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**Variational fractals**

# Variational Fractals

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In memoriam, Ennio de Giorgi

## 1. – Introduction

The aim of this paper is to prove suitable *scaled Poincaré inequalities* on a large class of variational fractals, that is, self-similar fractals that support an invariant energy form.

The class of *variational fractals* has been introduced in [23], as a general non-differentiable framework in which *both* standard fractal examples and Euclidean classic Laplacians can be analyzed at the same time. *Self-similarity* is the basic notion in this setting.

Scaled Poincaré inequalities are important because they produce a number of important structural properties of elliptic equations, like Hölder continuity of local solutions, estimates of fundamental solutions and energy decay of Saint-Venant type. A structural theory – based on *Dirichlet forms* – that generalizes the celebrated theory of De Giorgi has been carried out in a general non-differentiable setting by M. Biroli and the author in [4], ... , [7]. Indeed, the theory of Dirichlet forms - itself a culmination of the classic energy approach to potential theory - offers the natural setting for De Giorgi's renowned *truncation method*, incorporated in that theory as the so-called *Markovianity property* of the form. A review of "irregularities" in variational theories – from the point of view of fractals – can be found in [24].

As in the classic De Giorgi-Moser's theory, in the general theory too a fundamental role is played by *scaled Sobolev inequalities*. These in fact lead to the $L^\infty$ estimates of local solutions, that are the initial point for obtaining the basic Harnack inequalities. In [6], [7], the derivation of scaled Sobolev inequalities from scaled Poincaré inequalities – known in the theory of Lie groups after [30] and [28] – was extended to the non-differentiable setting of Dirichlet forms – possibly with *singular* local energy measures, as it is the case for most typical fractals – by relying on De Giorgi's truncation method. Clearly, scaled Poincaré inequalities offer a considerably simpler access to the whole theory.

The main contribution of this paper is to show that – in presence of self-similarity – by far an easier and more accessible starting point can be given to the theory. The new starting point is a single *global* Poincaré inequality, equivalent to the existence of a *spectral gap* – that is, the strict positivity of the first non-trivial eigenvalue – which is in turn a consequence, for example, of a Rellich property (compact imbedding in the $L^2$ space of the theory) of the space of functions of finite energy. From one side this shows that some important classic inequalities can be extended to general self-similar variational structures, possibly non-differentiable and fractal. From the other side, the result leads to a better understanding of the role of self-similarity in the classic Sobolev-Morrey theory itself.

One of the main features of the theory – as already mentioned – is that it applies, at the same time, to large families of self-similar fractals – like the so-called *nested fractals* of [21], [19], [11] – as well as to self-similar Euclidean sets, like cubes supporting the classic Laplacian. There is an essential ingredient that makes the building of such a "universal" theory possible. It is the "correct" definition of a *metric* inside the structure, which assumes the role of the Euclidean – or Riemannian – metric of the classic theory. The idea here, generally speaking, comes from physics. Such an intrinsic metric, in principle, cannot be simply derived from the *statics* of the structure, instead, it must be derived from its *dynamics*, that is, it must be obtained from the invariant energy form itself. A further discussion of this physical background can be found in [25].

For general Dirichlet forms, a variational metric – which reduces to the geodesic metric in Riemannian manifolds – was first introduced in [4]. It is indeed the main tool of the general structural theory mentioned before. However, this metric is not available on standard fractals, like for example the so-called *Sierpinski gasket*. In fact, the metric is related to a generalized "eiconal equation", that involves the *densities* of the local energies of the Dirichlet form with respect to the underlying volume mesure of the structure (the distance to a given point is defined to be a generalized *maximal sub-solution* of the eiconal equation, see [4]). Now, on typical fractals – like the gasket – such densities do not exist: The local energy – the "square of the gradient" of the potential field – due to the irregularities of the structure does not possess well defined pointwise values in a.e. sense. The way out to this uncomfortable situation – as first shown in [23] – is found by turning to a suitable *effective quasi-metric*. The essential – and determining – property enjoied by this quasi-distance is that its square shares the same scaling invariance (self-similarity) of the energy.

When endowed with this effective metric, the fractal $K$ becomes a *space of homogeneous type*, in the sense af abstract harmonic analysis, [10], [29]. In particular, the volume of an intrinsic ball $B_R$ of radius $R$ – that is, the measure $\mu(B_R)$, where $\mu$ is the invariant volume measure of $K$ – scales, up to equivalence, according to a power of the radius. The exponent $\nu > 0$ of this power law – the *homogeneous dimension* of $K$ – is uniquely determined by the basic structural constants of the variational fractal, as we further explain below. It is the homogeneous dimension $\nu$, and not the initial fractal (Hausdorff) dimension

$d_f$, the fundamental constant that governs all dynamical volume scalings in the fractal. For example, the energy in the intrinsic balls $B_R$ of $K$ shows the usual Euclidean-like scaling $R^{\nu-2}$, while the Poincaré inequalities on $B_R$ exhibit the usual $R^2$ scaling. This explains why the *same* exponent – the volume growth exponent $\nu$ – is at the same time the basic dynamical exponent, governing spectral asymptotics and standard diffusions, as well as the critical exponent of Sobolev-Morrey imbeddings.

