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Homogenization and Hyperbolicity

LUC TARTAR

In memory of Ennio De Giorgi

In the early 70s, working with François Murat on an academic question of optimal design, we had been led to rediscover something that Sergio Spagnolo had already done a few years before under the name of G-convergence, the name being a reminder of its relation with the convergence of Green kernels [Sp1], [Sp2]. Our ideas were quite different from his, and were not either along the line of his joint work with Ennio De Giorgi [DG&Sp], which was a first step towards creating the notion of Γ -convergence.

At a meeting in Roma in the Spring 1974, I had presented some results that we had obtained, and I boasted that our method was more powerful than the Italian one (I had forgotten to mention that these results had been obtained with François Murat [Ta1]). Last year, I was told that my claim had upset Ennio De Giorgi, and I had planned to apologize for that old comment on our next meeting, which unfortunately did not happen.

Although I have worked for many years on questions quite similar to some of those Ennio De Giorgi was interested in, I never really discussed with him about our different points of view on these questions. Despite my bold claim of 1974, I was too shy for initiating a discussion with him, but on at least one of my visits to Pisa, he had spontaneously told me about some question that he thought of interest. As this question was related to something that I had already started to think about a few years earlier, I gathered all the results that I knew about it when I was asked to write an article for a book dedicated to him [Ta2]. I had been struck on that occasion by the way Ennio De Giorgi was thinking about Mechanics, as he did not seem to know of any precise situation where the type of question that he was thinking about would apply, but he thought that it was important to study such questions. Indeed, it was.

As for myself, I had been a student at Ecole Polytechnique, at a time where the French education system was not yet destroyed, and therefore I had a course of “Mécanique Rationnelle” during the first year and a course of “Mécanique des Milieux Continus” during the second year, and it was indeed quite natural that we had been taught 18th Century Mechanics during the first year as it

only requires Ordinary Differential Equations, and that we had been taught 19th Century Mechanics during the second year as it requires some knowledge of Partial Differential Equations that we barely had at the time. After these two years at Ecole Polytechnique I had learned from Jacques-Louis Lions the state of the art in applying the methods of Functional Analysis for solving some linear and nonlinear partial differential equations, but although these equations were often connected to questions in Continuum Mechanics or Physics which I later made my goal to understand in a better way, I found that my thesis advisor had little interest in understanding more about Continuum Mechanics or Physics. As for Ennio De Giorgi, it had seemed to me that he had never really studied Continuum Mechanics or Physics, but I had felt that he had a deep philosophical interest in understanding what basic principles could be. As a consequence, the intuition of Ennio De Giorgi has helped him develop some interesting mathematical tools, which are often useful for some partial differential equations of Continuum Mechanics, but his approach has some limitations which must be overcome.

The notion of Γ -convergence that Ennio De Giorgi had developed was indeed an interesting new idea, which extended his work with Sergio Spagnolo [DG&Sp], but I did not use much that idea myself, as it did not fit so well into the program of research that I had already begun. François Murat had coined the word H-convergence to describe our approach, which differed from G-convergence by the fact that in considering sequences of solutions of $-\operatorname{div}(A^n \operatorname{grad}(u^n)) = f$ we were interested in the limit of both quantities $E^n = \operatorname{grad}(u^n)$ and $D^n = A^n \operatorname{grad}(u^n)$. That improvement on G-convergence is necessary in the case where A^n is not symmetric, but this different point of view is also adapted to treating more general equations, not necessarily elliptic, as one should not think that there is always an “energy” that should be minimized. One important difference between our points of view was then that Ennio De Giorgi was working with real functionals, the order relation on R playing an important role, while I wanted to work with partial differential equations together with a list of various interesting quantities to be identified, quantities which often happened to have a physical meaning (as E denoting an electric field, D a polarization field and $(E \cdot D)$ a density of electrostatic energy if one uses an interpretation in terms of Electrostatics, u being an electrostatic potential and f a density of charge). Because I had learned more about Continuum Mechanics, I knew of more general situations than the ones which Ennio De Giorgi had imagined, and it might be for that reason that I was often finding questions of Γ -convergence too restrictive, even in situations where I had failed to obtain results; one such example was the question of Homogenization in Finite Elasticity.

When I had first met John M. Ball at a meeting in Marseille in the Fall 1975, he had just succeeded in applying to realistic problems of Finite Elasticity the notion of quasiconvex functions in the sense of Morrey [Ba]. The advantage of using quasiconvex stored energy functions lies in their sequential weak lower semi-continuity, so useful for proving existence of solutions, but after learning about his approach, I thought that a more important question was to discover

which were the strain-stress relations such that for any sequence of equilibrium solutions u^n for which the strains and the stresses converge weakly, the weak limits automatically satisfy the same strain-stress relation. My idea was to use the Div-Curl lemma, which implies that

$$(1) \quad \sum_{j=1}^N \sigma_{ij}^n \frac{\partial u_k^n}{\partial x_j} \rightharpoonup \sum_{j=1}^N \sigma_{ij}^\infty \frac{\partial u_k^\infty}{\partial x_j} \text{ for every } i, k = 1, \dots, N,$$

while the monotonicity argument only used the information that

$$(2) \quad \sum_{i,j=1}^N \sigma_{ij}^n \frac{\partial u_i^n}{\partial x_j} \rightharpoonup \sum_{i,j=1}^N \sigma_{ij}^\infty \frac{\partial u_i^\infty}{\partial x_j}.$$

A year later, I knew of more general relations implied by the general theory of Compensated Compactness that I had developed with François Murat, but I could not discover a “natural” class, like that of monotone relations for equations of the form $-\text{div}(F(\text{grad}(u))) = f$ for which I had described Homogenization questions in my cours Peccot in the Spring 1977. I had then failed to find a “natural” class stable by Homogenization for Finite Elasticity, as I reported in the proceedings of a meeting in Rio de Janeiro in the Summer 1977 [Ta3], but as was pointed out to me much later (by Gianni Dal Maso, I think), one can apply Γ -convergence to solve that Homogenization problem in Finite Elasticity, and I must admit then that Γ -convergence is a powerful tool, but this result has a few defects, that I still do not know how to correct at the moment.

The first defect is that the Γ -convergence approach does not apply to the evolution problem. The second defect is that it only considers sequences of global minimizers and does not say what would happen to sequences of other equilibrium solutions. Of course, it says nothing about which strain-stress relations one could obtain at the limit, as one does not yet know which strain-stress relations are gradients of quasiconvex functions. The third defect is that the Γ -convergence approach assumes indeed that stored energy functions are quasiconvex and, apart from mathematical convenience, I have never heard any argument in favour of quasiconvexity. It is certainly a physical requirement that the evolution problem should have the finite propagation speed property, and plane wave solutions for the linearized problem do travel at finite speed if one imposes only the Legendre-Hadamard condition, equivalent to the rank-one convexity of the stored energy function (a condition weaker than quasiconvexity, and different from it in dimension $N \geq 3$, according to a counter-example of Vladimír Šverák [Sv]). The fourth defect is that the Γ -convergence approach only applies to hyperelastic materials, i.e. materials whose strain-stress relation is associated to a stored energy function, but that could be a purely academic remark, as it might be that all real materials are indeed hyperelastic (for the evolution problem to be well posed, for example).

