ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 25, nº 3-4 (1997), p. 807-828
<http://www.numdam.org/item?id=ASNSP_1997_4_25_3-4_807_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXV (1997), pp. 807-828

On the Behavior of the Solutions of the Navier-Stokes Equations at Vanishing Viscosity

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Dedicated to the Memory of Ennio De Giorgi

Abstract. In this article we establish partial results concerning the convergence of the solutions of the Navier-Stokes equations to that of the Euler equations. Namely, we prove convergence on any finite interval of time, in space dimension two, under a physically reasonable assumption. We consider the flow in a channel or the flow in a general bounded domain.

1. - Introduction

The large Reynolds number (or small viscosity) behavior of wall bounded flows is an outstanding problem in mathematics and physics. Many articles are devoted to this problem in the fluid mechanic and mathematical literatures. This includes the well-known works of W. Eckhaus [E], P. Fife [F], T. Kato [K1], J. L. Lions [Lo2], O. Oleinik [O] and M. I. Vishik and L. A. Lyusternik [VL] among the mathematical literature. More recent results include the work of M. Sammartino and R. E. Caflish [SC] who proved the convergence of the solutions of the Navier-Stokes equations to that of the Euler equations, for a small interval of time in a half plane, in the context of analytic solutions, and the work of Weinan E and B. Engquist [EE] who proved the blowing up of smooth solutions of the Prandtl equation for a certain class of compactly supported C^∞ initial data. Related works appear in [As] and [CW]; see also the recent work of G. J. Barenblatt and A. J. Chorin [BC1,2], [Ch]. In the mechanical literature see e.g. [BP], [Ba], [Ge], [Gr], [La] and [Vd].

In earlier works, we have studied the boundary layer behavior, for large Reynolds numbers, of the solutions of linearized Navier-Stokes equations of the Oseen type (see [TW1-4]); see also the work of S. N. Alekseenko [Al] for a related work in the case where the physical boundary is non-characteristic and the work of T. Yanagisawa and Z. Xin [YX] where they take the approach of Prandtl's equation for the linearized Navier-Stokes equations. For the nonlinear problem the difficulties are much more important and we aim in this article to derive a partial result. Namely we apply some of the technics developed in [TW1-4] to the full (nonlinear) Navier-Stokes equations in space dimension two. Under physically reasonable assumptions we prove that the solutions to the Navier-Stokes equations converge to the solutions of the Euler equations on any finite interval of time. The assumptions that we make on the solutions are either the boundedness at the wall of the gradient of the pressure, or we assume a moderate growth condition for the tangential derivative of the tangential flow near the wall. It is noted that in the work of W. E and B. Engquist the quantity which blows up in finite time is the tangential derivative of the tangential flow.

The article is organized as follows. In Section 2 we set the notations and state the main results in the case of a rectangular geometry (flow in a channel); then in Section 3 we give the proofs. In Section 4 we briefly show how to extend to more general domains the results established for the rectangular geometry.

The results presented in this article were announced in [TW5].

ACKNOWLEDGEMENTS. XW acknowledges helpful conversation with W.E who brought the work of Asano to our attention.

This work was supported in part by the National Science Foundation under Grant NSF-DMS-9400615, by ONR under grant NAVY-N00014-91-J-1140 and by the Research Fund of Indiana University. The second author was supported by an NSF postdoctoral position at the Courant Institute.

2. – The main results

We consider the Navier-Stokes equations in space dimension two in an infinite channel $\Omega_\infty = \mathbb{R} \times (0, 1)$:

$$\frac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon = f \quad \text{in } \Omega_\infty \times \mathbb{R}_+ , \tag{2.1}$$

$$\nabla \cdot u^\varepsilon = 0 \quad \text{in } \Omega_\infty \times \mathbb{R}_+ . \tag{2.2}$$

The velocity u^ε vanishes at the boundary $\partial\Omega_\infty$ of the channel (i.e. at $y = 0, 1$),

$$u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\infty \times \mathbb{R}_+ , \tag{2.3}$$

and periodicity (period 2π) is assumed in the horizontal (x) direction. We set

$$\Omega = (0, 2\pi) \times (0, 1), \quad \Gamma = (0, 2\pi) \times \{0, 1\} ,$$

and introduce the natural function spaces

$$V = \left\{ v \in (H^1_{\text{loc}}(\Omega_\infty))^2, \quad \operatorname{div} v = 0, \quad v \,|_{\partial\Omega_\infty} = 0 \right\} ,$$

$$H = \left\{ v \in (L^2_{\text{loc}}(\Omega_\infty))^2, \quad \operatorname{div} v = 0, \quad v_2 \,|_{\partial\Omega_\infty} = 0 \right\} ,$$

where periodicity in x (period 2π) is understood; these spaces are endowed with their usual scalar products and norms.

Equations (2.1)-(2.3) are supplemented with the initial condition

$$u^\varepsilon = u_0 \quad (\text{given in } V) \text{ at } \quad t = 0. \tag{2.4}$$

We intend to compare the solutions of (2.1)-(2.4) to the solutions (u^0, p^0) of the corresponding Euler equations, i.e.

$$\frac{\partial u^0}{\partial t} + (u^0 \cdot \nabla)u^0 + \nabla p^0 = f \quad \text{in} \quad \Omega_\infty \times \mathbb{R}_+, \tag{2.5}$$

$$\nabla \cdot u^0 = 0 \quad \text{in} \quad \Omega_\infty \times \mathbb{R}_+, \tag{2.6}$$

$$u^0 \cdot n = 0 \quad \text{on} \quad \partial\Omega_\infty \times \mathbb{R}_+, \tag{2.7}$$

$$u^0 = u_0 \quad \text{at} \quad t = 0, \tag{2.8}$$

where n is the unit outward normal, so that for $u^0 = (u_1^0, u_2^0)$, (2.7) is equivalent to $u_2^0 = 0$ at $y = 0, 1$. Furthermore space periodicity in x is understood.

The well-posedness and the regularity of the solutions of the Navier-Stokes equations are classical (see e.g. Lions [Lo1], Temam [T3,4]). This provides all the desired regularity for u^ε, p^ε provided f and u_0 are sufficiently smooth. For the Euler equations the existence and regularity can be derived by modifying a classical work of T. Kato [K2], and with further regularity derived as in [T1,2].

Our main result in the present article asserts that if the tangential derivative of p^ε (x−derivative on Γ) does not grow too fast when $\varepsilon \to 0$, then u^ε converges to u^0 as $\varepsilon \to 0$; namely we have the following:

THEOREM 1. *Let $(u^\varepsilon, p^\varepsilon)$ and (u^0, p^0) be the solutions of the Navier-Stokes and Euler equations above. We assume that $T > 0$ is fixed and that there exist two constants κ_1, δ independent of ε, $0 \le \delta < 1/2$, such that either*

$$\left\|p^\varepsilon\right\|_{L^2(0,T;H^{1/2}(\Gamma))} \le \kappa_1 \varepsilon^{-\delta}, \tag{2.9}$$

or

$$\left\|p_x^\varepsilon\right\|_{L^2(0,T;L^2(\Gamma))} = \left\|p^\varepsilon\right\|_{L^2(0,T;H^1(\Gamma))} \le \kappa_1 \varepsilon^{-\delta-1/4}. \tag{2.10}$$

Then, there exists a constant κ_2, independent of ε, such that

$$\left\|u^\varepsilon - u^0\right\|_{L^\infty(0,T;H)} \le \kappa_2 \varepsilon^{(1-2\delta)/15}. \tag{2.11}$$

REMARK 1. The convergence in (2.11) is in the strong (norm) topology of $L^\infty(0, T; L^2(\Omega)^2)$; as usual in boundary layer phenomena, convergence in the $L^\infty(\Omega)$ or $H^1(\Omega)$-norm is not expected (is not true in general), because of the discrepancy in the boundary values of u^ε and u^0.

REMARK 2. As mentioned in the introduction, it is expected on physical grounds that p^ε and $p_x^\varepsilon = \partial p^\varepsilon / \partial x$ remain bounded on and near Γ; therefore (2.9) and (2.10) are physically realistic hypotheses, since they even allow growth of p^ε or p_x^ε. Of course, a complete mathematical proof of the convergence of u^ε to u^0 along this line would necessitate proving these hypotheses. □

The proof of Theorem 1 is based on a related result which has some interest on its own. We state this result.

