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Evolution of Subsets of $\mathbb{C}^2$ and Parabolic Problem for the Levi Equation

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0. – Introduction

The most natural way to evolve a compact subset K of $\mathbb{C}^2$ is to include it into some family $E \subset \mathbb{C}^2 \times \overline{\mathbb{R}}^+$ such that $K = pr_{\mathbb{C}^2}(E \cap (\mathbb{C}^2 \times \{0\}))$. The subset $K_t = pr_{\mathbb{C}^2}(E \cap (\mathbb{C}^2 \times \{t\}))$ is then, by definition, the *evolution of K at the time t*. Obviously, the way to generate the family E depends upon the geometric or functional properties of K we want to be reflected by an evolution.

If K is the closure $\overline{\Omega}$ of a bounded domain Ω, we may also think to evolve it by evolving its boundary $\Gamma_\circ = b\Omega$ and, in the case when $\Gamma_\circ$ is smooth at least, we may expect that the evolution is described by a smooth function $u = u(z,t)$, i.e. $\Gamma_t = \{z \in \mathbb{C}^2 : u(z,t) = 0\}$. In this situation, we say that $\{\Gamma_t\}_{t\geq 0}$ is the evolution of $\Gamma_\circ$ by *Levi curvature* if the following holds true: for every positive s, the trajectory $z = z(t)$, of a point $p \in \Gamma_s$ is the solution of the problem

$$\begin{cases} \dot{z}(t) = k_L(z,t)\nu(z,t), t > s \\ z(s) = p \end{cases}$$

where $\nu = \nu(z,t)$ is the inner unit normal vector field to Γ_t and

$$k_L = -|\partial u|^{-3}\det\begin{pmatrix} 0 & u_1 & u_2 \\ u_{\bar{1}} & u_{\bar{1}1} & u_{\bar{1}2} \\ u_{\bar{2}} & u_{\bar{2}1} & u_{\bar{2}2} \end{pmatrix} = |\partial u|^{-1}(\delta_{\alpha\beta} - |\partial u|^{-2}u_{\bar{\alpha}}u_\beta)u_{\alpha\bar{\beta}}$$

is the Levi curvature of the hypersurface $u = 0$ [ST$_1$] ($u_\alpha = u_{z_\alpha}, u_{\bar{\alpha}} = u_{\bar{z}_\alpha}, \alpha = 1, 2$, $|\partial u|^2 = |u_1|^2 + |u_2|^2$).

Since $z(t) \in \Gamma_t$ for every $t > s$, we deduce that, where $\partial u \neq 0$, u is a solution of the parabolic equation

$$u_t = \mathcal{L}(u) = (\delta_{\alpha\beta} - |\partial u|^{-2}u_{\bar{\alpha}}u_\beta)u_{\alpha\bar{\beta}}.$$

This enables us to consider the following general definition.

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Let K be a compact subset of $\mathbb{C}^2$, the zero set of a continuous function $g : \mathbb{C}^2 \to \mathbb{R}$ which is constant for $|z| \gg 0$.

Let $u \in C^\circ(\mathbb{C}^2 \times \overline{\mathbb{R}}^+)$ be a *weak solution* (Section 1) of the parabolic problem corresponding to g:

$$\text{(P)} \qquad \begin{cases} u_t = \mathcal{L}(u) & \text{in } \mathbb{C}^2 \times \mathbb{R}^+ \\ u = g & \text{on } \mathbb{C}^2 \times \{0\} \\ u = \text{const} & \text{for } |z| + t \gg 0. \end{cases}$$

The family $\{K_t\}_{t \geq 0}$ of the subsets $K_t = \{z \in \mathbb{C}^2 : u(z, t) = 0\}$ (which actually depends only on K) is called the *evolution of K by Levi curvature*.

By definition, there exists a minimum time $t^* = t^*(K)$ with the property: $K_t = \emptyset$ for every $t > t^*$. It is called the *extinction time* of K.

Moreover, by virtue of the comparison principle (Theorem 1.1), if K' is a compact subset of K then $K'_t \subseteq K_t$ for every time t.

This approach is largely inspired by the paper of Evans and Spruck [ES] on evolution by *mean curvature*. The parabolic problem which governs the evolution by Levi curvature is generally similar to that of the evolution by mean curvature; with one crucial difference: even if K is locally the graph of a smooth function, the operator $\mathcal{L}$ is elliptic degenerate.

The aim of this paper is to study the geometric properties of the evolution, by Levi curvature, $\{K_t\}_{t \geq 0}$, in terms of $K = K_\circ$.

It is organized in four sections. In the first one we give the notion of weak solution (in the sense of *viscosity*) of $u_t = \mathcal{L}(u)$ and, using the Perron method, we prove the existence and unicity for the parabolic problem (P)(Theorem 1.4).

In the second one we study the effect of pseudoconvexity on evolution. The main result is that when Ω is a bounded weakly pseudoconvex domain of $\mathbb{C}^2$ then the evolution $\{\overline{\Omega}_t\}_{t \geq 0}$ of $\overline{\Omega}$ is contained in $\overline{\Omega}$ (Theorem 2.1).

As a consequence: the evolution of a compact subset K of a complex curve is contained in K; the one of a compact subset K of a Levi flat hypersurface X is contained in X (Corollary 2.3). The extinction time of a compact subset K of a totally real submanifold $M \subset \mathbb{C}^2$ is 0 i.e. $K_t = \emptyset$ for every positive time t (Theorem 3.4).

We conjecture that if a bounded domain Ω is not pseudoconvex then $\overline{\Omega}_t \not\subset \overline{\Omega}$ for some t. This gives rise to the following question: given a compact subset of $\mathbb{C}^2$ what kind of hull it is possible to produce by similar sort of evolution.

For an arbitrary pseudoconvex domain Ω it is not clear at all if the domains Ω_t are pseudoconvex as well.

This is the case when $b\Omega$ is strongly pseudoconvex. Indeed the evolution is then *stationary* i.e. the solution $u = u(z, t)$ of the corresponding parabolic problem is of the form $u(z, t) = v(z) + t$ where $\mathcal{L}(v) = 1$ in Ω and $v = 0$ on $b\Omega$ (Theorem 3.3). Thus, thanks to Corollary 3.2 of [ST$_1$], the domains $\{v < \text{const}\}$ are pseudoconvex.

Stationary evolutions are studied in Section 3.

Finally in Section 4 evolution of starshaped, in particular convex, sets is analyzed.

We prove that, if Ω is a starshaped bounded domain, for any time t the evolution of $\Gamma_\circ = b\Omega$ at the time t has no interior (Corollary 4.4). Moreover, if Ω is convex, the sets $\Gamma_t \cap \Omega$ are pairwise disjoint for $0 < t \leq t^*$ (Corollary 4.6) and this fact would imply that the evolution $\{\Gamma_t\}_{t\geq 0}$ is of stationary type, provided it were known that $\mathcal{E}_t^{\mathcal{L}}(b\Omega) \subset \Omega$.

1. – Properties of weak solutions. Existence

1. Let $U \subset \mathbb{C}^2 \times \mathbb{R}^+$ be an open subset and $u : U \to \mathbb{R}$ be an upper semicontinuous function; u is said to be a *weak subsolution* of $u_t = \mathcal{L}(u)$ if, for every (z°, t°) and ϕ smooth near (z°, t°) such that $u - \phi$ has a local maximum at (z°, t°), one has

$$\phi_t \leq (\delta_{\alpha\beta} - |\partial\phi|^{-2}\phi_{\bar\alpha}\phi_\beta)\phi_{\alpha\bar\beta}$$

at (z°, t°) if $\partial\phi(z^\circ, t^\circ) \neq 0$ and

$$\phi_t \leq (\delta_{\alpha\beta} - \bar\eta^\alpha \eta^\beta)\phi_{\alpha\bar\beta},$$

for some $\eta \in \mathbb{C}^2$ with $|\eta| \leq 1$, if $\partial\phi(z^\circ, t^\circ) = 0$; a lower semicontinuous function $u : U \to \mathbb{R}$ is said to be a *weak supersolution* if, for every (z°, t°) and ϕ smooth near (z°, t°) such that $u - \phi$ has a local minimum at (z°, t°), one has

$$\phi_t \geq (\delta_{\alpha\beta} - |\partial\phi|^{-2}\phi_{\bar\alpha}\phi_\beta)\phi_{\alpha\bar\beta}$$

at (z°, t°) if $\partial\phi(z^\circ, t^\circ) \neq 0$ and

$$\phi_t \geq (\delta_{\alpha\beta} - \bar\eta^\alpha \eta^\beta)\phi_{\alpha\bar\beta},$$

for some $\eta \in \mathbb{C}^2$ with $|\eta| \leq 1$, if $\partial\phi(z^\circ, t^\circ) = 0$.

A *weak solution* is a continuous function which is both a weak subsolution and a weak supersolution.

It is immediate to prove that uniform limits on compact subsets of sequences of weak subsolutions (weak supersolutions) are weak subsolutions (weak supersolutions) as well. Furthermore if $\Phi : \mathbb{R} \to \mathbb{R}$ is nondecreasing and continuous and u is a weak subsolution (weak supersolution) then $\Phi(u)$ is a weak subsolution (weak supersolution) as well. If Φ is continuous and u is a weak solution then $\Phi(u)$ is also a weak solution ([ES]).

2. Let us consider the cylinder $Q = \Omega \times (0, h)$ in $\mathbb{C}^2 \times \overline{\mathbb{R}}^+$, where Ω is a bounded domain of $\mathbb{C}^2$ and let

$$\Sigma = (\overline{\Omega} \times \{0\}) \cup (b\Omega \times (0, h)).$$

We have the following comparison principle

THEOREM 1.1. *Let $u, v \in C^0(\overline{Q})$ be respectively a weak subsolution and a weak supersolution in Q. If $u \leq v$ on $\overline{\Sigma}$ then $u \leq v$.*

Let us recall briefly the main properties of the regularization by "sup and inf" convolution ([ES], [Sl]).

Let $u : \mathbb{C}^2 \times \overline{\mathbb{R}}^+ \to \mathbb{R}$ be continuous and $|u|$ be bounded. Set, for $\epsilon > 0$ and $z \in \mathbb{C}^2$, $t \in \bar{\mathbb{R}}^+$

$$u^\epsilon(z,t) = \max\left\{u(\zeta,s) - \epsilon^{-1}(|z-\zeta|^2 + (t-s)^2), (\zeta,s) \in \mathbb{C}^2 \times \overline{\mathbb{R}}^+\right\}$$

and

$$u_\epsilon(z,t) = \min\left\{u(\zeta,s) + \epsilon^{-1}(|z-\zeta|^2 + (t-s)^2), (\zeta,s) \in \mathbb{C}^2 \times \overline{\mathbb{R}}^+\right\}.$$

The above definitions immediately imply that $u_\epsilon \leq u \leq u^\epsilon$ and $|u_\epsilon|$, $|u^\epsilon|$ are bounded by $\sup|u|$; u_ϵ, u^ϵ are Lipschitz and $u^\epsilon \searrow u$, $u_\epsilon \nearrow u$ uniformly on compact subsets as $\epsilon \to 0$. Moreover, the functions

$$u^\epsilon(z,t) + \epsilon^{-1}(|z|^2 + t^2)$$

and

$$u_\epsilon(z,t) - \epsilon^{-1}(|z|^2 + t^2)$$

are respectively convex and concave; in particular they are twice differentiable a.e.

