omega^{j,k} + R\partial_j\left[\rho(\theta - \bar{\theta})\right] + R\bar{\theta}\partial_j[\rho - \bar{\rho}],$$

which will be used in Section 2.

The paper is organized as follows: In Section 2 we bound the L^4- and L^∞-norms of $\rho - \bar{\rho}$ from above by $A(T)$ using equation (2.5) for ρ which gives the exponential decay of $\rho - \bar{\rho}$. In Sections 3 and 4 we prove Theorems 1.1 and 1.2, respectively.

We shall make repeated use of various standard inequalities. The first one of these is Young's inequality

$$(1.14) \quad |ab| \le \delta^{-p/q}\frac{|a|^p}{p} + \delta\frac{|b|^q}{q}, \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \qquad 0 < \delta \le 1.$$

As a consequence of Young's inequality and Gagliado-Nirenberg's inequality (see [8], [24], also see [13, Lemma 2.1]) we have

$$(1.15) \quad \|u\|_{L^p} \le C(p, n)\|u\|^{n/p - n/2 + 1}\|\nabla u\|^{n/2 - n/p} \quad (\le C(p, n)(\|u\| + \|\nabla u\|))$$

for $u \in H^1$, $p \ge 2$ when $n = 2$ and $p \in [2, 6]$ when $n = 3$, and

$$(1.16) \qquad \|u\|_{L^\infty} \le C(p, n)\|u\|^{1 - n/4}\|\nabla^2 u\|^{n/4}$$

for $u \in H^2$, where $C(p, n)$ in (1.15)-(1.16) is a positive constant depending only on p and n. It is easy to see that

$$(1.17) \qquad z^b \le z^a + z^c \qquad \forall z \ge 0, c \ge b \ge a \ge 0.$$

Throughout this paper the same letter C will denote various positive constants which depend only on μ, λ, c_V, κ, $\bar{\rho}$, $\bar{\theta}$ (and also on α in the case of Theorem 1.2).

2. – L^4- and L^∞-estimates for ρ

Let (ρ, v, θ) be a solution of (1.1)-(1.4) on $\mathbb{R}^n \times [0, T]$ for some positive T which satisfies $(1.9)_1$ on $[0, T]$; let $e_0 \leq 1$. We assume that ρ, θ satisfy (recall $\phi(t) = \min\{t, 1\}$)

$$(2.1) \quad |\rho(x, t) - \bar\rho|, \quad \phi(t)|\theta(x, t) - \bar\theta| \leq \min\{\bar\rho, \bar\theta\}/2 \quad \text{for all } x \in \mathbb{R}^n, t \in [0, T].$$

In this section we show that under (2.1) $\sup_{0 \leq t \leq T} \|\rho(t) - \bar\rho\|_{L^\infty}$ and $\int_0^T \|\rho(t) - \bar\rho\|^4 dt$ are bounded from above by powers of e_0 and $A(T)$. We begin with the following lemma which gives the equivalence of integrals in Eulerian and Lagrangian coordinates, and will frequently be used in the sequel.

LEMMA 2.1. *Fix* $t_0 \in [0, T]$ *and define the system of particle trajectories* $x(y, t)$ *by*

$$\frac{dx}{dt}(y, t) = v(x(y, t), t), \quad x(y, t_0) = y.$$

Let $g \geq 0$ *be an integrable function, and let* $t \in [0, T]$; *then each of the integrals*

$$\int g(x(y, t), t) dy \quad \text{and} \quad \int g(x, t) dx$$

is bounded by C *times the other.*

Lemma 2.1 follows easily from the assumption (2.1) and its proof can be found in [9, Lemma 3.2].

Using equations (1.12)-(1.13) and applying a suitable Fourier multiplier theorem, we can bound the L^p-norm of ∇v and ∇F as follows:

LEMMA 2.2.

$$(2.2) \qquad \|\nabla v(t)\|_{L^p} \leq C \|(F(t), \omega(t), \rho(t) - \bar\rho, \theta(t) - \bar\theta)\|_{L^p},$$

$$(2.3) \qquad \|\nabla F(t)\|_{L^p} \leq C \|\dot{v}(t)\|_{L^p}.$$

PROOF. Taking Fourier tranforms for (1.13) and (1.12) respectively, we obtain

$$(\lambda + 2\mu)\widehat{\partial_i v_j} = \frac{\xi_j \xi_i}{|\xi|^2} \widehat{F} + (\lambda + 2\mu)\frac{\xi_k \xi_i}{|\xi|^2} \widehat{\omega^{j,k}} + R\frac{\xi_j \xi_i}{|\xi|^2} \left(\widehat{\rho(\theta - \bar\theta)} + \bar\theta\widehat{(\rho - \bar\rho)} \right),$$

$$\widehat{\partial_i F} = \frac{\xi_j \xi_i}{|\xi|^2} \widehat{\rho \dot{v}_j}.$$

The multiplier $m(\xi) = \xi_j \xi_i / |\xi|^2$ is in $C^\infty(\mathbb{R}^n \setminus \{0\})$ and satisfies $|D^\alpha m(\xi)| \leq C(|\alpha|)|\xi|^{-\alpha}$. Therefore it follows from the Marcinkiewicz multiplier theorem [28, page 96] that the linear mapping $v \to \mathcal{F}^{-1}(m\hat{v})$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$, where $\mathcal{F}^{-1}$ is the Fourier inverse transform. $\qquad\square$

In the following lemma we employ a certain exponential decay of $\rho - \bar{\rho}$ to estimate $\int_0^T \|\rho(t) - \bar{\rho}\|_{L^4}^4 dt$.

LEMMA 2.3.

$$(2.4) \qquad \sup_{0 \le t \le T} \|\rho(t) - \bar{\rho}\|_{L^4}^4 + \int_0^T \|\rho(t) - \bar{\rho}\|_{L^4}^4 dt \le C(e_0^2 + A^2(T)).$$

PROOF. Using (1.1), we may write (1.11) in the form

$$(2.5) \qquad (\lambda + 2\mu)\frac{d}{dt}(\rho - \bar{\rho}) + R\bar{\theta}\rho(\rho - \bar{\rho}) = -\rho F - R\rho^2(\theta - \bar{\theta}).$$

Multiplying (2.5) by $(\rho - \bar{\rho})^3$, using (2.1) and Young's inequality, we obtain

$$\frac{d}{dt}(\rho - \bar{\rho})^4 + C^{-1}(\rho - \bar{\rho})^4 \le C(F^4 + (\theta - \bar{\theta})^4).$$

We integrate the above inequality over a fixed particle path $x(y, t)$ to arrive at

$$(2.6) \qquad \begin{aligned} (\rho - \bar{\rho})^4(x(y, t), t) + \frac{1}{C}\int_0^t (\rho - \bar{\rho})^4(x(y, s), s)ds &\le (\rho_0(y) - \bar{\rho})^4 \\ &+ C\int_0^t \left\{F^4 + (\theta - \bar{\theta})^4\right\}(x(y, s), s)ds \end{aligned}$$

for $t \in [0, T]$. Applying (1.15) and (2.3), recalling the definition of F and $A(T)$, we have

$$(2.7) \qquad \begin{aligned} \int_0^t \left(\|F\|_{L^4}^4 + \|\theta - \bar{\theta}\|_{L^4}^4\right) ds &\le \int_0^t \left(\|F\|^{4-n}\|\nabla F\|^n + \|\theta - \bar{\theta}\|^{4-n}\|\nabla\theta\|^n\right) ds \\ &\le C\int_0^t \left\{(\|\nabla v\| + \|\rho - \bar{\rho}\| + \|\theta - \bar{\theta}\|)^{4-n}\|\dot{v}\|^{n-2}\|\dot{v}\|^2 \right. \\ &\left. + \|\theta - \bar{\theta}\|^{4-n}\|\nabla\theta\|^{n-2}\|\nabla\theta\|^2\right\} ds \\ &\le CA(T)\int_0^t (\|\dot{v}\|^2 + \|\nabla\theta\|^2)ds \le CA^2(T), \quad t \in [0, T]. \end{aligned}$$

Integrating (2.6) over $\mathbb{R}^n$ with respect to y, applying Lemma 2.1, and using (2.7), we obtain (2.4). $\square$

To derive pointwise bounds for $\rho - \bar{\rho}$ we first consider the case $T \le 1$. We integrate (2.5) along particle trajectories and make use of (2.1) to obtain

$$(2.8) \qquad \begin{aligned} \|\rho(t) - \bar{\rho}\|_{L^\infty} &\le Ce_0 + C\int_0^t \|\rho(s) - \bar{\rho}\|_{L^\infty}ds \\ &+ C\int_0^t (\|F\|_{L^\infty} + \|\theta - \bar{\theta}\|_{L^\infty})ds, \qquad 0 \le t \le T \le 1, \end{aligned}$$

where the second integral on the right-hand side of (2.8) can be bounded as follows, using Sobolev's imbedding theorem ($W^{m,p} \hookrightarrow L^\infty$, $mp > n$), (2.3), (1.15), and Hölder's inequality:

$$
\int_0^t (\|F\|_{L^\infty} + \|\theta - \bar\theta\|_{L^\infty})ds \le C \int_0^t (\|F\|_{L^4} + \|\dot v\|_{L^4} + \|\theta - \bar\theta\|_{H^2})ds
$$

$$
\le C \int_0^t \left\{ \|(\nabla v, \rho - \bar\rho, \theta - \bar\theta)\|^{(4-n)/4} \|\dot v\|^{n/4} + \|\dot v\|^{(4-n)/4} \|\nabla \dot v\|^{n/4} \right\} ds
$$

(2.9)
$$
+ C \left(\int_0^t \|(\theta - \bar\theta, \nabla\theta, \Delta\theta)\|^2 ds \right)^{1/2} \le C A^{(4-n)/8} \left(\int_0^t \|\dot v\|^2 ds \right)^{n/8}
$$

$$
+ C \left(\int_0^t \|\dot v\|^2 ds \right)^{(4-n)/8} \left(\int_0^t \phi^{-n/4}(s)ds \right)^{1/2}
$$

$$
\times \left(\int_0^t \phi \|\nabla \dot v\|^2 ds \right)^{n/8} + C A^{1/2} \le C A^{1/2}(T), \qquad 0 \le t \le T \le 1.
$$