Of course some of these criticisms may seem unfair as even now very little is known by any approach for what concerns hyperbolic systems in more

than one space variable, but when I was writing [Ta3] twenty years ago in the Summer 1977 I had also criticized my own approach for not being able to take into account entropy conditions, and it was only in the Fall 1977 that I had understood how to treat entropy conditions, in the same way than constitutive relations [Ta4]. I doubt that Ennio De Giorgi knew much about hyperbolic systems of conservation laws, and he was probably not aware of my criticisms anyway, but it would have been interesting to know his insight on that subject, because among the equations of Continuum Mechanics hyperbolic systems of conservation laws play a crucial role, and because the subject is considered difficult enough to have scared away many good mathematicians, so that any new idea would be welcome. I had often thought that the class of quasilinear hyperbolic systems as described by Peter Lax was too large, but until recently I had no idea about what conditions to add to the hyperbolicity of the linearized problem. Recently, I had an idea which was explored for the case of Linearized Elasticity by one of my students, Sergio Gutiérrez [Gu], and there might be interesting extensions to nonlinear problems.

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Which materials should one use in Linearized Elasticity?

Linearized Elasticity consists in looking for a displacement u satisfying the equilibrium equations

$$(3) \quad - \sum_{j=1}^N \frac{\partial \sigma_{ij}}{\partial x_j} = f_i \text{ for } i = 1, \dots, N,$$

together with some boundary conditions, the symmetric Cauchy stress tensor σ satisfying the constitutive relations

$$(4) \quad \sigma_{ij} = \sum_{k,l=1}^N C_{ijkl} \varepsilon_{kl} \text{ for } i, j = 1, \dots, N,$$

and the linearized strain tensor ε being defined by

$$(5) \quad \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \text{ for } i, j = 1, \dots, N.$$

Of course, one assumes that C_{ijkl} is unchanged by permuting i and j or by permuting k and l (one adds invariance under the exchange of ij and kl for having a quadratic stored energy function).

A linear system of partial differential equations is said to be elliptic if, for constant coefficients, the equation is easily solved by Fourier transform, and in the case of (3)-(4)-(5) it means that for every $\xi \in R^N \setminus 0$ the linear mapping $\widehat{u} \mapsto \widehat{v}$ is invertible, where for $i = 1, \dots, N$ one has

$$(6) \quad \widehat{v}_i = \sum_{j=1}^N \xi_j \left(\sum_{k,l=1}^N C_{ijkl} \frac{1}{2} (\widehat{u}_k \xi_l + \widehat{u}_l \xi_k) \right).$$

Using the symmetry in k and l , (6) means

$$(7) \quad \begin{aligned} \widehat{v}_i &= \sum_{k=1}^N A_{ik}(\xi) \widehat{u}_k \text{ for } i = 1, \dots, N \\ A_{ik}(\xi) &= \sum_{j,l=1}^N C_{ijkl} \xi_j \xi_l \text{ for } i, k = 1, \dots, N, \end{aligned}$$

and so ellipticity consists in imposing that the *acoustic tensor* $\mathcal{A}(\xi)$ is invertible for every $\xi \neq 0$. The *strong ellipticity condition* consists in imposing that $\mathcal{A}(\xi)$ is positive definite for every $\xi \neq 0$, i.e. there exists $\alpha > 0$ such that

$$(8) \quad \sum_{i,j,k,l=1}^N C_{ijkl} \lambda_i \xi_j \lambda_k \xi_l \geq \alpha |\lambda|^2 |\xi|^2 \text{ for all } \lambda, \xi \in R^N,$$

the Legendre-Hadamard condition.

The *very strong ellipticity condition* consists in imposing that there exists $\alpha > 0$ such that

$$(9) \quad \sum_{i,j,k,l=1}^N C_{ijkl} M_{ij} M_{kl} \geq \alpha |M|^2 \text{ for all symmetric matrices } M.$$

In the case of isotropic materials,

$$(10) \quad \begin{aligned} \sigma_{ij} &= 2\mu \varepsilon_{ij} + \lambda \delta_{ij} \sum_{k=1}^N \varepsilon_{kk} \text{ for } i, j = 1, \dots, N, \text{ i.e.} \\ C_{ijkl} &= \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \lambda \delta_{ij} \delta_{kl}, \end{aligned}$$

the acoustic tensor is

$$(11) \quad \mathcal{A}(\xi) = \mu |\xi|^2 I + (\lambda + \mu) \xi \otimes \xi,$$

so that ellipticity means that the Lamé parameters λ, μ satisfy $\mu \neq 0$ and

$2\mu + \lambda \neq 0$, strong ellipticity means that they satisfy $\mu > 0$ and $2\mu + \lambda > 0$ while very strong ellipticity means that they satisfy $\mu > 0$ and $2\mu + N\lambda > 0$.

The general theory of Homogenization (as I have developed it with François Murat) applied to Linearized Elasticity uses Lax-Milgram lemma, i.e. it uses V -ellipticity for a space V satisfying $(H_0^1(\Omega))^N \subset V \subset (H^1(\Omega))^N$. In order to talk about the effective properties of a mixture (corresponding to a sequence of coefficients C_{ijkl}^n), it is important that these properties be local, and certainly the effective coefficients should not depend upon which boundary conditions are used, i.e. they should be independent of which space V is used. This poses no problem if one mixes only materials whose strain-stress relation is very strongly elliptic with the same α , and coefficients can be chosen to be discontinuous in that case, but there is a problem if one uses other materials: there are examples, using materials whose strain-stress relation is only strongly elliptic, for which one only knows how to describe the case of Dirichlet conditions, i.e. $V = (H_0^1(\Omega))^N$, and for some particular class of discontinuous coefficients. One may think that I could certainly do better if I was more clever, but there is actually a deeper problem here.

Independently of creating a general theory of Homogenization, one can compute the effective coefficients C_{ijkl}^{eff} for layered mixtures, under the simple condition that if the layers are perpendicular to ξ , all the materials used are such that their acoustic tensors satisfy $\mathcal{A}(\xi) \geq \alpha I$ for the same $\alpha > 0$. I had heard in 1975 that Mc Connell had computed the formulas for layered materials in Linearized Elasticity, analogous to the formulas that François Murat had obtained for a diffusion equation, but with more technical computations of Linear Algebra [MC]. I had noticed later that one could explain all Homogenization questions for layered media, for many linear equations not necessarily elliptic, by an application of the Div-Curl lemma, and I had first described the method in a nonlinear setting in the Spring 1979 when, as a consultant for INRIA with Georges Duvaut, we had been asked questions about effective elastic properties of mixtures of steel and rubber. I only mentioned the method in writing in [Ta5], and as I had chosen to illustrate my general method by an example of Linearized Elasticity, I was given later an earlier reference concerning Linearized Elasticity (by Backus if my memory is correct) by R. Kohn, who obviously had not understood what I had said.