If u is a weak subsolution, u^ϵ is a weak subsolution in $\mathbb{C}^2 \times (\sigma(\epsilon), +\infty)$, where $\sigma(\epsilon) = C\epsilon^{\frac{1}{2}}$ and C is a constant depending only on u. Moreover

$$u_t^\epsilon \leq \mathcal{L}(u^\epsilon)$$

at each point of twice differentiability of u^ϵ, where $\partial u^\epsilon \neq 0$. Similarly, if u is a weak supersolution u_ϵ is also a supersolution and

$$u_{\epsilon,t} \geq \mathcal{L}(u_\epsilon)$$

at each point of twice differentiability of u_ϵ, where $\partial u_\epsilon \neq 0$.

Now let us assume that $u \in C^0(\mathbb{C}^2 \times \overline{\mathbb{R}}^+)$ is a weak subsolution (respectively a weak supersolution) in a domain D of $\mathbb{C}^2 \times \mathbb{R}^+$ and let

$$D_\epsilon = \left\{z \in D : \operatorname{dist}(z, bD) > 2\sup|u|\epsilon^{1/2}\right\}.$$

Then u^ϵ (respectively u_ϵ) is a weak subsolution (respectively a weak supersolution) in D_ϵ. This can be seen as follows. Let $u^\epsilon - \phi$ have a local maximum at $(z^\circ, t^\circ) \in D_\epsilon$, ϕ smooth, and $(\zeta^\circ, s^\circ) \in \mathbb{C}^2 \times \mathbb{R}^+$ be such that

$$u^\epsilon(z^\circ, t^\circ) = u(\zeta^\circ, s^\circ) - \epsilon^{-1}(|z^\circ - \zeta^\circ|^2 + (t-s)^2).$$

Set

$$\psi(z,t) = \phi(z + z^\circ - \zeta^\circ, t + t^\circ - s^\circ).$$

Then $(\zeta^\circ, s^\circ) \in D$ and for all (z,t) near (z°, t°) and all $(\zeta, s) \in \mathbb{C}^2 \times \overline{\mathbb{R}}^+$ we have

$$\begin{aligned} u(\zeta,s) - \epsilon^{-1}(|z-\zeta|^2 + (t-s)^2) - \phi(z,t) \le u^\epsilon(z,t) - \phi(z,t) \le u(\zeta^\circ, s^\circ) \\ - \epsilon^{-1}(|z^\circ - \zeta|^2 + (t-s)^2) \\ - \phi(z^\circ, t^\circ); \end{aligned}$$

in particular, for $z = \zeta + z^\circ - \zeta^\circ$, $t = s + t^\circ - s^\circ$ and (ζ, s) near (ζ°, s°), we have

$$u(\zeta,t) - \psi(\zeta,s) \le u(\zeta^\circ, s^\circ) - \psi(\zeta^\circ, s^\circ),$$

i.e. $u - \psi$ has a strict local maximum at (ζ°, s°). Since

$$\psi_\alpha(\zeta^\circ, s^\circ) = \phi_\alpha(z^\circ, t^\circ), \ \psi_{\alpha\bar\beta}(z^\circ, t^\circ) = \phi_{\alpha\bar\beta}(\zeta^\circ, s^\circ)$$

we obtain

$$\phi_t \ge (\delta_{\alpha\beta} - |\partial\phi|^{-2}\phi_{\bar\alpha}\phi_\beta)\phi_{\alpha\bar\beta}$$

at (z°, t°) if $\partial\phi(z^\circ, t^\circ) \ne 0$ and

$$\phi_t \ge (\delta_{\alpha\beta} - \bar\eta^\alpha \eta^\beta)\phi_{\alpha\bar\beta}$$

with $\eta \in \mathbb{C}^2$, $|\eta| \le 1$, if $\partial\phi(z^\circ, t^\circ) = 0$.

PROOF OF THEOREM 1.1. We may assume that $u < v$ on $\overline{\Sigma}$ and, for a contradiction, that $\max_{\overline{Q}} (u - v) = a > 0$.

Extend u, v by continuous and bounded functions in such a way to have $u < v$ in $(\mathbb{C}^2 \setminus \Omega) \times [0,h]$ and u, v constant for $|z| \gg 0$.

Since $\max_{\overline{Q}} (u - v) = a > 0$ we have

$$\max_{\mathbb{C}^2 \times [0,h]} \left(u - v - \alpha(h-t)^{-1}\right) \ge \frac{a}{2}$$

for α small enough and consequently

$$(\star) \qquad \max_{\mathbb{C}^2 \times [0,h]} \left(u^\epsilon - v_\epsilon - \alpha(h-t)^{-1}\right) \ge \frac{a}{2}$$

for all $\epsilon \le \epsilon_0$.

Now define for $z,\ z + \zeta \in \mathbb{C}^2$, $t,\ t + s \in [0,h]$

$$\begin{aligned} \Phi_{\epsilon,\delta}(z,\zeta,t,s) = u^\epsilon(z+\zeta, t+s) - v_\epsilon(z,t) \\ - \alpha(h-t)^{-1} - \delta^{-1}(|\zeta|^4 + s^4). \end{aligned}$$

$\Phi_{\epsilon,\delta}$ is negative outside of a compact subset $K \subset \mathbb{C}^2 \times [0, h)$ and $\Phi_{\epsilon,\delta} \geq \Phi_{\epsilon',\delta}$, $\Phi_{\epsilon,\delta'} \geq \Phi_{\epsilon,\delta}$ provided $\epsilon' \leq \epsilon$, $\delta' \leq \delta$. Moreover, because of $(\star)$,

$$\max_K \Phi_{\epsilon',\delta} \geq \frac{a}{4}.$$

Let $p_{\epsilon,\delta} = (z', \zeta', t', s')$ be a maximum point for $\Phi_{\epsilon,\delta}$; by definition $|\zeta'|$, $|s'| \leq C\delta^{1/4}$ (where C depends only on u and v). We claim that for ϵ, δ near 0, (z', t') and (z', ζ', t', s') belong to Q.

To prove this we consider a limit point $(z^\circ, \zeta^\circ, t^\circ, s^\circ)$ of the bounded set $\{p_{\epsilon,\delta}\}$. Then, since $\Phi_{\epsilon,\delta}(p_{\epsilon,\delta}) \geq a/4$, $\delta^{-1}(|\zeta'|^4 + s'^4)$ must be bounded as $\epsilon, \delta \to 0$ and this forces ζ° and s° to be 0. It follows that

$$u(z^\circ, t^\circ) - v(z^\circ, t^\circ) - \alpha(h - t^\circ)^{-1} - B \geq \frac{a}{4}$$

for some positive constant B; since $u < v$ in $\mathbb{C}^2 \setminus Q$ this proves our claim. At this point, thanks to the fact that $(h - t)^{-1}$ is not increasing, we can follow step by step the proof of Theorem 3.2 in [ES].

Corollary 1.2. *Let $D \subset \mathbb{C}^2 \times \overline{\mathbb{R}}^+$ be a bounded domain and $u, v \in C^\circ(\overline{D})$ be respectively a weak subsolution and a weak supersolution in D. Then, if $u \leq v$ on bD, we have $u \leq v$.*

Corollary 1.3. *Let $u, v \in C^\circ(\mathbb{C}^2 \times \overline{\mathbb{R}}^+)$ be respectively a weak subsolution and a weak supersolution in $\mathbb{C}^2 \times \mathbb{R}^+$, and suppose that u and v are constant for $|z| + t \gg 0$. Then, if $u \leq v$ for $t = 0$, we have $u \leq v$. Moreover, if u and v are weak solutions we have*

$$\max_{\mathbb{C}^2 \times \overline{\mathbb{R}}^+} |u - v| = \max_{\mathbb{C}^2 \times \{0\}} |u - v|.$$

3. The above results permit us to use the Perron method to prove the following existence theorem

Theorem 1.4. *Let $g : \mathbb{C}^2 \to \mathbb{R}$ be continuous and constant for $|z| \gg 0$. Then the parabolic problem corresponding to g has a unique weak solution.*

Proof. Unicity immediately follows from the comparison principle.

In order to prove the existence of a solution u we fix $R_\circ$ such that $g(z) = C \in \mathbb{R}$ for $|z| \geq R_\circ$ and we choose $M > 0$ and $R > 0$ such that: $g(z) \geq C - M$ for $|z| \leq R_\circ$ and $-R^2 + R_\circ^2 \leq -M$. Consequently, $-R^2 + |z|^2 + C \leq g(z)$ for all $|z| \leq R$. Then

$$u_1(z, t) = \min\,(-R^2 + |z|^2 + C + t, C)$$

is a continuous subsolution in $\mathbb{C}^2 \times \mathbb{R}^+$ with $u_1(z, 0) \leq g(z)$ and $u_1(z, t) = C$ for $|z|^2 + t \geq R^2$.

This can be seen as follows. Consider a function $\chi_\epsilon \in C^\infty(\mathbb{R})$ with the properties: $\chi_\epsilon(s) = 0$ for $s \geq -\epsilon/3$ and $\chi_\epsilon(s) = s$ for $s \leq -\epsilon$ and define

$$w(z, t) = \chi_\epsilon(-R^2 + |z|^2 + t) + C.$$

w is a weak subsolution in $D = \{|z|^2 + t < R^2\}$, which is constant ($= C$) in a neighbourhood of bD in $\overline{D}$. Then extend it by C on the rest of $\mathbb{C}^2 \times \mathbb{R}^+$ and let $\epsilon \to 0$ to obtain the conclusion.

Now let us set $W = B(0, R+1) \times (0, R+1)$, where $B(0, R+1)$ is the ball $\{|z| < R+1\}$ and let $\mathcal{F}$ be the class of the functions u satisfying

(i) $u : \overline{W} \to [-\infty, +\infty)$ is upper semicontinuous and $u_{|W}$ is a weak subsolution

(ii) $u(z, t) \leq C$ for $|z|^2 + t > R^2$ and $u(z, 0) \leq g(z)$ for $|z| < R+1$.

Observe that, in view of the comparison principle,

$$\sup u \leq \max (C, \sup g).$$

Let us denote by $u_\circ : \overline{W} \to \mathbb{R}$ the function

$$(z, t) \mapsto \sup \{u(z, t) : u \in \mathcal{F}\}$$

and by $u_\circ^*$ its upper semicontinuous envelope. Then, since $u_{1|\overline{W}} \in \mathcal{F}$ and $u_1 = C$ for $|z|^2 + t \geq R^2$, we have $u_\circ = u_\circ^* = C$ whenever $R^2 \leq |z|^2 + t^2 \leq (R+1)^2$.

ASSERTION 1. $u^*_{\circ|W}$ is a weak subsolution.

The proof is the same as in [ST_1], Lemma 4.1.