The paper is organized as follows. In Section 2 we review main notions and properties of self-similar fractals, prove a "finite overlapping" property (Theorem 2.1), that is the analogue of an asymptotic density result due to Hutchinson, [15], and introduce the notion of *boundary*. In Section 3, we define a family of quasi-metrics, indexed by a real parameter $\delta > 0$, and show that each one of them gives the fractal the structure of a space of homogeneous type of dimension $\nu = d_f/\delta$ (Theorem 3.1). In Section 4 we introduce the notion of variational fractal and show that the parameter $\delta$ can be so chosen – in a given variational fractal – so that the invariant energy $E$ scales with an exponent $\nu - 2$ (Theorem 4.1). Such a $\delta$ is uniquely determined by the basic constants of the fractal, namely the similitudes factors $\alpha_1, \ldots, \alpha_N$ and the energy "renormalization" factors $\rho_1, \ldots, \rho_N$, assumed to be of the type $\rho_i = \alpha_i^{-d_f \sigma}$ for some constant $\sigma < 1$ (Lemma 4.1). We also prove "change of variable" formulas for the local energy (Theorem 4.5). In Section 5 we introduce some capacity notions and inequalities and we prove our main result, the scaled Poincaré inequalities on the balls of the intrinsic metric (Theorem 5.1). The main step here is the proof of a Poincaré inequality accross two contiguous copies of $K$ (Lemma 5.3), the main technical difficulty being that the two distinct copies intersect on a set of measure zero. In Section 6 we state the imbedding inequalities obtained from [6], [7]. In the final Section 7 we briefly describe some basic examples.

## 2. – Selfsimilar fractals

By $\mathbb{R}^D$, $D \geq 1$, we denote the $D$-dimensional Euclidean space, by $B_e(x, r)$, the balls $\{y \in \mathbb{R}^D : | x - y | < r\}$, $x \in \mathbb{R}^D$, $r > 0$, by $\operatorname{diam}_e A$ the diameter of $A \subset \mathbb{R}^D$. We suppose that $\Psi = \{\psi_1, \ldots, \psi_N\}$ is a given set of *contractive similitudes*, that is,

$$(2.1) \qquad | \psi_i(x) - \psi_i(y) | = \alpha_i^{-1} | x - y |, \quad x, y \in \mathbb{R}^D,$$

$i = 1, \ldots, N$, where we assume $\alpha_1 \geq \ldots \geq \alpha_N > 1$. Then, there exists a

unique (non empty) compact set $K$ in $\mathbb{R}^D$, which is *invariant* for $\Psi$, that is,

$$K = \Psi(K) := \bigcup_{i=1}^{N} \psi_i(K).$$

The real number $d_f > 0$ uniquely defined by the relation

$$\sum_{i=1}^{N} \alpha_i^{-d_f} = 1$$

is the *similarity (or fractal) dimension* of $K$. Moreover, there exists a unique Borel regular measure $\mu$ with unit mass which is *invariant* for $\Psi$, that is,

$$\int_K \varphi \, d\mu = \sum_{i=1}^{N} \alpha_i^{-d_f} \int_K \varphi \circ \psi_i \, d\mu$$

for every integrable $\varphi : K \longrightarrow \mathbb{R}$, and $\mu$ is supported on $K$. More specific metric information on $K$ and $\mu$ are available when the following *open set condition* is also satisfied:

$$(2.2) \qquad \bigcup_{i=1}^{N} \psi_i(U) \subset U, \quad \psi_i(U) \cap \psi_j(U) = \emptyset \quad \text{if} \quad i \neq j,$$

for some given non-empty bounded open subset $U$ of $\mathbb{R}^D$. By $c_1$ we shall denote the radius of a Euclidean ball contained in $U$ and by $c_2$ the radius of a ball containing $U$.

Under this assumption, the Hausdorff dimension of $K$ is equal to $d_f$, $0 < H^{d_f}(K) < \infty$, where $H^{d_f}$ is the Hausdorff $d_f$-dimensional measure in $\mathbb{R}^D$, and

$$\mu \equiv (H^{d_f}(K))^{-1} H^{d_f} \llcorner K.$$

For every similitude $\psi : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ such that $\psi : K \longrightarrow K$, we then have

$$\int_{\psi(K)} f(y)\mu(dy) = (\text{Lip}\,\psi)^{d_f} \int_K f \circ \psi(x)\mu(dx)$$

for all integrable $f$ on $\psi(K)$. We will use the notation $\psi_{i_1,\dots,i_n} := \psi_{i_1} \circ \psi_{i_2} \circ \dots \circ \psi_{i_n}$, $A_{i_1,\dots,i_n} := \psi_{i_1,\dots,i_n}(A)$, for arbitrary $n$-ples of indeces $i_1,\dots,i_n \in \{1,\dots,N\}$ and arbitrary $A \subset K$, with the convention $\psi_{i_1,\dots,i_n} = \text{Id}$, $A_{i_1,\dots,i_n} \equiv A$ for $n = 0$. For every sequence $i_1,\dots,i_n,\dots \in \{1,\dots,N\}$ we have $K \supset K_{i_1} \supset K_{i_1,\dots,i_n} \supset \dots$, $U \supset U_{i_1} \supset U_{i_1,\dots,i_n} \supset \dots$ and for every $n \geq 1$

$$K = \bigcup_{i_1,\dots,i_n=1}^{N} K_{i_1,\dots,i_n},$$

with $\mu(K_{i_1,\dots,i_n} \cap K_{j_1,\dots,j_n}) = 0$ whenever $(i_1,\dots,i_n) \neq (j_1,\dots,j_n)$, $n \geq 1$, hence