Let us come back to Linearized Elasticity, and assume that the Homogenization theory can be extended to a class \mathcal{C} included in the class \mathcal{S} of all materials with a strong ellipticity strain-stress relation, and including the class $\mathcal{V}\mathcal{S}$ of all materials with a very strong ellipticity strain-stress relation, i.e. we assume that the effective coefficients of any mixture of materials from \mathcal{C} correspond to a material in the class \mathcal{C} . In particular, if the materials involved in a layered mixture all belong to \mathcal{C} , the formula giving the effective coefficients of the mixture should give a material in \mathcal{C} , and that remark gives us a way to identify some materials in \mathcal{S} that cannot belong to any such extension \mathcal{C} . Indeed, if one mixes a material with coefficients $C_0 \in \mathcal{S}$ with a material with coefficients $C_* \in \mathcal{V}\mathcal{S} \subset \mathcal{C}$ and one finds a material with coefficients C_1 outside

\mathcal{S} , we can be sure that C_0 does not belong to any extension \mathcal{C} , and we will say that C_0 has an index of badness 1 (the coefficients outside \mathcal{S} having an index of badness 0); if C_0 is such that C_1 has an index of badness 1, then it does not belong to any extension \mathcal{C} and has an index of badness 2, and so on.

In his thesis [Gu], Sergio Gutiérrez studied in dimension 2 or 3, which isotropic materials could be rejected by this criterium. In dimension $N = 2$, he found that the materials satisfying $\mu > 0, 2\mu + \lambda > 0, 2\mu + 2\lambda < 0$ (i.e. isotropic materials corresponding to strongly elliptic but not very strongly elliptic strain-stress relations, but without the boundary between these) have an index of badness 1. In dimension $N = 3$, he found that the materials satisfying $\mu > 0, 2\mu + \lambda > 0, 2\mu + 2\lambda < 0$ have an index of badness 1, and those satisfying $\mu > 0, 2\mu + 2\lambda \geq 0, 2\mu + 3\lambda < 0$ have an index of badness 2 (or at most 2) [Gu].

Of course, one reason why I want to reject everything outside \mathcal{S} is that \mathcal{S} corresponds to materials for which plane waves propagate at finite speed, and therefore, at least for the isotropic materials considered by Sergio Gutiérrez, I want to reject the materials corresponding to strongly elliptic but not very strongly elliptic strain-stress relations because if they existed one could construct some unrealistic materials by one or two laminations (notice that I also assume that all materials in $\mathcal{V}\mathcal{S}$ are accepted).

It would be a more convincing argument if one could avoid the limiting procedure of the laminations, and show for example that with a particular interface between two materials with coefficients C_0 and C_* the evolution problem is ill posed (or is well posed but does not have the finite propagation speed property, because of surface waves propagating too fast along the interface).

The mathematical theorem is that no class \mathcal{C} which is stable by Homogenization and contains all the (stable) class $\mathcal{V}\mathcal{S}$ can contain any isotropic material whose strain-stress relation is strongly elliptic but not very strongly elliptic (apart from the limiting cases perhaps), and probably it cannot contain any such anisotropic material either (but I do not know if Sergio Gutiérrez has extended his computations to that case), i.e. \mathcal{C} must probably coincide with $\mathcal{V}\mathcal{S}$. The physical interpretation of that result which I propose is that isotropic (and probably anisotropic) materials whose strain-stress relation is strongly elliptic but not very strongly elliptic should be considered unrealistic. One should notice that this argument of rejection goes much farther than the usual arguments invoking "Thermodynamics", which only impose $\mu > 0$ for isotropic materials, I believe.

If one could extend this criterium to Finite Elasticity, for example in the hyperelastic case, accepting the convex functions and rejecting the functions which do not satisfy the Legendre-Hadamard condition, and using the formula for layered mixtures as a way to create new materials, one might discover a natural class stable by Homogenization: if only convex functions remain it would mean that there is no generalization of Homogenization in the sense that I have imagined, while if all quasiconvex functions remain, I would have fallen into my own trap, but I would have learned how quasiconvexity could be considered a perfectly valid hypothesis from a physical point of view.

Homogenization of the wave equation

Let us consider a sequence of solutions u^n of wave equations

$$(12) \quad \rho^n \frac{\partial^2 u^n}{\partial t^2} - \operatorname{div}(A^n \operatorname{grad}(u^n)) = f^n \text{ in } \Omega \times (0, T),$$

such that

$$(13) \quad u^n \rightharpoonup u^\infty \text{ in } H_{\text{loc}}^1(\Omega \times (0, T)) \text{ weak,}$$

and assume that ρ^n and A^n are independent of t and satisfy

$$(14) \quad \begin{aligned} \rho^n &\rightharpoonup \rho^\infty \text{ in } L_{\text{loc}}^\infty(\Omega) \text{ weak } \star, \\ A^n &\text{ H-converges to } A^{\text{eff}}, \end{aligned}$$

and that f^n satisfies

$$(15) \quad f^n \rightarrow f^\infty \text{ in } H_{\text{loc}}^{-1}(\Omega \times (0, T)) \text{ strong.}$$

Then u^∞ satisfies the wave equation

$$(16) \quad \rho^\infty \frac{\partial^2 u^\infty}{\partial t^2} - \operatorname{div}(A^{\text{eff}} \operatorname{grad}(u^\infty)) = f^\infty \text{ in } \Omega \times (0, T).$$

Indeed for every $\varphi \in \mathcal{D}(0, T)$, the sequence U^n defined by

$$(17) \quad U^n(x) = \int_0^T u^n(x, t) \varphi(t) dt \text{ for a.e. } x \in \Omega,$$

converges in $H_{\text{loc}}^1(\Omega)$ weak to U^∞ defined from u^∞ in a similar way, and satisfies an equation

$$(18) \quad \begin{aligned} -\operatorname{div}(A^n \operatorname{grad}(U^n)) &= g^n \text{ in } \Omega, \\ g^n &\rightarrow g^\infty \text{ in } H_{\text{loc}}^{-1}(\Omega) \text{ strong,} \end{aligned}$$

so that

$$(19) \quad A^n \operatorname{grad}(U^n) \rightharpoonup A^{\text{eff}} \operatorname{grad}(U^\infty) \text{ in } (L_{\text{loc}}^2(\Omega))^N \text{ weak,}$$

and therefore, by varying φ , one has

$$(20) \quad A^n \operatorname{grad}(u^n) \rightharpoonup A^{\text{eff}} \operatorname{grad}(u^\infty) \text{ in } (L_{\text{loc}}^2(\Omega \times (0, T)))^N \text{ weak.}$$

Notice that this argument does not use any symmetry for A^n , but in (14) it is implicitly assumed that there exists $\alpha > 0$ such that $(A^n \eta, \eta) \geq \alpha |\eta|^2$ a.e. for all $\eta \in \mathbb{R}^N$; the argument does not use any positivity for ρ^n either, so

that I should not even speak of wave equations at this level. In order to say something relevant to wave equations, I assume that

$$(21) \quad \rho^n(x) \geq \rho_- > 0 \text{ a.e. } x \in \Omega,$$

and that all A^n are symmetric, and one then has a wave equation, for which existence and uniqueness are known if one uses the boundary conditions corresponding to a space V such that $H_0^1(\Omega) \subset V \subset H^1(\Omega)$, if initial conditions are of the form

$$(22) \quad \begin{aligned} u^n(\cdot, 0) &= v^n \in V, \text{ with } v^n \rightharpoonup v^\infty \text{ in } V \text{ weak,} \\ \rho^n \frac{\partial u^n}{\partial t}(\cdot, 0) &= w^n \in L^2(\Omega), \text{ with } w^n \rightharpoonup w^\infty \text{ in } L^2(\Omega) \text{ weak,} \end{aligned}$$

and if the sequence f^n satisfies

$$(23) \quad \begin{aligned} f^n &\text{ stays bounded in } L^1(0, T; L^2(\Omega)) \\ f^n &\rightharpoonup f^\infty \text{ in } \mathcal{D}'(\Omega), f^\infty \in L^1(0, T; L^2(\Omega)). \end{aligned}$$

Then under these hypotheses, the limit u^∞ satisfies the equation with the coefficients ρ^∞ and A^{eff} , the right hand side f^∞ , and the initial conditions v^∞, w^∞ .