ASSERTION 2. For all $z^\circ \in \overline{B}(0, R+1)$

$$\lim_{(z,t)\to(z^\circ,0)} u_\circ(z, t) = \lim_{(z,t)\to(z^\circ,0)} u_\circ^*(z, t) = g(z).$$

To prove this we fix $\epsilon > 0$ and we choose two functions ϕ, ψ in $C^2(\overline{B}(0, R+1))$ in such a way to have

$$g(z) - \epsilon < \phi(z) < g(z) < \psi(z) < g(z) + \epsilon,$$

for all $z \in \overline{B}(0, R+1)$. Let λ be a constant such that

$$|\phi_{\alpha\bar{\beta}}(z)\xi^\alpha\bar{\xi}^\beta| \leq \lambda|\xi|^2, \quad |\psi_{\alpha\bar{\beta}}(z)\xi^\alpha\bar{\xi}^\beta| \leq \lambda|\xi|^2$$

for all $z \in \overline{B}(0, R+1)$ and $\xi \in \mathbb{C}^2$. Then

$$u_+(z, t) = \lambda t + \psi(z), \quad u_-(z, t) = -\lambda t + \phi(z)$$

are respectively a weak supersolution and a weak subsolution in W; moreover they are continuous on $\overline{W}$, $u_- \in \mathcal{F}$ and $u_{|bW} \leq u_{+|bW}$ for all $u \in \mathcal{F}$. In view of the comparison principle for u, u_+ and the definition of $\mathcal{F}$, we deduce that $u_- \leq u_\circ \leq u_0^* \leq u_+$ in $\overline{W}$ and consequently, since u_-, u_+ are continuous, that

$$g(z) - \epsilon \leq u_-(z_\circ, 0) \leq \liminf_{(z,t)\to(z^\circ,0)} u_\circ(z,t) \leq \limsup_{(z,t)\to(z^\circ,0)} u_\circ^*(z,t)$$
$$\leq u_+(z_\circ, 0) \leq g(z_\circ) + \epsilon.$$

Assertion 2 follows, ϵ being arbitrary.

Thanks to the fact that $u_\circ^*(z,t) = C$ whenever $R^2 \leq |z|^2 + t \leq (R+1)^2$ Assertions 1 and 2 imply that $u_\circ^* \in \mathcal{F}$, therefore $u_\circ^* = u_\circ$ is continuous at every point of bW and $u_\circ(z,0) = g(z)$ if $|z| \leq R+1$. Thus all the hypotheses of the Walsh Lemma ([W]) are satisfied and consequently $u_\circ$ is continuous in $\overline{W}$.

Finally $u_\circ$ is a weak solution in W. For if not there exist $(z^\circ, t^\circ) \in W$ and $\phi \in C^\infty(W)$ with the properties: (z°, t°) is a strict local minimum point for $u_\circ - \phi_\circ$, $u_\circ(z^\circ, t^\circ) = \phi(z^\circ, t^\circ)$ and

$$\phi_t < (\delta_{\alpha\beta} - |\partial\phi|^{-2}\phi_{\bar{\alpha}}\phi_\beta)\phi_{\alpha\bar{\beta}}$$

at (z°, t°) if $\partial\phi(z^\circ, t^\circ) \neq 0$ and

$$\phi_t < (\delta_{\alpha\beta} - \bar{\eta}^\alpha\eta^\beta)\phi_{\alpha\bar{\beta}}$$

for all $\eta \in \mathbb{C}^2$ with $|\eta| \leq 1$ if $\partial\phi(z^\circ, t^\circ) = 0$. Hence, for $\epsilon > 0$ small enough, $u = \max\,(u_\circ, \phi + \epsilon)$ is a subsolution belonging to $\mathcal{F}$: contradiction. Therefore $u_\circ$ is a solution and $u_\circ = C$ on a neighbourhood of $bW \setminus B(0, R+1) \times \{0\}$; consequently, the function

$$u(z,t) = \begin{cases} u_\circ(z,t) & \text{if } (z,t) \in \overline{W} \\ C & \text{if } (z,t) \in \mathbb{C}^2 \times \overline{\mathbb{R}}^+ \setminus \overline{W} \end{cases}$$

is the solution of the parabolic problem.

2. – Evolution of a compact subset: geometric properties

1. Let $K \subset \mathbb{C}^2$ be a compact subset, the zero set $\{g = 0\}$ of a continuous function $g : \mathbb{C}^2 \to \mathbb{R}$ which is constant for $|z| \gg 0$, u a weak solution of the parabolic problem corresponding to g and $\{K_t\}_{t\geq 0}$ the evolution of K. We also use the notation $K_t = \mathcal{E}_t^{\mathcal{L}}(K)$. Then the semigroup property

$$\mathcal{E}_{t'}^{\mathcal{L}}\left(\mathcal{E}_t^{\mathcal{L}}(K)\right) = \mathcal{E}_{t+t'}^{\mathcal{L}}(K)$$

holds true and there exists a time $t^* = t^*(K)$, the extinction time of K, such that $\mathcal{E}_t^{\mathcal{L}}(K) = \emptyset$ for $t > t^*$. If K and K' are compact and $K \subset K'$, then, using the comparison principle, we derive that

$$\mathcal{E}_t^{\mathcal{L}}(K) \subset \mathcal{E}_t^{\mathcal{L}}(K')$$

for all $t \geq 0$. Moreover, by the same argument as in [ES], it can be shown that the evolution $\{K_t\}_{t\geq 0}$ does not depend upon the choice of the particular function g with the above properties.

THEOREM 2.1. *Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain with $b\Omega$ smooth of class C^3. Then*

$$\mathcal{E}_t^{\mathcal{L}}(\overline{\Omega}) \subset \overline{\Omega}.$$

The above regularity assumption is probably excessive; in fact the theorem follows from the more general Lemma and the Remark given next.

LEMMA 2.2. *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain. Suppose that there is a continuous function $\rho : W \to \mathbb{R}$, where W is an open neighbourhood of $\overline{\Omega} \subset W$, a positive constant M such that*

$$\overline{\Omega} = \{z \in W : \rho(z) \leq 0\}$$

and the inequality

$$\mathcal{L}(\rho)(z) \geq -M\rho(z)$$

holds in the weak sense in $W \setminus \overline{\Omega}$. Then

$$\mathcal{E}_t^{\mathcal{L}}(\overline{\Omega}) \subset \overline{\Omega}$$

for all $t \geq 0$.

REMARK 2.1. Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain, W an open neighbourhood of $\overline{\Omega}$ and $\rho : W \to \mathbb{R}$ a C^3-regular defining function for $b\Omega$ such that

$$\Omega = \{z \in W : \rho(z) < 0\}$$

and $\partial\rho(z) \neq 0$ for $z \in b\Omega$. Then ρ satisfies the above Lemma condition in $W^* \setminus \overline{\Omega}$, with some positive constant M, where $W^* \subset W$ is a compact neighbourhood of $b\Omega$.

PROOF. (Sketch) Following [S], let $r(z)$ denote the minimum distance of z to $b\Omega$ and p_z the point in $b\Omega$ such that $|z - p_z| = r(z)$. Take a tubular neighbourhood W^* of $b\Omega$ in W such that, for every $z \in W^*$, the segment $[z, p_z]$ is contained in W^*. We can assume W^* compact, $W^* \subset W$ and that $\partial\rho(z) \neq 0$ on W^*. Then $\mathcal{L}(\rho)(z)$ is a Lipschitz function on W^*, with some Lipschitz constant C. Since Ω is weakly pseudoconvex, $\mathcal{L}(\rho)(z) \geq 0$ for $z \in b\Omega$. Thus, for $z \in W^* \setminus \overline{\Omega}$,

$$\mathcal{L}(\rho)(z) \geq \mathcal{L}(\rho)(p_z) - C|z - p_z| \geq -Cr(z).$$

Since ρ is positive and C^1 in $W^* \setminus \overline{\Omega}$, $\rho = 0$ and $\partial\rho(z) \neq 0$ on $b\Omega$, there is a constant C_1 such that $r(z) \leq C_1\rho(z)$ for $z \in W^* \setminus \Omega$. Consequently

$$\mathcal{L}(\rho)(z) \geq -(CC_1)\rho(z),$$

for $z \in W^* \setminus \Omega$, which yields the condition with $M = CC_1$.

PROOF OF LEMMA 2.2. We need the following technical fact. Let $W \subset \mathbb{C}^2$ be open and $\rho : W \to \mathbb{R}$ be a weak continuous solution of the inequality

$$\mathcal{L}(\rho)(z) \geq -h(z)$$

where $h : W \to \mathbb{R}^+$ is a continuous positive function. Suppose that χ is a continuous increasing function $\mathbb{R} \to \mathbb{R}$ with $\chi' \in L^\infty(\mathbb{R})$ and $0 \leq \chi' \leq 1$. Then

$$\mathcal{L}(\chi \circ \rho)(z) \geq -h(z),$$

in the weak sense.

Now let ρ be as in the statement. We may suppose, without loss of generality (by replacing, if necessary, ρ by $\max(\rho, 0)$), that

$$\overline{\Omega} = \{z \in W : \rho(z) = 0\}.$$

Let now, for $\delta > 0$, $W_\delta = \{z \in W : \rho(z) < \delta\}$. There is a $\delta_\circ > 0$ such that for $\delta \leq \delta_\circ$, $\overline{W}_\delta \subset W$. Denote by g^δ, $0 \leq \delta \leq \delta_\circ$, the function

$$g^\delta(z) = \begin{cases} \rho(z) & \text{if } z \in W_\delta \\ \delta & \text{if } z \in \mathbb{C}^2 \setminus W_\delta. \end{cases}$$

Clearly g^δ is continuous on $\mathbb{C}^2$. Set

$$u_\epsilon^\delta(z, t) = g^\delta(z) - \epsilon t,$$

for $(z, t) \in \mathbb{C}^2 \times \overline{\mathbb{R}}^+$.

ASSERTION: $u_\epsilon^\delta = u_\epsilon^\delta(z, t)$ is a weak subsolution of the parabolic equation

$$\mathcal{L}(w) \geq -M\delta$$

provided $M\delta \leq \epsilon$.

Let now $u^\delta : \mathbb{C}^2 \times \overline{\mathbb{R}}^+ \to \mathbb{R}$ be the weak solution of the parabolic problem corresponding to g^δ.

By the comparison principle (note that $\lim_{t \to +\infty} u_\epsilon^\delta(z, t) = -\infty$) we have $u_\epsilon^\delta(z, t) \leq u^\delta(z, t)$, hence, with K a compact in $\overline{\Omega}$, $t > 0$,

$$\begin{aligned} \mathcal{E}_t^{\mathcal{L}}(K) \subset \mathcal{E}_t^{\mathcal{L}}(\overline{\Omega}) &= \{z \in \mathbb{C}^2 : u^\delta(z, t) = 0\} = \{z \in \mathbb{C}^2 : u^\delta(z, t) \leq 0\} \\ &\subset \{z \in \mathbb{C}^2 : u_\epsilon^\delta(z, t) \leq 0\} = \{z \in \mathbb{C}^2 : g^\delta(z) \leq \epsilon t\}. \end{aligned}$$

Fix now an arbitrary $\delta > 0$, and let $\epsilon = M\delta$. Consider $t \in (0, 1/2M)$, and z with $g^\delta(z) \leq \epsilon t$. Then $z \in W_\delta$. Indeed, $\epsilon t < \epsilon/2M \leq \delta/2$ and $g^\delta(z) = \delta$ outside of W_δ. Hence $g^\delta(z) = \rho(z) < \delta/2$, i.e. $\mathcal{E}^t(K) \subset W_{\delta/2}$ for all $t \in (0, 1/2M)$. Seeing that $\delta > 0$ was arbitrary and M independent on the choice of d, we obtain that $\mathcal{E}_t^{\mathcal{L}}(K) \subset \overline{\Omega}$ for $0 \leq t < 1/2M$; in particular $\mathcal{E}_t^{\mathcal{L}}(\overline{\Omega}) \subset \overline{\Omega}$ for

$0 \le t < 1/2M$. By the semigroup property of $\mathcal{E}_t^{\mathcal{L}}$ we conclude that $\mathcal{E}_t^{\mathcal{L}}(\overline{\Omega}) \subset \overline{\Omega}$ for all $t \ge 0$.

In order to prove the Assertion we show first that

$$\mathcal{L}(\rho^\delta)(z) \ge -M\delta$$

in $\mathbb{C}^2$ (in the weak sense).