THEOREM 2. *Let* $(u^\varepsilon, p^\varepsilon)$ *and* (u^0, p^0) *be the solutions of the Navier- Stokes and Euler equations above and let* $T > 0$ *be fixed. We assume that there exist two constants* α *and* κ_3*, independent of* ε*,* $3/4 \leq \alpha < 1$*, such that either*

$$\|u^\varepsilon_{2x}\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} \leq \kappa_3 \varepsilon^{3/2-2\alpha}, \tag{2.12}$$

or

$$\|u^\varepsilon_{1x}\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} \leq \kappa_3 \varepsilon^{1/2-\alpha}, \tag{2.13}$$

or

$$\|u^\varepsilon_{1xx}\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} \leq \kappa_3 \varepsilon^{3/2-3\alpha}, \tag{2.14}$$

or

$$\|u^\varepsilon_{2xx}\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} \leq \kappa_3 \varepsilon^{7/2-5\alpha}, \tag{2.15}$$

where Γ_r *is the* $r-$ *neighborhood of* Γ *in* Ω*.*

Then there exists a constant κ_4 *independent of* ε *such that*

$$\left\|u^\varepsilon - u^0\right\|_{L^\infty(0,T;H)} \leq \begin{cases} \kappa_4 \varepsilon^{(1-\alpha)/3} & \text{in the case of (2.13)} \\ \kappa_4 \varepsilon^{2(1-\alpha)/5} & \text{in the case of (2.12) and (2.14)} \\ \kappa_4 \varepsilon^{5(1-\alpha)/11} & \text{in the case of (2.15)}. \end{cases} \tag{2.16}$$

REMARK 3. Theorem 2 can be extended without any change to the three-dimensional case. This is not the case however of Theorem 1 which uses an orthogonality property of the nonlinear term (see (3.3)) valid only in space dimension 2.

REMARK 4. On physical grounds, we expect that the normal component u_2^ε of the velocity does not display a boundary layer phenomenon since $u_2^\varepsilon = u_2^0 = 0$ on Γ, although its normal derivative $\partial u_2^\varepsilon/\partial y$ might display such a boundary layer. Thus the assumption (2.12) is physically reasonable as well, although this is less transparent than for hypotheses (2.9) and (2.10) in Theorem 1.

We also learn from boundary layer theory that although the velocity may exhibit a large variation in the normal direction, the variation in the x-direction is expected to be of lower order. Hence (2.12)-(2.15) are all reasonable from this point of view. Indeed assumptions (2.12)-(2.15) are a subset of the hypotheses for deriving the Prandtl equation for boundary layer (see e.g. Landau and Lifstchitz [LL]).

REMARK 5. Conditions (2.12) and (2.13) are also close to be mathematically necessary for (2.16) to be true. Indeed by the energy equation for u^ε and u^0,

$$\begin{aligned} |u^\varepsilon(T)|_H^2 + 2\varepsilon \int_0^T |u^\varepsilon|_V^2\, dt &= |u_0|_H^2 + 2\int_0^T (f, u^\varepsilon)_H dt\,, \\ \left|u^0(T)\right|_H^2 &= |u_0|_H^2 + 2\int_0^T (f, u^0)_H dt\,. \end{aligned}$$

Hence if e.g. the second condition (2.16) occurs then, as $\varepsilon \to 0$,

$$\begin{aligned} \varepsilon \int_0^T |u^\varepsilon|_V^2\, dt &= \varepsilon \int_0^T |\nabla u^\varepsilon|_{L^2(\Omega)}^2\, dt \\ &\leq \kappa\ \varepsilon^{2(1-\alpha)/5}\,, \end{aligned}$$

which implies (2.13) with a different α, and κ is a constant independent of ε; the proof is similar for (2.12). In fact T. Kato [K1] proved that for u^ε to converge to u^0 in $L^\infty(0, T; H)$ strongly it is necessary and sufficient that ∇u^ε converges to zero in $L^2(0, T; L^2(\Gamma_{\varepsilon^\alpha}))$. Notice that Kato's result involves the whole gradient whereas Theorem 2 involves only the tangential derivative.

REMARK 6. It is interesting to observe a heuristic connection between (2.14) and Kolmogorov's dissipation length (wave number). We take the extreme case of $\alpha = 1$ in (2.14) and suppose that k is the highest effective wave number (in the x-direction). We also assume that the left and right-hand sides of (2.14) are of the same magnitude. Then

$$u_1^\varepsilon \sim \sum_{j=1}^{k} u_{1j}^\varepsilon(y) e^{ikx}\,,$$

and hence, with $\alpha = 1$,

$$\begin{aligned} \varepsilon^{3/2-3\alpha} &\sim \left\|u_{1xx}^\varepsilon\right\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} \\ &\sim k^2 \left\|u_1^\varepsilon\right\|_{L^2(0,T;L^2(\Omega))} \\ &\sim k^2\,. \end{aligned}$$

Hence

$$k \sim \varepsilon^{-3/4}\,,$$

which is exactly the Kolmogorov dissipation wave number (so far as the dependence on the kinematic viscosity ε is concerned).

3. – Proof of the main results

Theorem 1 is a consequence of Theorem 2. Hence we first prove Theorem 1 assuming that Theorem 2 is valid.

PROOF OF THEOREM 1. We multiply (2.1) by $-\Delta u^\varepsilon$, integrate over Ω and notice that

$$\int_\Omega f \cdot \Delta u^\varepsilon = -\int_\Omega \nabla f \cdot \nabla u^\varepsilon + \int_{y=1} f \frac{\partial}{\partial y} u^\varepsilon - \int_{y=0} f \frac{\partial}{\partial y} u^\varepsilon ;$$

hence thanks to the trace theorem and the Agmon inequality,

$$\begin{aligned} \left| \int_\Omega f \cdot \Delta u^\varepsilon \right| &\leq |\nabla f| \left|\nabla u^\varepsilon\right| + |f|_{L^2(\Gamma)} \left| \frac{\partial}{\partial y} u^\varepsilon \right|_{L^2(\Gamma)} \\ &\leq |\nabla f| \left|\nabla u^\varepsilon\right| + \kappa \left|\nabla u^\varepsilon\right|^{1/2} \left|\nabla u^\varepsilon\right|^{1/2}_{H^1(\Omega)} \\ &\leq \frac{1}{2} \left|\nabla u^\varepsilon\right|^2 + \frac{1}{2} |\nabla f|^2 + \kappa \left|\nabla u^\varepsilon\right|^{1/2} \left|\Delta u^\varepsilon\right|^{1/2} \\ &\leq \frac{\varepsilon}{4} \left|\Delta u^\varepsilon\right|^2 + \frac{1}{2} \left|\nabla u^\varepsilon\right|^2 + \kappa \varepsilon^{-1/2} , \end{aligned} \tag{3.1}$$

where κ is a constant independent of ε.

On the other hand, since Δu^ϵ is divergence free

$$\int_\Omega \nabla p^\varepsilon \cdot \Delta u^\varepsilon = -\int_{y=1} p^\varepsilon \Delta u_2^\varepsilon \, dx + \int_{y=0} p^\varepsilon \Delta u_2^\varepsilon \, dx .$$

Therefore

$$\begin{aligned} \left| \int_\Omega \nabla p^\varepsilon \cdot \Delta u^\varepsilon \right| &\leq \left|p^\varepsilon\right|_{H^{1/2}(\Gamma)} \left|\Delta u_2^\varepsilon\right|_{H^{-1/2}(\Gamma)} \\ &\leq \text{(by the trace theorem for divergence free functions)} \\ &\leq \kappa \left|p^\varepsilon\right|_{H^{1/2}(\Gamma)} \left|\Delta u^\varepsilon\right| \\ &\leq \frac{\varepsilon}{4} \left|\Delta u^\varepsilon\right|^2 + \frac{\kappa}{\varepsilon} \left|p^\varepsilon\right|^2_{H^{1/2}(\Gamma)} . \end{aligned} \tag{3.2}$$

We also notice that

$$\int_\Omega (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \Delta u^\varepsilon = 0 , \tag{3.3}$$

which can be proved exactly as in the purely periodic case (see e.g. [T4]).