Inserting (2.9) into (2.8) and applying Gronwall's inequality, we conclude

(2.10)
$$
\sup_{0 \le t \le T} \|\rho(t) - \bar\rho\|_{L^\infty} \le C(e_0 + A^{1/2}(T)), \quad T \le 1.
$$

In the case of $T > 1$ we multiply (2.5) by $\rho - \bar\rho$ and use (2.1) to find that

$$
\frac{d}{dt}(\rho - \bar\rho)^2 + C^{-1}(\rho - \bar\rho)^2 \le C(\|F\|_{L^\infty}^2 + \|\theta - \bar\theta\|_{L^\infty}^2), \quad t \in [1, T].
$$

Multiplying this by $e^{t/C}$, integrating over a fixed particle path $x(y, t)$ ($1 \le t \le T$), we obtain by (2.1), (2.10), and arguments similar to those used in (2.9) that

$$
\|\rho(t) - \bar\rho\|_{L^\infty}^2 \le C\|\rho(1) - \bar\rho\|_{L^\infty}^2
$$

$$
+ C \int_1^t e^{-(t-s)/C}(\|F\|_{L^4}^2 + \|\dot v\|_{L^4}^2 + \|\theta - \bar\theta\|_{L^\infty}^2)ds
$$

$$
\le C\|\rho(1) - \bar\rho\|_{L^\infty}^2
$$

(2.11)
$$
+ C \sup_{1 \le s \le T} (\|F\|^{(4-n)/2} \|\dot v\|^{n/2} + \|\theta - \bar\theta\|_{H^1}^2 + \|\Delta\theta\|^2)(s)
$$

$$
+ C \int_1^t (\|\dot v\|^2 + \|\nabla \dot v\|^2)ds
$$

$$
\le C(e_0^2 + A(T)), \quad 1 \le t \le T.
$$

Combining (2.10) and (2.11), we thus have proved

LEMMA 2.4.

$$
\sup_{0 \le t \le T} \|\rho(t) - \bar\rho\|_{L^\infty} \le C(e_0 + A^{1/2}(T)).
$$

3. – Proof of Theorem 1.1

In this section all the assumptions described in the first paragraphy of Section 2 will continue to hold. The main result of this section is the estimate for A in Theorem 3.4 below, from which Theorem 1.1 easily follows.

From (2.2) with $p = 4$, (2.7), and Lemma 2.3, we easily get

$$(3.1) \qquad \int_0^t \|\nabla v(s)\|_{L^4}^4 ds \le C(e_0^2 + A^2(T)) + C \int_0^t \|\omega\|_{L^4}^4 ds, \qquad t \in [0, T].$$

In the following lemma we estimate the vorticity term on the right-hand side of (3.1).

LEMMA 3.1.

$$(3.2) \quad \sup_{0 \le t \le T} \phi^{3-n}(t) \|\omega(t)\|_{L^4}^4 + \int_0^T \phi^{3-n}(\|\omega\|_{L^4}^4 + \|\omega\|_{L^6}^6) ds \le C(e_0^2 + A^{4/3}(T) + A^3(T)).$$

PROOF. Differentiating (1.2) with respect to x_k, one sees that

$$\rho \frac{d}{dt} \partial_k v_j = \mu \Delta \partial_k v_j - \rho \partial_k v \cdot \nabla v_j - \dot{v}_j \partial_k \rho + (\lambda + \mu) \partial_k \partial_j (\text{div } v) - R \partial_k \partial_j (\rho \theta).$$

Interchanging k and j and subtracting, we obtain

$$(3.3) \qquad \rho \frac{d}{dt} w^{j,k} = \mu \Delta \omega^{j,k} + \rho(\partial_j v \cdot \nabla v_k - \partial_k v \cdot \nabla v_j) + (\dot{v}_k \partial_j \rho - \dot{v}_j \partial_k \rho).$$

For simplicity we denote $\eta = \omega^{j,k}$. Using (2.1), (1.15) with $n = 2$, and the transport theorem

$$(3.4) \qquad\qquad \frac{d}{dt} \int \rho f dx = \int \rho \dot{f} dx,$$

and keeping in mind that $\|\eta\|_{L^p} \le C\|\nabla v\|_{L^p}$, we see that

$$\int_0^t \phi^{3-n} \int \rho \frac{d}{dt} \eta^4 dx \, ds \ge \int_0^t \frac{d}{dt}\left(\rho \phi^{3-n} \int \eta^4 dx\right) ds - (3-n) \int_0^t \|\eta\|_{L^4}^4 ds$$

$$\ge \phi^{3-n} \int \rho \eta^4 dx - Ce_0^2 - C \int_0^t \|\eta\|^2 \|\nabla \eta\|^2 ds$$

$$\ge \frac{\bar{\rho}}{2} \phi^{3-n}(t) \|\eta(t)\|_{L^4}^4 - C(e_0^2 + A^2).$$

Theorefore, multiplying (3.3) by $\phi^{3-n}(\omega^{j,k})^3$ and integrating over $\mathbb{R}^n \times (0, t)$ ($0 \le t \le T$), we integrate by parts with respect to x and make use of (2.1) to arrive at

$$(3.5) \quad \begin{aligned} \phi^{3-n}(t) \|\eta(t)\|_{L^4}^4 &+ \int_0^t \int \phi^{3-n} \eta^2 |\nabla \eta|^2 dx \, ds \le C(e_0^2 + A^2) \\ &+ C \int_0^t \int \phi^{3-n}(|\nabla v|^2 |\eta|^3 + |\dot{v}| \, |\eta|^2 |\nabla \eta| + |\eta|^3 |\nabla \dot{v}|) dx \, ds. \end{aligned}$$

Note that by virtue of (1.17), (1.15), and the definition of F,

$$\int_0^T \phi^{3-n} \int \eta^4 dx\, ds \leq \int_0^T \phi^{3-n} \int (|\eta|^{2+4/n} + \eta^6) dx\, ds$$

$$\leq C \int_0^T (\|\eta\|^{4/n} \|\nabla \eta\|^2 + \phi^{3-n} \|\eta\|_{L^6}^6) ds$$

$$\leq C A^{1+2/n} + C \int_0^T \phi^{3-n} \|\eta\|_{L^6}^6 ds\,.$$

So, adding $\int_0^t \int \phi^{3-n}(\eta^4 + \eta^6) dx\, ds$ to (3.5) on both sides, applying (1.14) with $p = 2$, (2.2) with $p = 4$, (2.4), and (2.7), we find that

$$\phi^{3-n}(t)\|\eta(t)\|_{L^4}^4 + \int_0^t \int \phi^{3-n}(\eta^4 + \eta^6 + \eta^2 |\nabla \eta|^2) dx\, ds$$

$$\leq C(e_0^2 + A^2 + A^{1+2/n})$$

$$+ \delta \int_0^t \phi^{3-n} \|\nabla v\|_{L^4}^4 ds + C\delta^{-1} \int_0^t \phi^{3-n} \|\eta\|_{L^6}^6 ds$$

$$+ C \int_0^t \int \phi^{3-n}(|\dot{v}|^2 \eta^2 + |\eta|^3 |\nabla \dot{v}|) dx\, ds$$

(3.6)

$$\leq C(e_0^2 + A^{5/3} + A^2) + C\delta \int_0^t \phi^{3-n} \|\omega\|_{L^4}^4 ds$$

$$+ C\delta^{-1} \int_0^t \phi^{3-n} \|\eta\|_{L^6}^6 ds \qquad (0 < \delta < 1)$$

$$+ C \left(\int_0^t \phi^{3-n} \|\dot{v}\|_{L^3}^3 ds \right)^{2/3} \left(\int_0^t \phi^{3-n} \|\eta\|_{L^6}^6 ds \right)^{1/3}$$

$$+ C A^{1/2} \left(\int_0^t \phi^{3-n} \|\eta\|_{L^6}^6 ds \right)^{1/2}.$$

Noting that $\phi^{2(3-n)/n} = \phi^{3-n}$ for $n = 2, 3$, we get from (1.15) and (1.14) that

$$\int_0^t \phi^{3-n} \|\eta\|_{L^6}^6 ds \leq C \int_0^t \phi^{3-n} \|\eta\|_{L^4}^{6-n} \|\eta|\nabla \eta| \|^{n/2} ds$$

$$\leq C A^{1/2} \int_0^t \|\eta\|_{L^4}^{4-n} \phi^{(3-n)/2} \|\eta|\nabla \eta| \|^{n/2} ds$$

(3.7)

$$\leq C A^{1/2} \left(\int_0^t \|\nabla v\|_{L^4}^4 ds \right)^{(4-n)/4} \left(\int_0^t \phi^{2(3-n)/n} \|\eta|\nabla \eta| \|^2 ds \right)^{n/4}$$

$$\leq C A^{(6-n)/4} \left(\int_0^t \phi^{3-n} \|\eta|\nabla \eta| \|^2 ds \right)^{n/4}$$

and

(3.8)

$$\int_0^t \phi^{3-n} \|\dot{v}\|_{L^3}^3 ds \leq C A^{1/2} \int_0^t \phi^{(3-n)/2} \|\dot{v}\|^{(4-n)/2} \|\nabla \dot{v}\|^{n/2} ds$$

$$\leq C A^{1/2} \int_0^t (\|\dot{v}\|^2 + \phi^{2(3-n)/n} \|\nabla \dot{v}\|^2) ds \leq C A^{3/2}.$$

Inserting (3.7) and (3.8) into (3.6), using (1.14), and (1.17), we deduce

$$
\phi^{3-n}(t)\|\eta(t)\|_{L^4}^4 + \int_0^t \phi^{3-n}(\|\eta\|_{L^4}^4 + \|\eta\|_{L^6}^6)ds \le C\delta^{-4}(e_0^2 + A^{4/3} + A^3)
$$

(3.9)

$$
+ C\delta \int_0^t \phi^{3-n}\|\omega\|_{L^4}^4 ds
$$

for all $t \in [0, T]$. Recalling $\eta = \omega^{j,k}$, we sum over j, k in (3.9) and take δ appropriately small to obtain the lemma.   □

As a consequence of (3.1) and Lemma 3.1 we have

(3.10)
$$
\int_0^T \|\nabla v(s)\|_{L^4}^4 ds \le C(e_0^2 + A^{4/3}(T) + A^3(T)).
$$

In fact, in the case of $n = 2$ it follows from (1.15) that $\int_0^T \|\omega\|_{L^4}^4 ds \le C \int_0^T \|\omega\|^2\|\nabla\omega\|^2 ds \le CA^2(T)$, which combined with Lemma 3.1 and (3.1) gives (3.10).