Applying the technical fact with $\chi(s) = \min(s, \delta')$ for $0 < \delta' < \delta$, we obtain that

$$\mathcal{L}(\rho^{\delta'})(z) = \mathcal{L}(\chi \circ \rho)(z) \ge -\rho(z) \ge -M\delta$$

in W_δ, in the weak sense. On the other hand, for $\delta' < \delta$, $\overline{W}_{\delta'} \subset W_\delta$ and $\rho^{\delta'}$ is constant on $\mathbb{C}^2 \setminus \overline{W}_{\delta'}$, hence

$$\mathcal{L}(\rho^{\delta'}) \ge -M\delta$$

on $\mathbb{C}^2$ for all $\delta' < \delta$. Since $\rho^{\delta'} \to \rho^\delta$, uniformly, we obtain the conclusion.

To verify the Assertion fix $\delta, \epsilon > 0$ with $M\delta \le \epsilon$ and $(z^\circ, t^\circ) \in \mathbb{C}^2 \times \mathbb{R}^+$.

Take a smooth test function ϕ with $\phi(z,t) \ge u_\epsilon^\delta(z,t)$ and $\phi(z^\circ, t^\circ) \ge u_\epsilon^\delta(z^\circ, t^\circ)$. Evidently $\phi_t(z^\circ, t^\circ) = -\epsilon$. Set now $\psi(z) = \phi(z, t^\circ) + \epsilon t^\circ$. Then $\psi(z) \ge g^\delta(z)$, $\psi(z^\circ) = g^\delta(z^\circ)$. By what is preceding, we have, in case $\partial\psi(z^\circ) \ne 0$, $\mathcal{L}(\psi)(z^\circ) \ge -M\delta$,

$$\mathcal{L}(\phi)(z^\circ, t^\circ) = \mathcal{L}(\psi)(z^\circ) \ge -M\delta \ge -\epsilon = \phi_t(z^\circ, t^\circ),$$

or, in case $\partial\psi(z^\circ) = 0$, for some vector $\eta \in \mathbb{C}^2$, $|\eta| \le 1$,

$$(\delta_{\alpha\beta} - \bar\eta^\alpha \eta^\beta)\phi_{\alpha\bar\beta}(z^\circ, t^\circ) = (\delta_{\alpha\beta} - \bar\eta^\alpha\eta^\beta)\psi_{\alpha\bar\beta}(z^\circ) \ge -M\delta \ge -\epsilon = \phi_t(z^\circ, t^\circ).$$

The technical fact is proved as follows. Since χ can be approximated uniformly on compact subsets of $\mathbb{R}$ by smooth functions with the required properties, we assume, with loss of generality, that $\chi : \mathbb{R} \to \mathbb{R}$, $\chi \in C^\infty(\mathbb{R})$, $0 < \chi'(s) \le 1$; hence $\chi^{-1} \in C^\infty(\mathbb{R})$ a.e. . Let ψ be a smooth test function for $\mathcal{L}(\chi \circ \rho) \ge -h$, i.e.

$$\psi(z) \ge (\chi \circ \rho)(z), \psi(z^\circ) = (\chi \circ \rho)(z^\circ);$$

then $\psi^* = \chi^{-1} \circ \rho$ is a test function too, i.e.

$$\psi^*(z) \ge \rho(z), \psi^*(z^\circ) = \rho(z^\circ).$$

In case $\partial\psi(z^\circ) \ne 0$ we have $\partial\psi^*(z^\circ) \ne 0$ and, by virtue of the hypothesis, $\mathcal{L}(\psi^*)(z^\circ) \ge -h(z^\circ)$, hence

$$\begin{aligned}\mathcal{L}(\psi)(z) = \mathcal{L}(\chi \circ \psi^*)(z^\circ) &= \chi'(\psi^*(z^\circ))\mathcal{L}(\psi^*)(z^\circ)\\ &\ge -\chi'(\psi^*(z^\circ))h(z^\circ) > -h(z^\circ).\end{aligned}$$

If $\partial\psi(z^\circ) = 0$, then $\partial\psi^*(z^\circ) = 0$ and there is a vector $\eta \in \mathbb{C}^2$, $|\eta| \le 1$, with

$$(\delta_{\alpha\beta} - \bar\eta^\alpha\eta^\beta)\psi^*_{\alpha\bar\beta}(z^\circ) \ge -h(z^\circ).$$

Note that, because $\psi^*_\alpha(z^\circ) = 0$, $\alpha = 1, 2$, we have

$$(\delta_{\alpha\beta} - \bar{\eta}^\alpha \eta^\beta)\psi_{\alpha\bar{\beta}}(z^\circ) = \chi'(\psi^*(z^\circ))(\delta_{\alpha\beta} - \bar{\eta}^\alpha \eta^\beta)\psi^*_{\alpha\bar{\beta}}(z^\circ) \\ \geq -\chi'(\psi^*(z^\circ))h(z^\circ) \geq -h(z^\circ).$$

The proof of the Lemma is now complete.

COROLLARY 2.3. *Let K be a compact subset of $\mathbb{C}^2$, K^S its Stein hull, the intersection of all Stein neighbourhoods of K. Then*

(i) $\mathcal{E}^{\mathcal{L}}_t(K) \subset K^S$ *for every $t \geq 0$ (in particular, if K is a Stein compact, $\mathcal{E}^{\mathcal{L}}_t(K) \subset K$ for all $t \geq 0$;*
(ii) *if K belongs to a complex curve, $\mathcal{E}^{\mathcal{L}}_t(K) \subset K$ for all $t \geq 0$;*
(iii) *if K belongs to a Levi flat hypersurface X, $\mathcal{E}^{\mathcal{L}}_t(K) \subset X$ for all $t \geq 0$.*

PROOF. To prove (ii) we apply the theorem of Siu: X has a Stein base of neighbourhoods [Si], [D]. To prove (iii) we consider two pseudoconvex bounded domains Ω_1, Ω_2 with the property: $K \subset b\Omega_\alpha$, $\alpha = 1, 2$ and $\overline{\Omega}_1 \cap \overline{\Omega}_2 \subset X$ ([GS]).

REMARK 2.2 Let U be a bounded pseudoconvex domain of $\mathbb{C}^2$ and g a continuous function $\mathbb{C}^2 \to \mathbb{R}$ such that $g = M = \sup_{\mathbb{C}^2} g$ in $\mathbb{C}^2 \setminus U$. Let u be the weak solution of the parabolic problem corresponding to g. Then u is constant $(= M)$ in $(\mathbb{C}^2 \setminus U) \times \overline{\mathbb{R}}^+$.

PROOF. In view of the comparison principle we have $u \leq M$. Let $u(z^\circ, t^\circ) = c < M$ and set $K_{t^\circ} = \{z \in \mathbb{C}^2 : u(z, t^\circ) = u(z^\circ, t^\circ)\}$. K_{t° is the evolution of $K_\circ = \{z \in \mathbb{C}^2 : u(z, 0) = g(z) = c\}$ at the time t° which is contained in U by virtue of the hypothesis. Since U is pseudoconvex, $K_{t^\circ} \subset U$ and $(z^\circ, t^\circ) \in U \times \overline{\mathbb{R}}^+$.

2. Let us assume now that K is the boundary $\Gamma_\circ$ of a bounded domain $\Omega \subset \mathbb{C}^2$ and let $\{\Gamma_t\}_{t\geq 0}$ be its evolution. We say that the evolution is *strictly contracting* (respectively *weakly contracting*) if, for every $t > 0$, $\Gamma_t \subset \Omega$ (respectively $\Gamma_t \subset \overline{\Omega}$). $\{\Gamma_t\}_{t\geq 0}$ is said to be *stationary* if, for every $t \geq 0$,

$$\Gamma_t = \{z \in \Omega : v(z) = -t\}$$

where v is a weak solution of the *stationary problem* associated with the evolution: $\mathcal{L}(v) = 1$ in Ω and $v = 0$ on $b\Omega$.

For every weak solution of the stationary problem we have

$$\{z \in \Omega : v(z) = -t\} \subset \Gamma_t.$$

Moreover

PROPOSITION 2.4. *Let $v \in C^0(\overline{\Omega})$ be a weak solution of the stationary problem and extend it by* 0 *on the rest of* $\mathbb{C}^2$. *Then*

$$u(z,t) = \begin{cases} \min(0, v(z)+t) & \text{if } (z,t) \in \overline{\Omega} \times \overline{\mathbb{R}}^+ \\ 0 & \text{if } (z,t) \in (\mathbb{C}^2 \setminus \overline{\Omega}) \times \overline{\mathbb{R}}^+ \end{cases}$$

is a weak solution of $u_t = \mathcal{L}(u)$ (and $u = v$ for $t = 0$).

PROOF. Let

$$D = \{(z,t) \in \Omega \times \overline{\mathbb{R}}^+ : u(z,t) < 0\}.$$

Consider a function $\chi_\epsilon \in C^\infty(\mathbb{R})$ with the properties: $\chi_\epsilon(s) = 0$ for $s \geq -\epsilon/3$ and $\chi_\epsilon(s) = s$ for $s \leq -\epsilon$ and define $w(z,t) = \chi_\epsilon(v(z)+t)$. Then w is a weak solution in $\Omega \times \mathbb{R}^+$ which is vanishing on a neighbourhood of bD in $\overline{D}$. Then extend it by 0 on the rest of $\mathbb{C}^2 \times \overline{\mathbb{R}}^+$ and let $\epsilon \to 0$ to obtain the conclusion.

COROLLARY 2.5. *Let $v \in C^0(\overline{\Omega})$ be a weak solution of the stationary problem. Set $N_t = \{z \in \Omega : v(z) = -t\}$, $t \geq 0$. Then*
(a) *for every $t_\circ, t > 0$*

$$\mathcal{E}_t^{\mathcal{L}}(N_{t_\circ}) = N_{t+t_\circ};$$

(b) *for every $t \geq 0$*

$$N_t \subset \mathcal{E}_t^{\mathcal{L}}(b\Omega) = \mathcal{E}_t^{\mathcal{L}}(N_\circ).$$

PROOF. Part (a) of the statement follows from the above proposition. In order to prove part (b) we take a continuous function $g : \mathbb{C}^2 \to \mathbb{R}$ satisfying: $g = 0$ in $\overline{\Omega}$, $g > 0$ outside and g constant for $|z| \gg 0$. Let u be the solution of the parabolic problem corresponding to g. Since the evolution of a level set does not depend on the choice of the function, part (a) implies that for all $t, \epsilon > 0$

$$\{z \in \mathbb{C}^2 : u(z,t) = -\epsilon\} = \mathcal{E}_t^{\mathcal{L}}(\{g = -\epsilon\}) = N_{t+\epsilon}.$$

Now fix $t > 0$ and let z° be an arbitrary point of N_t. Then, by what is preceding, we have $u(z^\circ, t-\epsilon) = 0$ for all $0 < \epsilon < t$; hence $u(z^\circ, t) = 0$ and therefore $z^\circ \in \mathcal{E}_t^{\mathcal{L}}(b\Omega)$.