Combining (3.1)-(3.3) we deduce

$$\frac{d}{dt} \left|\nabla u^\varepsilon\right|^2 + \varepsilon \left|\Delta u^\varepsilon\right|^2 \leq \left|\nabla u^\varepsilon\right|^2 + \kappa \varepsilon^{-1/2} + \frac{1}{\varepsilon} \left|p^\varepsilon\right|^2_{H^{1/2}(\Gamma)} . \tag{3.4}$$

Hence under the assumption (2.9), we have

$$\|u^\varepsilon\|_{L^2(0,T;(H^2(\Omega))^2)} \le \kappa\varepsilon^{-1-\delta}. \tag{3.5}$$

Thanks to Poincaré's inequality

$$\begin{aligned} \|u^\varepsilon_{2x}\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} &\le \varepsilon^\alpha \left\|u^\varepsilon_{2xy}\right\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} \\ &\le \kappa\varepsilon^{-1-\delta+\alpha}. \end{aligned} \tag{3.6}$$

Thus (2.12) is verified with $\alpha = \frac{5}{6} + \frac{\delta}{3}$ and hence thanks to Theorem 2, (2.16), we deduce (2.11).

Another way to estimate $\int_\Omega \nabla p^\varepsilon \cdot \Delta u^\varepsilon$ is the following. Notice that since $u_2^\varepsilon = 0$ on Γ

$$\begin{aligned} \int_{y=0} p^\varepsilon \cdot \Delta u_2^\varepsilon = & \int_{y=0} p^\varepsilon \frac{\partial^2 u_2^\varepsilon}{\partial y^2}\, dx \\ = & -\int_{y=0} p^\varepsilon \frac{\partial^2 u_1^\varepsilon}{\partial x \partial y}\, dx \\ = & \int_{y=0} p_x^\varepsilon u_{1y}^\varepsilon \, dx . \end{aligned}$$

Similarly

$$\int_{y=1} p^\varepsilon \cdot \Delta u_2^\varepsilon = \int_{y=1} p_x^\varepsilon u_{1y}^\varepsilon \, dx .$$

Hence we deduce that

$$\begin{aligned} \left|\int_\Omega \nabla p^\varepsilon \cdot \Delta u^\varepsilon\right| &\le |p_x^\varepsilon|_{L^2(\Gamma)} \left|u_y^\varepsilon\right|_{L^2(\Gamma)} \\ &\le \kappa \,|p_x^\varepsilon|_{L^2(\Gamma)} \left|\nabla u^\varepsilon\right|^{1/2} \left|\Delta u^\varepsilon\right|^{1/2} \\ &\le \frac{\varepsilon}{4}\left|\Delta u^\varepsilon\right|^2 + \frac{1}{2}\left|\nabla u^\varepsilon\right|^2 + \kappa\varepsilon^{-1/2}\,|p_x^\varepsilon|^2_{L^2(\Gamma)} . \end{aligned} \tag{3.7}$$

Combining (3.1), (3.3) and (3.7) we find

$$\frac{d}{dt}\left|\nabla u^\varepsilon\right|^2 + \varepsilon\left|\Delta u^\varepsilon\right|^2 \le \left|\nabla u^\varepsilon\right|^2 + \kappa\varepsilon^{-1/2} + \kappa\varepsilon^{-1/2}\,|p_x^\varepsilon|^2_{L^2(\Gamma)} . \tag{3.8}$$

Thus under the assumption (2.10) we have (3.5) again and hence (2.11). This completes the proof of Theorem 1. □

Observe that we have

$$\int_\Omega \nabla p^\varepsilon \cdot \Delta u^\varepsilon = \int_{y=1} p_x^2 u_{1y}^\varepsilon \, dx - \int_{y=0} p_x^\varepsilon u_{1y}^\varepsilon \, dx \, .$$

Thus if $p_x^\varepsilon \geq 0$ at the boundary and if there is no "adverse flow" at the boundary, i.e. $u_{1y}^\varepsilon \geq 0$ at $y = 0$ and $u_{1y}^\varepsilon \leq 0$ at $y = 1$ we deduce

$$\int_\Omega \nabla p^\varepsilon \cdot \Delta u^\varepsilon \leq 0 \, .$$

This together with (3.1) and (3.3) implies

$$\frac{d}{dt} \left|\nabla u^\varepsilon\right|^2 + \varepsilon \left|\Delta u^\varepsilon\right|^2 \leq \left|\nabla u^\varepsilon\right|^2 + \kappa \varepsilon^{-1/2} \, , \tag{3.9}$$

which further implies

$$\left\|u^\varepsilon\right\|_{L^2(0,T;(H^2(\Omega))^2)} \leq \kappa \varepsilon^{-3/4} \, . \tag{3.10}$$

Next we proceed with the proof of Theorem 2.

PROOF OF THEOREM 2. Observe that, thanks to Poincaré's inequality and the identity

$$u_{1xx}^\varepsilon = -u_{2xy}^\varepsilon \, ,$$

the assumption (2.14) implies assumption (2.12). Thus we need only to prove the theorem under the assumptions (2.12), (2.13) and (2.15).

Let us recall a class of divergence free functions which agrees with $-u^0$ on the wall of the channel, i.e. $y = 0, 1$ (see e.g. R. Temam and X. Wang [TW1]). Let

$$\rho \in C^\infty([0, \infty)), \quad \rho(0) = 1, \quad \int_0^1 \rho(s) ds = 0, \quad \operatorname{supp} \rho \subset [0, 1] \, .$$

Consider for $\alpha \in \left(\frac{1}{2}, 1\right)$ the stream line functions[(1)]

$$\psi^{\varepsilon,\alpha}(t; x, y) = -u_1^0(t; x, 0) \int_0^y \rho\left(\frac{s}{\varepsilon^\alpha}\right) ds + u_1^0(t; x, 1) \int_0^{1-y} \rho\left(\frac{s}{\varepsilon^\alpha}\right) ds \, . \tag{3.11}$$

We then define

$$\begin{aligned}
\varphi^{\varepsilon,\alpha}(t; x, y) &= \nabla^\perp \psi^{\varepsilon,\alpha}(t; x, y) \\
&= \left\{\psi_y^{\varepsilon,\alpha}, -\psi_x^{\varepsilon,\alpha}\right\} \\
&= \left\{-u_1^0(t; x, 0)\rho\left(\frac{y}{\varepsilon^\alpha}\right) - u_1^0(t; x, 1)\rho\left(\frac{1-y}{\varepsilon^\alpha}\right), \right. \\
&\qquad \left. u_{1x}^0(t; x, 0) \int_0^y \rho\left(\frac{s}{\varepsilon^\alpha}\right) ds - u_{1x}^0(t; x, 1) \int_0^{1-y} \rho\left(\frac{s}{\varepsilon^\alpha}\right) ds \right\} .
\end{aligned} \tag{3.12}$$

[(1)]This α is not the same as in Theorem 2; note that α does not appear in the statement of Theorem 1.

It is easy to see that

$$\text{div } \varphi^{\varepsilon,\alpha} = 0, \tag{3.13}$$

and

$$\varphi^{\varepsilon,\alpha} = -u^0 \quad \text{at } y = 0, 1. \tag{3.14}$$

Next we consider $w^\varepsilon = u^\varepsilon - u^0 - \varphi^{\varepsilon,\alpha}$ ($\in V$); then w^ε satisfies the equation

$$\begin{aligned} &\frac{\partial w^\varepsilon}{\partial t} - \varepsilon \Delta w^\varepsilon + (u^\varepsilon \cdot \nabla) w^\varepsilon + (w^\varepsilon \cdot \nabla) u^0 + \nabla(p^\varepsilon - p^0) \\ &\quad = -\frac{\partial \varphi^{\varepsilon,\alpha}}{\partial t} + \varepsilon \Delta u^0 + \varepsilon \Delta \varphi^{\varepsilon,\alpha} - (u^\varepsilon \cdot \nabla) \varphi^{\varepsilon,\alpha} - (\varphi^{\varepsilon,\alpha} \cdot \nabla) u^0, \end{aligned} \tag{3.15}$$

$$\nabla \cdot w^\varepsilon = 0, \tag{3.16}$$

$$w^\varepsilon = 0 \qquad \text{on } \Gamma \times \mathbb{R}_+, \tag{3.17}$$

$$w^\varepsilon = -\varphi^{\varepsilon,\alpha} \qquad \text{at } t = 0. \tag{3.18}$$