I am interested now in using the finite propagation speed property of the wave equation, and I will assume that $\Omega = R^N$ (in order to avoid reflection effects on $\partial\Omega$), and I want to use the simple fact that if c^n is the *maximum speed of propagation* for the wave equation with ρ^n, A^n and c^{eff} is the maximum speed of propagation for the wave equation with $\rho^\infty, A^{\text{eff}}$, then one must have

$$(24) \quad c^{\text{eff}} \leq \liminf_{n \rightarrow \infty} c^n.$$

Indeed if the initial data v^∞, w^∞ have their support in a compact set K , one can choose $v^n = v^\infty, w^n = w^\infty$ for all n , and one has $u^n(x, t) = 0$ if $\text{dist}(x, K) > t c^n$; if a subsequence satisfies $c^n \rightarrow c^\infty$, then one has $u^\infty(x, t) = 0$ if $\text{dist}(x, K) > t c^\infty$, and therefore $c^{\text{eff}} \leq c^\infty$.

A more precise statement is obtained by replacing c^n by $c^n(\xi)$ the *maximum speed of propagation in the direction ξ* (with $|\xi| = 1$), i.e. using the fact that if the initial data have their support included in the strip $\{x : z_1 \leq (x, \xi) \leq z_2\}$, then at any time $t > 0$ the solution u^n is 0 outside the strip $\{x : z_1 - t c^n(\xi) \leq (x, \xi) \leq z_2 + t c^n(\xi)\}$.

Of course, a precise expression of $c(\xi)$ will be needed, and a good choice of the sequence ρ^n must be made, as this choice has no effect on A^{eff} , but might constrain the limit $c^{\text{eff}}(\xi)$ and therefore give some indirect information on A^{eff} .

Let φ be a smooth function in $R^N \times (0, T)$, and let us multiply (13) by

$$(25) \quad \varphi \frac{\partial u^n}{\partial t},$$

giving

$$\begin{aligned}
 (26) \quad & \frac{\partial}{\partial t} \left(\frac{\varphi \rho^n}{2} \left| \frac{\partial u^n}{\partial t} \right|^2 + \frac{\varphi}{2} (A^n \text{grad}(u^n) \cdot \text{grad}(u^n)) \right) - \text{div} \left(\varphi \frac{\partial u^n}{\partial t} A^n \text{grad}(u^n) \right) \\
 & = \varphi \frac{\partial u^n}{\partial t} f^n + \frac{\partial \varphi}{\partial t} \left(\frac{\rho^n}{2} \left| \frac{\partial u^n}{\partial t} \right|^2 + \frac{1}{2} (A^n \text{grad}(u^n) \cdot \text{grad}(u^n)) \right) \\
 & \quad - \frac{\partial u^n}{\partial t} \left(A^n \text{grad}(u^n) \cdot \text{grad}(\varphi) \right).
 \end{aligned}$$

In the case where $f^n = 0$, we want the right hand side of (26) to be ≤ 0 for all functions u^n not necessarily solutions of any equation, and this is done by choosing φ such that

$$\begin{aligned}
 (27) \quad & \frac{\partial \varphi}{\partial t} \leq 0 \\
 & (A^n \text{grad}(\varphi) \cdot \text{grad}(\varphi)) \leq \rho^n \left| \frac{\partial \varphi}{\partial t} \right|^2.
 \end{aligned}$$

The preceding inequality shows that

$$(28) \quad c^n(\xi) \leq \sup_x \sqrt{\frac{(A^n \xi \cdot \xi)}{\rho^n}}.$$

Indeed, choosing φ of the form $\varphi_0((x, \xi) - \kappa t)$ with $\kappa > 0$, (27) becomes

$$\begin{aligned}
 (29) \quad & \varphi'_0 \geq 0 \\
 & (A^n \xi \cdot \xi) \leq \rho^n \kappa^2 \text{ where } \varphi'_0 \neq 0,
 \end{aligned}$$

and by taking $\varphi_0(z) = 0$ for $z \leq z_2$ and $0 < \varphi_0(z) \leq 1$ and φ_0 increasing for $z > z_2$, one deduces that for $t > 0$ one has

$$(30) \quad \int_{R^N} \varphi \left(\frac{\rho^n}{2} \left| \frac{\partial u^n}{\partial t} \right|^2 + \frac{1}{2} (A^n \text{grad}(u^n) \cdot \text{grad}(u^n)) \right) dx \leq 0,$$

and therefore $u^n(x, t) = 0$ for $(x, \xi) - \kappa t > z_2$, once κ is such that (29) holds. Choosing instead a function $\varphi_0((x, \xi) + \kappa t)$, one deduces that $u^n(x, t) = 0$ for $(x, \xi) + \kappa t < z_1$, hence the upper estimate for $c^n(\xi)$.

Of course, the energy estimate for the wave equation is first derived for smooth coefficients and smooth initial data (and smooth right hand side), in which case the solution is more regular and all integrations by parts are valid, and the result obtained is kept at the limit in the case of discontinuous coefficients, which we are handling.

If one has $A^n \rightharpoonup A^\infty$ in $L^\infty(R^N; \mathcal{L}_s(R^N, R^N))$ weak \star , and if one chooses $\rho^n = (A^n \xi \cdot \xi)$ so that $\rho^\infty = (A^\infty \xi \cdot \xi)$, then $c^n(\xi) \leq 1$ and therefore $c^{\text{eff}}(\xi) \leq 1$, and one deduces that

$$(31) \quad (A^{\text{eff}} \xi \cdot \xi) \leq (A^\infty \xi \cdot \xi) \text{ a.e. } ,$$

if one has improved the analog of (29) into the more precise relation

$$(32) \quad c^{\text{eff}}(\xi) = \sup_x \sqrt{\frac{(A^{\text{eff}}\xi.\xi)}{\rho^\infty}}.$$

As the coefficients of the wave equation that I consider may be discontinuous, the only proof that I could think of consists in using a blow-up argument, a technique which Ennio De Giorgi had introduced for a quite different purpose, that of studying the regularity of solutions of elliptic equations.