A partial converse of Proposition 2.2 is provided by the following

PROPOSITION 2.6. *Let u be the weak solution of the parabolic problem corresponding to g and let N be the zero set of u. If $\Gamma_t \cap \Gamma_{t'} = \emptyset$ for $t \neq t'$, then N is the compact graph of a continuous function $v : X \to (-\infty, 0]$, $X \subset \mathbb{C}^2$, such that $\overline{\Omega} \subset X$ and $v < 0$ in Ω, $v = 0$ on $b\Omega$. Moreover $v_o = v_{|\Omega}$ is a weak solution of the stationary problem and Ω is Stein.*

PROOF. Let $g^{-1}(0) = b\Omega$, where $g(z)$ is constant for $|z| \gg 0$, and

$$X = \{z \in \mathbb{C}^2 : (z,t) \in N, \text{ for some } t \geq 0\}.$$

Clearly N and X are compact and $b\Omega \subset X$,

$$N \cap (\mathbb{C}^2 \times \{0\}) = N \cap (X \times \{0\}) = b\Omega \times \{0\}.$$

By definition

$$N = \bigcup_{0 \le t \le t^*} \Gamma_t \times \{t\}, \quad X = \bigcup_{0 \le t \le t^*} \Gamma_t.$$

Consider an arbitrary $z^\circ \in X$. By these formulae there exists $t^\circ \in \overline{\mathbb{R}}^+$ such that $(z^\circ, t^\circ) \in N$ if and only if $z^\circ \in \Gamma_{t^\circ}$. Since distinct Γ_{t°'s are distjoint t° is unique. Denote $v(z^\circ) = -t^\circ$. This defines the function $v : X \to [-t^*, 0]$. Since the graph of v is the compact set N, v is a continuous function with the properties: $v^{-1}(0) = b\Omega = \Gamma_\circ$, $v^{-1}(-t) = \Gamma_t$. To relate the domain X of v to $\overline{\Omega}$, assume (without loss of generality) that $g < 0$ on Ω, $g > 0$ on $\mathbb{C}^2 \setminus \overline{\Omega}$ and $g = 1$ for $|z| \gg 0$. Fix $z^\circ \in \Omega$. Then the function $t \to u(z^\circ, t)$ is negative at $t = 0$ and $u = 1$ for $t \gg 0$, hence there exists t° such that $u(z^\circ, t^\circ) = 0$, i.e. $v(z^\circ) = -t^\circ$ and $z^\circ \in X$. Thus $\overline{\Omega} \subset X$. (The question of equality will be addressed soon.)

It remains to show that v satifies the stationary equation $\mathcal{L}(v) = 1$ in Ω. Let $g^*(z) = v(z)$ for $z \in \overline{\Omega}$, and $g^*(z) = g(z)$ for $z \in \mathbb{C}^2 \setminus \overline{\Omega}$; in particular $g(z) > 0$ on $\mathbb{C}^2 \setminus \overline{\Omega}$. Let $u^*(z, t)$ be the solution of the parabolic problem corresponding to g^*: $\mathcal{L}(u^*) = u^*_t$, $u^*(z, 0) = g^*(z)$, $u^*(z, t)$ constant for $|z| \gg 0$.

Let

$$H = \{(z, t) \in \mathbb{C}^2 \times \overline{\mathbb{R}}^+ : u^*(z, t) < 0\},$$

consider $(z, t) \in H$ and let $-c = u^*(z, t)$. Then $c > 0$ and

$$\{z' \in \mathbb{C}^2 : u^*(z', t) = -c\} = \mathcal{E}_t^{\mathcal{L}}\big(\{g^*(z') = -c\}\big) = \mathcal{E}_t^{\mathcal{L}}(\Gamma_c) = \Gamma_{c+t}.$$

Consequently $z \in \Gamma_{c+t}$, i.e. $v(z) = -c - t$ and $u^*(z, t) = -c = v(z) + t$. Thus the parabolic equation

$$\mathcal{L}(v(z) + t) = \frac{\partial(v(z) + t)}{\partial t}$$

holds in H and (by a simple exercise on the weak solutions) $\mathcal{L}(v) = 1$ in Ω (if $z \in \Omega$ then, for some t, $(z, t) \in H$).

Observe finally that $\{v < 0\} = \Omega$ and that v is a subsolution to the homogeneous equation $\mathcal{L}(v) = 0$ in Ω. Hence, by [ST$_1$], Corollary 3.2, Ω is pseudoconvex.

REMARK 2.3 In view of Theorem 2.1, if $b\Omega$ is of class C^3 then $X = \overline{\Omega}$.

We conjecture that for an arbitrary bounded domain Ω which is not pseudoconvexe $\Gamma_t \not\subset \overline{\Omega}$ for some t.

3. – Stationary evolution

1. Let Ω be a bounded domain defined by $\{\rho < 0\}$, where ρ is smooth and strictly p.s.h. in a neighbourhood of $\overline{\Omega}$ and $\partial\rho \neq 0$ on $b\Omega$.

THEOREM 3.1. *Let $g \in C^\circ(b\Omega)$. The Dirichlet problem $\mathcal{L}(u) = 1$ in $\overline{\Omega}$ and $u = g$ on $b\Omega$ has a unique weak solution $u \in C^\circ(\overline{\Omega})$. If g belongs to $C^{2,\alpha}(b\Omega)$ then $u \in Lip\ (\overline{\Omega})$.*

PROOF. The Perron method applies in the present case as well. We define the class $\mathcal{F}$ of the upper semicontinuous subsolutions u of $\mathcal{L}(u) = 1$ in $\overline{\Omega}$ such that $u \leq g$ on $b\Omega$. $\mathcal{F}$ is not empty (if g_1 is a smooth function on $\overline{\Omega}$ and $g_1 \leq g$, $g_1 + \lambda\rho$ is a member of $\mathcal{F}$ provided $\lambda \gg 0$). Let $u_\circ : \overline{\Omega} \to \mathbb{R}$ be the function

$$(z, t) \mapsto \sup \{u(z, t) : u \in \mathcal{F}\}$$

and $u_\circ^*$ its upper semicontinuous envelope. Then, taking into account the comparison principle (which can be established using the same device as in the parabolic case), it is possible to show that $u_\circ$ is the desired solution.

We observe that, to make the Perron method applicable, P-regularity of the boundary $b\Omega$ suffices [ST$_1$]: $\overline{\Omega}$ is said to be *P-regular* if for $z^\circ \in b\Omega$ and $r > 0$ there is a continuous function ψ in $B(z^\circ, r) \cap \overline{\Omega}$ such that $\psi(z^\circ) = 0$, $\psi(z) < 0$ for $z = z^\circ$ and ψ is p.s.h in $B(z^\circ, r) \cap \overline{\Omega}$.

In order to establish the existence of Lipschitz solutions we approximate $\mathcal{L}$ by the uniformly elliptic operators

$$\mathcal{L}_\epsilon(u) = ((1 + \epsilon)\delta_{\alpha\beta} - (|\partial u|^2 + \epsilon)^{-1} u_{\bar{\alpha}} u_\beta) u_{\alpha\bar{\beta}}$$

and we consider the approximated problem $\mathcal{L}_\epsilon(u) = 1$ in $\overline{\Omega}$ and $g = 1$ on $b\Omega$, $\epsilon > 0$, for which we derive the following a priori estimates:

(i) $\max\limits_{\overline{\Omega}} |u| \leq \max\limits_{b\Omega} |u| + r^2$

(ii) $\max\limits_{\overline{\Omega}} |\partial u| \leq C$

where r is the radius of the smallest ball B containing $\overline{\Omega}$ and C is a constant depending only on g (for $\epsilon \to 0$).

Indeed let $v(z) = -|z - a|^2$ where a is the center of B, σ be a positive number and $u \in C^2(\Omega) \cap C^\circ(\overline{\Omega})$ be the solution of $\mathcal{L}_\epsilon(u) = 1$ with boundary value g. One has $\mathcal{L}_\epsilon(v) < \mathcal{L}_\epsilon(u)$; by virtue of the comparison principle $u(z) - v(z) \leq \max\limits_{b\Omega} (u - v)$ and consequently

$$\begin{aligned} u(z) &\leq \max_{b\Omega} |u| + \max_{z', z'' \in \overline{\Omega}} |v(z') - v(z'')| \\ &\leq \max_{b\Omega} |u| + r^2. \end{aligned}$$

The same argument applied to u and $v(z) = |z - a|^2$ yelds

$$u(z) \geq -\max_{b\Omega} |u| - r^2.$$

To obtain the estimate for the gradient we proceed as in [ST$_1$] (Lemma 2.2). We derive for $w = |\partial u|^2$ the elliptic equation

$$b_{\alpha\bar{\beta}} w_{\alpha\bar{\beta}} + b_\alpha w_\alpha = b_{\alpha\bar{\beta}} w_{\alpha\gamma} w_{\bar{\beta}\bar{\gamma}} + b_{\alpha\bar{\beta}} w_{\alpha\bar{\gamma}} w_{\bar{\beta}\gamma}$$

where $b_{\alpha\bar{\beta}} = (1+\epsilon)(|\partial u|^2 + \epsilon)\delta_{\alpha\beta} - u_{\bar{\alpha}} u_\beta$.

Since the matrix $(b_{\alpha\bar{\beta}})$ is positive definite , again by virtue of the maximum principle, we have

$$\max_{\overline{\Omega}} |\partial u| = \max_{b\Omega} |\partial u|.$$

Thus, to conclude, it is enough to bound the outward normal derivative $\partial u/\partial \nu$ along $b\Omega$. As in [ST$_1$], without any assumption on the Levi curvature of $b\Omega$, we obtain the estimate

$$\frac{\partial v}{\partial \nu} \le \frac{\partial u}{\partial \nu} \le \frac{\partial v'}{\partial \nu}$$

where $v = g - a\rho$, $v' = g + a\rho$ (here g denotes a smooth extension of the boundary value and a a sufficiently large constant).

Now the classical PDE theory gives, for every $0 < \epsilon < 1$, the existence of a unique solution $u_\epsilon \in C^{2,\alpha}(\overline{\Omega})$ of the approximated problem [LSU]; moreover, in view of the a priori estimates, $\{u_\epsilon\}$ is a bounded subset of $C^1(\overline{\Omega})$. Let $\{\epsilon_\nu\}$ be a sequence such that $\epsilon_\nu \to 0$ and $\{u_{\epsilon_\nu}\} \to u$ in $Lip\ (\overline{\Omega})$ as $\nu \to +\infty$: u is a weak solution of our problem. For if $u - \psi$ has a strict local maximum at $z^\circ \in \Omega$ and $\psi \in C^\infty$ then there exists a sequence $z^{\epsilon_\nu} \to z^\circ$ such that $u_{\epsilon_\nu} - \psi$ has a local maximum at z^{ϵ_ν}. Hence, if $\partial\psi(z^\circ) \neq 0$, one has

$$1 = \mathcal{L}_{\epsilon_\nu}(u_{\epsilon_\nu})(z^{\epsilon_\nu}) \le \mathcal{L}_{\epsilon_\nu}(\psi)(z^{\epsilon_\nu})$$

and, letting $\nu \to \infty$, $1 \le \mathcal{L}(\psi)(z^\circ)$. If $\partial\psi(z^\circ) = 0$ we set

$$\eta^\nu = (|\partial\psi(z^{\epsilon_\nu})|^2 + \epsilon)^{-\frac{1}{2}} \partial\psi(z^{\epsilon_\nu})$$

to obtain

$$(\delta_{\alpha\beta} - \bar{\eta}_\alpha \eta_\beta)\psi_{\alpha\bar{\beta}}(z^\circ) \ge 1$$

for some η belonging to the closure of $\{\eta^\nu\}$.

Thus u is a weak subsolution. The proof that u is a weak supersolution is similar.

2. We also have estimates of solutions. In order to state this let us denote $\lambda_1(z) \le \lambda_2(z)$ the eigenvalues of the matrix $(\rho_{\alpha,\bar{\beta}})$ at $z \in \overline{\Omega}$ and set

$$\lambda_1 = \min\ \lambda_1(z), \lambda_2 = \max\ \lambda_2(z)$$

in $\overline{\Omega}$.