$$\mu(K) = \sum_{i_1,\dots,i_n=1}^{N} \mu(K_{i_1,\dots,i_n}) .$$

Following [21], we will call $K_{i_1,\dots,i_n}$, $n \geq 1$, $i_1,\dots,i_n \in \{1,\dots,N\}$, a *n-complex*. Two complexes $K_{i_1,\dots,i_m}$, $K_{j_1,\dots,j_n}$, are *distinct* if $(i_1,\dots,i_m) \neq (j_1,\dots,j_n)$, $m \geq 1$, $n \geq 1$. We have $K_{i_1,\dots,i_n} \subset \overline{U}_{i_1,\dots,i_n}$ for every $i_1,\dots,i_n \in \{1,\dots,N\}$ and $K_{i_1,\dots,i_n} \cap U_{j_1,\dots,j_n} = \emptyset$ for every $(i_1,\dots,i_n) \neq (j_1,\dots,j_n)$, $n \geq 1$. Moreover

$$\text{diam}_e\, K_{i_1,\dots,i_n} = \alpha_{i_1}^{-1} \cdots \alpha_{i_n}^{-1} \text{diam}_e\, K , \qquad \mu(K_{i_1,\dots,i_n}) = \alpha_{i_1}^{-d_f} \cdots \alpha_{i_n}^{-d_f} .$$

For all previous properties of self-similar fractals we refer to [15]. We shall use them all freely throghout the paper.

Let $i_1,\dots,i_n,\dots \in \{1,\dots,N\}$ be an arbitrary sequence of indeces. For every integer $n \geq 1$ we have $\alpha_{i_1}^{-1} \cdots \alpha_n^{-1} \text{diam}_e\, K < \alpha_{i_1}^{-1} \cdots \alpha_{n-1}^{-1} \text{diam}_e\, K$. Let $0 < R \leq \text{diam}_e\, K$. For every sequence $i_1,\dots,i_n,\dots \in \{1,\dots,N\}$, there exists a <u>least</u> integer $m \geq 0$, such that $\text{diam}_e\, K_{i_1,\dots,i_m} \equiv \alpha_{i_1}^{-1} \cdots \alpha_{i_m}^{-1} \text{diam}_e\, K \leq R$, ($m = 0$ if and only if $R = \text{diam}_e\, K$). The set of all finite sequences $(i_1,\dots,i_m)$ obtained in this way, in correpondence of a given $0 < R \leq \text{diam}_e\, K$, will be denoted by $I_R$. The following lemma is easily proved:

LEMMA 2.1. *Let $0 < R \leq \text{diam}_e\, K$ and let $I_R$ be defined as above. Then, for every $(i_1,\dots,i_m) \in I_R$ we have:*

$$\alpha_1^{-1} R < \alpha_{i_1}^{-1} \cdots \alpha_{i_m}^{-1} \text{diam}_e\, K \leq R ,$$

*and $0 \leq m_* \leq m \leq m^*$, where $m^*$ is the least integer greater or equal to $\log_{\alpha_N}(R^{-1} \text{diam}_e\, K)$, $m_*$ the largest integer smaller or equal to $\log_{\alpha_1}(R^{-1} \text{diam}_e\, K)$.*

Now let $0 < R \leq \text{diam}_e\, K$. By $G_R$ we denote the set:

$$G_R = \{K_{i_1,\dots,i_m} : (i_1,\dots,i_m) \in I_R\} .$$

Note that $G_R = \{K\}$ if $R = \text{diam}_e\, K$. From Lemma 2.1 we get

LEMMA 2.2. *Let $0 < R \leq \text{diam}_e\, K$ and let $G_R$ be defined as above. Then, for every $K_{i_1,\dots,i_m} \in G_R$ we have $\alpha_1^{-1} R < \text{diam}_e\, K_{i_1,\dots,i_m} \leq R$, with $0 \leq m_* \leq m \leq m^*$ if $m \geq 1$. Moreover, for every pair $K_{i_1,\dots,i_{m'}}$, $K_{j_1,\dots,j_{m''}} \in G_R$, $m' \geq 1$, $m'' \geq 1$, $(i_1,\dots,i_{m'}) \neq (j_1,\dots,j_{m''})$, we have $\mu(K_{i_1,\dots i_{m'}} \cap K_{j_1,\dots,j_{m''}}) = 0$, $U_{i_1,\dots,i_{m'}} \cap U_{j_1,\dots,j_{m''}} = \emptyset$.*

PROOF. The first part of the lemma is an immediate consequence of Lemma 2.1. Let $(i_1,\dots,i_{m'}) \in I_R$, $m' \geq 1$, $(j_1,\dots,j_{m''}) \in I_R$, $m'' \geq 1$, with $(i_1,\dots,i_{m'}) \neq (j_1,\dots,j_{m''})$, and let, say, $m' \leq m''$. *A priori* only two alternative cases may occur: (a), $i_1 = j_1$, $i_{m'} = j_{m'}$, and then $m' < m''$;