The following lemma is motivated by the second law of thermodynamics and embodies the dissipative character of viscosity and thermal diffusion (cf. (3.14) below).

LEMMA 3.2.

(3.11)
$$
\sup_{0 \le t \le T} \|(\rho - \bar\rho, v, \theta - \bar\theta)(t)\|^2 + \int_0^T (\|\nabla v\|^2 + \|\nabla\theta\|^2)ds
$$
$$
\le C(e_0^2 + A^{4/3}(T) + A^3(T)).
$$

PROOF. We first consider the case $T \le 1$. Multiplying (1.2) and (1.3) by v and $\theta - \bar\theta$ in $L^2(\mathbb{R}^n \times (0, t))$ respectively, and integrating by parts with respect to x, we employ (3.4), (2.1), Cauchy-Schwarz's inequality, and (2.7), (3.10) to deduce that

$$
\|v(t)\|^2 + \|\theta(t) - \bar\theta\|^2 + \int_0^t (\|\nabla v\|^2 + \|\nabla\theta\|^2)ds
$$

$$
\le Ce_0^2 + C\int_0^t (\|\rho - \bar\rho\|^2 + \|\theta - \bar\theta\|^2)ds
$$

(3.12)

$$
+ C \int_0^t \int (|\nabla v|^4 + |\theta - \bar\theta|^4)dx\,ds
$$

$$
\le C(e_0^2 + A^{4/3} + A^3) + C\int_0^t (\|\rho - \bar\rho\|^2 + \|\theta - \bar\theta\|^2)ds, \; 0 \le t \le T \le 1.
$$

If we multiply the equation $\dot\rho = -\rho\,\mathrm{div}\,v$ (eq. (1.1)) by $\rho(\rho - \bar\rho)$ and integrate, we obtain

$$
\|\rho(t) - \bar\rho\|^2 \le C\int_0^t \|\rho - \bar\rho\|^2 ds + \frac{1}{2}\int_0^t \|\nabla v\|^2 ds.
$$

Adding the above inequality to (3.12) and applying Gronwall's inequality, we obtain

$$(3.13) \quad \|(\rho - \bar\rho, v, \theta - \bar\theta)(t)\|^2 + \int_0^t \|(\nabla v, \nabla \theta)\|^2 ds$$
$$\leq C(e_0^2 + A^{4/3}(T) + A^3(T)), \ 0 \leq t \leq T \leq 1.$$

It is easy to see by integration by parts that $2 \int D \cdot D dx = \int (|\nabla v|^2 + (\operatorname{div} v)^2) dx$. So using equations (1.1)-(1.3) and (3.4), integrating by parts, we get after a straightforward calculation that

$$
\begin{aligned}
(3.14) \quad & \frac{d}{dt} \int \left\{ \rho \frac{|v|^2}{2} + R\bar\theta \left(\bar\rho - \rho \log \frac{\bar\rho}{\rho} - \rho \right) + c_V \rho \left(\theta - \bar\theta \log \frac{\theta}{\bar\theta} - \bar\theta \right) \right\} dx \\
& = \int \left\{ -\mu |\nabla v|^2 - (\lambda + \mu)(\operatorname{div} v)^2 - \kappa \bar\theta \frac{|\nabla \theta|^2}{\theta^2} \right. \\
& \qquad \left. + \left(1 - \frac{\bar\theta}{\theta} \right) (\lambda (\operatorname{div} v)^2 + 2\mu D \cdot D) \right\} dx \\
& = -\int \left\{ \frac{\bar\theta}{\theta} (\lambda (\operatorname{div} v)^2 + 2\mu D \cdot D) + \kappa \bar\theta \frac{|\nabla \theta|^2}{\theta^2} \right\} dx.
\end{aligned}
$$

When $T \geq 1$, we integrate (3.14) over $(1, t)$ ($t \in [1, T]$), keep in mind that $\lambda (\operatorname{div} v)^2 + 2\mu D \cdot D \geq (\lambda + 2\mu/n)(\operatorname{div} v)^2 \geq 0$, utilise (2.1), the mean value theorem, and (3.13) to infer that

$$(3.15) \quad \|(v, \rho - \bar\rho, \theta - \bar\theta)(t)\|^2 + \int_1^t (\|\nabla v\|^2 + \|\nabla \theta\|^2) ds \leq C \|(v, \rho - \bar\rho, \theta - \bar\theta)(1)\|^2$$
$$\leq C(e_0^2 + A^{4/3}(T) + A^3(T)) \qquad \forall t \in [1, T].$$

Combining (3.13) and (3.15), we obtain (3.11).   □

We now proceed to get Sobolev-norm estimates for ρ, v, θ. We shall make use of the following transport theorem:

$$(3.16) \quad \int_0^t \int f \dot g \, dx \, ds = \int fg \, dx \Big|_0^t - \int_0^t \int \dot f g \, dx \, ds - \int_0^t \int fg \operatorname{div} v \, dx \, ds,$$

which follows from (3.4), (1.1) and a simple calculation.

Multiply (1.2) and (1.3) by $\dot v$ and $\dot \theta$ respectively, and integrate over $\mathbb{R}^n \times (0, t)$. If we integrate by parts, take into account that $\partial_i \dot f = \frac{d}{dt} \partial_i f + (\partial_i v) \cdot \nabla f$,

use (2.1), Cauchy-Schwarz's inequality, (3.16), and (1.14), we obtain

$$
\begin{aligned}
\frac{\bar{\rho}}{4} \int_0^t & (\|\dot{v}\|^2 + c_V \|\dot{\theta}\|^2) ds \leq -\frac{1}{2} \int_0^t \int \frac{d}{dt} \{ \mu |\nabla v|^2 \\
& + (\lambda + \mu)(\operatorname{div} v)^2 + \kappa |\nabla \theta|^2 \} dx \, ds \\
& - R \int_0^t \int \nabla(\rho\theta)\dot{v} \, dx \, ds + C \int_0^t \int (\theta^2 |\nabla v|^2 \\
& + |\nabla v|^4 + |\nabla v| \, |\nabla\theta|^2 + |\nabla v|^3) dx \, ds \\
& \leq C e_0^2 - \frac{1}{2}(\mu\|\nabla v(t)\|^2 + \kappa\|\nabla\theta(t)\|^2) + R \int_0^t \int (\rho\theta - \bar{\rho}\bar{\theta}) \operatorname{div} \dot{v} \, dx \, ds \\
& + C \int_0^t \int (|\theta - \bar{\theta}|^4 + |\nabla v|^2 + |\nabla v|^4 + |\nabla\theta|^{8/3}) dx \, ds, \ 0 \leq t \leq T .
\end{aligned}
$$

(3.17)

Utilising (2.1), (3.16), (1.1), (1.14), and (1.17), (2.7), (3.10)-(3.11), we deduce

$$
\begin{aligned}
R \int_0^t \int (\rho\theta - \bar{\rho}\bar{\theta}) \operatorname{div} \dot{v} \, dx \, ds & \leq R \int_0^t \int (\rho\theta - \bar{\rho}\bar{\theta})\frac{d}{dt} \operatorname{div} v \, dx \, ds \\
& + C \int_0^t \int (|\theta| + 1)|\nabla v|^2 dx \, ds \\
& \leq R \int (\rho\theta - \bar{\rho}\bar{\theta}) \operatorname{div} v \, dx \Big|_0^t - R \int_0^t \int \rho\dot{\theta} \operatorname{div} v \, dx \, ds \\
& + C \int_0^t \int (|\theta - \bar{\theta}| + 1)|\nabla v|^2 dx \, ds \\
& \leq C(e_0^2 + \|(\rho - \bar{\rho}, \theta - \bar{\theta})(t)\|^2) + \frac{\mu}{4}\|\nabla v(t)\|^2 + \frac{c_V\bar{\rho}}{8} \int_0^t \|\dot{\theta}\|^2 ds \\
& + C \int_0^t \int (|\theta - \bar{\theta}|^4 + |\nabla v|^2 + |\nabla v|^4) dx \, ds \\
& \leq C(e_0^2 + A^{4/3} + A^3) + \frac{\mu}{4}\|\nabla v(t)\|^2 + \frac{c_V\bar{\rho}}{8} \int_0^t \|\dot{\theta}\|^2 ds
\end{aligned}
$$

(3.18)

and

$$
\begin{aligned}
\int_0^t \int |\nabla\theta|^{8/3} dx \, ds & \leq \int_0^t \int (|\nabla\theta|^2 + |\nabla\theta|^{2(2+n)/n}) dx \, ds \\
& \leq C(e_0^2 + A^{4/3} + A^3) + C \int_0^t \|\nabla\theta\|^{4/n} \|\Delta\theta\|^2 ds \\
& \leq C(e_0^2 + A^{4/3} + A^3 + A^{1+2/n}) \leq C(e_0^2 + A^{4/3} + A^3) .
\end{aligned}
$$

(3.19)

Substituting (3.18) and (3.19) into (3.17), and applying (2.7), (3.10)-(3.11), one gets

(3.20) $\quad \displaystyle\sup_{0 \leq t \leq T} (\|\nabla v\|^2 + \|\nabla\theta\|^2)(t) + \int_0^T (\|\dot{v}\|^2 + \|\dot{\theta}\|^2) ds \leq c(e_0^2 + A^{4/3}(T) + A^3(T)) .$

It follows easily from equation (1.3), (2.1), (3.20), (3.10)-(3.11), and (2.7) that

$$(3.21) \qquad \int_0^T \|\Delta\theta\|^2 ds \le \int_0^T \int \left\{ |\dot\theta|^2 + |\nabla v|^2 + |\theta - \bar\theta|^4 + |\nabla v|^4 \right\} dx\, ds$$
$$\le C(e_0^2 + A^{4/3}(T) + A^3(T)).$$