Let us assume then that $c^{\text{eff}}(\xi) \leq 1$, and let us deduce that $(A^{\text{eff}}\xi.\xi) \leq \rho^\infty$ a.e. Taking the origin at a Lebesgue point of both A^{eff} and ρ^∞ , we rescale the wave equation by considering coefficients A_s, ρ_s defined by

$$(33) \quad A_s(x) = A^{\text{eff}}\left(\frac{x}{s}\right), \quad \rho_s(x) = \rho^\infty\left(\frac{x}{s}\right) \text{ a.e. } x \in R^N,$$

and the wave equation with coefficients ρ_s, A_s corresponds to a speed $c_s(\xi)$, which obviously coincides with $c^{\text{eff}}(\xi)$, as one sees easily by rescaling the solution and the initial data in a similar way. Letting s tend to ∞ , the coefficients converge strongly (in L^p_{loc} for every $p < \infty$), to their frozen value at 0, and as the speed cannot increase at the limit, one has $c_\infty \leq \lim c_s(\xi) = c^{\text{eff}}(\xi) \leq 1$, but the speed of propagation c_∞ for a wave equation with constant coefficient is easily computed, but as only an inequality is needed, it is enough to consider solutions of the form $u((x.\xi) - \kappa t)$, which satisfy a one dimensional equation whose explicit solutions have been known since D'Alembert.

The preceding argument gives then a new proof of a classical upper bound for A^{eff} , but I do not know of a similar derivation of a lower bound for A^{eff} based on hyperbolicity.

Homogenization of the time dependent Linearized Elasticity system

If instead of the wave equations (12), we consider now time dependent Linearized Elasticity systems

$$(34) \quad \begin{aligned} \rho^n \frac{\partial^2 u_i^n}{\partial t^2} - \sum_{j=1}^N \frac{\partial \sigma_{ij}^n}{\partial x_j} &= f_i^n \text{ in } \Omega \times (0, T) \text{ for } i = 1, \dots, N \\ \sigma_{ij}^n &= \sum_{k,l=1}^N C_{ijkl}^n \varepsilon_{kl}^n \text{ for } i, j = 1, \dots, N \\ \varepsilon_{ij}^n &= \frac{1}{2} \left(\frac{\partial u_i^n}{\partial x_j} + \frac{\partial u_j^n}{\partial x_i} \right) \text{ for } i, j = 1, \dots, N, \end{aligned}$$

such that

$$(35) \quad u_i^n \rightharpoonup u_i^\infty \text{ in } H_{loc}^1(\Omega \times (0, T)) \text{ for } i = 1, \dots, N,$$

and assume that ρ^n and all the C_{ijkl}^n are independent of t and satisfy

$$(36) \quad \begin{aligned} \rho^n &\rightharpoonup \rho^\infty \text{ in } L^\infty(\Omega) \text{ weak } \star, \\ C^n &\text{ H-converges to } C^{\text{eff}}, \end{aligned}$$

and that

$$(37) \quad f_i^n \rightarrow f_i^\infty \text{ in } H_{loc}^{-1}(\Omega \times (0, T)) \text{ strong, for } i = 1, \dots, N,$$

then u^∞ satisfies the equation

$$(38) \quad \begin{aligned} \rho^\infty \frac{\partial^2 u_i^\infty}{\partial t^2} - \sum_{j=1}^N \frac{\partial \sigma_{ij}^\infty}{\partial x_j} &= f_i^\infty \text{ in } \Omega \times (0, T) \text{ for } i = 1, \dots, N \\ \sigma_{ij}^\infty &= \sum_{k,l=1}^N C_{ijkl}^{\text{eff}} \varepsilon_{kl}^\infty \text{ for } i, j = 1, \dots, N. \end{aligned}$$

Notice that this argument does not use the symmetry for C_{ijkl}^n under the exchange of ij and kl , but in (36) it is implicitly assumed that the very strong ellipticity condition (9) holds with the same $\alpha > 0$; the argument does not use any positivity for ρ^n either. If one assumes then that the coefficients ρ^n satisfy (21) and that the coefficients C_{ijkl}^n are invariant under the exchange of ij and kl , then the evolution equation is well posed if one uses the boundary conditions corresponding to a space V such that $(H_0^1(\Omega))^N \subset V \subset (H^1(\Omega))^N$ for which Korn's inequality holds (which is automatically the case if $\Omega = R^N$ or if Ω is a bounded open set with Lipschitz boundary); of course, one needs to use initial conditions analogous to (22) and hypotheses on f^n analogous to (23).

We consider now the case $\Omega = R^N$ in order to discuss finite propagation speeds, and the same argument (24), and its analog for the $c^n(\xi)$ hold, as they are not based on which particular equation is used, as long as it has the finite propagation speed property.

Let φ be a smooth function on $R^N \times (0, T)$, and let us multiply the first equation #i of (34) by

$$(39) \quad \varphi \frac{\partial u_i^n}{\partial t}$$

and sum in i , giving

$$(40) \quad \begin{aligned} &\frac{\partial}{\partial t} \left(\frac{\varphi \rho^n}{2} \sum_{i=1}^N \left| \frac{\partial u_i^n}{\partial t} \right|^2 + \frac{\varphi}{2} \sum_{i,j=1}^N \sigma_{ij}^n \frac{\partial u_i^n}{\partial x_j} \right) - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(\varphi \sigma_{ij}^n \frac{\partial u_i^n}{\partial t} \right) \\ &= \varphi \sum_{i=1}^N f_i^n \frac{\partial u_i^n}{\partial t} + \frac{\partial \varphi}{\partial t} \left(\frac{\rho^n}{2} \sum_{i=1}^N \left| \frac{\partial u_i^n}{\partial t} \right|^2 + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij}^n \frac{\partial u_i^n}{\partial x_j} \right) \\ &\quad - \sum_{i,j=1}^N \frac{\partial \varphi}{\partial x_j} \left(\sigma_{ij}^n \frac{\partial u_i^n}{\partial t} \right). \end{aligned}$$

In the case where $f^n = 0$, we want the right hand side of (40) to be ≤ 0 for all functions u^n not necessarily solutions of any equation. In order to do this, we introduce the symmetric bilinear forms B_x^n defined on $N \times N$ matrices by

$$(41) \quad B_x^n(M, P) = \sum_{i,j,k,l=1}^N C_{ijkl}^n(x) M_{ij} P_{kl},$$

and the condition that the right hand side of (40) be ≤ 0 for all functions u^n corresponds to

$$(42) \quad \frac{\partial \varphi}{\partial t} (\rho^n |\eta|^2 + B_x^n(M, M)) - 2B_x^n(M, \eta \otimes \text{grad}(\varphi)) \leq 0,$$

for all $\eta \in R^N$ and all symmetric matrices M , because of the symmetry of σ^n . Using then the fact that, for a.e. $x \in R^N$, the form B_x^n is positive definite when restricted to symmetric matrices, and using the other symmetries of the coefficients C_{ijkl}^n , (42) means

$$(43) \quad \begin{aligned} & \frac{\partial \varphi}{\partial t} \leq 0, \\ & B(\eta \otimes \text{grad}(\varphi), \eta \otimes \text{grad}(\varphi)) \leq \rho^n |\eta|^2 \left| \frac{\partial \varphi}{\partial t} \right|^2 \text{ for all } \eta \in R^N. \end{aligned}$$