(b), there exists a largest integer $p \geq 0$, such that the first $p$ indices of both $(i_1, \ldots, i_{m'})$, $(j_1, \ldots, j_{m''})$ are equal, and $p < m'$. In the first case we have $\alpha_{j_1}^{-1} \cdots \alpha_{j_{m'}}^{-1} \operatorname{diam}_e K \equiv \alpha_{i_1}^{-1} \cdots \alpha_{i_{m'}}^{-1} \operatorname{diam}_e K \leq R$: since $m' < m''$, this is in contradiction with the definition of $m''$. Thus, only case (b) can actually occur. Then, $(i_1, \ldots, i_{m'}) \equiv (i_1, \ldots, i_p, i_{p+1}, \ldots, i_{m'})$, $(j_1, \ldots, j_{m''}) \equiv (i_1, \ldots, i_p, j_{p+1}, \ldots, j_{m'}, \ldots, j_{m''})$, $i_{p+1} \neq j_{p+1}$. Therefore,

$$
\begin{aligned}
K_{i_1,\ldots,i_{m'}} \cap K_{j_1,\ldots,j_{m''}} &= K_{i_1,\ldots,i_p i_{p+1},\ldots,i_{m'}} \cap K_{i_1,\ldots,i_p j_{p+1},\ldots,j_{m'},\ldots,j_{m''}} \\
&= \psi_{i_1,\ldots,i_p}\left(\psi_{i_{p+1},\ldots,i_{m'}}(K) \cap \psi_{j_{p+1},\ldots,j_{m'},\ldots,j_{m''}}(K)\right) \\
&\subset \psi_{i_1,\ldots,i_p}(K_{i_{p+1}} \cap K_{j_{p+1}}).
\end{aligned}
$$

Since $i_{p+1} \neq j_{p+1}$, we have

$$
\begin{aligned}
\mu(K_{i_1,\ldots,i_{m'}} \cap K_{j_1,\ldots,j_{m''}}) &\leq \mu\left(\psi_{i_1,\ldots,i_p}(K_{i_{p+1}} \cap K_{j_{p+1}})\right) \\
&= \alpha_{i_1}^{-d_f} \cdots \alpha_{i_p}^{-d_f} \mu(K_{i_{p+1}} \cap K_{j_{p+1}}) = 0.
\end{aligned}
$$

Similarly, $U_{i_1,\ldots,i_{m'}} \cap U_{j_1,\ldots,j_{m''}} \subset \psi_{i_1,\ldots,i_p}(U_{i_{p+1}} \cap U_{j_{p+1}}) = \emptyset$.

We now show that a self-similar fractal (satisfying the open set condition) enjoys the following *finite overlapping property* (for a related asymptotic density result, see [15], Theorem 5.3 (1) (i)).

THEOREM 2.1. *Let $K$ be a self-similar fractal* – (2.1), (2.2) – *and let*

$$
M = \left(1 + 2\frac{c_2}{\operatorname{diam}_e K}\right)^D \left(\alpha_1^{-1} \frac{c_1}{\operatorname{diam}_e K}\right)^{-D}.
$$

*Then, for every $x \in K$ and $0 < R \leq \operatorname{diam}_e K$, the family*

$$
G_{x,R} = \left\{K_{i_1,\ldots,i_m} : K_{i_1,\ldots,i_m} \in G_R, \ K_{i_1,\ldots,i_m} \cap B_e(x, R) \neq \emptyset\right\},
$$

*contains at most $M$ distinct complexes and*

$$
K \cap B_e(x, R) \subset \bigcup_{G_{x,R}} K_{i_1,\ldots,i_m},
$$

PROOF. Let $x \in K$ and $0 < R \leq \operatorname{diam}_e K$ be fixed. If $R = \operatorname{diam}_e K$, then $G_R = \{K\}$ and the theorem is obvious. Now let $0 < R < \operatorname{diam}_e K$. Let $m^* \geq 1$ be the least integer such that $\alpha_N^{-m^*} \operatorname{diam}_e K \leq R$. We have

$$
K = \bigcup_{i_1,\ldots,i_{m^*}=1}^{N} K_{i_1,\ldots,i_{m^*}}
$$

therefore,

$$K \cap B_e(x, R) = \bigcup_{i_1,\ldots,i_{m*}=1}^{N} K_{i_1,\ldots,i_{m*}} \cap B_e(x, R).$$