In order to bound $\int_0^T \|\nabla\omega\|^2 ds$, we note that integration by parts gives the following identity:

$$\int \{ \rho(\partial_j v_l \partial_l v_k - \partial_k v_l \partial_l v_j) + \dot v_k \partial_j \rho - \dot v_j \partial_k \rho \} f\, dx$$
$$= \int (\partial_j v_l \partial_l v_k - \partial_k v_l \partial_l v_j - \partial_j \dot v_k + \partial_k \dot v_j)\rho f\, dx + \int \rho(\dot v_j \partial_k f - \dot v_k \partial_j f) dx$$
$$= \int \rho f \frac{d}{dt}\omega^{j,k} + \int \rho(\dot v_j \partial_k f - \dot v_k \partial_j f).$$

Therefore, multiplying (3.3) by $\omega^{j,k}$, integrating, and summing over j, k, we utilise Cauchy-Schwarz's inequality, (2.1) and (3.20) to arrive at

$$(3.22) \qquad \int_0^T \|\nabla\omega(s)\|^2 ds \le C \int_0^T \|\dot v\|^2 ds \le C(e_0^2 + A^{4/3}(T) + A^3(T)).$$

In the following lemma we derive estimate for $\phi^{3-n}\|\dot v\|^2$ and $\int_0^T \phi^{3-n}\|\nabla\dot v\|^2 ds$ appearing in the definition of A.

LEMMA 3.3.

$$(3.23) \quad \sup_{0\le t\le T} \phi^{3-n}(t)\|\dot v(t)\|^2 + \int_0^T \phi^{3-n}(s)\|\nabla\dot v(s)\|^2 ds \le C(e_0^2 + A^{4/3}(T) + A^3(T)).$$

PROOF. We apply the operator $\frac{d}{dt} = \partial_t + v \cdot \nabla$ to (1.2) and use (1.1) to see that

$$(3.24) \quad \begin{aligned} \rho\ddot v_j &= \mu\{\Delta\partial_t v_j + \mathrm{div}(v\Delta v_j)\} + (\lambda+\mu)\{\partial_j\,\mathrm{div}\,\partial_t v + \mathrm{div}(v\partial_j\,\mathrm{div}\,v)\} \\ &\quad - R\{\partial_j\partial_t(\rho\theta) + \mathrm{div}(v\partial_j(\rho\theta))\} \equiv I_j^1 + I_j^2 + I_j^3, \quad j = 1, \cdots, n. \end{aligned}$$

It is easy to see that by integration by parts,

$$\int I_j^1 \dot v_j dx = -\mu \int (\nabla\partial_t v_j \cdot \nabla\dot v_j + v_k\Delta v_j\partial_k\dot v_j)dx$$
$$= -\mu \int \{|\nabla\dot v|^2 - (\partial_k v_j\nabla v_k \cdot \nabla\dot v_j + v_k\partial_k\nabla v_j \cdot \nabla\dot v_j)$$
$$(3.25) \qquad - (\nabla v_k \cdot \nabla v_j\partial_k\dot v_j + v_k\nabla v_j \cdot \partial_k\nabla\dot v_j)\}dx$$
$$= -\mu \int \{|\nabla\dot v|^2 - \partial_k v_j\nabla v_k \cdot \nabla\dot v_j + \partial_k v_k\nabla v_j \cdot \nabla\dot v_j$$
$$- \nabla v_k \cdot \nabla v_j\partial_k\dot v_j\}dx.$$

In the same manner we have

$$(3.26) \quad \int I_j^2 \dot{v}_j dx = -(\lambda + \mu) \int \{|\operatorname{div} \dot{v}|^2 - \partial_k v_j \partial_j v_k \operatorname{div} \dot{v}$$

$$+ (\operatorname{div} v)^2 \operatorname{div} \dot{v} - \partial_j v_k \partial_k \dot{v}_j \operatorname{div} v\} dx$$

and

$$(3.27) \quad \int I_j^3 \dot{v}_j dx = -R \int \left\{ \frac{d}{dt}(\rho\theta) \operatorname{div} \dot{v} + \rho\theta(\operatorname{div} v \operatorname{div} \dot{v} - \partial_j v_k \partial_k \dot{v}_j) \right\} dx.$$

Now multiply (3.24) by $\phi^{3-n} \dot{v}_j$ in $L^2(\mathbb{R}^n \times (0, t))$ and sum over j. If we make use of (3.25)-(3.27), (3.4), Cauchy-Schwarz's inequality, and (2.1), (2.7), (3.10)-(3.11), and (3.20), we obtain

$$\phi^{3-n}(t)\|\dot{v}(t)\|^2 + \int_0^t \phi^{3-n}\|\nabla\dot{v}\|^2 ds$$

$$\leq Ce_0^2 + C \int_0^t \int \{|\nabla v|^4 + |\dot{v}|^2 + |\dot{\theta}|^2 + \theta^2 |\nabla v|^2\} dx \, ds$$

$$\leq Ce_0^2 + C \int_0^t \int \{|\nabla v|^4 + |\dot{v}|^2 + |\dot{\theta}|^2 + |\nabla v|^2 + |\theta - \bar{\theta}|^4\} dx \, ds$$

$$\leq C(e_0^2 + A^{4/3}(T) + A^3(T)) \qquad \forall\, t \in [0, T],$$

which yields (3.23). The proof is complete. $\qquad \square$

Next we bound $\phi(t)\|\dot{\theta}(t)\|^2$. Applying the operator $\frac{d}{dt} = \partial_t + v \cdot \nabla$ to (1.3) and using (1.1), we find that

$$c_V \rho \ddot{\theta} = \kappa\{\Delta \partial_t \theta + \operatorname{div}(v\Delta\theta)\} + (-R\rho\theta \operatorname{div} v$$

$$+ \lambda(\operatorname{div} v)^2 + 2\mu D \cdot D) \operatorname{div} v$$

$$(3.28) \qquad + R\rho\{\theta(\operatorname{div} v)^2 - \dot{\theta} \operatorname{div} v - \theta \operatorname{div} \dot{v} + \theta \partial_k v_l \partial_l v_k\}$$

$$+ 2\lambda(\operatorname{div} \dot{v} - \partial_k v_l \partial_l v_k) \operatorname{div} v$$

$$+ 2\mu D_{ij}(\partial_i \dot{v}_j + \partial_j \dot{v}_i - \partial_i v_k \partial_k v_j - \partial_j v_k \partial_k v_i).$$

We multiply (3.28) by $\phi\dot{\theta}$ and integrate over $\mathbb{R}^n \times (0, t)$. Applying an argument similar to the one used in (3.25) for the first term on the right-hand side of (3.28), using (3.4), (2.1), we infer

$$\phi(t)\|\dot{\theta}(t)\|^2 + \int_0^t \phi\|\nabla\dot{\theta}\|^2 ds \leq C \int_0^t \|\dot{\theta}\|^2 ds$$

$$+ C \int_0^t \int \phi\{(|\theta| |\nabla v|^2 + |\nabla v|^3 + |\dot{\theta}| |\nabla v|$$

$$(3.29) \qquad + |\theta| |\nabla\dot{v}| + |\nabla v| |\nabla\dot{v}|)|\dot{\theta}| + |\nabla v|^2 |\nabla\theta|^2\} dx \, ds$$

$$\leq C \int_0^t (\|\dot{\theta}\|^2 + \|\nabla v\|_{L^4}^4 + \phi\|\nabla\dot{v}\|^2) ds$$

$$+ C \int_0^t \int \phi\{(|\nabla v|^2 + (\theta - \bar{\theta})^2)|\dot{\theta}|^2 + |\nabla\theta|^4\} dx \, ds$$

for $t \in [0, T]$. Note that by virtue of (1.15), (1.17), and Lemma 3.1,

$$\int_0^t \int |\omega|^{n+2} dx\, ds \leq \begin{cases} C \int_0^t \|\omega\|^2 \|\nabla\omega\|^2 ds, & \text{if } n = 2 \\[4mm] \int_0^t (\|\omega\|_{L^4}^4 + \|\omega\|_{L^6}^6) ds, & \text{if } n = 3 \end{cases} \leq C(e_0^2 + A^{4/3} + A^3);$$

hence, it follows from Lemma 2.2, (2.1), (1.15), and Lemma 2.3 that

$$\begin{aligned} (3.30) \quad & \int_0^t \int (|\nabla v|^{n+2} + |\theta - \bar\theta|^{n+2}) dx\, ds \\ & \leq C \int_0^t (\|(F, \theta - \bar\theta, \omega)\|_{L^{n+2}}^{n+2} + \|\rho - \bar\rho\|_{L^4}^4) ds \\ & \leq C \int_0^t (\|F\| + \|\theta - \bar\theta\|)^{(2n-n^2+4)/2} (\|\dot v\| + \|\nabla\theta\|)^{n^2/2} ds \\ & \quad + C(e_0^2 + A^{4/3} + A^3) \leq C(e_0^2 + A^{4/3} + A^3) \\ & \quad + C A^{(2n-n^2+4)/4} \int_0^t (\|\dot v\|^2 + \|\nabla\theta\|^2)^{n^2/4} ds \\ & \leq C(e_0^2 + A^{4/3} + A^3). \end{aligned}$$

Thus, the second integral term on the right-hand side of (3.29) can be bounded as follows, using (1.14)-(1.15) and (3.30).

$$\begin{aligned} (3.31) \quad & \int_0^t \int \phi\{(|\nabla v|^2 + (\theta - \bar\theta)^2)|\dot\theta|^2 + |\nabla\theta|^4\} dx\, ds \\ & \leq C \int_0^t \int (|\nabla v|^{n+2} + |\theta - \bar\theta|^{n+2}) dx\, ds \\ & \quad + C \int_0^t \phi^{1+2/n} \int |\dot\theta|^{2+4/n} dx\, ds + C \int_0^t \phi \|\nabla\theta\|^{4-n} \|\Delta\theta\|^n ds \\ & \leq C(e_0^2 + A^{4/3} + A^3) + C \int_0^t \phi^{1+2/n} \|\dot\theta\|^{4/n} \|\nabla\dot\theta\|^2 ds \\ & \quad + C A \int_0^t \|\Delta\theta\|^2 ds \leq C(e_0^2 + A^{4/3} + A^3). \end{aligned}$$