We multiply (3.15) by w^ε and integrate over Ω; the contribution of the pressure terms vanishes and there remains:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|w^\varepsilon|^2 + \varepsilon |\nabla w^\varepsilon|^2 \le\; & \left|\nabla u^0\right|_\infty |w^\varepsilon|^2 + \varepsilon \left|\nabla u^0\right| |\nabla w^\varepsilon| \\ & + \varepsilon \left|\nabla \varphi^{\varepsilon,\alpha}\right| |\nabla w^\varepsilon| + \left|\frac{\partial \varphi^{\varepsilon,\alpha}}{\partial t}\right| |w^\varepsilon| \\ & + \left|\nabla u^0\right|_\infty |\varphi^{\varepsilon,\alpha}| |w^\varepsilon| - \int_\Omega (u^\varepsilon \cdot \nabla) \varphi^{\varepsilon,\alpha} w^\varepsilon. \end{aligned} \tag{3.19}$$

Observe that, by the expression (3.12) of $\varphi^{\varepsilon,\alpha}$:

$$\left\| \frac{\partial \varphi^{\varepsilon,\alpha}}{\partial t} \right\|_{L^\infty(0,T;H)} \le \kappa \varepsilon^{\alpha/2}, \tag{3.20}$$

$$\left\| \varphi^{\varepsilon,\alpha} \right\|_{L^\infty(0,T;H)} \le \kappa \varepsilon^{\alpha/2}, \tag{3.21}$$

$$\left\| \varphi^{\varepsilon,\alpha} \right\|_{L^\infty(0,T;(H^1(\Omega))^2)} \le \kappa \varepsilon^{-\alpha/2}, \tag{3.22}$$

and similar bounds are valid for any x-derivative of those expressions. Then we write

$$\begin{aligned} \left| \int_\Omega (u^\varepsilon \cdot \nabla) \varphi^{\varepsilon,\alpha} w^\varepsilon \right| \le & \left| \int_\Omega u_1^\varepsilon \varphi_{1x}^{\varepsilon,\alpha} w_1^\varepsilon \right| + \left| \int_\Omega u_2^\varepsilon \varphi_{1y}^{\varepsilon,\alpha} w_1^\varepsilon \right| \\ & + \left| \int_\Omega u_1^\varepsilon \varphi_{2x}^{\varepsilon,\alpha} w_2^\varepsilon \right| + \left| \int_\Omega u_2^\varepsilon \varphi_{2y}^{\varepsilon,\alpha} w_2^\varepsilon \right|; \end{aligned} \tag{3.23}$$

we observe that $\varphi^{\varepsilon,\alpha}$ is supported in $\Gamma_{\varepsilon^\alpha}$. Thus we deduce

$$
\begin{aligned}
\left|\int_\Omega u_1^\varepsilon \varphi_{1x}^{\varepsilon,\alpha} w_1^\varepsilon\right| &\le |u_1^\varepsilon|_{L^2(\Gamma_{\varepsilon^\alpha})} |\varphi_{1x}^{\varepsilon,\alpha}|_\infty |w_1^\varepsilon| \\
&\le \left(u_1^\varepsilon = w_1^\varepsilon + u_1^0 + \varphi_1^{\varepsilon,\alpha}\right) \\
&\le \left(|w_1^\varepsilon| + \left|u_1^0\right|_{L^2(\Gamma_{\varepsilon^\alpha})} + |\varphi_1^{\varepsilon,\alpha}|\right) |\varphi_{1x}^{\varepsilon,\alpha}|_\infty |w_1^\varepsilon| \\
&\le \text{(Thanks to (3.12) and } u_1^0 \text{ being smooth)} \\
&\le \kappa |w_1^\varepsilon|^2 + \kappa\varepsilon^\alpha .
\end{aligned} \tag{3.24}
$$

Similarly,

$$
\left|\int_\Omega u_1^\varepsilon \varphi_{2x}^{\varepsilon,\alpha} w_2^\varepsilon\right| \le \kappa |w^\varepsilon|^2 + \kappa\varepsilon^\alpha , \tag{3.25}
$$

$$
\left|\int_\Omega u_2^\varepsilon \varphi_{2y}^{\varepsilon,\alpha} w_2^\varepsilon\right| \le \kappa |w^\varepsilon|^2 + \kappa\varepsilon^\alpha . \tag{3.26}
$$

For the second term on the right-hand side of (3.23),

$$
\begin{aligned}
\left|\int_\Omega u_2^\varepsilon \varphi_{1y}^{\varepsilon,\alpha} w_1^\varepsilon\right| &\le \left|\int_\Omega u_{2y}^\varepsilon \varphi_1^{\varepsilon,\alpha} w_1^\varepsilon\right| + \left|\int_\Omega u_2^\varepsilon \varphi_1^{\varepsilon,\alpha} w_{1y}^\varepsilon\right| \\
&\le \left|\int_\Omega u_{1x}^\varepsilon \varphi_1^{\varepsilon,\alpha} w_1^\varepsilon\right| + \left|\int_\Omega u_2^\varepsilon \varphi_1^{\varepsilon,\alpha} w_{1y}^\varepsilon\right| \\
&\le |u_{1x}^\varepsilon|_{L^2(\Gamma_{\varepsilon^\alpha})} |\varphi_1^{\varepsilon,\alpha}|_\infty |w_1^\varepsilon|_{L^2(\Gamma_{\varepsilon^\alpha})} \\
&\quad + |u_2^\varepsilon|_{L^2(\Gamma_{\varepsilon^\alpha})} |\varphi_1^{\varepsilon,\alpha}|_\infty \left|w_{1y}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})} \\
&\le \text{(by Poincaré's inequality and } u_{2y}^\varepsilon = -u_{1x}^\varepsilon) \\
&\le \kappa\varepsilon^\alpha \left|u_{1x}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})} \left|w_{1y}^\varepsilon\right| \\
&\le \frac{\varepsilon}{8} |\nabla w^\varepsilon|^2 + \kappa\varepsilon^{2\alpha-1} |u_{1x}^\varepsilon|^2_{L^2(\Gamma_{\varepsilon^\alpha})} .
\end{aligned} \tag{3.27}
$$

Combining (3.19)-(3.27) with the Cauchy-Schwarz inequality we deduce

$$
\frac{d}{dt} |w^\varepsilon|^2 + \varepsilon |\nabla w^\varepsilon|^2 \le \kappa |w^\varepsilon|^2 + \kappa\varepsilon^{1-\alpha} + \kappa\varepsilon^\alpha + \kappa\varepsilon^{2\alpha-1} |u_{1x}^\varepsilon|^2_{L^2(\Gamma_{\varepsilon^\alpha})} . \tag{3.28}
$$

Now if (2.13) holds with $\alpha = \alpha_0$, then for $\alpha \ge \alpha_0$

$$
\begin{aligned}
\|w^\varepsilon\|_{L^\infty(0,T;H)} &\le \kappa\varepsilon^{(1/2)-(\alpha/2)} + \kappa\varepsilon^{\alpha-\alpha_0} + \kappa\varepsilon^{\alpha/2} \\
&\le \kappa\varepsilon^{(1-\alpha_0)/3} \text{ for } \alpha = \frac{1}{3}(1+2\alpha_0) > \alpha_0 .
\end{aligned} \tag{3.29}
$$

When this is combined with (3.12) we deduce (2.16).