The preceding inequality shows that

$$(44) \quad c^n(\xi) \leq \sup_{x, |\eta|=1} \sqrt{\frac{(\mathcal{A}^n(\xi)\eta, \eta)}{\rho^n}},$$

and the blow-up argument mentioned before, and the use of functions of the form $u((x, \xi) - \kappa t)$, shows that one has equality in (44). If one defines then ρ^n by

$$(45) \quad \rho^n(x) = \sup_{|\eta|=1} (\mathcal{A}^n(\xi)\eta, \eta),$$

one has $c^n(\xi) = 1$, and therefore $c^{\text{eff}}(\xi) \leq 1$, i.e.

$$(46) \quad \sup_{|\eta|=1} (\mathcal{A}^{\text{eff}}(\xi)\eta, \eta) \leq \rho^\infty.$$

In the special case of isotropic materials, the acoustic tensor has the form (11), and therefore (45) corresponds to the choice

$$(47) \quad \rho^n = 2\mu^n + \lambda^n$$

and therefore, in the case where the effective material is itself isotropic, (46) means

$$(48) \quad 2\mu^{\text{eff}} + \lambda^{\text{eff}} \leq 2\mu^\infty + \lambda^\infty,$$

where the index ∞ corresponds to weak \star limits, but (48) is not as precise as the classical upper bound for the effective coefficients, which corresponds to

$$(49) \quad B_x^{\text{eff}}(M, M) \leq B_x^\infty(M, M) \text{ for all symmetric } M \text{ a.e. } x,$$

and as

$$(50) \quad B^n(M, M) = 2\mu^n \text{trace}(M^T M) + \lambda^n (\text{trace}(M))^2 \\ \text{for all symmetric } M \text{ a.e. } x,$$

in the case of isotropic materials, (49) corresponds to the bounds

$$(51) \quad \mu^{\text{eff}} \leq \mu^\infty \\ 2\mu^{\text{eff}} + N\lambda^{\text{eff}} \leq 2\mu^\infty + N\lambda^\infty,$$

which imply (48).

Homogenization of Maxwell's system

In 1981, during a meeting in New York, I was told about the following result of Schulgasser: in dimension $N = 3$, if an isotropic mixture with conductivity $a(x)$ has an (isotropic) effective conductivity $a^{\text{eff}}(x)$ and if the isotropic mixture with conductivity $b(x) = \frac{1}{a(x)}$ has an (isotropic) effective conductivity $b^{\text{eff}}(x)$, then one has $a^{\text{eff}}b^{\text{eff}} \geq 1$. I had immediately found a proof based on the finite propagation speed property for Maxwell's system, and I had mentioned it to George Papanicolaou and Joseph Keller. I was a little upset to discover later that George had written his own proof of that result (in an article with Kesten, I think), without mentioning mine, as I thought that if he had already done it before I had mentioned mine, he should have told me; however, his proof worked for $N \geq 3$, while mine had only been thought in connection with Maxwell's equation, in dimension $N = 3$.

My idea was to consider Maxwell's equation in a material with dielectric permittivity $\varepsilon(x) = a(x)$ and magnetic susceptibility $\mu(x) = b(x)$, and as the local velocity of Light $c(x)$ is defined by $\varepsilon\mu c^2 = 1$, one would have $c(x) = 1$, and therefore $c^{\text{eff}} \leq 1$, proving then that $\varepsilon^{\text{eff}}\mu^{\text{eff}} \geq 1$, i.e. $a^{\text{eff}}b^{\text{eff}} \geq 1$.

If we consider a sequence of solutions of Maxwell's system

$$\begin{aligned}
 (52) \quad & \operatorname{div}(B^n) = 0, \quad \frac{\partial B^n}{\partial t} + \operatorname{curl}(E^n) = 0 \text{ in } \Omega \times (0, T) \\
 & \operatorname{div}(D^n) = \rho^n, \quad -\frac{\partial D^n}{\partial t} + \operatorname{curl}(H^n) = j^n \text{ in } \Omega \times (0, T) \\
 & D^n = \varepsilon^n E^n, \quad B^n = \mu^n H^n \text{ in } \Omega \times (0, T),
 \end{aligned}$$

such that

$$\begin{aligned}
 (53) \quad & B^n \rightharpoonup B^\infty, D^n \rightharpoonup D^\infty, E^n \rightharpoonup E^\infty, H^n \rightharpoonup H^\infty \\
 & \text{in } (H^1_{\text{loc}}(\Omega \times (0, T)))^3 \text{ weak,}
 \end{aligned}$$

and assume that ε^n and μ^n are independent of t and satisfy

$$(54) \quad \varepsilon^n \text{ H-converges to } \varepsilon^{\text{eff}}, \quad \mu^n \text{ H-converges to } \mu^{\text{eff}},$$

and that

$$\begin{aligned}
 (55) \quad & \rho^n \rightarrow \rho^\infty \text{ in } H^{-1}_{\text{loc}}(\Omega \times (0, T)) \text{ strong,} \\
 & j^n \rightarrow j^\infty \text{ in } (H^{-1}_{\text{loc}}(\Omega \times (0, T)))^3 \text{ strong,}
 \end{aligned}$$

then one has

$$\begin{aligned}
 (56) \quad & \operatorname{div}(B^\infty) = 0, \quad \frac{\partial B^\infty}{\partial t} + \operatorname{curl}(E^\infty) = 0 \text{ in } \Omega \times (0, T) \\
 & \operatorname{div}(D^\infty) = \rho^\infty, \quad -\frac{\partial D^\infty}{\partial t} + \operatorname{curl}(H^\infty) = j^\infty \text{ in } \Omega \times (0, T) \\
 & D^\infty = \varepsilon^{\text{eff}} E^\infty, \quad B^\infty = \mu^{\text{eff}} H^\infty \text{ in } \Omega \times (0, T).
 \end{aligned}$$

Notice that this argument does not use the symmetry for ε^n and μ^n , but in (54) it is implicitly assumed that there exists $\alpha > 0$ such that $(\varepsilon^n \eta, \eta) \geq \alpha |\eta|^2$ and $(\mu^n \eta, \eta) \geq \alpha |\eta|^2$ a.e. for all $\eta \in R^N$, although in that first part of the argument one could change the sign of all μ^n for example. One should notice also that after integration in t against a smooth function φ , one obtains equations which are not of the form $-\operatorname{div}(A^n \operatorname{grad}(u^n)) = g^n$, as E^n and H^n are not curl free, and the full force of the Div-Curl lemma is needed.