Inserting (3.31) into (3.29), employing (3.20), (3.10), and Lemma 3.3, we obtain

$$(3.32) \quad \sup_{0 \leq t \leq T} \phi(t) \|\dot\theta(t)\|^2 + \int_0^T \phi \|\nabla\dot\theta\|^2 ds \leq C(e_0^2 + A^{4/3}(T) + A^3(T)).$$

Finally, we estimate $\phi\|\Delta\theta\|$ to close the estimates for A. From (1.3), (2.1), (3.32), (3.20), and Lemma 2.2, (1.15), (3.2), and Lemma 2.3 we get

$$
\begin{aligned}
\phi^2(t)\|\Delta\theta(t)\|^2 &\le C(\phi^2\|\dot\theta\|^2 + \|\nabla v\|^2 + \phi^2\|\nabla v\|_{L^4}^4)(t) \\
&\le C(e_0^2 + A^{4/3} + A^3) + C\phi^2(\|F\|_{L^4}^4 + \|\theta - \bar\theta\|_{L^4}^4) \\
&\quad + C(\phi^{3-n}\|\omega\|_{L^4}^4 + \|\rho - \bar\rho\|_{L^4}^4) \\
&\le C(e_0^2 + A^{4/3} + A^3) + C\phi^2(\|F\|^{4-n}\|\dot v\|^n \\
&\quad + \|\theta - \bar\theta\|^{4-n}\|\nabla\theta\|^n) \le C(e_0^2 + A^{4/3} + A^3) \quad \forall\, t \in [0, T] .
\end{aligned}
$$
(3.33)

Recalling the definition of $A(T)$, we combine the estimates (3.2), (3.10)-(3.11), (3.20)-(3.23), and (3.32)-(3.33) to obtain

THEOREM 3.4. *We have*

$$
A(T) \le C(e_0^2 + A^{4/3}(T) + A^3(T)) .
$$

Now we are able to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Note that $A(0) < 2Ce_0^2$ mit C being the same constant as in Theorem 3.4. Thus it follows from Theorem 3.4 that

$$
A(T) < 2Ce_0^2
$$
(3.34)

provided $e_0 \le \min\{1, 1/(2^6 C^5)\}$. By virtue of Lemma 2.4, Sobolev's imbedding theorem, and (3.34), we have

$$
\begin{aligned}
\sup_{0 \le t \le T} (\|\rho - \bar\rho\|_{L^\infty} + \phi\|\theta - \bar\theta\|_{L^\infty})(t) &\le C(e_0 + A^{1/2}(T)) \\
&\quad + C \sup_{0 \le t \le T} \phi\|(\theta - \bar\theta, \nabla\theta, \Delta\theta)(t)\| \\
&\le C(e_0 + A^{1/2}(T)) \le \tilde C e_0 \le \min\{\bar u, \bar\theta\}/3
\end{aligned}
$$
(3.35)

provided $e_0 \le \min\{1, 1/(2^6 C^5), \bar u/(3\tilde C), \bar\theta/(3\tilde C)\} =: \epsilon$.

For the initial data satisfying $e_0 \le \epsilon$ we have proved that under (2.1) the estimate (3.35) holds. Since (3.35) is valid for $t = 0$, by virtue of the continuity of ρ and θ, (3.35) remains valid (for all $T \ge 0$). Hence (3.34) holds (for all $T \ge 0$). Thus we have shown that if $e_0 \le \epsilon$, then we have the estimates (3.34)-(3.35) (for all $T \ge 0$). As a result of (3.34)-(3.35), Sobolev's imbedding theorem, (1.15), and (2.3), we have (cf. the derivation of (3.33))

$$
\begin{aligned}
\phi^2(t)(\|\nabla v\|_{L^4}^4 + \|v\|_{L^\infty}^4)(t) &\le C\phi^2(\|v\|_{L^4}^4 + \|\nabla v\|_{L^4}^4) \\
&\le C\phi^2(\|v\|^{4-n}\|\nabla v\|^n + \|\nabla v\|_{L^4}^4) \\
&\le C(e_0^2 + A^{4/3} + A^3) \le Ce_0^2 \quad \forall\, t \in [0, T],
\end{aligned}
$$
(3.36)

and for $t \geq 1$,

$$
(3.37) \quad
\begin{aligned}
\|(F, \theta - \bar{\theta})(t)\|_{L^\infty}^{2+4/n} &\leq C\|(F, \dot{v}, \theta - \bar{\theta}, \nabla\theta)\|_{L^{2+4/n}}^{2+4/n} \\
&\leq C\|(\dot{v}, \nabla\dot{v}, \nabla\theta, \Delta\theta)(t)\|^2 .
\end{aligned}
$$

To complete the proof of Theorem 1.1 it remains to show the asymptotic behavior as $T \to \infty$. To this end we multiply (2.5) by $e^{\gamma t}\mathrm{sgn}(\rho - \bar{\rho})(\rho - \bar{\rho})^{1+4/n}$ with $\gamma = R\bar{\theta}\bar{\rho}/(2\lambda + 4\mu)$ and integrate over $[1, T]$, utilise (3.34)-(3.35), and (3.37) to arrive at

$$
\|\rho(T) - \bar{\rho}\|_{L^\infty}^{2+4/n} \leq Ce^{-\gamma T} + C\int_1^T e^{-\gamma(T-t)}\|(F, \theta - \bar{\theta})(t)\|_{L^\infty}^{2+4/n}dt
$$

$$
\leq Ce^{-\gamma T} + Ce^{-\gamma T/2} + C\int_{T/2}^T \|(\dot{v}, \nabla\dot{v}, \nabla\theta, \Delta\theta)\|^2 dt \to 0 \quad \text{as } T \to \infty .
$$

It is easy to see by (3.34)-(3.35) that $\|(\nabla v, \nabla\theta)\|^2$ and $\frac{d}{dt}\|(\nabla v, \nabla\theta)\|^2$ are in $L^1(1, \infty)$. Therefore, $\|\nabla v(T)\|$ and $\|\nabla\theta(T)\|$ tend to zero as $T \to \infty$. Since $\|(v, \theta - \bar{\theta})(t)\|$, $\|(v, \theta - \bar{\theta})(t)\|_{L^\infty}$ are uniformly bounded for all $t \geq 1$, we see by (1.15) that for any $q > 2$, $\|(v, \bar{\theta})(T)\|_{L^q} \to 0$ as $T \to \infty$.   □

We end this section by giving a lower bound for $\theta(x, t)$ in the case of $\theta > 0$. In view of $\lambda(\mathrm{div}\,v)^2 + 2\mu D \cdot D \geq (\lambda + 2\mu/n)(\mathrm{div}\,v)^2$, we may write (1.3) in the form

$$
c_V\rho\dot{\theta} \geq \kappa\Delta\theta + (\lambda + 2\mu/n)\left[\mathrm{div}\,v - \frac{R\rho\theta}{2(\lambda + 2\mu/n)}\right]^2 - \frac{R^2\rho^2\theta^2}{4(\lambda + 2\mu/n)}
$$

$$
\geq \kappa\Delta\theta - nR^2\rho^2\theta^2/(4n\lambda + 8\mu) .
$$

If we multiply the above inequality by $-l\theta^{-l-1}$ ($l \geq 1$ integer) in $L^2(\mathbb{R}^n \times (0, t))$ ($0 \leq t \leq 1$), integrate by parts, and use (3.4), (3.35), we find that

$$
(3.38) \quad \|\theta(t)^{-1}\|_{L^l}^l \leq C\|(\theta_0)^{-1}\|_{L^l}^l + Cl\int_0^1 (1 + \|\theta - \bar{\theta}\|_{L^\infty})\|\theta^{-1}\|_{L^l}^l ds, \quad t \in [0, 1] .
$$

Applying Gronwall's inequality to (3.38), and utilising (2.9), (3.34), one concludes

$$
\|\theta(t)^{-1}\|_{L^l}^l \leq Ce^{Cl}\|(\theta_0)^{-1}\|_{L^l}^l \quad \text{for any} \quad t \in [0, 1] ,
$$

which, by taking the $1/l$-th power and then passing to the limit as $l \to \infty$, yields $\sup_{0 \leq t \leq 1} \|\theta(t)^{-1}\|_{L^\infty} \leq C\min_{\mathbb{R}^n}\theta^0$. This together with (3.35) gives

$$
(3.39) \quad \theta(x, t) \geq \theta^* \quad \text{for all} \quad x \in \mathbb{R}^n, \ t \in [0, T] ,
$$

where θ^* is a positive constant which depends only on the same quantities as the constant C in the introduction and $\min_{\mathbb{R}^n}\theta^0$.