An alternative estimate of the second term on the right-hand side of (3.23) is the following

$$\left|\int_\Omega u_2^\varepsilon \varphi_{1y}^{\varepsilon,\alpha} w_1^\varepsilon\right| \le \left|\int_\Omega u_2^\varepsilon u_1^\varepsilon \varphi_{1y}^{\varepsilon,\alpha}\right| + \left|\int_\Omega u_2^\varepsilon u_1^0 \varphi_{1y}^{\varepsilon,\alpha}\right| + \left|\int_\Omega u_2^\varepsilon \varphi_{1y}^{\varepsilon,\alpha} \varphi_1^{\varepsilon,\alpha}\right| . \tag{3.30}$$

Observe that

$$\begin{aligned}
\left|\int_\Omega u_2^\varepsilon u_1^0 \varphi_{1y}^{\varepsilon,\alpha}\right| &\le \left|\int_\Omega u_{2y}^\varepsilon u_1^0 \varphi_1^{\varepsilon,\alpha}\right| + \left|\int_\Omega u_2^\varepsilon u_{1y}^0 \varphi_1^{\varepsilon,\alpha}\right| \\
&\le \left|\int_\Omega u_{1x}^\varepsilon u_1^0 \varphi_1^{\varepsilon,\alpha}\right| + |u_2^\varepsilon| \left|u_{1y}^0\right|_\infty |\varphi_1^{\varepsilon,\alpha}| \\
&\le \left|\int_\Omega u_1^\varepsilon u_{1x}^0 \varphi_1^{\varepsilon,\alpha}\right| + \left|\int_\Omega u_1^\varepsilon u_1^0 \varphi_{1x}^{\varepsilon,\alpha}\right| + |u_2^\varepsilon| \left|u_{1y}^0\right|_\infty |\varphi_1^{\varepsilon,\alpha}| \\
&\le |u^\varepsilon| \left(\left|u_{1x}^0\right|_\infty + \left|u_1^0\right|_\infty + \left|u_{1y}^0\right|_\infty\right) \left(|\varphi_1^{\varepsilon,\alpha}| + |\varphi_{1x}^{\varepsilon,\alpha}|\right) \\
&\le \text{(Thanks to (3.21))} \\
&\le \kappa \varepsilon^{\alpha/2} ,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\left|\int_\Omega u_2^\varepsilon \varphi_{1y}^{\varepsilon,\alpha} \varphi_1^{\varepsilon,\alpha}\right| &= \frac{1}{2}\left|\int_\Omega u_{2y}^\varepsilon (\varphi_1^{\varepsilon,\alpha})^2\right| \\
&= \frac{1}{2}\left|\int_\Omega u_{1x}^\varepsilon (\varphi_1^{\varepsilon,\alpha})^2\right| \\
&= \left|\int_\Omega u_1^\varepsilon \varphi_1^{\varepsilon,\alpha} \varphi_{1x}^{\varepsilon,\alpha}\right| \\
&\le \text{(Thanks to (3.21) and (3.22))} \\
&\le \kappa \varepsilon^{\alpha/2} .
\end{aligned} \tag{3.32}$$

It remains to estimate $\int_\Omega u_2^\varepsilon u_1^\varepsilon \varphi_{1y}^{\varepsilon,\alpha}$. We shall only estimate the integral in the region $\{y < \varepsilon^\alpha\}$; the counterpart in $\{1 - y < \varepsilon^\alpha\}$ is estimated in a similar way.

$$\begin{aligned}
\left|\int_{\{y<\varepsilon^\alpha\}} u_2^\varepsilon u_{1y}^\varepsilon \varphi_{1y}^{\varepsilon,\alpha}\right| &= \varepsilon^{-\alpha} \left|\int_{\{y<\varepsilon^\alpha\}} u_2^\varepsilon u_1^\varepsilon u_1^0(t,x,0) \rho'\left(\frac{y}{\varepsilon^\alpha}\right)\right| \\
&\le \varepsilon^{-\alpha} \left|u_1^0\right|_\infty |u_1^\varepsilon|_{L^2(\Gamma_{\varepsilon^\alpha})} \left|u_2^\varepsilon \rho'\left(\frac{y}{\varepsilon^\alpha}\right)\right|_{L^2(\{y<\varepsilon^\alpha\})} \\
&\le \kappa \left|u_{1y}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})} \left|u_2^\varepsilon \rho'\left(\frac{y}{\varepsilon^\alpha}\right)\right|_{L^2(\{y<\varepsilon^\alpha\})}
\end{aligned} \tag{3.33}$$

We define

$$\tilde{\rho}(y) = \int_y^{\varepsilon^\alpha} \left(\rho'\left(\frac{s}{\varepsilon^\alpha}\right)\right)^2 ds$$
$$= \varepsilon^\alpha \int_{y/\varepsilon^\alpha}^1 (\rho'(s))^2 ds\,.$$

Then

$$\tilde{\rho}(\varepsilon^\alpha) = 0, \quad \tilde{\rho}'(y) = -\left(\rho'\left(\frac{y}{\varepsilon^\alpha}\right)\right)^2 ;$$

thus

$$\begin{aligned}
\int_{\{y<\varepsilon^\alpha\}} (u_2^\varepsilon)^2 \left(\rho'\left(\frac{y}{\varepsilon^\alpha}\right)\right)^2 &= 2\int_{\{y<\varepsilon^\alpha\}} u_2^\varepsilon u_{2y}^\varepsilon \tilde{\rho}(y) \\
&= -2\int_{\{y<\varepsilon^\alpha\}} u_2^\varepsilon u_{1x}^\varepsilon \tilde{\rho}(y) \\
&= 2\int_{\{y<\varepsilon^\alpha\}} u_{2x}^\varepsilon u_1^\varepsilon \tilde{\rho}(y) \qquad (3.34)\\
&\le 2\left|u_{2x}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})} \left|u_1^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})} |\tilde{\rho}|_\infty \\
&\le \kappa\varepsilon^{2\alpha} \left|u_{2x}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})} \left|u_{1y}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})} .
\end{aligned}$$

Combining (3.33) and (3.34) and using (3.22), we deduce

$$\begin{aligned}
\left|\int_\Omega u_2^\varepsilon u_1^\varepsilon \varphi_{1y}^{\varepsilon,\alpha}\right| &\le \kappa\varepsilon^\alpha \left|u_{1y}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})}^{3/2} \left|u_{2x}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})}^{1/2} \\
&\le \frac{\varepsilon}{12}\left|u_{1y}^\varepsilon\right|^2 + \kappa\varepsilon^{4\alpha-3}\left|u_{2x}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})}^2 \qquad (3.35)\\
&= \frac{\varepsilon}{12}\left|w_{1y}^\varepsilon + u_{1y}^0 + \varphi_{1y}^{\varepsilon,\alpha}\right|^2 + \kappa\varepsilon^{4\alpha-3}\left|u_{2x}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})}^2 \\
&\le \frac{\varepsilon}{4}\left|w_{1y}^\varepsilon\right|^2 + \kappa\varepsilon^{1-\alpha} + \kappa\varepsilon^{4\alpha-3}\left|u_{2x}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})}^2 .
\end{aligned}$$

Therefore, thanks to (3.19)-(3.26), (3.30)-(3.35),

$$\frac{d}{dt}\left|w^\varepsilon\right|^2 + \varepsilon\left|\nabla w^\varepsilon\right|^2 \le \left|w^\varepsilon\right|^2 + \kappa\varepsilon^{1-\alpha} + \kappa\varepsilon^{\alpha/2} + \kappa\varepsilon^{4\alpha-3}\left|u_{2x}^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})}^2 . \qquad (3.36)$$

From (3.36) we conclude that if (2.12) holds with $\alpha = \alpha_0$, we have for $\alpha \ge \alpha_0$

$$\begin{aligned}
\left\|w^\varepsilon\right\|_{L^\infty(0,T;H)} &\le \kappa\varepsilon^{1/2-(\alpha/2)} + \kappa\varepsilon^{2\alpha-2\alpha_0} \qquad (3.37)\\
&\le \kappa\varepsilon^{2(1-\alpha_0)/5} \text{ for } \alpha = \frac{1+4\alpha_0}{5} > \alpha_0 .
\end{aligned}$$

When this is combined with (3.12) we deduce (2.16) again.