Assuming now that ε^n and μ^n are symmetric, one can then prove existence and uniqueness of solutions under suitable initial data and boundary conditions (as described in the book of Georges Duvaut & Jacques-Louis Lions [Du&Li] for example). In order to study the finite propagation effects, I choose $\Omega = R^N$, and for a smooth function φ on $R^N \times (0, T)$, I multiply the equations in (52) by

φE^n and $-\varphi H^n$ and, after some algebraic manipulations with the ε_{ijk} symbol for cross products and curl, this gives

$$\begin{aligned}
 (57) \quad -\varphi(j^n \cdot E^n) &= \varphi\left(\frac{\partial B^n}{\partial t} \cdot H^n\right) + \varphi(\operatorname{curl}(E^n) \cdot H^n) \\
 &\quad + \varphi\left(\frac{\partial D^n}{\partial t} \cdot H^n\right) - \varphi(\operatorname{curl}(H^n) \cdot E^n) \\
 &= \frac{\partial}{\partial t} \left(\frac{\varphi(B^n \cdot H^n)}{2} + \frac{\varphi(D^n \cdot E^n)}{2} \right) \\
 &\quad - \frac{1}{2} \frac{\partial \varphi}{\partial t} \left((B^n \cdot H^n) + (D^n \cdot E^n) \right) \\
 &\quad + \operatorname{div}(\varphi(E^n \times H^n)) - (\operatorname{grad}(\varphi) \cdot E^n \times H^n).
 \end{aligned}$$

In the case where $j^n = 0$, I want to have

$$(58) \quad \frac{\partial \varphi}{\partial t} ((\mu^n H \cdot H) + (\varepsilon^n E \cdot E)) + 2(\operatorname{grad}(\varphi) \cdot E \times H) \leq 0$$

for all $E, H \in R^N$, and using functions $\varphi((x \cdot \xi) - \kappa t)$, one deduces that

$$(59) \quad c^n(\xi) \leq \sup_{x, E \neq 0, H \neq 0} \frac{2(\xi \cdot E \times H)}{(\mu^n H \cdot H) + (\varepsilon^n E \cdot E)},$$

and in the isotropic case (59) gives a bound independent of ξ

$$(60) \quad \sqrt{\varepsilon^n \mu^n} c^n(\xi) \leq 1.$$

The blow-up argument mentioned before and the use of functions of the form $E((x \cdot \xi) - \kappa t)$, $H((x \cdot \xi) - \kappa t)$, shows that one has equality in (59).

In the case of anisotropic $a^{\text{eff}} = \varepsilon^{\text{eff}}$, $b^{\text{eff}} = \mu^{\text{eff}}$, one needs to express that $c^{\text{eff}}(\xi) \leq 1$ for all ξ , and one such formula is

$$(61) \quad (a^{\text{eff}} E \cdot E)(b^{\text{eff}} H \cdot H) \geq |E \times H|^2 \text{ for all } E, H \in R^3.$$

Comments

The preceding examples consisted in looking for informations about the effective coefficients of a first equation by introducing a second equation of a different type, though not completely alien to the first, and certainly there is a lot more to be done in this direction. I have gathered these examples here because they are all related to the hyperbolic character of the new equation, but one may as well use a second equation which is not hyperbolic, and I hope in the near future to write down other results where hyperbolicity plays no role.

The last example is reminiscent of a method developed by David Bergman, who had considered the case of isotropic materials when the effective material is also isotropic [Be]. Needless to say, there is a lot more to be done in order to understand more general cases.

In the last years, I have never failed in my talks to warn about the defects of Linearized Elasticity, and as I have used this linear system here in order to explain some mathematical questions, I hope that it is clear to the reader that I have only considered these equations in connection with mathematical questions of hyperbolicity, which are not so well understood in nonlinear situations. One should understand the discussions using Linearized Elasticity as a mere training ground for the more realistic time dependent Finite Elasticity, which is a quasilinear system for which not enough is known. One important defect of Linearized Elasticity is that it assumes the gradient of the deformation to be near Identity, while in fact it is only near a rotation, and it gives unrealistic results in corners where the gradient of the solution of Linearized Elasticity often becomes infinite; it is often mentioned in that respect that one can still deduce some interesting results in corners due to the existence of an invariant of both Finite Elasticity and Linearized Elasticity, but this argument is valid for homogeneous bodies.

Can one explain spectroscopy?

After Sergio Spagnolo had successfully studied sequences of elliptic or parabolic problems

$$(62) \quad \begin{aligned} & -\operatorname{div}\left(A^n(x)\operatorname{grad}(u^n)\right) = f^n \text{ in } \Omega \\ & \frac{\partial u^n}{\partial t} - \operatorname{div}\left(A^n(x, t)\operatorname{grad}(u^n)\right) = f^n \text{ in } \Omega \times (0, T), \end{aligned}$$

with general bounded positive definite matrices A^n , he stumbled on the apparently similar case of second order “wave” equations with general bounded and symmetric positive definite matrices A^n

$$(63) \quad \frac{\partial^2 u^n}{\partial t^2} - \operatorname{div}\left(A^n(x, t)\operatorname{grad}(u^n)\right) = f^n \text{ in } \Omega \times (0, T),$$

because all available theorems giving existence and uniqueness of solutions (and satisfying suitable boundary conditions and initial data) assumed that the coefficients A^n were regular in t ; he worked then at understanding if one could define solutions of (63) without regularity hypotheses on the coefficients. I have already described the case where A^n does not depend on t , and I want to discuss now why it is important to study some cases of time dependent coefficients, in connection with classical experiments, which have resulted in the development of non classical theories, the experiments of spectroscopy.

In an experiment of spectroscopy, one sends Light through a gas, and therefore a natural "classical" setting for this problem is to consider Maxwell's system (for describing Light), and to study the question of multiple scattering of Light on the "atoms" or "molecules" constituting this gas. This is not an easy problem, and one stumbles immediately on a first difficulty due to the fact that the "atoms" or "molecules" constituting the gas, whatever they are, are supposed to be moving, and one should therefore develop a theory for Maxwell's system (or for the scalar wave equation as a first step), with coefficients that do vary with t , but those variations with respect to t should certainly not be arbitrary.

Let us consider, as a first step, the case of a rigid and homogeneous object traveling with fixed velocity V in an infinite and uniform medium; in that case the coefficients are of the form $A(x - Vt)$, and as A is discontinuous at the surface of the object, these coefficients do not satisfy the usual hypothesis that their derivative with respect to t is bounded, but one can easily get rid of this difficulty by performing a Galilean transformation in the case of the wave equation, or by performing a Lorentz transformation in the case of Maxwell's system, so that one is back to the case where the coefficients are independent of t . In the case of the wave equation in all R^N , that I would like to take as a model, the energy estimate would therefore be obtained by (formally) multiplying (63) by

$$(64) \quad \frac{\partial u}{\partial t} + \sum_{i=1}^N V_i \frac{\partial u}{\partial x_i},$$

and the coefficients would be partially regular and satisfy

$$(65) \quad \left(\frac{\partial}{\partial t} + \sum_{i=1}^N V_i \frac{\partial}{\partial x_i} \right) A \text{ is bounded.}$$

Of course, it would certainly be better if the velocity V stayed much smaller than the characteristic velocity of the wave equation itself, which in the case of Maxwell's system would mean that V stays much below the velocity of Light c . If there were a large number m of some of these rigid and homogeneous objects, with various shapes, each traveling at its own constant velocity, then the existence and uniqueness of a solution would follow easily from the finite propagation speed property of the wave equation, as long as collisions between these objects would be avoided. In that case, the energy estimate would still

be obtained by (formally) multiplying (63) by (64), and that will correspond to the coefficients satisfying (65), but V would be variable now.