4. – Proof of Theorem 1.2

In this section we prove Theorem 1.2 using the a priori estimeates (3.34)-(3.35). Throughout this section we assume that $e_0 \leq \epsilon$ with ϵ being the same as in (3.35), such that (3.34)-(3.35) hold. We start with the following definition:

$$
\begin{aligned}
(4.1) \quad B(T) := & \sup_{0 \leq t \leq T} \left(\|\nabla\rho\|^2 + \|\nabla\rho\|_{L^\alpha}^2 \right)(t) + \int_0^{\min\{1,T\}} \|\theta - \bar\theta\|_{L^\infty}^2 ds \\
& + \left[\int_0^{\min\{1,T\}} \left(\|\nabla\rho\| + \|\nabla\rho\|_{L^\alpha} + \|\nabla v\|_{L^\infty} \right) ds \right]^2 \\
& + \int_1^{\max\{1,T\}} \left(\|\nabla\rho\|^2 + \|\nabla\rho\|_{L^\alpha}^2 \right) ds .
\end{aligned}
$$

It is easy to see that by (3.34), (1.15), and Hölder's inequality,

$$
\begin{aligned}
(4.2) \quad & \int_0^{\min\{1,T\}} \|\dot v\|_{L^\alpha} ds + \left\{ \int_1^{\max\{1,T\}} \left(\|\dot v\|_{L^\alpha}^2 + \|\nabla\theta\|_{L^\alpha}^2 \right) ds \right\}^{1/2} \\
& \leq C \int_0^{\min\{1,T\}} \|\dot v\|^{n/\alpha - n/2 + 1} \left(\phi \|\nabla\dot v\|^2 \right)^{n/4 - n/(2\alpha)} \phi^{n/(2\alpha) - n/4} ds \\
& \quad + C \left\{ \int_1^{\max\{1,T\}} \|(\dot v, \nabla\dot v, \nabla\theta, \Delta\theta)\|^2 ds \right\}^{1/2} \\
& \leq C \left(\int_0^T \|\dot v\|^2 \right)^{n/(2\alpha) - (n-2)/4} \left(\int_0^T \phi\|\nabla\dot v\|^2 \right)^{n/4 - n/(2\alpha)} \\
& \quad \times \left(\int_0^{\min\{1,T\}} \phi(s)^{n/\alpha - n/2} ds \right)^{1/2} \\
& \quad + C A^{1/2}(T) \leq C A^{1/2}(T) \leq c e_0 \leq C E_0 .
\end{aligned}
$$

Differentiating (2.5) with respect to x_j, one gets

$$
\begin{aligned}
(4.3) \quad (2\mu + \lambda) \frac{d}{dt} \partial_j \rho + R\bar\theta \rho \partial_j \rho = & -R\bar\theta(\rho - \bar\rho)\partial_j\rho - (2\mu + \lambda)\partial_j v_k \partial_k \rho \\
& - F\partial_j\rho - \rho\partial_j F - 2R\rho(\theta - \bar\theta)\partial_j\rho - R\rho^2\partial_j\theta .
\end{aligned}
$$

Let $p \geq 2$ and $m \in \mathbb{N}$. Recalling the definition of F, we multiply (4.3) by $\operatorname{sgn}(\partial_j\rho)\rho|\partial_j\rho|^{p-1}\|\rho^{1/p}\partial_j\rho\|_{L^p}^{m-p}$, integrate over Ω with respect to x, and use (3.35), (2.3), Hölder's inequality, and (1.14) to arrive at

$$
\begin{aligned}
(4.4) \quad & \frac{(2\mu + \lambda)}{m} \frac{d}{dt} \|\rho^{1/p}\partial_j\rho\|_{L^p}^m + \frac{R\bar\theta\bar\rho}{4} \|\rho^{1/p}\partial_j\rho\|_{L^p}^m \\
& \leq C\|(\rho - \bar\rho, \nabla v, F, \theta - \bar\theta)\|_{L^\infty} \|\partial_j\rho\|_{L^p}^m + C(\|\partial_j F\|_{L^p}^m + \|\partial_j\theta\|_{L^p}^m) \\
& \leq C\|(\rho - \bar\rho, \nabla v, \theta - \bar\theta)\|_{L^\infty} \|\partial_j\rho\|_{L^p}^m + C(\|\dot v\|_{L^p}^m + \|\partial_j\theta\|_{L^p}^m) .
\end{aligned}
$$

We first consider the case $T \leq 1$. Take $m = 1$, $p = 2$ respectively $p = \alpha$ in (4.4), then integrate with respect to t over $(0, t)$ $(0 \leq t \leq T \leq 1)$ and sum over. Recalling the definition of $B(T)$, we employ (1.15), (4.2), and Cauchy-Schwarz's inequality, (3.34)-(3.35), Hölder's inequality to obtain

$$
\sup_{0 \leq t \leq T} (\|\nabla \rho\| + \|\nabla \rho\|_{L^\alpha})(t) + \int_0^T (\|\nabla \rho\| + \|\nabla \rho\|_{L^\alpha}) ds \leq C(E_0 + E_0 B^{1/2})
$$

$$
(4.5) \qquad + C B^{1/2} \int_0^T \|(\nabla v, \theta - \bar{\theta})\|_{L^\infty} ds + C \int_0^T (\|(\dot{v}, \nabla \theta)\| + \|(\dot{v}, \nabla \theta)\|_{L^\alpha}) \, ds
$$

$$
\leq C(E_0 + B) + C \int_0^T \|(\nabla \theta, \Delta \theta)\| ds \leq C(E_0 + B(T)), \quad 0 \leq t \leq T \leq 1 .
$$

When $T \geq 1$, we note that by Sobolev's imbedding theorem, (1.15), equation (1.2), and the elliptic regularity for the operator $\mu \Delta + (\mu + \lambda) \nabla \operatorname{div}$, and (3.34)-(3.35),

$$
\|\nabla v(t)\|_{L^\infty} \leq C(\|\nabla v\|_{L^\alpha} + \|\nabla^2 v\|_{L^\alpha})(t)
$$

$$
\leq C(\|\nabla v\| + \|\nabla^2 v\| + \|\nabla^2 v\|_{L^\alpha})(t)
$$

$$
(4.6) \qquad \leq C E_0 + C(\|\dot{v}\| + \|\nabla(\rho \theta)\| + \|\dot{v}\|_{L^\alpha} + \|\nabla(\rho \theta)\|_{L^\alpha})(t)
$$

$$
\leq C E_0 + C\{\|(\dot{v}, \nabla \theta, \Delta \theta, \nabla \dot{v}, \nabla \rho)\| + \|\nabla \rho\|_{L^\alpha}\}(t)
$$

$$
\leq C(E_0 + B^{1/2}(T) + \|\nabla \dot{v}(t)\|) \quad \forall t \in [1, T] .
$$

Recalling the definition of $B(T)$, we take $m = 2$, $p = 2$ respectively $p = \alpha$ in (4.4), integarte then over $(1, t)$ $(1 \leq t \leq T)$, sum over, and use (4.5)-(4.6), (4.2), (3.34)-(3.35) to obtain

$$
\sup_{1 \leq t \leq T} (\|\nabla \rho\|^2 + \|\nabla \rho\|_{L^\alpha}^2)(t) + \int_1^T (\|\nabla \rho\|^2 + \|\nabla \rho\|_{L^\alpha}^2) ds \leq C(\|\nabla \rho\|^2 + \|\nabla \rho\|_{L^\alpha}^2)(1)
$$

$$
+ C \int_1^T \{(E_0 + B^{1/2} + \|\nabla \dot{v}\|)(\|\nabla \rho\|^2 + \|\nabla \rho\|_{L^\alpha}^2)
$$

$$
+ \|(\nabla \dot{v}, \nabla \theta)\|^2 + \|(\dot{v}, \nabla \theta)\|_{L^\alpha}^2\} ds
$$

$$
\leq C(E_0^2 + B^{3/2} + B^2 + E_0 B) \leq C(E_0^2 + B^{3/2}(T) + B^2(T)),
$$

which together with (4.5) yields

$$
\sup_{0 \leq t \leq T} (\|\nabla \rho\|^2 + \|\nabla \rho\|_{L^\alpha}^2)(t) + \left\{ \int_0^{\min\{1, T\}} (\|\nabla \rho\| + \|\nabla \rho\|_{L^\alpha}) ds \right\}^2
$$

$$
(4.7) \qquad + \int_1^{\max\{1, T\}} (\|\nabla \rho\|^2 + \|\nabla \rho\|_{L^\alpha}^2) ds
$$

$$
\leq C(E_0^2 + B^{3/2}(T) + B^2(T)), \quad T \geq 0 .
$$

It follows from Sobolev's imbedding theorem and (3.34) that

$$(4.8) \quad \int_0^{\min\{1,T\}} \|\theta - \bar\theta\|_{L^\infty}^2 ds \le C \int_0^{\min\{1,T\}} (\|\theta - \bar\theta\|^2 + \|\nabla\theta\|^2 + \|\Delta\theta\|^2) ds \le C E_0^2 .$$

By arguments similar to those used for (4.6) and (4.5) we have

$$
\begin{aligned}
\int_0^{\min\{1,T\}} \|\nabla v\|_{L^\infty} ds &\le C \int_0^{\min\{1,T\}} \|(\nabla v, \nabla^2 v)\|_{L^\alpha} ds \\
&\le C \int_0^{\min\{1,T\}} (\|(\nabla v, \Delta v)\| + \|\nabla^2 v\|_{L^\alpha}) ds \\
(4.9) \qquad &\le C e_0 + C \int_0^{\min\{1,T\}} (\|(\dot v, \nabla(\rho\theta), v)\| + \|(\dot v, \nabla(\rho\theta), v)\|_{L^\alpha}) ds \\
&\le C E_0 + C \int_0^{\min\{1,T\}} (1 + \|\theta - \bar\theta\|_{L^\infty})(\|\nabla\rho\| + \|\nabla\rho\|_{L^\alpha}) ds \\
&\le C(E_0 + B^{3/4}(T) + B(T)) ,
\end{aligned}
$$

where we have also used (4.7).

Combining (4.7)-(4.9) and (3.35), we obtain $B(T) \le C(E_0^2 + B^{3/2}(T) + B^2(T))$, which by taking into account $B(0) < 2C E_0^2$ gives

LEMMA 4.1.

$$(4.10) \qquad\qquad\qquad B(T) < 2C E_0^2$$

provided $E_0 \le \min\{1/(8C^2), \epsilon\} =: \epsilon_1$ *with* ϵ *being the same as in* (3.35).

From integration by parts, (1.3), and (3.35), it follows easily that $\phi^2(t)\|\Delta v(t)\|^2 \le \mu^{-2}\phi^2\|\mu\Delta v + (\mu + \lambda)\nabla \operatorname{div} v\|^2 \le c(\phi^2\|\dot v\|^2 + \|\nabla\rho\|^2 + \|\nabla\theta\|^2)$. Hence by (3.34) and (4.10),

$$(4.11) \qquad \sup_{0 \le t \le T} \phi^2(t)\|\Delta v(t)\|^2 + \int_0^T \phi^2\|\Delta v\|^2 ds \le C E_0^2 \le C .$$

In the sequel we estimate higher order derivatives of ρ, v, θ. Let $0 < b < \min\{1, T\}$ be an arbitary but fixed number. *In the calculations that follow we denote by* Λ *a generic constant which only depends on the same quantities as* C, b *and* $\|(\rho - \bar\rho, v, \theta - \bar\theta)(\cdot, b)\|_{H^3}$.