THEOREM 3.2. *Let $u \in C^\circ(\overline{\Omega})$ be a weak solution of $\mathcal{L}(u) = 1$ in $\overline{\Omega}$. Then the following estimate holds true:*

$$\lambda_1^{-1}\rho(z) + \min_{b\Omega} u \le u(z) \le \lambda_2^{-1}\rho(z) + \max_{b\Omega} u.$$

In particular, the stationary problem has a unique weak solution $u \in Lip\ (\overline{\Omega})$ such that

$$\sup_{\overline{\Omega}} |\partial u| \le \lambda_1^{-1} \sup_{b\Omega} |\partial \rho|.$$

PROOF. Set $v = \lambda_1^{-1}\rho + m$ where $m = \min_{b\Omega} u$ and prove that v is a weak subsolution of $\mathcal{L}(u) = 1$. This is actually trivial where $\partial v \neq 0$ since there $\mathcal{L}(v) = \lambda_1^{-1}\mathcal{L}(\rho) \ge 1$ (i.e. v is a classical subsolution). Now let z° be a critical point for v and $\{z^\nu\} \subset \Omega$ be a sequence of points such that $\partial v(z^\nu) \neq 0$ and $z^\nu \to z^\circ$ as $\nu \to +\infty$. Set

$$\eta^\nu = |\partial v(z^\nu)|^{-1} \partial v(z^\nu).$$

Then (upon passing to a subsequence) we have $\eta^\nu \to \eta$, with $|\eta| = 1$ and

$$(\delta_{\alpha\beta} - \bar{\eta}_\alpha \eta_\beta) v_{\alpha\bar{\beta}}(z^\circ) \ge 1.$$

If φ is a smooth function and $v - \varphi$ has a local maximum at z° then, since the matrix $(v_{\alpha\bar{\beta}} - \varphi_{\alpha\bar{\beta}})$ is definite negative at z°, we derive

$$(\delta_{\alpha\beta} - \bar{\eta}_\alpha \eta_\beta) \varphi_{\alpha\bar{\beta}}(z^\circ) \ge 1.$$

This shows that v is a subsolution and, owing to the comparison principle, that $v \le u$ in $\overline{\Omega}$.

The proof of the right-hand side inequality is similar.

If u is the weak solution of the stationary problem, the preceding estimate implies

$$|\partial u| \le \lambda_1^{-1} |\partial \rho|$$

on $b\Omega$ and the conclusion follows in view of the maximum principle for $|\partial u|$.

3. Boundaries of strictly pseudoconvex domains evolve in stationary way. More generally, let Ω be a bounded domain in $\mathbb{C}^2$ such that $\overline{\Omega} = \overset{o}{\overline{\Omega}}$ and W be an open neighbourhood of $\overline{\Omega}$. Assume that there exists a continuous function $h : W \setminus \overline{\Omega} \to \mathbb{R}^+$ such that: h is weak subsolution of $\mathcal{L}(h) = 1$, $h(z) \to 0$ as $z \to z^\circ$ and

$$D^+ h(z^\circ) = \lim \sup_{z \to z^\circ} |z - z^\circ|^{-1} h(z) = +\infty$$

for every $z^\circ \in b\Omega$.

Under these hypotheses we have

THEOREM 3.3. *Let $v \in C^\circ(\overline{\Omega})$ be a weak solution of the stationary problem and for every $t \geq 0$, let*

$$\Gamma_t = \{z \in \overline{\Omega} : v(z) + t = 0\}$$

and

$$\overline{\Omega}_t = \{z \in \overline{\Omega} : v(z) + t \leq 0\}.$$

Then $\{\Gamma_t\}_{t\geq 0}$ and $\{\overline{\Omega}_t\}_{t\geq 0}$ are respectively the evolution of $\Gamma_\circ = b\Omega$ and $\overline{\Omega}_\circ = \overline{\Omega}$. In particular

$$t^*(b\Omega) = t^*(\overline{\Omega}) = \| v \|_{C^\circ(\overline{\Omega})} .$$

PROOF. We may assume that h is continuous in $\overline{W} \setminus \overline{\Omega}$ and constant, $h = c_\circ$, on $b\Omega$. Clearly $v(z) + t$ is a weak solution of $u_t = \mathcal{L}(u)$ in $\overline{\Omega} \times \mathbb{R}^+$ and, consequently, $|v(z) + t|$ is a weak solution too. Let us choose c_1, c_2, $0 < c_2 < c_1 < c_\circ$ and set

$$W_j = \{z \in W \setminus \overline{\Omega} : h(z) < c_j\} \cup \overline{\Omega}, \ \ j = 1, 2.$$

The domains W_1, W_2 are pseudoconvex and $\overline{\Omega} \subset W_2$, $\overline{W}_2 \subset W$. Let

$$M > \max\,(c_\circ, \sup\ v)$$

and $\varphi : [0, c_\circ] \to \mathbb{R}$ be a continuous function such that: $\varphi(\xi) = \xi$ for $0 \leq \xi \leq c_2$, $\varphi(\xi) = M$ for $c_1 \leq \xi \leq c_\circ$ and $\varphi(\xi)$ is linear for $\xi \in [c_2, c_1]$. In particular, $\varphi(\xi) \geq \xi$ for all $\xi \in [0, c_\circ]$. Finally we define

$$g(z) = \begin{cases} |v(z)| & \text{if } z \in \overline{\Omega} \\ \varphi(h(z)) & \text{if } z \in W \setminus \overline{\Omega} \\ M & \text{if } z \in \mathbb{C}^2 \setminus \overline{W}. \end{cases}$$

Clearly, g is continuous, bounded and constant for $|z| \gg 0$. Moreover, $g(z) = |v(z)|$ for $z \in \overline{\Omega}$, $g(z) \geq h(z)$ for $z \in \overline{W} \setminus \overline{\Omega}$ and

$$\sup_{\mathbb{C}^2} g = M = g(\zeta)$$

for $\zeta \in \mathbb{C}^2 \setminus \overline{W}_1$.

Let us consider the weak solution of the parabolic problem corresponding to g. We have $u(z, t) = M$ for all $z \in \mathbb{C}^2$ and $t \geq t_\circ$ and, thanks to Remark 2.1, $u(z, t) = M$ on $(\mathbb{C}^2 \setminus W_1) \times \overline{\mathbb{R}}^+$. Moreover, if $V = W \times (0, t_\circ)$, we have $u = M$ on $bV \setminus (\mathbb{C}^2 \times \{0\})$.

Now we are going to define a weak subsolution $u_\circ$ of $u_t = \mathcal{L}(u)$ to be compared with u:

$$u_\circ(z, t) = \begin{cases} |v(z) + t| & \text{if } z \in \overline{\Omega}, 0 \leq t \leq t_\circ \\ h(z) + t & \text{if } z \in W \setminus \overline{\Omega}, 0 \leq t \leq t_\circ. \end{cases}$$

Clearly $u_\circ$ is continuous in $\overline{V}$ and

$$0 \leq u_\circ(z, t) \leq M + t_\circ$$

for $(z,t) \in \overline{V}$. In order to prove that $u_\circ$ is a weak subsolution in V, let us consider $\varphi \in C^\infty(V)$ and (z°, t°) with the following properties: $u_\circ(z^\circ, t^\circ) = \varphi(z^\circ, t^\circ)$ and $u_\circ(z,t) \le \varphi(z,t)$ near (z°, t°). Then $z^\circ \notin b\Omega$ for otherwise

$$+\infty > |\partial\varphi(z^\circ, t^\circ)| \ge D^+\varphi(\cdot, t^\circ)(z^\circ) \ge D^+h(z^\circ) = +\infty$$

which is a contradiction.

Thus $z^\circ \in V \setminus b\Omega$ and it is then clear, by definition of $u_\circ$, that $\varphi_t \ge \mathcal{L}(\varphi)$ at (z°, t°). This proves that $u_\circ$ is a weak subsolution.

Let $\alpha > M^{-1}(M + t_\circ)$. By what is preceding we deduce that $u_\circ \le \alpha u$ on bV and consequently, in view of the comparison principle, $0 \le u_\circ \le \alpha u$ on $\overline{V}$ (since αu is a weak solution too).

Now, if $z \in \Gamma_t$, one has $v(z) = -t$, $u_\circ(z,t) = 0$ and therefore $\Gamma_t \subset \{z \in \overline{\Omega} : v(z) = -t\}$ for all $t > 0$. Since we already proved the opposite (Corollary 2.5) we obtain the first part of the statement.

Finally, the inclusion $\overline{\Omega}_t \subset \overline{\Omega}$ follows from Theorem 2.1 since $\overline{\Omega}$ has a Stein base of neighbourhoods (thanks to the existence of h). On the other hand we have

$$\begin{aligned}\overline{\Omega}_t = \{z \in \overline{\Omega} : v(z) \le -t\} &= \bigcup_{s \ge t} \Gamma_s = \bigcup_{s\ge t} \mathcal{E}_t^{\mathcal{L}}(\Gamma_{s-t}) \\ &\subset \mathcal{E}_t^{\mathcal{L}}\left(\bigcup_{s\ge t} \Gamma_{s-t}\right) = \mathcal{E}_t^{\mathcal{L}}(\overline{\Omega})\end{aligned}$$

namely $\overline{\Omega}_t = \mathcal{E}_t^{\mathcal{L}}(\overline{\Omega})$.

This ends the proof.

REMARK 3.1 A bounded strictly pseudoconvex domain with a C^2-boundary satisfies the hypotheses of the above theorem.

PROOF. There exists a C^2 function ψ, strictly p.s.h on a domain V, with the properties: $\overline{\Omega} \subset V$, $\overline{\Omega} = \{\psi \le 0\}$, $\partial\psi(z) \ne 0$ for all $z \in b\Omega$. We may also assume that

$$\psi_{\alpha\bar{\beta}}\xi^\alpha\bar{\xi}^\beta \ge |\xi|^2$$

in V. This implies that $\mathcal{L}(\psi) \le 1$ in V.

Let us define

$$W = \{z \in \Omega : \psi(z) < 1/4\}$$

and

$$h(z) = \psi(z)^{1/2},$$

$z \in W \subset \overline{\Omega}$. Then we have

$$\mathcal{L}(h) = \frac{1}{2}\psi(z)^{-1/2}\mathcal{L}(\psi) \ge 1$$

if $z \in W \setminus \overline{\Omega}$ and $h(z) \to 0$, $D^+h \to 0$, as $z \to z^\circ$, if $z^\circ \in b\Omega$.

4. For V open in $\mathbb{C}^2$ denote by $\mathcal{P}(V) = C(V) \cap PSH(V)$, the class of all functions continuous and p.s.h on V. Let $K \subset V$ be compact. We say that K is $\mathcal{P}(V) - convex$, if for every $z_\circ \in V \setminus K$, there is a function $\phi \in \mathcal{P}(V)$ such that $\phi(z) \leq 0$ for $z \in K$ and $\phi(z_\circ) > 0$.

Following [S], a compact subset K of $\mathbb{C}^2$ is said to be $\mathcal{B} - regular$ if for every positive M there exists a smooth p.s.h function λ such that: $0 < \lambda < 1$ and, near K,

$$\lambda_{\alpha\bar{\beta}}(z)\xi^\alpha\bar{\xi}^\beta \geq M|\xi|^2.$$

Then as for "instantaneous disappearance" we have

THEOREM 3.4. *Let K be a compact subset of $\mathbb{C}^2$. Suppose that K is $\mathcal{B}$-regular and $\mathcal{P}(V)$-convex for some open neighbourhood V. Then the extinction time of K is 0, i.e. $\mathcal{E}_t^{\mathcal{L}}(K) = \emptyset$ for $t > 0$.*

PROOF. (a) We first prove that there is a function $\psi \in \mathcal{P}(V)$ such that $\psi(z) \geq 0$ for $z \in V$ and $\psi^{-1}(0) = K$ (and this holds for every $\mathcal{P}(V)$-convex compact subset of $\mathbb{C}^n$).