Could one find a class of coefficients depending on (x, t) , generalizing the preceding case of a finite number of moving rigid inclusions avoiding collisions, for which existence, uniqueness and uniform bounds could be obtained? Could one extend that theory to the case of elastic bodies, for which one might have to introduce effects of electrostriction and magnetostriction? Could one answer questions of Homogenization for these classes of equations, and would they explain the effects that physicists have observed?

When I had written what I knew about memory effects in [Ta2], I had forgotten to mention that (around 1980) I had guessed that the absorption and spontaneous emission effects that physicists had invented, in order to explain what they were observing in experiments of spectroscopy, were but the sign of an effective equation having a memory effect. Mathematicians know that the fact that an equation has solutions is a different matter from the fact of choosing a proof of this fact among all the different proofs that have been found; on the contrary, physicists seem to believe that if one proof relies on a probability argument then it shows that Nature follows probabilistic rules. I could not tell what equations one should consider for describing these phenomena, but as a first step towards studying the propagation of waves in a material with moving objects in it, I had chosen to study first order hyperbolic equations with variable coefficients.

One part of the challenge is that one has no idea about what these moving objects are, as most of what physicists claim about particles, atoms, molecules, result from some dogmatic theory like Quantum Mechanics. I think that some of these dogmas, which had been thought natural at some time, are just mistakes on the path of discovery, and that they will be replaced by more realistic ones, probably after a more complete mathematical theory of microstructures in solutions of partial differential equations will have been developed. Anyway, I hope that for performing the Homogenization program that I have sketched, it will not be necessary to know exactly what these objects are. The concept of H-measures, that I have introduced a few years ago, permits to compute in a more accurate way some effective coefficients and it also permits to describe the evolution of some microstructures in solutions of hyperbolic systems in a way which explains why these microstructures can behave like "particles" [Ta6]. After I had successfully created this concept, Graeme Milton had pointed out to me that in order to compute scattering coefficients I needed information on three-point correlations, while H-measures are only related to two-point correlations, and therefore his feeling was that one needs a more precise mathematical tool to describe questions about spectroscopy.

Up to this point I have avoided the question of "collisions" between these moving objects, because in order to take them into account one must certainly forget about a "classical" point of view, and instead of using a Boltzmann like theory, one should dare look at reality: there are no "particles" out there, only waves. I have shown in [Ta7] a computation, done with Patrick Gérard, which shows the importance of interaction in the case of multiple scales, and the

same computation (related to the classical effect of beats) contains the qualitative answer to a question which had puzzled me for years: why is it that the rays of absorption of Hydrogen are attributed to the electron, and what are protons doing then?

Obviously, in order to carry out the program that I have tried to sketch here, one must be ready to question a lot of things that one has been told by physicists, but one must also question a lot of things that one has been told by mathematicians concerning Physics. As I have written a few months ago in the conclusion of [Ta7], “the transition to the new era might be difficult for many who may see their preferred equation lose part of its scientific interest, although one should remember that obsolete problems may still contain quite interesting Mathematics, but one should not lure students into working on an obsolete problem without having explained to them what one is really looking for.” One could certainly question what I have been teaching for all these years, about how Physics should be explained through the study of microstructures of solutions of partial differential equations; a few mathematicians have tried to attribute to themselves part of the program of research which I have expanded through the years, and although I cannot accept such a dishonest behaviour which should be unheard of among scientists, I could almost forgive a thief who would do a good job of leading the young researchers in the right direction, but that is rarely the case.

It is important to do one’s duty, and although a mathematician’s duty might be different from that of a physicist, I agree with my colleague Robert Griffiths when he writes “any scientist . . . is under obligation to God to seek after the truth, and God holds him accountable for the quality of his work”. I think that Ennio De Giorgi would have agreed too.

REFERENCES

- [Ba] J. M. BALL, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal. **63** (1977), 337-403.
- [Be] D. J. BERGMAN, *Bulk physical properties of composite media*, “Les Méthodes de l’Homogénéisation: théorie et Applications en Physique”. Ecole d’Eté d’Analyse Numérique C.E.A.-E.D.F.-INRIA (Bréau-sans-Nappe, 1983), 1-128, Collection de la Direction des Etudes et Recherches d’Electricité de France 57, Eyrolles, Paris, 1985.
- [DG&Sp] E. DE GIORGI – S. SPAGNOLO, *Sulla convergenza degli integrali dell’energia per operatori ellittici del 2° ordine*, Boll. Un. Mat. Ital. **8** (1973), 391-411.
- [Du&Li] G. DUVAUT – J.-L. LIONS, “Les Inéquations on Mécanique et en Physique”, Paris, Dunod, 1972.
- [Gr] R. B. GRIFFITHS, *Is Theology a Science?*, September 1980, J. American Scientific Affiliation.
- [Gu] S. GUTIÉRREZ, “Laminations in Linearized Elasticity and a Lusin type Theorem for Sobolev Spaces”, PhD thesis, Carnegie-Mellon University, May 1997.

- [MC] W. H. MC CONNELL, *On the Approximation of Elliptic Operators with Discontinuous Coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **3** (1976), 121-137.
- [Sp1] S. SPAGNOLO, *Sul limite delle soluzioni di problemi di Cauchy relativi all' equazione del calore*, Ann. Scuola Norm. Sup. Pisa **21** (1967), 657-699.
- [Sp2] S. SPAGNOLO, *Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22** (1968), 571-597.
- [Sv] V. ŠVERÁK, *Rank-one convexity does not imply quasiconvexity*, Proc. Roy. Soc. Edinburgh **120** (1992), 185-189.
- [Ta1] L. TARTAR, *Convergence d'opérateurs différentiels*, in: "Atti Giorni Analisi Convessa e Applicazioni" (Roma, 1974), 101-104.
- [Ta2] L. TARTAR, *Nonlocal effects induced by homogenization*, "Partial differential equations and the calculus of variations", Vol. II, 925-938. Progr. Nonlinear Differential Equations Appl., 2, Birkhäuser Boston, Boston, MA., 1989. (Book dedicated to Ennio De Giorgi).
- [Ta3] L. TARTAR, *Nonlinear constitutive relations and homogenization*, "Contemporary developments in continuum mechanics and partial differential equations" (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), pp. 472-484. North-Holland Math. Studies, 30, North-Holland, Amsterdam, 1978.
- [Ta4] L. TARTAR, *Equations hyperboliques non linéaires*, Séminaire Goulaouic-Schwartz (1977/1978), Exp. No. 18, 18 pp., Ecole Polytech., Palaiseau, 1978.
- [Ta5] L. TARTAR, *Remarks on homogenization*, "Homogenization and effective moduli of materials and media" (Minneapolis, Minn., 1984/1985), 228-246, IMA Vol. Math. Appl., 1, Springer, New York-Berlin, 1986.
- [Ta6] L. TARTAR, *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations*, Proc. Royal Soc. Edinburgh A **115** (1990), 193-230.
- [Ta7] L. TARTAR, *Approximation of H-measures*, Conference in honour of Roland Glowinski (Tours, May 1997); text finished too late to be included in the proceedings.

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