Using (1.15), (3.34)-(3.35), (4.10)-(4.11), we argue similarly to (4.6) to obtain

$$
\begin{aligned}
\int_b^T \|\nabla v\|_{L^\infty}^{4\alpha/(\alpha n - 2n)} ds &\le C \int_b^T (\|\nabla v\|_{L^\alpha} + \|\nabla^2 v\|_{L^\alpha})^{4\alpha/(\alpha n - 2n)} ds \\
&\le C \int_b^T \{\|\nabla v\|^{4\alpha/(\alpha n - 2n)-2}\|\Delta v\|^2 ds + \|(\dot v, v, \nabla(\rho\theta))\|_{L^\alpha}^{4\alpha/(\alpha n - 2n)}\} ds \\
(4.12) \qquad &\le \Lambda + \Lambda \int_b^T \{\|(\dot v, v, \nabla\theta)\|^{4\alpha/(\alpha n - 2n)-2}\|(\nabla\dot v, \nabla v, \Delta\theta)\|^2 \\
&\qquad + \|\nabla\rho\|_{L^\alpha}^{4\alpha/(\alpha n - 2n)}\} ds \le \Lambda .
\end{aligned}
$$

In the same manner we have

$$\int_b^T \|(\theta - \bar{\theta}, \rho - \bar{\rho})\|_{L^\infty}^{4\alpha/(\alpha n - 2n)} ds \leq C$$

(4.13)

$$\times \int_b^T \|(\theta - \bar{\theta}, \nabla\theta, \rho - \bar{\rho}, \nabla\rho)\|_{L^\alpha}^{4\alpha/(\alpha n - 2n)} ds \leq \Lambda .$$

Recalling the definition of I_j^1 in (3.24), we integrate by parts to infer

$$\int I_j^1 \ddot{v}_j dx = \mu \int \left(\frac{d}{dt} \Delta v_j + \text{div } v \, \Delta v_j \right) \ddot{v}_j dx \leq -\frac{\mu}{2} \frac{d}{dt} \|\nabla \dot{v}_j\|^2$$

(4.14)

$$+ C\|\nabla v\|_{L^\infty}(\|\Delta v\| \, \|\ddot{v}\| + \|\nabla\dot{v}\|^2) \leq -\frac{\mu}{2} \frac{d}{dt} \|\nabla\dot{v}_j\|^2$$

$$+ C(1 + \|\nabla v\|_{L^\infty}^{2\alpha/(\alpha n - 2n)})(\|\Delta v\| \, \|\ddot{v}\| + \|\nabla\dot{v}\|^2), \quad t \in [b, T].$$

In the same way we obtain

$$\int I_j^2 \ddot{v}_j dx = -\frac{\lambda + \mu}{2} \frac{d}{dt} \| \text{div } \dot{v}\|^2$$

(4.15)

$$+ C(1 + \|\nabla v\|_{L^\infty}^{2\alpha/(\alpha n - 2n)})(\|\Delta v\| \, \|\ddot{v}\| + \|\nabla\dot{v}\|^2), \quad t \in [b, T],$$

and

$$\|I_j^3(t)\|^2 = R^2 \|\frac{d}{dt}(\partial_j(\rho\theta)) + \text{div } v \, \partial_j(\rho\theta)\|^2$$

$$\leq \Lambda(1 + \|\nabla v\|_{L^\infty}^{4\alpha/(\alpha n - 2n)})\|(\nabla\rho, \nabla\theta)\|^2 + \Lambda\|\nabla\rho\|_{L^\alpha}^2 \|\dot{\theta}\|_{L^{\frac{2\alpha}{\alpha - 2}}}^2$$

(4.16)

$$+ \Lambda\|(\nabla\dot{\theta}, \Delta v)\|^2$$

$$\leq \Lambda(1 + \|\nabla v\|_{L^\infty}^{4\alpha/(\alpha n - 2n)})\|(\nabla\rho, \nabla\theta, \Delta v)\|^2$$

$$+ \Lambda\|(\nabla\dot{\theta}, \dot{\theta})\|^2, \quad t \in [b, T],$$

where we have also used (3.35), (4.10), equation (1.1), Hölder's inequality and (1.15) in (4.16).

If we now multiply (3.24) by $\ddot{v}_j$, and integrate over $\mathbb{R}^n \times (b, t)$ ($b \leq t \leq T$), make use of (4.14)-(4.16), and (3.34)-(3.35), (4.10)-(4.11), we obtain

(4.17) $\|\nabla\dot{v}(t)\|^2 + \int_b^t \|\ddot{v}\|^2 ds \leq \Lambda + \Lambda \int_b^t \|\nabla v\|_{L^\infty}^{4\alpha/(\alpha n - 2n)} \|\nabla\dot{v}\|^2 ds, \quad t \in [b, T].$

It should be pointed out here that the derivation of (4.17) is formal because of the lack of regularity in some steps. However, the regirous derivation can be achived by using difference quotient (with respect to t) and taking to the limit (cf. [25, pp. 145-163]), or by using mollifiers (cf. [18], [13]).

Applying Gronwall's inequality to (4.17) and using (4.12), we get

LEMMA 4.2.

$$\sup_{b\le t\le T}\|\nabla\dot v(t)\|^2+\int_b^T\|\ddot v(s)\|^2ds\le\Lambda\,.$$

We apply $\frac{d}{dt}$ to (1.2), utilise the elliptic regularity, (3.34)-(3.35), and (4.10)-(4.12) to deduce (cf. (4.16))

$$\begin{aligned}
(4.18)\quad \int_b^T\|\Delta\dot v\|^2ds\le{}&\Lambda\int_b^T\{\|(\ddot v,\nabla\dot\theta)\|^2+\|\nabla\rho\|_{L^\alpha}^2\|\dot\theta\|_{L^{\frac{2\alpha}{\alpha-2}}}^2\\
&+(1+\|\nabla v\|_{L^\infty}^{4\alpha/(\alpha n-2n)})\|(\dot v,\nabla\rho,\nabla\theta,\nabla v,\Delta v)\|^2\}ds\le\Lambda\,.
\end{aligned}$$

Differentiate (4.3) with respect to x_j, sum over j, and multiply by $\rho\Delta\rho$ in $L^2(\mathbb{R}^n\times(b,T))$. If we use (1.12), Hölder's inequality, and (3.34)-(3.35), (1.15), (4.10)-(4.11), we conclude

$$\begin{aligned}
(4.19)\quad \|\Delta\rho(t)\|^2+\int_b^t\|\Delta\rho\|^2ds\le{}&\Lambda+\Lambda\int_b^t\{\|(\nabla v,\rho-\bar\rho,\theta-\bar\theta)\|_{L^\infty}^{4\alpha/(\alpha n-2n)}\|\Delta\rho\|^2\\
&+\|\nabla\rho\|_{L^\alpha}^2\|(\nabla\rho,\Delta v,\dot v,\nabla\theta)\|_{L^{\frac{2\alpha}{\alpha-2}}}^2+\|\nabla\dot v\|^2+\|\Delta\theta\|^2\}ds\\
\le{}&\Lambda\delta^{-n/(\alpha-n)}+\Lambda\int_b^t[\|(\nabla v,\rho-\bar\rho,\theta-\bar\theta)\|_{L^\infty}^{4\alpha/(\alpha n-2n)}\\
&+\|\nabla\rho\|_{L^\alpha}^2]\|\Delta\rho\|^2ds+\delta\int_b^t\|\nabla\Delta v\|^2ds\quad t\in[b,T]\quad(0<\delta\le1)\,,
\end{aligned}$$

where we have also used the following inequality for $\|\Delta v\|_{L^{\frac{2\alpha}{\alpha-2}}}$:

$$\|\cdot\|_{L^{\frac{2\alpha}{\alpha-2}}}\le C\|\cdot\|^{1-n/\alpha}\|\nabla\cdot\|^{n/\alpha}\le C\delta^{-n/(2\alpha-2n)}\|\cdot\|+\delta^{1/2}\|\nabla\cdot\|\quad(0<\delta\le1)\,.$$

Taking ∇ on both sides of (1.2) and applying the elliptic regularity, similarly to (4.19) we obtain

$$\begin{aligned}
(4.20)\quad \int_b^t\|\nabla\Delta v\|^2ds\le{}&\Lambda\int_b^t\|(\nabla(\rho\dot v),\nabla^2(\rho\theta),\nabla v)\|^2ds\\
\le{}&\Lambda+\Lambda\int_b^t(\|\nabla\rho\|_{L^\alpha}^2\|(\dot v,\nabla\theta)\|_{L^{\frac{2\alpha}{\alpha-2}}}^2+\|\Delta\rho\|^2)ds\\
\le{}&\Lambda+\Lambda\int_b^t\|\Delta\rho\|^2ds,\quad t\in[b,T]\,.
\end{aligned}$$

Multiplying (4.19) by $\Lambda+1$, adding then the resulting inequality to (4.20), and choosing $\delta=1/[2(\Lambda+1)]$, we apply Gronwall's inequality, and use (4.12)-(4.13), Lemma 4.1 to conclude

$$(4.21)\quad \sup_{b\le t\le T}\|\Delta\rho(t)\|^2+\int_b^T(\|\Delta\rho\|^2+\|\nabla\Delta v\|^2)ds\le\Lambda\,.$$

As a consequence of (4.21) we have

LEMMA 4.3.
$$\sup_{b \leq t \leq T} (\|\nabla \Delta v(t)\| + \|\nabla v(t)\|_{L^\infty}) \leq \Lambda.$$

In fact, Lemma 4.3 can be shown as follows, using equation (1.2), (1.15), (3.34)-(3.35), (4.10)-(4.11), and Lemma 4.2 (cf. (4.20)):

$$
\begin{aligned}
(4.22) \quad & \|\nabla \Delta v(t)\|^2 + \|\nabla v\|_{L^\infty}^2 \leq \Lambda \|(\nabla v, \Delta v, \nabla \Delta v)\| \\
& \leq \Lambda + \Lambda \left\{ \|\nabla \rho\|_{L^\alpha}^2 \|(\dot{v}, \nabla \theta)\|_{L^{\frac{2\alpha}{\alpha-2}}}^2 + \|(\nabla \dot{v}, \Delta \rho, \Delta \theta, \nabla v)\|^2 \right\} \leq \Lambda.
\end{aligned}
$$