We could also replace u_{2x}^ε by u_{2xx}^ε in a suitable way. Indeed we notice

$$\begin{aligned}\int_{\Gamma_{\varepsilon\alpha}} (u_{2x}^\varepsilon)^2 &= -\int_{\Gamma_{\varepsilon\alpha}} u_{2xx}^\varepsilon u_2^\varepsilon \le |u_2^\varepsilon|_{L^2(\Gamma_{\varepsilon\alpha})} |u_{2xx}^\varepsilon|_{L^2(\Gamma_{\varepsilon\alpha})} \\ &\le \varepsilon^\alpha \left|u_{2y}^\varepsilon\right| |u_{2xx}^\varepsilon|_{L^2(\Gamma_{\varepsilon\alpha})} .\end{aligned}$$

Thus (3.35) can be rewritten as

$$\begin{aligned}\left|\int_\Omega u_2^\varepsilon u_1^\varepsilon \varphi_{1y}^{\varepsilon\alpha}\right| &\le \kappa\varepsilon^{5\alpha/4} \left|\nabla u^\varepsilon\right|^{7/4} |u_{2xx}^\varepsilon|_{L^2(\Gamma_{\varepsilon\alpha})}^{1/4} \\ &\le \frac{\varepsilon}{12} \left|\nabla u^\varepsilon\right|^2 + \kappa\varepsilon^{10\alpha-7} |u_{2xx}^\varepsilon|_{L^2(\Gamma_{\varepsilon\alpha})}^2 \\ &\le \frac{\varepsilon}{4} \left|\nabla w^\varepsilon\right|^2 + \kappa\varepsilon^{1-\alpha} + \kappa\varepsilon^{10\alpha-7} |u_{2xx}^\varepsilon|_{L^2(\Gamma_{\varepsilon\alpha})}^2 .\end{aligned} \tag{3.38}$$

Hence

$$\frac{d}{dt}\left|w^\varepsilon\right|^2 + \varepsilon\left|\nabla w^\varepsilon\right|^2 \le \left|w^\varepsilon\right|^2 + \kappa\varepsilon^{1-\alpha} + \kappa\varepsilon^{\alpha/2} + \kappa\varepsilon^{10\alpha-7} |u_{2xx}^\varepsilon|_{L^2(\Gamma_{\varepsilon\alpha})}^2 . \tag{3.39}$$

If (2.15) holds, namely

$$\left\|u_{2xx}^\varepsilon\right\|_{L^\infty(0,T;L^2(\Gamma_{\varepsilon\alpha}))} \le \kappa\varepsilon^{7/2-5\alpha_0},$$

we then deduce

$$\begin{aligned}\left\|w^\varepsilon\right\|_{L^\infty(0,T;H)} &\le \kappa\varepsilon^{1/2-(\alpha/2)} + \kappa\varepsilon^{5\alpha-5\alpha_0} \\ &\le \kappa\varepsilon^{5(1-\alpha_0)/11} \text{ for } \alpha = \frac{1+10\alpha_0}{11} > \alpha_0 .\end{aligned} \tag{3.40}$$

This further implies, thanks to (3.12)

$$\left\|u^\varepsilon - u^0\right\|_{L^\infty(0,T;H)} \le \kappa\varepsilon^{5(1-\alpha_0)/11} . \tag{3.41}$$

In a similar fashion we may state results based on assumptions on the growth rate of $\|\frac{\partial^k u_i^\varepsilon}{\partial x^k}\|_{L^2(0,T;L^2(\Gamma_{\varepsilon\alpha}))}$ for $i = 1, 2$, $k = 3, 4, \dots$, provided that the solution is regular enough for this kind of norm to be finite (say e.g. under certain compatibility conditions on the data, see Temam [T3]).

This completes the proof of Theorem 2. □

We now state and prove a corollary which might has interest on its own.

COROLLARY 1. *Let $(u^\varepsilon, p^\varepsilon)$ and (u^0, p^0) be the solutions of the Navier-Stokes and Euler equations above. Let $T > 0$ be fixed and assume that $\omega^\varepsilon = \operatorname{curl} u^\varepsilon$ satisfies the property*

$$\|\omega_x^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_6 \varepsilon^{-(3/2)+\delta}, \tag{3.42}$$

for some $0< \delta <1/4$ and a constant κ_6 independent of ε. Then we have

$$\left\|u^\varepsilon - u^0\right\|_{L^\infty(0,T;H)} \leq \kappa_7 \varepsilon^{\delta/6}, \tag{3.43}$$

for some constant κ_7 independent of ε.

PROOF. Recall the relationship between the velocity field u^ε and the vorticity ω^ε given by

$$\begin{aligned} \Delta u_2^\varepsilon &= \omega_x \quad \text{in} \quad \Omega, \\ u_2^\varepsilon &= 0 \quad \text{on} \quad \Gamma. \end{aligned}$$

Thus (3.42) implies

$$\|u_2^\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq \kappa \varepsilon^{-(3/2)+\delta},$$

and hence

$$\begin{aligned} \|u_{1x}^\varepsilon\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} &\leq \varepsilon^\alpha \left\|u_{1xy}^\varepsilon\right\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} \\ &\leq \kappa \varepsilon^{-(3/2)+\delta+\alpha}. \end{aligned}$$

Thus (2.13) is satisfied with $\alpha = 1 - \delta/2$.

This completes the proof of the corollary. □

REMARK 7. We may also localize (3.45) in the following way

$$\|\omega_x\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))} \leq \kappa \varepsilon^{-(3/2)+\delta}. \tag{3.44}$$

The proof of (3.44) is then based on utilizing a cut-off function in the proof of Corollary 1.

4. – Curved boundary case

The purpose of this section is to briefly describe how the results in the previous sections can be generalized to domains with curved boundaries.

It is then necessary to introduce a local orthogonal curvilinear coordinate system in the neighborhood of a given smooth domain Ω in $\mathbb{R}^2$. We assume that

$$\partial\Omega = \bigcup_{j=1}^{n} \Gamma_j$$

where the Γ_js are the connected components of $\partial\Omega$ which are smooth Jordan curves in $\mathbb{R}^2$, with Ω lying locally on one side of Γ_j.

Let $\delta > 0$ be chosen in such a way that all normals to $\partial\Omega$ do not intersect in $\Gamma_{3\delta}$. Let (ξ_1, ξ_2) be the natural orthogonal curvilinear coordinate system in $\Gamma_{2\delta}$, i.e. ξ_2 denotes the algebraic distance from the point to $\partial\Omega$ and ξ_1 is the abscissa on the curve at distance ξ_2 to $\partial\Omega$ ($\xi_2 > 0$ in Ω, $\xi_2 < 0$ outside Ω). It is easily checked that

$$ds^2 = (h(\xi_1, \xi_2))^2 (d\xi_1)^2 + (d\xi_2)^2 ,$$

with $h > 0$ and smooth in $\Gamma_{2\delta}$.

Let $\{e_1, e_2\}$ form a local normal basis such that e_1 and e_2 are unit vectors in the positive ξ_1, ξ_2 directions; hence on $\partial\Omega$, $e_1 = \tau(\xi_1, 0)$, $e_2 = n(\xi_1, 0)$. Let

$$u^\varepsilon = u_\tau^\varepsilon e_1 + u_n^\varepsilon e_2 .$$

Then

$$\begin{aligned}
(u^\varepsilon \cdot \nabla)\, u^\varepsilon &= \left(u^\varepsilon \cdot \nabla u_\tau^\varepsilon + \frac{u_n^\varepsilon}{h} u_\tau^\varepsilon \frac{\partial h}{\partial \xi_2}\right) e_1 \\
&\quad + \left(u^\varepsilon \cdot \nabla u_n^\varepsilon - \frac{u_\tau^\varepsilon}{h} u_\tau^\varepsilon \frac{\partial h}{\partial \xi_2}\right) e_2 \\
&= \left(\frac{u_\tau^\varepsilon}{h}\frac{\partial u_\tau^\varepsilon}{\partial \xi_1} + u_n^\varepsilon \frac{\partial u_\tau^\varepsilon}{\partial \xi_2} + \frac{u_n^\varepsilon u_\tau^\varepsilon}{h}\frac{\partial h}{\partial \xi_2}\right) e_1 \qquad (4.1)\\
&\quad + \left(\frac{u_\tau^\varepsilon}{h}\frac{\partial u_n^\varepsilon}{\partial \xi_1} + u_n^\varepsilon \frac{\partial u_n^\varepsilon}{\partial \xi_2} - \frac{(u_\tau^\varepsilon)^2}{h}\frac{\partial h}{\partial \xi_2}\right) e_2
\end{aligned}$$

$$\operatorname{div} u^\varepsilon = \frac{1}{h}\left(\frac{\partial u_\tau^\varepsilon}{\partial \xi_1} + \frac{\partial (h u_n^\varepsilon)}{\partial \xi_2}\right) = 0 ,$$

i.e.

$$\frac{\partial u_\tau^\varepsilon}{\partial \xi_1} + h \frac{\partial u_n^\varepsilon}{\partial \xi_2} + u_n^\varepsilon \frac{\partial h}{\partial \xi_2} = 0 . \qquad (4.2)$$