By Lemma 2.2, for every $m$-complex $K_{i_1,\ldots,i_m} \in G_R$ we have $\alpha_1^{-1} R < \mathrm{diam}_e\, K_{i_1,\ldots,i_m} \leq R$, $1 \leq m_* \leq m \leq m^*$. In particular, for every $i_1,\ldots,i_{m*} \in \{1,\ldots,N\}$, we have $K_{i_1,\ldots,i_{m*}} \subset K_{i_1,\ldots,i_m}$ with $K_{i_1,\ldots,i_m} \in G_R$, therefore $K \cap B_e(x, R) \subset \bigcup_{G_R} K_{i_1,\ldots,i_m} \cap B_e(x, R)$, hence also $K \cap B_e(x, R) \subset \bigcup_{G_{x,R}} K_{i_1,\ldots,i_m} \cap B_e(x, R)$. We now prove that there exist at most $M' \leq M$ distinct $m$-complexes $K_{i_1^l,\ldots,i_m^l} \in G_{x,R}$, $m \geq 1$, $l = 1,\ldots,M'$. By the open set condition, every $m$-complex $K_{i_1,\ldots,i_m}$, $i_1,\ldots,i_m \in \{1,\ldots,N\}$, $m \geq 1$, is contained in the closure $\overline{U}_{i_1,\ldots,i_m}$ in $\mathbb{R}^D$ of the open set $U_{i_1,\ldots,i_m}$. It suffices then to prove that there exist $M' \leq M$ distinct sets $\overline{U}_{i_1,\ldots,i_m}$, with $(i_1,\ldots,i_m) \in I_R$, $m \geq 1$, such that $\overline{U}_{i_1,\ldots,i_m} \cap B(x, R) \neq \emptyset$. Each one of such sets, $U_{i_1,\ldots,i_m} = \psi_{i_1,\ldots,i_m}(U)$, contains a Euclidean ball of radius $\alpha_{i_1}^{-1} \cdot \ldots \cdot \alpha_{i_m}^{-1} c_1 = \mathrm{diam}_e\, K_{i_1,\ldots,i_m} c_1 / \mathrm{diam}_e\, K \geq R \alpha_1^{-1} c_1 / \mathrm{diam}_e\, K$ and is contained in a Euclidean ball of radius $\alpha_{i_1}^{-1} \cdot \ldots \cdot \alpha_{i_m}^{-1} c_2 = \mathrm{diam}_e\, K_{i_1,\ldots,i_m} c_2 / \mathrm{diam}_e\, K \leq R c_2 / \mathrm{diam}_e\, K$, where we have taken Lemma 2.2 into account. Moreover, by Lemma 2.2, any two distinct sets $U_{i_1,\ldots,i_{m'}}$, $U_{j_1,\ldots,j_{m''}}$, $(i_1,\ldots,i_{m'}) \neq (j_1,\ldots,j_{m''}) \in I_R$, $m' \geq 1$, $m'' \geq 1$ are disjoint. Let $M'$ be the number of all such distinct $U_{i_1,\ldots,i_{m'}}$. By the previous properties, there exists $M'$ points whose mutual distance is $\geq R\alpha_1^{-1} c_1 / \mathrm{diam}_e\, K$, while all of them are contained in a Euclidean ball of radius $R(1 + 2c_2 / \mathrm{diam}_e\, K)$. Therefore, $M' \leq [\alpha_1^{-1} c_1 / \mathrm{diam}_e\, K]^{-D} [1 + 2c_2 / \mathrm{diam}_e\, K]^D$. We conclude that at most $M$ distinct sets $\overline{U}_{i_1,\ldots,i_m}$ meet $B_e(x, R)$, where $M$ is the constant in the statement of the theorem.

We now define the *boundary* $\Gamma$ of $K$ by setting

$$\Gamma = \bigcup_{i \neq j=1}^{N} \psi_i^{-1}(K_i \cap K_j).$$

LEMMA 2.3. $\Gamma$ *is a compact subset of* $K \cap \partial U$. *Moreover,* $\mu(\Gamma) = 0$.

PROOF. We have $K_i \subset \overline{U}_i$ for every $i = 1,\ldots,N$ and $K_i \cap U_j = \emptyset$ whenever $i \neq j$. Therefore, $K_i \cap K_j \subset \overline{U}_i \cap \overline{U}_j$, while $K_i \cap K_j \cap U_i = K_i \cap K_j \cap U_j = \emptyset$. Thus, $K_i \cap K_j \subset \partial U_i \cap \partial U_j$, hence $K_i \cap K_j \subset K_i \cap \partial U_i \cap K_j \cap \partial U_j$. Therefore, $\psi_i^{-1}(K_i \cap K_j) \subset \psi_i^{-1}(K_i) \cap \psi_i^{-1}(\partial U_i)$ and since $\psi_i^{-1} \partial U_i = \partial \psi_i^{-1} U_i$ we obtain $\psi_i^{-1}(K_i \cap K_j) \subset K \cap \partial U$ for every $i \neq j$. Moreover, since $\psi_i^{-1}(K_i \cap K_j)$ is compact for every $i, j = 1,\ldots,N$, $\Gamma$ is compact. Let us now prove that $\mu(\Gamma) = 0$. In fact, $\mu(\Gamma) \leq \sum_{i \neq j=1}^{N} \mu(\psi_i^{-1}(K_i \cap K_j)) = \sum_{i \neq j=1}^{N} \alpha_i^{d_f} \mu(K_i \cap K_j) = 0$.

It is easy to see that $\Gamma$ is also characterized as the set $\Gamma$ such that

$$K_i \cap K_j = \Gamma_i \cap \Gamma_j \quad \text{for every pair} \quad i \neq j.$$

In the following, we shall assume that this relation scales down by self-similarity to every pair of distinct $n$-complexes, that is, that for every $n \geq 1$ and every $(i_1, \ldots, i_n) \neq (j_1, \ldots, j_n)$, we have

$$(2.3) \qquad K_{i_1,\ldots,i_n} \cap K_{j_1,\ldots,j_n} = \Gamma_{i_1,\ldots,i_n} \cap \Gamma_{j_1,\ldots,j_n} \, .$$

REMARK 2.1. On nested fractals, $\Gamma$ coincides with the set $F$ of the essential fixed points of the given similitudes $\{\psi_1, \ldots, \psi_N\}$ and (2.3) is satisfied. In fact, by definition of nested fractals, [21], $K_{i_1,\ldots,i_n} \cap K_{j_1,\ldots,j_n} = F_{i_1,\ldots,i_n} \cap F_{j_1,\ldots,j_n}$, hence $\Gamma \equiv F$. On the other hand, if $\{\psi_1, \ldots, \psi_N\}$ are the $N = \alpha^D$ similitudes of $\mathbb{R}^D$ that decompose the unit cube $Q = [0,1]^D$ in $N$ coordinate (closed) cubes of side $\alpha^{-1}$, where $\alpha$ is a given integer $\geq 2$, then (2.3) is satisfied, with $\Gamma$ the union of all $\ell$-dimensional faces of $Q$, $\ell \leq D - 1$, that is, $\Gamma = \partial Q$.