Next we estimate $\|\nabla \dot{\theta}\|$ and $\|\nabla \Delta \rho\|^2$. First note that in (3.28),

$$\Delta \partial_t \theta + \operatorname{div}(v \Delta \theta) = \frac{d}{dt} \Delta \theta + \Delta \theta \operatorname{div} v = \Delta \dot{\theta} - 2 \nabla v \cdot \nabla^2 \theta - \Delta v \cdot \nabla \theta + \Delta \theta \operatorname{div} v.$$

So multiplying (3.28) by $\ddot{\theta}$ in $L^2(\mathbb{R}^n \times (b, t))$, employing a partial integration, and Cauchy-Schwarz's inequality, we obtain by (3.4), Lemma 4.3, (3.34)-(3.35), (1.15), (4.21) that

$$
\begin{aligned}
(4.23) \quad & \int_b^t \|\ddot{\theta}\|^2 ds + \|\nabla \dot{\theta}(t)\|^2 \leq \Lambda + \Lambda \int_b^t \|\Delta v\|_{L^\alpha}^2 \|\nabla \theta\|_{L^{\frac{2\alpha}{\alpha-2}}}^2 ds \\
& \leq \Lambda + \Lambda \int_b^t \|(\Delta v, \nabla \Delta v)\|^2 \|(\nabla \theta, \Delta \theta)\|^2 ds \leq \Lambda, \quad t \in [b, T].
\end{aligned}
$$

Taking ∇ on both sides of (1.3), applying Lemma 4.3 and (4.23), we get in the same manner as in the derivation of (4.22) that

$$
\begin{aligned}
(4.24) \quad & \|\nabla \theta(t)\|_{L^\infty}^2 + \|\nabla \Delta \theta(t)\|^2 \leq \Lambda + \Lambda \|\nabla \Delta \theta\|^2 \\
& \leq \Lambda(\|\nabla \dot{\theta}\|^2 + \|\nabla \rho\|_{L^\alpha}^2 \|\dot{\theta}\|_{L^{\frac{2\alpha}{\alpha-2}}}^2) + \Lambda \\
& \leq \Lambda + \Lambda \|\nabla \dot{\theta}(t)\|^2 \leq \Lambda, \quad t \in [b, T].
\end{aligned}
$$

Combining (4.23) and (4.24), we obtain

LEMMA 4.4.
$$\sup_{b \leq t \leq T} (\|\nabla \dot{\theta}\|^2 + \|\nabla \Delta \theta\|^2 + \|\nabla \theta\|_{L^\infty}^2)(t) + \int_b^T \|\ddot{\theta}(s)\|^2 ds \leq \Lambda.$$

Applying $\frac{d}{dt}$ respectively ∇ to (1.3) and utilising Lemma 4.4, we obtain analogously to (4.18) and (4.24) that

$$(4.25) \quad \int_b^T (\|\Delta \dot{\theta}\|^2 + \|\nabla \Delta \theta\|^2)(s) ds \leq \Lambda + \Lambda \int_b^T (\|\ddot{\theta}\|^2 + \|\nabla \dot{\theta}\|^2) ds \leq \Lambda.$$

Applying the operator Δ to (4.3), multiplying by $\partial_j \Delta \rho$ in $L^2(\mathbb{R}^n \times (b, T))$ ($b \leq t \leq T$), summing over j, and utilising Lemma 4.4, following the same arguments as used for (4.18)-(4.19) and (4.24), we can show the following lemma, the proof of which is therefore omitted here.

LEMMA 4.5.

$$\sup_{b \leq t \leq T} \left(\|\nabla \Delta \rho\|^2 + \|\nabla \rho\|_{L^\infty}^2 \right)(t) + \int_b^T \|\nabla \Delta \rho\|^2 ds \leq \Lambda \,.$$

Now we are able to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Recalling $\partial_t = \frac{d}{dt} - v \cdot \nabla$ and $\partial_t \rho = -\operatorname{div}(\rho v)$ (i.e. (1.1)), we combine (3.34)-(3.35), (3.36), (3.39), and (4.11), (4.18), (4.21), (4.25), and Lemmas 4.1-4.5 to obtain the following a priori estimates.

THEOREM 4.6. *Under the conditions of Theorem* 1.2 *we have*

(4.26) $\Lambda^{-1} \leq \rho(x, t), \theta(x, t) \leq \Lambda$ *for all* $x \in \mathbb{R}^n, t \in [b, T]$,

(4.27) $\displaystyle\sup_{b \leq t \leq T} \{\|(\rho - \bar\rho, v, \theta - \bar\theta)\|_{H^3} + \|\partial_t \rho\|_{H^2} + \|(\partial_t v, \partial_t \theta)\|_{H^1}\}(t) \leq \Lambda \,,$

and

(4.28) $\displaystyle\int_b^T \|(\nabla \rho, \nabla v, \nabla \theta, \partial_t \rho, \partial_t v, \partial_t \theta)\|_{H^2}^2 ds \leq \Lambda$

provided $E_0 \leq \epsilon_1$ *with* ϵ_1 *being defined in Lemma* 4.1.

The following local existence theorem can be shown by the standard contraction mapping argument (see for example [22], [29], in particular [18, Theorem 5.2], [13, Theorem 2.2]).

THEOREM 4.7 (Local existence). *Suppose* $(\rho_0 - \bar\rho, v_0, \theta_0 - \bar\theta) \in H^3$ *and* $\inf \rho_0(x) > 0$. *Then there exists a positive constant* t_0 *such that the Cauchy problem* (1.1)-(1.4) *has a unique solution* (ρ, v, θ) *on* $\mathbb{R}^n \times [0, t_0]$ *satisfying*

$$(\rho - \bar\rho, v, \theta - \bar\theta) \in C^0([0, t_0], H^3), \quad \partial_t \rho \in C^0([0, t_0], H^2) \,,$$
$$(\partial_t v, \partial_t \theta) \in C^0([0, t_0], H^1), \quad (v, \theta - \bar\theta) \in L^2([0, t_0], H^4) \,,$$

and for some $\nu > 0$ *independent of* t_0,

$$\rho(x, t) \geq \frac{1}{2} \inf \rho_0(x) \qquad \text{for all } x \in \mathbb{R}^n, t \in [0, t_0],$$

(4.29) $\displaystyle\sup_{0 \leq t \leq t_0} \|(\rho - \bar\rho, v, \theta - \bar\theta)(t)\|_{H^3}$

$$+ \int_0^{t_0} \|(\nabla v, \nabla \theta)\|_{H^3}^2 ds \leq \nu \|(\rho_0 - \bar\rho, v_0, \theta_0 - \bar\theta)\|_{H^3} \,.$$

If in addition $\inf \theta_0 > 0$, *then* $\theta(x, t) > 0$ *for all* $x \in \mathbb{R}^n, t \in [0, t_0]$.

The positivity of θ, which is not given in [18, Theorem 5.2], can be easily verified as follows: multiplying (1.3) by $1/\rho$ and integrating along particle trajectories, using (4.29), Sobolev's imbedding theorem, and Cauchy-Schwarz's inequality, one obtains that for all $x \in \mathbb{R}^n, t \in [0, t_0]$,

$$\theta(x, t) \geq \inf \theta_0 - C_1 \int_0^{t_0} (\|\Delta \theta\|_{H^2} + \|(\theta - \bar{\theta}, \nabla v)\|_{H^2}^2 + 1) ds$$
$$\geq \inf \theta_0 - C_1(\sqrt{t_0} + t_0) > 0,$$

provided t_0 appropriately small, where C_1 is a positive constant depending only on $c_V, \kappa, R, \lambda, \mu, \nu, \inf \rho_0$, and $\|(\rho_0 - \bar{\rho}, v_0, \theta_0 - \bar{\theta})\|_{H^3}$.

Under the conditions of Theorem 1.2 we have a unique local solution (ρ, v, θ) by Theorem 4.7. In view of the a priori estimates (4.26)-(4.27) we thus can apply Theorem 4.7 to continue the local solution globally in time. Moreover, the estimates (4.26)-(4.27) hold for any $T > 0$. This proves the existence and uniqueness in Theorem 1.2. To complete the proof it remains to show (1.10). To this end we use (4.28) and the identity $\frac{d}{dt} \int f dx = \int \partial_t f dx + \int \text{div}(vf) dx$ to deduce

$$\int_1^\infty \left| \frac{d}{dt} \|(\Delta \rho, \Delta v, \Delta \theta)\|^2 \right| ds \leq \int_1^\infty \|(\Delta \partial_t \rho, \Delta \partial_t v, \Delta \partial_t \theta, \Delta \rho, \Delta v, \Delta \theta)\|^2 ds < \infty,$$

which combined with (4.28) implies $\|(\Delta \rho, \Delta v, \Delta \theta)(t)\|^2 \to 0$ as $t \to \infty$. This together with (1.16) and (4.27) yields

$$\|(\rho - \bar{\rho}, v, \theta - \bar{\theta})(t)\|_{L^\infty} \leq C \|(\rho - \bar{\rho}, v, \theta - \bar{\theta})(t)\|^{1-n/4}$$
$$\times \|(\Delta \rho, \Delta v, \Delta \theta)(t)\|^{n/4} \to 0 \quad \text{as } t \to \infty.$$

The proof of Theorem 1.2 is complete.   □

NOTE ADDED IN PROOF. Recently, Hoff [35] proved the global existence of weak solutions when (ρ_0, θ_0, v_0) is close to $(\bar{\rho}, \bar{\theta}, 0)$ in $L^2 \cap L^\infty \times L^2 \times H^s \cap L^4$, where $s = 0$ for $n = 2$ and $s > 1/3$ for $n = 3$, (The L^p norms must be weighted when $n = 2$.) and λ/μ satisfies some (smallness) condition.

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