For the forms of other common differential operators one may consult, for instance, the Appendix of [Ba]. Now let τ be the unit tangential vector on $\partial\Omega$ (counterclockwise on an outer boundary and clockwise on an inner boundary), and define the stream line function $\psi^{\varepsilon,\alpha}$ as (see R. Temam and X. Wang [TW2])

$$\psi^{\varepsilon,\alpha}(\xi_1,\xi_2) = u^0(t;\xi_1,0)\cdot\tau(\xi_1,0)\int_0^{\xi_2}\rho\left(\frac{s}{\varepsilon^\alpha}\right)ds\,, \tag{4.3}$$

for $\xi_2\in(-2\delta,2\delta)$ and $\delta<\varepsilon^{1/2}$, $\alpha\geq\frac{1}{2}$. The velocity field $\varphi^{\varepsilon,\alpha}$ is defined as

$$\begin{aligned}\varphi^{\varepsilon,\alpha}(\xi_1,\xi_2) &= \operatorname{curl}\ \psi^{\varepsilon,\alpha}(\xi_1,\xi_2)\\ &= u^0(\xi_1,0)\cdot\tau(\xi_1,0)\rho\left(\frac{\xi_2}{\varepsilon^\alpha}\right)e_1\\ &\quad -\frac{1}{h(\xi_1,\xi_2)}\frac{\partial}{\partial\xi_1}(u^0(\xi_1,0)\cdot\tau(\xi_1,0))\int_0^{\xi_2}\rho\left(\frac{s}{\varepsilon^\alpha}\right)ds\,e_2\,.\end{aligned} \tag{4.4}$$

Again $w^\varepsilon = u^\varepsilon - u^0 - \varphi^{\varepsilon,\alpha}$ satisfies the equations (3.15)-(3.18) with the new $\varphi^{\varepsilon,\alpha}$ defined above.

It is easily checked that (3.19)-(3.22) remain valid. Then instead of (3.23) we write:

$$\begin{aligned}\int_\Omega(u^\varepsilon\cdot\nabla)\varphi^{\varepsilon,\alpha}w^\varepsilon &= \int_\Omega(u^\varepsilon\cdot\nabla)\varphi^{\varepsilon,\alpha}\cdot u^\varepsilon - \int_\Omega(u^\varepsilon\cdot\nabla)\varphi^{\varepsilon,\alpha}\cdot u^0\\ &= -\int_\Omega(u^\varepsilon\cdot\nabla)u^\varepsilon\cdot\varphi^{\varepsilon,\alpha} + \int_\Omega(u^\varepsilon\cdot\nabla)u^0\cdot\varphi^{\varepsilon,\alpha}\\ &= -\int_\Omega u^0(\xi_1,0)\cdot\tau(\xi_1,0)\rho\left(\frac{\xi_2}{\varepsilon^\alpha}\right)\\ &\qquad\cdot\left(\frac{u^\varepsilon_\tau}{h}\frac{\partial u^\varepsilon_\tau}{\partial\xi_1} + u^\varepsilon_n\frac{\partial u^\varepsilon_\tau}{\partial\xi_2} + u^\varepsilon_n u^\varepsilon_\tau\frac{\partial h}{\partial\xi_2}\right)\\ &\quad -\int_\Omega\frac{1}{h}\frac{\partial}{\partial\xi_1}(u^0(\xi_1,0)\cdot\tau(\xi_1,0))\int_0^{\xi_2}\rho\left(\frac{s}{\varepsilon^\alpha}\right)ds\\ &\qquad\cdot\left(\frac{u^\varepsilon_\tau}{h}\frac{\partial u^\varepsilon_n}{\partial\xi_1} + u^\varepsilon_n\frac{\partial u^\varepsilon_n}{\partial\xi_2} - \frac{(u^\varepsilon_\tau)^2}{h}\frac{\partial h}{\partial\xi_2}\right) + \int_\Omega(u^\varepsilon\cdot\nabla)u^0\cdot\varphi^{\varepsilon,\alpha}\end{aligned} \tag{4.5}$$

$$\begin{aligned}\left|\int_\Omega(u^\varepsilon\cdot\nabla)u^0\varphi^{\varepsilon,\alpha}\right| &\leq |u^\varepsilon|\left|\nabla u^0\right|_\infty|\varphi^{\varepsilon,\alpha}|\\ &\leq \kappa\varepsilon^{\alpha/2}\,,\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&\left|\int_\Omega u^0(\xi_1,0)\cdot\tau(\xi_1,0)\rho\left(\frac{\xi_2}{\varepsilon^\alpha}\right)\frac{u_\tau^\varepsilon}{h}\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right| \\
&\quad\le \kappa\left|u_\tau^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})}\left|\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right|_{L^2(\Gamma_{\varepsilon^\alpha})} \\
&\quad\le \kappa\varepsilon^\alpha\left|\frac{\partial u_\tau^\varepsilon}{\partial\xi_2}\right|\left|\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right|_{L^2(\Gamma_{\varepsilon^\alpha})} \\
&\quad\le \kappa\varepsilon^\alpha\left(\left|\nabla w^\varepsilon\right|+\left|\nabla u^0\right|+\left|\nabla\varphi^{\varepsilon,\alpha}\right|\right)\left|\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right|_{L^2(\Gamma_{\varepsilon^\alpha})} \\
&\quad\le \frac{\varepsilon}{8}\left|\nabla w^\varepsilon\right|^2+\kappa\varepsilon^{2\alpha-1}\left|\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right|^2_{L^2(\Gamma_{\varepsilon^\alpha})}+\kappa\varepsilon^{1-\alpha}\,,
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
&\left|\int_\Omega u^0(\xi_1,0)\cdot\tau(\xi_1,0)\rho\left(\frac{\xi_2}{\varepsilon^\alpha}\right)u_n^\varepsilon u_\tau^\varepsilon\frac{\partial h}{\partial\xi_2}\right| \\
&\quad\le \kappa\left|u^\varepsilon\right|^2_{L^2(\Gamma_{\varepsilon^\alpha})} \\
&\quad\le \kappa\varepsilon^{2\alpha}\left|\nabla u^\varepsilon\right|^2 \\
&\quad\le \kappa\varepsilon^{2\alpha}\left(\left|\nabla w^\varepsilon\right|^2+\left|\nabla u^0\right|^2+\left|\nabla\varphi^{\varepsilon,\alpha}\right|^2\right) \\
&\quad\le \frac{\varepsilon}{8}\left|\nabla w^\varepsilon\right|^2+\kappa\varepsilon^\alpha\,,
\end{aligned}
\tag{4.8}
$$

$$
\left|\int_\Omega\frac{1}{h}\frac{\partial}{\partial\xi_1}(u^0(\xi_1,0)\cdot\tau(\xi_1,0))\int_0^{\xi_2}\rho\left(\frac{s}{\varepsilon^\alpha}\right)ds\cdot\frac{(u_\tau^\varepsilon)^2}{h}\frac{\partial h}{\partial\xi_2}\right|\le\kappa\varepsilon^\alpha\,,
\tag{4.9}
$$

$$
\begin{aligned}
&\left|\int_\Omega\frac{1}{h}\frac{\partial}{\partial\xi_1}(u^0(\xi_1,0)\cdot\tau(\xi_1,0))\int_0^{\xi_2}\rho\left(\frac{s}{\varepsilon^\alpha}\right)ds\cdot\left(\frac{u_\tau^\varepsilon}{h}\frac{\partial u_n^\varepsilon}{\partial\xi_1}+u_n^\varepsilon\frac{\partial u_n^\varepsilon}{\partial\xi_2}\right)\right| \\
&\quad\le \kappa\varepsilon^\alpha\left|u^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})}\left|\nabla u^\varepsilon\right| \\
&\quad\le \kappa\varepsilon^{2\alpha}\left|\nabla u^\varepsilon\right|^2 \\
&\quad\le \kappa\varepsilon^{2\alpha-1}\,,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
&\left|\int_\Omega u^0(\xi_1,0)\cdot\tau(\xi_1,0)\rho\left(\frac{\xi_2}{\varepsilon^\alpha}\right)u_n^\varepsilon\frac{\partial u_\tau^\varepsilon}{\partial\xi_2}\right|\\
&\le\kappa\left|u_n^\varepsilon\right|_{L^2(\Gamma_{\varepsilon^\alpha})}\left|\nabla u^\varepsilon\right|\\
&\le\kappa\varepsilon^\alpha\left|\frac{\partial u_n^\varepsilon}{\partial\xi_2}\right|_{L^2(\Gamma_{\varepsilon^\alpha})}\left|\nabla u^\varepsilon\right|\\
&\le\text{(Thanks to (4.2))}\\
&\le\kappa\varepsilon^\alpha\left|\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right|_{L^2(\Gamma_{\varepsilon^\alpha})}\left(\left|\nabla w^\varepsilon\right|+\left|\nabla u^0\right|+\left|\nabla\varphi^{\varepsilon,\alpha}\right|\right)\\
&\le\frac{\varepsilon}{8}\left|\nabla w^\varepsilon\right|^2+\kappa\varepsilon^{2\alpha-1}\left|\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right|^2_{L^2(\Gamma_{\varepsilon^\alpha})}+\kappa\varepsilon^{1-\alpha}\,.
\end{aligned}
\tag{4.11}
$$