For every $\zeta \in V \setminus K$, choose a function $\psi \in \mathcal{P}(V)$ such that $\psi(\zeta) > 0$, $\psi_{|K} \leq 0$ and put $\psi^\zeta = \max(\psi, 0)$. Then

$$\psi^\zeta(\zeta) > 0, \ \psi^\zeta_{|K} = 0, \ \psi^\zeta_{|V} \geq 0$$

and $\inf \psi^\zeta > 0$ on a neighbourhood B^ζ of ζ in $V \setminus K$.

The family $\{B^\zeta\}_{\zeta \in V\setminus K}$ is an open covering of $V \setminus K$ which admits a countable subcovering $\{B^{\zeta_m}\}$. For simplicity denote $B^{\zeta_m} = B_m$ and $\psi^{\zeta_m} = \psi_m$. Let α_m be positive factors such that

$$\sup \left\{\alpha_m \psi_m(z) : z \in \overline{B}_1 \cup \cdots \overline{B}_m\right\} \leq 2^{-m}.$$

It is easy to see that the series

$$\sum_0^{+\infty} \alpha_m \psi_m$$

converges uniformly on compact subsets of V and that it defines a positive continuous p.s.h function $\psi : V \to \mathbb{R}$ such that $\psi^{-1}(0) = K$.

(b) Now fix $V_\circ$, a neighbourhood of K in $\mathbb{C}^2$, in such a way that K is $\mathcal{P}(V_\circ)$-convex and fix $0 < \epsilon < 1$. We will define a weak subsolution v^ϵ of the parabolic equation.

To this scope we consider a neighbourhood V of K, contained in $V_\circ$, such that there is a smooth, p.s.h function $\lambda : V \to (0, \epsilon)$ with

$$\lambda_{\alpha\bar{\beta}}(z)\xi^\alpha\bar{\xi}^\beta \geq |\xi|^2$$

for $z \in V$, $\xi \in \mathbb{C}^2$.

Note that K is $\mathcal{P}(V)$-convex because $V \subset V_\circ$. Let $\psi : V \to \mathbb{R}^+$ be a function belonging to $\mathcal{P}(V)$ like in (a). Multiplying it, if needed, by a positive constant, we can assume that the set $W = \{\psi \leq 1\}$ is compact. Let

$$g(z) = \begin{cases} \min(\psi(z), 1) & \text{if } z \in V \\ 1 & \text{if } z \notin V; \end{cases}$$

$g : \mathbb{C}^2 \to \mathbb{R}$ is continuous and constant on $\mathbb{C}^2 \setminus W$. Let u^ϵ be the weak solution of the parabolic problem corresponding to g and T be the minimum time such that $u^\epsilon(z,t)$ is constant (=1) for $t \geq T$, all z (we note in passing that all the object U, λ, W, g, T depend on ϵ, but we will suppress this in our notation). Let now

$$v^\epsilon(z,t) = \psi(z) + \lambda(z) - \epsilon + t$$

for $(z,t) \in \overline{W} \times [0,T]$. Clearly v^ϵ is a (local) weak subsolution of the parabolic problem in $H = W \times (0,T)$ which is continuous on $\overline{H}$.

Note that

$$v^\epsilon(z,0) \leq g(z)$$

for $z \in \overline{W}$ and that

$$v^\epsilon(z,t) \leq 1 + T$$

for $z \in \overline{H} \setminus \mathbb{C}^2 \times \{0\}$.

Therefore

$$v^\epsilon(z,t) \leq (1+T)u^\epsilon(z,t)$$

for $(z,t) \in bH$. Since $(1+T)u^\epsilon$ is a weak solution, in view of the comparison principle we obtain that

$$v^\epsilon(z,t) \leq (1+T)u^\epsilon(z,t)$$

for (z,t) in $\overline{H}$.

Thus, for $0 \leq t \leq T$,

$$\mathcal{E}_t^{\mathcal{L}}(K) = \{z \in \overline{W} : (1+T)u^\epsilon(z,t) \leq 0\} \subset \{z \in \overline{W} : v^\epsilon(z,t) \leq 0\}.$$

If $\mathcal{E}_t^{\mathcal{L}}(K) \neq \emptyset$, then for $z \in \mathcal{E}_t^{\mathcal{L}}(K)$, $v^\epsilon(z,t) \leq 0$ i.e.

$$\psi(z) + t \leq \epsilon - \lambda(z) < \epsilon.$$

Since $\psi(z) \geq 0$, this implies that $t < \epsilon$ i.e. that $\mathcal{E}_t^{\mathcal{L}}(K) = \emptyset$ for all $t \geq \epsilon$. But since $\epsilon > 0$ is arbitrary, this proves the theorem.

REMARK 3.2 This is the case if K is a compact subset of a totally real submanifold $M \subset \mathbb{C}^2$.

4. – Evolution of starshaped and convex sets

1. To study the effect of homotety in space variables on the solutions of our parabolic equation we consider the following standard transformation of functions:

$$(H_k u)(z,t) = u(kz, k^2 t), \ k > 0.$$

If K is a compact subset of $\mathbb{C}^2$ and α a real number, we denote

$$\alpha K = \{\alpha z \in \mathbb{C}^2 : z \in K\}.$$

PROPOSITION 4.1. *If u is a local solution of the parabolic equation $u_t = \mathcal{L}(u)$ then so are $H_k u$, $k > 0$. Moreover, if*

$$K = \Gamma_t = \mathcal{E}_t^{\mathcal{L}}(\Gamma_\circ)$$

then, for every $s > 0$,

$$\mathcal{E}_t^{\mathcal{L}}(s\Gamma_\circ) = s\Gamma_t.$$

PROOF. (Sketch).The first part of the statement is evident if u is a C^2 solution. If u is a weak subsolution, $h > 0$ fixed, and ψ a smooth test function, i.e. $\psi(z,t) \geq H_k u(z,t)$ with equality at (z°, t°), then $H_{k^{-1}}\psi$ is a test function for u, i.e. $H_{k^{-1}}\psi(z,t) \geq u(z,t)$ with equality at $(k^{-1}z^\circ, k^{-1}t^\circ)$. Hence $H_{k^{-1}}\psi$ satsfies the required inequality at $(k^{-1}z^\circ, k^{-1}t^\circ)$, which implies that ψ satisfies the definition at (z°, t°).

Let now $g^{-1}(0) = K$, g constant for $|z| \gg 0$. Let u be the solution of the parabolic problem corresponding to g. Then $\Gamma_t = \{z \in \mathbb{C}^2 : u(z,t) = 0\}$. Let $g^*(z) = g(s^{-1}z)$. Then

$$g^{*-1}(0) = sK, \ H_{s^{-1}}u(z,0) = g^*$$

and so
$$\begin{aligned} \mathcal{E}_t^{\mathcal{L}}(sK) = \mathcal{E}_t^{\mathcal{L}}(s\Gamma_\circ) &= \{z \in \mathbb{C}^2 : H_{s^{-1}}u(z,t) = 0\} \\ &= \{z \in \mathbb{C}^2 : u(s^{-1}z, s^{-2}t) = 0\} = s\Gamma_{s^{-2}t}. \end{aligned}$$

2. A bounded domain Ω of $\mathbb{C}^2$ is said to be *strictly starshaped* with respect to $z^\circ \in \Omega$ if, for every half-straight line $l(z^\circ)$ starting from z°, the intersection $l(z^\circ) \cap b\Omega$ consists of exactly one point.

If Ω is strictly starshaped with respect to $0 \in \mathbb{C}^n$, we let $p(z) = s$ if $s > 0$ and $z \in sb\Omega$, and $p(0) = 0$. We call p the *gauge function* of Ω with respect to 0.

THEOREM 4.2. *Let $\Omega \subset \mathbb{C}^2$ be strictly starshaped with respect to 0 and let $\{\Gamma_t\}_{t\geq 0}$ be the evolution of $\Gamma_0 = b\Omega$. Let t^* be its extinction time. Then*

(a) $\left(\alpha^{-1/2}\Gamma_\alpha\right) \cap \left(\beta^{-1/2}\Gamma_\beta\right) = \emptyset$ *for all* $0 < \alpha < \beta \leq t^*$;

(b) $\bigcup\{\alpha^{-1/2}\Gamma_\alpha : 0 < \alpha \leq t^*\} = \mathbb{C}^2$;

(c) *there is a unique continuous function μ such that*

$$\mu\left(\alpha^{-1/2}\Gamma_\alpha\right) = \alpha^{-1/2}, \ \ \alpha > 0;$$

(d) $\lim_{|z|\to+\infty} \frac{\mu(z)}{p(z)} = 1.$

PROOF. For a finite positive C consider

$$g^C = \min\,(p(z), C)$$

and let u^C denote the unique solution of the parabolic problem corresponding to g^C.

Now the fı ıction u^C can be easily decribed via its level sets. If $s < C$, then $\{p(z) = s\} = s\Gamma_\circ$, and so, for $t > 0$,

$$\{z \in \mathbb{C}^2 : u^C(z,t) = s\} = \mathcal{E}_t^{\mathcal{L}}(\Gamma_\circ) = s\Gamma_{s^{-2}t},$$

i.e.

$$\text{(1)} \qquad u^C = s \text{ on } s\Gamma_{s^{-2}t} \times \{t\},\ \text{if } 0 < s < C, 0 \le t \le t^*;$$

otherwise $u^C(z,t) = C$.

Consider now $0 < \alpha < \beta \le t^*$ and choose $t > 0$ small enough so that

$$s_1 = \alpha^{-1/2}t^{1/2} < C, \quad s_2 = \beta^{-1/2}t^{1/2} < C.$$

Then

$$s_1\Gamma_{s_1^{-2}t} = \alpha^{-1/2}t^{1/2}\Gamma_\alpha, \quad s_2\Gamma_{s_2^{-2}t} = \beta^{-1/2}t^{1/2}\Gamma_\beta,$$

and by (1),

$$\text{(2)} \qquad u^C = s_1 = \alpha^{-1/2}t^{1/2}$$

on

$$\left(\alpha^{-1/2}t^{1/2}\Gamma_\alpha\right) \times \{t\},$$

whenever $\alpha^{-1/2}t^{1/2} < C$, $t > 0$ and

$$\text{(3)} \qquad u^C = s_2 = \beta^{-1/2}t^{1/2}$$

on

$$\left(\beta^{-1/2}t^{1/2}\Gamma_\beta\right) \times \{t\},$$

whenever $\beta^{-1/2}t^{1/2} < C$, $t > 0$.

Since $s_1 \ne s_2$ it follows that the sets $\alpha^{-1/2}t^{1/2}\Gamma_\alpha$ and $\beta^{-1/2}t^{1/2}\Gamma_\beta$ are disjoint, being different sublevel sets of the continuous function $u^C(\cdot, t)$. By (3)

$$\{(z,t) \in \mathbb{C}^2 \times \mathbb{R}^+ : u^C(z,t) < C\} = \bigcup_{C^{-2}t < \alpha \le t^*} \left(\alpha^{-1/2}t^{1/2}\Gamma_\alpha\right) \times \{t\},$$

and, whenever $C < C'$, $u^C = u^{C'}$ on the set $\{u^C < C\}$.