For sake of brevity, the invariant set $K$ of a given family of similitudes $\Psi = \{\psi_1, \ldots, \psi_N\}$ satisfying (2.1), (2.2), (2.3) will be referred to in the following as a *self-similar fractal*.

# 3. – Homogeneous fractal spaces

We begin this section by recalling the notion of *space of homogeneous type*, or simply, *homogeneous space*, from Coifman and Weiss, [10]. We give this definition in a slightly different form:

DEFINITION 2.1. A *homogeneous space* is a pair $K = (K, d)$, where $K$ is a topological space and $d$ is a quasi-distance on $K$, such that:
 (j) The (quasi-)balls $B(x, r)$ form a basis of open neighborhoods of $x$,
 (jj) There exist a constant $\nu > 0$ and a constant $c > 0$, such that for every $r > 0$ and for every $0 < \varepsilon < 1$ the ball $B(x, r)$ contains at most $c\varepsilon^{-\nu}$ points $z_i$'s whose mutual distance is greater or equal than $\varepsilon r$.

Note that every $y \in B(x, r)$ belongs to some ball $B(z_i, \varepsilon r)$, therefore $B(x, r) \subset \bigcup_{i=1}^{n} B(z_i, \varepsilon r)$, $n = c\varepsilon^{-\nu}$. An important case in which the homogeneity condition (jj) is satisfied is considered in the following:

LEMMA 3.1. *Let a measure $\mu$ exist on $K$, such that for some constant $\nu > 0$ and $c_0 > 0$ and for every $0 < r \leq R$*

$$0 < c_0 \left(\frac{r}{R}\right)^{\nu} \mu\big(B(x, R)\big) \leq \mu\big(B(c, r)\big) \, .$$

*Then, (jj) holds, with the same constant $\nu$ and $c := c_0^{-1}[2c_T^2(3/2 + c_T)]^{\nu}$, where $c_T \geq 1$ is the constant occurring in the quasi-triangle inequality satisfied by $d$.*

PROOF. Let $x_1, \ldots, x_q \in B(x, r)$ with $d(x_i, x_j) > \varepsilon r$ for $i \neq j$. The balls $B(x_i, \varepsilon r/2c_T)$ are disjoint and all contained in $B(x, R)$, where $R := (c_T + 1/2)r$. In fact, if $y \in B(x_i, \varepsilon r/2c_T) \cap B(x_j, \varepsilon r/2c_T)$, then $d(x_i, x_j) \leq c_T[d(x_i, y) + d(x_j, y)] < c_T \varepsilon r/c_T = \varepsilon r$, contradicting $d(x_i, x_j) > \varepsilon r$; moreover, if $y \in B(x_i, \varepsilon r/2c_T)$ then $d(x, y) < c_T(r + \varepsilon r/2c_T) = (c_T + 1/2)r$. Therefore, $B(x_i, \varepsilon r/2c_T) \subset B(x, (c_T + 1/2)r)$, thus $\sum_{i=1}^{q} \mu(B(x_i, \varepsilon r/2c_T)) \leq \mu(B(x, R))$. On the other hand, if $R' := c_T(3/2 + c_T)r$ then $B(x, R) \subset B(x_i, R')$. In fact, if $y \in B(x, R)$ then $d(x_i, y) < c_T(r + R) = c_T(3/2 + c_T)r$. Therefore, $\mu(B(x, R)) \leq \mu(B(x_i, R'))$. Since $\varepsilon r/2c_T \leq r/2 < R' = c_T(3/2 + c_T)r$, by the assumption of the lemma we have $\mu(B(x_i, R')) \leq c/\varepsilon^\nu \mu(B(x_i, \varepsilon r/2c_T))$ where $c := c_0^{-1}[2c_T^2(3/2 + c_T)]^\nu$. In fact,

$$
\mu(B(x_i, R')) \leq c_0^{-1} \mu\left(B\left(x_i, \frac{\varepsilon r}{2c_T}\right)\right) \left(\frac{2c_T R'}{\varepsilon r}\right)^\nu
$$

$$
= c_0^{-1} \left[\varepsilon^{-1} 2c_T^2 \left(\frac{3}{2} + c_T\right)\right]^\nu \mu\left(B\left(x_i, \frac{\varepsilon r}{2c_T}\right)\right)
$$

Therefore,

$$
\sum_{i=1}^{q} \mu\left(B\left(x_i, \frac{\varepsilon r}{2c_T}\right)\right) \leq \mu(B(x, R)) \leq \mu(B(x_i, R')) \leq \frac{c}{\varepsilon^\nu} \mu\left(B\left(x_i, \frac{\varepsilon r}{2c_T}\right)\right),
$$

and by summing over $i = 1, \ldots, q$

$$
q \sum_{i=1}^{q} \mu\left(B\left(x_i, \frac{\varepsilon r}{2c_T}\right)\right) \leq \frac{c}{\varepsilon^\nu} \sum_{i=1}^{q} \mu\left(B\left(x_i, \frac{\varepsilon r}{2c_T}\right)\right),
$$

thus, $q \leq c\varepsilon^{-\nu}$.