Combining (4.5)-(4.11) we deduce

$$
\left|\int_\Omega(u^\varepsilon\cdot\nabla)\varphi^{\varepsilon,\alpha}w^\varepsilon\right|\le\frac{\varepsilon}{4}\left|\nabla w^\varepsilon\right|^2+\kappa\varepsilon^{1-\alpha}+\kappa\varepsilon^{2\alpha-1}\left|\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right|^2_{L^2(\Gamma_{\varepsilon^\alpha})}\,.
\tag{4.12}
$$

This further implies, thanks to (3.19)-(3.22),

$$
\frac{d}{dt}\left|w^\varepsilon\right|^2+\varepsilon\left|\nabla w^\varepsilon\right|\le\kappa\left|w^\varepsilon\right|^2+\kappa\varepsilon^{1-\alpha}+\kappa\varepsilon^{2\alpha-1}\left|\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right|^2_{L^2(\Gamma_{\varepsilon^\alpha})}\,.
\tag{4.13}
$$

Using then the Gronwall lemma we conclude that Theorem 2 is valid under the following assumption similar to (2.13):

$$
\left\|\frac{\partial u_\tau^\varepsilon}{\partial\xi_1}\right\|_{L^2(0,T;L^2(\Gamma_{\varepsilon^\alpha}))}\le\kappa\varepsilon^{1/2-\alpha}\,.
\tag{4.14}
$$

For Theorem 1, we observe that (3.2) can be rewritten as

$$
\begin{aligned}
\int_\Omega\nabla p^\varepsilon\cdot\Delta u^\varepsilon&=-\int_\Gamma p^\varepsilon\cdot\Delta u^\varepsilon\cdot e_2\\
&\le\left|p^\varepsilon\right|_{H^{1/2}(\Gamma)}\left|\Delta u^\varepsilon\cdot n\right|_{H^{-1/2}(\Gamma)}\\
&\le\left|p^\varepsilon\right|_{H^{1/2}(\Gamma)}\left|\Delta u^\varepsilon\right|\\
&\le\frac{\varepsilon}{4}\left|\Delta u^\varepsilon\right|^2+\frac{1}{\varepsilon}\left|p^\varepsilon\right|^2_{H^{1/2}(\Gamma)}\,.
\end{aligned}
\tag{4.15}
$$

Thus the first part of Theorem 1 remains true without change.

Now for the second part of Theorem 1, we observe that

$$\begin{aligned}\Delta u^\varepsilon = &\frac{\partial}{\partial \xi_1}\left(\frac{1}{h}\right)\frac{\partial(hu_n^\varepsilon)}{\partial \xi_2}e_2 + \frac{1}{h}\frac{\partial^2(hu_n^\varepsilon)}{\partial \xi_2^2}e_2\\ &+\frac{\partial}{\partial \xi_1}\left(\frac{1}{h}\right)\frac{\partial u_\tau^\varepsilon}{\partial \xi_1}e_2 + \frac{1}{h}\frac{\partial^2 u_\tau^\varepsilon}{\partial \xi_1\partial \xi_2}e_2\\ &-\frac{1}{h}\frac{\partial}{\partial \xi_1}\left(\frac{1}{h}\right)\frac{\partial(hu_\tau^\varepsilon)}{\partial \xi_2}e_2 - \frac{1}{h^2}\frac{\partial^2(hu_\tau^\varepsilon)}{\partial \xi_1\partial \xi_2}e_2\\ &+\frac{1}{h}\frac{\partial}{\partial \xi_1}\left(\frac{1}{h}\right)\frac{\partial(hu_n^\varepsilon)}{\partial \xi_1}e_2 + \frac{1}{h^2}\frac{\partial^2 u_n^\varepsilon}{\partial \xi_2^2}e_2\\ &+\text{a scalar function times } e_1 .\end{aligned}$$

Thus

$$\begin{aligned}\int_\Omega \nabla p^\varepsilon\cdot\Delta u^\varepsilon = -\int_\Gamma p^\varepsilon\Bigg(&\frac{\partial}{\partial \xi_1}\left(\frac{1}{h}\right)\frac{\partial(hu_n^\varepsilon)}{\partial \xi_2} + \frac{1}{h}\frac{\partial^2(hu_n^\varepsilon)}{\partial \xi_2^2}\\ &+\frac{\partial}{\partial \xi_2}\left(\frac{1}{h}\right)\frac{\partial u_\tau^\varepsilon}{\partial \xi_1} + \frac{1}{h}\frac{\partial^2 u_\tau^\varepsilon}{\partial \xi_1\partial \xi_2} - \frac{1}{h}\frac{\partial}{\partial \xi_1}\left(\frac{1}{h}\right)\frac{\partial(hu_\tau^\varepsilon)}{\partial \xi_1}\\ &-\frac{1}{h^2}\frac{\partial^2(hu_\tau^\varepsilon)}{\partial \xi_1\partial \xi_2} + \frac{1}{h}\frac{\partial}{\partial \xi_1}\left(\frac{1}{h}\right)\frac{\partial u_n^\varepsilon}{\partial \xi_1} + \frac{1}{h^2}\frac{\partial^2 u_n^\varepsilon}{\partial \xi_1^2}\Bigg)d\Gamma .\end{aligned}\tag{4.16}$$

Observe that

$$\begin{aligned}-\int_\Gamma p^\varepsilon\frac{1}{h}\frac{\partial(hu_n^\varepsilon)}{\partial \xi_2^2}d\Gamma = &\int_\Gamma p^\varepsilon\frac{1}{h}\frac{\partial^2(u_\tau^\varepsilon)}{\partial \xi_1\partial \xi_2}d\Gamma\\ &=-\int_\Gamma\left(\frac{\partial p^\varepsilon}{\partial \xi_1}\frac{1}{h} + p^\varepsilon\frac{\partial}{\partial \xi_1}\left(\frac{1}{h}\right)\right)\frac{\partial u_\tau^\varepsilon}{\partial \xi_2}d\Gamma\\ &\le \kappa\left|p^\varepsilon\right|_{H^1(\Gamma)}\left|\nabla u^\varepsilon\right|^{1/2}\left|\Delta u^\varepsilon\right|^{1/2}\\ &\le \frac{\varepsilon}{16}\left|\Delta u^\varepsilon\right|^2 + \kappa\left|\nabla u^\varepsilon\right|^2 + \kappa\varepsilon^{-1/2}\left|p^\varepsilon\right|^2_{H^1(\Gamma)} .\end{aligned}\tag{4.17}$$

Similar estimates hold for the other terms on the right-hand side of (4.16). Thus we conclude that

$$\left|\int_\Omega \nabla p^\varepsilon\cdot\Delta u^\varepsilon\right| \le \frac{\varepsilon}{2}\left|\Delta u^\varepsilon\right|^2 + \kappa\left|\nabla u^\varepsilon\right|^2 + \kappa\varepsilon^{-1/2}\left|p^2\right|^2_{H^1(\Gamma)}\tag{4.18}$$

and hence

$$\frac{d}{dt}\left|\nabla u^{\varepsilon}\right|^{2}+\varepsilon\left|\Delta u^{\varepsilon}\right|^{2}\leq\kappa\left|\nabla u^{\varepsilon}\right|^{2}+\kappa\varepsilon^{-1/2}\left|p^{\varepsilon}\right|_{H^{1}(\Gamma)}^{2}. \tag{4.19}$$

Hence the second part of Theorem 1 remains valid as well.

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