Since the sets $\{u^C < C\}$, for $C > 0$, form an increasing family of open sets, it follows that the family $\{u^C\}_{C\geq 0}$, defines a continuous function u on

$$H = \bigcup_{C>0} \left\{(z,t) \in \mathbb{C}^2 \times \mathbb{R}^+ : t > 0, u^C(z,t) < C\right\},$$

which is a local solution of the parabolic equation there.

Moreover $H = \mathbb{C}^2 \times \mathbb{R}^+$. Indeed, for every $R > 0$, there is a positive C such that the ball $\overline{B}(0, R)$ is contained in

$$\{z \in \mathbb{C}^2 : p(z) < C\},$$

hence

$$\mathcal{E}_t^{\mathcal{L}}(\overline{B}(0, R)) \times \{t\} = \left\{u^C < C\right\}.$$

But

$$\mathcal{E}_t^{\mathcal{L}}\left(\overline{B}(0, R)\right) = \overline{B}(0, \sqrt{R^2 - t}),$$

and

$$\bigcup_{R,t>0} \overline{B}(0, \sqrt{R^2 - t}) \times \{t\} = \mathbb{C}^2 \times \mathbb{R}^+.$$

We conclude that u is a well-defined continuous (local) weak solution of the parabolic equation on $\mathbb{C}^2 \times \mathbb{R}^+$. (Note that it is unbounded and its properties as $t \to 0$ are still to be determined.)

Let now $\mu(z) = u(z, 1)$, $z \in \mathbb{C}^2$. Then $\mu : \mathbb{C}^2 \to \mathbb{R}$ is a continuous function uniquely determined by the condition

$$\mu\left(\alpha^{-1/2}\Gamma_\alpha\right) = \alpha^{-1/2}, \quad \alpha > 0. \tag{4}$$

It follows a posteriori that

$$\bigcup_{0<\alpha\leq t^*} \alpha^{-1/2}\Gamma_\alpha = \mathbb{C}^2.$$

Using (1), (2), (3) and (4) we can recover u^C in terms of μ:

$$u(z,t) = t^{1/2}\mu(t^{-1/2}z), \quad z \in \mathbb{C}^2, \ t > 0,$$
$$u^C(z,t) = \min\left(C, t^{1/2}\mu(t^{-1/2}z)\right), \quad z \in \mathbb{C}^2, \ t > 0.$$

On the other hand we know that

$$u^C(z, 0) = \min\ (p(z), C)$$

and that u^C is continuous on $\mathbb{C}^2 \times \overline{\mathbb{R}}^+$.

Consider now an arbitrary sequence $\{\zeta_m\} \subset \mathbb{C}^2$, $|\zeta_m| \to +\infty$ and let

$$t_m = p(\zeta_m)^{-2},\ z_m = p(\zeta_m)^{-1}\zeta.$$

Then $t_m \to 0$ and $\{z_m\} \subset b\Omega = \Gamma_\circ$. Consider an arbitrary subsequence $\{z_{m_k}\}$ that has a limit $z_\circ$ and fix $C > 1$. Then, as $(z_{m_k}, t_{m_k}) \to (z^\circ, t^\circ)$ and since $p(z^\circ) = 1$, we have

$$\begin{aligned} 1 &= \min(C, p(z^\circ)) = u^C(z^\circ, 0) = \lim_{k\to+\infty} u^C(z_{m_k}, t_{m_k}) \\ &= \lim_{k\to+\infty} \min\left(C, t_{m_k}^{1/2}\mu(t_{m_k}^{-1/2} z_{m_k})\right) = \lim_{k\to+\infty} \min\left(p(\zeta_{m_k})^{-1}\mu(\zeta_{m_k}), C\right). \end{aligned}$$

It follows that only a finite number of terms

$$\frac{\mu(z_{n_k})}{p(z_{n_k})}$$

can exceed C, and

$$\lim_{k\to+\infty} \frac{\mu(z_{n_k})}{p(z_{n_k})} = 1.$$

We conclude that

$$\lim_{|z|\to+\infty} \frac{\mu(z)}{p(z)} = 1.$$

REMARK 4.1 We do not say that the subsets Γ_α themselves are disjoint. This is probably false if $b\Omega$ is not pseudonvex.

Thanks to the above proposition we may define the function $\mu : \mathbb{C}^2 \to \mathbb{R}^+$ setting, for $z \in \mathbb{C}^2$, $\mu(z) = \alpha^{-1/2}$ if $z \in \alpha^{-1/2}\Gamma_\alpha$.

Let us observe that, for a starshaped domain, this function is actually depending upon the point z°. Moreover

LEMMA 4.3. *Under the hypotheses of the above theorem, the function μ constructed there is continuous and is a weak solution of*

$$\mathcal{L}(\mu) = \frac{1}{2}\mu - Re\, z_\alpha \mu_{z_\alpha}.$$

Moreover, $\mu(z) \geq 1/\sqrt{t^*}$, for all $z \in \mathbb{C}^2$ and

$$\lim_{|z|\to\infty} \frac{\mu(z)}{p(z)} = 1$$

where $p(z)$ is the gauge function of Ω with respect to 0 (i.e. $p(z) = s$, $s > 0$, if and only if $s^{-1}z \in b\Omega$ and $p(0) = 0$).

We omit the proof of this lemma.

COROLLARY 4.4. *For all* $\alpha > 0$, $\overset{o}{\Gamma}_\alpha = \emptyset$.

PROOF. Suppose that Γ_α contains a ball $B(z^\circ, R)$, $R > 0$. Then $\mu = \mu(z^\circ)$ on $B(z^\circ, R)$.

Consider the test function

$$\psi(z) = \mu(z^\circ) + C|z - z^\circ|^2, \; C > 0.$$

Then $\psi \geq \mu$ in $B(z^\circ, R)$ and $\psi(z^\circ) = \mu(z^\circ)$.

On the other hand, with $\eta \in \mathbb{C}^2$, $|\eta| \leq 1$,

$$(\delta_{\alpha\beta} - \bar{\eta}^\alpha \eta^\beta)\psi_{\alpha\bar{\beta}} = (2 - |\eta|^2)C$$

bears no relation to

$$\frac{1}{2}\psi(z^\circ) - Re\, z^\circ_\alpha \psi_\alpha(z^\circ) = \frac{1}{2}\mu(z^\circ).$$

This is a contradiction.

3. We assume from now on that Ω is a bounded convex domain; we retain the notation introduced in the starshaped case. In particular $0 \in \Omega$ and $p(0) = 0$. Of course p is now convex.

Since Ω is starshaped with respect to every point $z^\circ \in \Omega$ and since

$$\mathcal{E}_t^{\mathcal{L}}(\Gamma_\circ - z^\circ) = \mathcal{E}_t^{\mathcal{L}}(\Gamma_\circ) - z^\circ = \Gamma_t - z^\circ,$$

part (a) of Theorem 4.2 implies immediately that

$$\alpha^{-1/2}(\Gamma_\alpha - z^\circ) \cap \beta^{-1/2}(\Gamma_\beta - z^\circ) = \emptyset \tag{5}$$

for every $z^\circ \in \Omega$ and $0 < \alpha < \beta \leq t^*$.

More precisely

$$\beta^{-1/2}(\overline{\Omega}_\beta - z^\circ) \subset \alpha^{-1/2}(\Omega_\alpha - z^\circ). \tag{6}$$

This fact gives

PROPOSITION 4.5. *Suppose that* Ω *is convex and that* $0 \in \Omega$. *Then*

(a) $\mu(z + \xi) \leq \mu(z) + p(\xi)$ *for all* $z, \xi \in \mathbb{C}^2$.

More precisely, for all $z \in \mathbb{C}^2$, $z^\circ \in \Omega$ *and* $s > 0$, *it holds*

(b) $\mu(z + sz^\circ) < \mu(z) + s$.

PROOF. It is evident that (a) follows from (b) by taking

$$z^\circ = (1 - 1/n)p(\xi)^{-1}\xi, \; s = p(\xi),$$

and passing to the limit.

In order to prove part (b) let us fix $z \in \mathbb{C}^2$, $z^\circ \in \Omega$ and $s > 0$.

Denote $\beta^{-1/2} = \mu(z)$ and consider arbitrary positive α such that $\alpha^{-1/2} \geq \beta^{-1/2} + s$. Then $z \in \beta^{-1/2}\Gamma_\beta$ and $z - \beta^{-1/2}z^\circ \in \beta^{-1/2}\left(\Gamma_\beta - z^\circ\right)$. Since $\alpha^{-1/2} > \beta^{-1/2}$, we get by (5)

$$z - \beta^{-1/2}z^\circ \notin \alpha^{-1/2}\left(\Gamma_\alpha - z^\circ\right)$$

i.e.

$$z + (\alpha^{-1/2} - \beta^{-1/2})z^\circ \notin \alpha^{-1/2}\Gamma_\alpha.$$

Considering now arbitrary $z^\circ \in \Omega$ (and z still fixed), we get

$$z + (\alpha^{-1/2} - \beta^{-1/2})\Omega \cap \alpha^{-1/2}\Gamma_\alpha = \emptyset,$$

and since Ω is starshaped with respect to 0,

$$z + s\Omega \subset z + (\alpha^{-1/2} - \beta^{-1/2})\Omega.$$

Hence,

$$(z + s\Omega) \cap \alpha^{-1/2}\Gamma_\alpha = \emptyset$$

for every α such that $\alpha^{-1/2} \geq \beta^{-1/2}+s$, i.e., for all such α, $\alpha^{-1/2} \notin \mu\left(z+s\Omega\right)$.

Hence

$$\mu\left(z + s\Omega\right) \subset [0, \beta^{-1/2} + s]$$

i.e., for each $z^\circ \in \Omega$, $\mu(z + sz^\circ) < \beta^{-1/2} + s = \mu(z) + s$.

REMARK 4.2 In particular, performing a translation of Ω in such a way as to have $0 \in \Gamma_{t^*}$, we derive from (a):

(c) if $0 < s < 1$, for all $0 \leq t \leq (t^{*-1/2} + s)^2$, $\Gamma_t \cap s\Omega = \emptyset$;

using as point z° every point of Ω we also obtain

$$\Gamma_t \cap \bigcup_{z^\circ \in \Gamma_{t^*}} \{s(\Omega - z^\circ) + z^\circ\} = \emptyset.$$

(c) can be rewritten in the following way: if $0 \in \Gamma_{t^*}$ then

$$\Gamma_t \cap s_\circ\Omega = \emptyset$$

where $s_\circ = s_\circ(t) = t^{-1/2} - t^{*-1/2}$.

COROLLARY 4.6. *For $0 < t \leq t^*$ the sets $\Gamma_t \cap \Omega$ are pairwise disjoint.*

PROOF. Suppose $0, \alpha < \beta \leq t^*$ and $z \in \Gamma_\alpha \bigcap \Gamma_\beta \bigcap \Omega$. By Proposition 4.5, part (b),

$$\alpha^{-1/2} = \mu(\alpha^{-1/2}z) = \mu\left(\beta^{-1/2}z + (\alpha^{-1/2} - \beta^{-1/2})z\right) < \beta^{-1/2} + (\alpha^{-1/2} - \beta^{-1/2})$$
$$= \alpha^{-1/2}.$$

This is a contradiction.

REMARK 4.3. We note that if we could prove that $\mathcal{E}_t^{\mathcal{L}}\{b\Omega\} \subset \Omega$ for every $t > 0$ the last corollary would imply that the evolution $\{\Gamma_t\}_{t\geq 0}$ of $\Gamma_\circ = b\Omega$ is of stationary type.

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