If in addition the opposite inequality $\mu(B(x, r)) \leq c_0'\left(\frac{r}{R}\right)^\nu \mu(B(x, R))$ also holds, for some constant $c_0'$ independent of $r$, then we say that $\nu$ is the *homogeneous dimension* of $K$, relative to the quasi-metric $d$.

In the rest of this section we shall consider a given self-similar fractal $K$, as defined at the end of Section 2. For every $\delta > 0$, we define the *quasi-distance* $d : K \times K \longrightarrow R$ by setting

(3.1)                    $d(x, y) = | x - y |^\delta, \quad x, y \in K$.

The *(quasi-) balls* associated with $d$ will be denoted by $B(x, r)$, that is, $B(x, r) := \{y \in K : d(x, y) < r\}$, $x \in K$, $r > 0$. For every $x \in K$ and every $r > 0$, we have $B(x, r) = B_e(x, r^{1/\delta}) \cap K$. For every $E \subset K$ the diameter $E$ with respect to the quasi-metric $d$ will be denoted by diam $E$. We have diam $E = (\text{diam}_e E)^\delta$.

LEMMA 3.2. *Let $K$ be a self-similar fractal. Let $\delta > 0$ be given and let $d$ be the quasi-metric (3.1) on $K$. Then, for every $x \in K$ and every $0 < r \leq R \leq \operatorname{diam} K$ we have*

$$M^{-1}\alpha_1^{-d_f}\mu(B(x,R))\left(\frac{r}{R}\right)^{\nu} \leq \mu(B(x,r)) \leq M\alpha_1^{d_f}\mu(B(x,R))\left(\frac{r}{R}\right)^{\nu},$$

*where $d_f$ is the fractal dimension of $K$ and $\nu = d_f/\delta$. Moreover, the first inequality above holds for every $0 < r \leq R$.*

*The constant $M$ here is the same constant $M$ occurring in Theorem 2.1, depending only on $D$, $\alpha_1$, $\operatorname{diam}_e K$, $c_1$, $c_2$.*

PROOF. We have $\operatorname{diam} K = (\operatorname{diam}_e K)^{\delta}$. Let $x \in K$, $0 < r \leq \operatorname{diam} K$ and $\ell := r^{1/\delta}$ Then, $0 < \ell = r^{1/\delta} \leq (\operatorname{diam} K)^{1/\delta} = \operatorname{diam}_e K$. By the finite overlapping property of Theorem 2.1,

$$K \cap B_e(x,\ell) \subset \bigcup_{l=1}^{M'} K_{i_1^l,\ldots,i_m^l}$$

with $\alpha_1^{-1}\ell < \operatorname{diam}_e K_{i_1^l,\ldots,i_m^l} \leq \ell$ for every $l = 1,\ldots,M'$. It is not restrictive to assume below $M' = M$. Since $x \in K \cap B_e(x,\ell)$, $x \in K_{i_1^*,\ldots,i_m^*} := K_{i_1^{l*},\ldots,i_m^{l*}}$, for some $K_{i_1^{l*},\ldots,i_m^{l*}} \in G_{x,\ell}$. Moreover, $\operatorname{diam}_e K_{i_1^*,\ldots,i_m^*} \leq \ell$, therefore, $K_{i_1^*,\ldots,i_m^*} \subset K \cap B_e(x,\ell)$. Thus, $K_{i_1^*,\ldots,i_m^*} \subset B(x,r) \subset \bigcup_{l=1}^{M} K_{i_1^l,\ldots,i_m^l}$ hence $\mu(K_{i_1^*,\ldots,i_m^*}) \leq \mu(B(x,r)) \leq \sum_{l=1}^{M} \mu(K_{i_1^l,\ldots,i_m^l})$. Since for every $l = 1,\ldots,M$ we have $\alpha_1^{-d_f}\ell^{d_f}(\operatorname{diam}_e K)^{-d_f} < \mu(K_{i_1^l,\ldots,i_m^l}) \leq \ell^{d_f}(\operatorname{diam}_e K)^{-d_f}$, then for every $0 < r \leq \operatorname{diam} K$ we find $\alpha_1^{-d_f}[r/\operatorname{diam} K]^{d_f/\delta} \leq \mu(B(x,r)) \leq M[r/\operatorname{diam} K]^{d_f/\delta}$. If $\nu := d_f/\delta$, then $\mu(B(x,r)) \leq M[R/\operatorname{diam} K]^{\nu}(r/R)^{\nu} \leq M\alpha_1^{d_f}\mu(B(x,R))(r/R)^{\nu}$ for $0 < r \leq R \leq \operatorname{diam} K$ and $\mu(B(x,r)) \geq \alpha_1^{-d_f}[R/\operatorname{diam} K]^{\nu}(r/R)^{\nu} \geq \alpha_1^{-d_f}M^{-1}\mu(B(x,R))(r/R)^{\nu}$. Therefore, $M^{-1}\alpha_1^{-d_f}\mu(B(x,R))(r/R)^{\nu} \leq \mu(B(x,r)) \leq M\alpha_1^{d_f}\mu(B(x,R))(r/R)^{\nu}$. If $0 < r \leq \operatorname{diam} K \leq R$, then $B(x,R) = K \cap B_e(x,R^{1/\delta})$, $R^{1/\delta} \geq (\operatorname{diam} K)^{1/\delta} = \operatorname{diam}_e K$ and since $x \in K$, we find $B(x,R) = K$ and $\mu(B(x,R)) = \mu(K) = 1$. Thus, $\mu(B(x,r)) \geq M^{-1}\alpha_1^{-d_f}\mu(B(x,R))(r/R)^{\nu}$. If $0 < \operatorname{diam} K \leq r \leq R$, then $\mu(B(x,r)) = \mu(B(x,R)) = 1$ and $\mu(B(x,r)) = 1 \geq M^{-1}\alpha_1^{-d_f}\mu(B(x,R))(r/R)^{\nu}$. This concludes the proof.

By applying Lemma 3.1 and Lemma 3.2 we finally obtain the

THEOREM 3.1. *Let $K$ be a self-similar fractal and let $d$ be the quasi-metric (3.1) on $K$, for a given $\delta > 0$. Then, $(K,d)$ is a homogeneous space of dimension $\nu = d_f/\delta$, where $d_f$ is the fractal dimension of $K$.*

Note that if we choose $\delta = 1$, then $d$ is just the restriction to $K$ of the Euclidean metric of $\mathbb{R}^D$. In this special case, the (homogeneous) dimension $\nu$ of $K \equiv (K, d)$ coincides with the fractal dimension $d_f$ of $K$. We shall introduce below *non-Euclidean* quasi-metrics of variational nature. Notice also that in the results of the present Section 3, as well as in Theorems 4.1 to 4.4 of the following Section 4, assumption (2.3) does not play any role.

## 4. – Variational fractals

We consider a self-similar fractal $K$ – in the sense of Section 2 – associated with a given set $\Psi = \{\psi_1, \dots, \psi_N\}$ of similitudes of $\mathbb{R}^D$ satisfying (2.1), (2.2), (2.3), $N \geq 2$. Uniquely associated with $\Psi$, as seen in Section 2, there exists also an invariant volume measure $\mu$ supported on $K$. We now *assume* – in addition – that a suitable *invariant energy form* $E$ also exists on $K$, which enjoys the following properties:

(4.1)          $E$ is a strongly local, regular, symmetric Dirichlet form with domain $D[E]$ in $L^2(K, \mu)$, $E \neq 0$;

(4.2)          $$E[u] = \sum_{i=1}^{N} \rho_i\, E[u \circ \psi_i] \quad \text{for every} \quad u \in D[E],$$

where $\rho_i = \mu(K_i)^\sigma$ for some given real constant $\sigma < 1$, independent of $i = 1, \dots, N$.

For definitions and properties concerning Dirichlet forms we refer to [12], for a short review see also [22]. We only notice that regularity here means that the space $D[E] \cap C(K)$ is dense both in $C(K)$ for the uniform norm and in $D[E]$ for the intrinsic norm $\|u\| := (E(u, u) + \|u\|^2_{L^2(K, \mu)})^{1/2}$. In (4.2) and in the following, we use the notation $E[u] = E(u, u)$ for every $u \in D[E]$. Following [23], we now give the

DEFINITION 4.1 A triple $(K, \mu, E)$ as above – with assumptions (2.1), (2.2), (2.3), (4.1), (4.2) – will be called a *variational fractal*.

The constants $N$, $\alpha \equiv (\alpha_1, \dots, \alpha_N)$ and $\sigma$ – occurring in (2.1) and (4.2) – are the *structural constants* of a given variational fractal. They are the basic "physical" constants governing the scaling of volume and length in the Euclidean metric of $\mathbb{R}^D$ and the scaling of the energy.

LEMMA 4.1. *Let $K$ be a variational fractal, with given structural constants $N$, $\alpha$ and $\sigma$. Then, there exists one and only one constant $\delta > 0$, such that both identities below hold:*

(4.3)                                 $$d(x, y) \equiv |x - y|^\delta,$$

(4.4)
$$d^2(x, y) \equiv \sum_{i=1}^{N} \rho_i d^2\big(\psi_i(x), \psi_i(y)\big)$$

*for every $x, y \in K$. Such a $\delta$ is uniquely determined by the identity*

(4.5)
$$\sum_{i=1}^{N} \rho_i \alpha_i^{-2\delta} = 1,$$

*and is given by*

(4.6)
$$\delta = d_f(1 - \sigma)/2,$$

*where $d_f$ is the fractal dimension of $K$.*

PROOF. By replacing $d(x, y) = | x - y |^{\delta}$ in the identity (4.4) and by taking (2.1) into account, we find $| x - y |^{2\delta} = \sum_{i=1}^{N} \rho_i^{-2\delta} | x - y |^{2\delta}$. Therefore (4.5) holds and this uniquely determines the constant $\delta$. By (4.2), we have $\sum_{i=1}^{N} \alpha_i^{-(d_f\sigma+2\delta)} = 1$. This implies, by the very definition of $d_f$, that $d_f\sigma + 2\delta = d_f$, hence $\delta$ is given by (4.6).

The quasi-metric (4.3), with $\delta$ determined by (4.5), will be also called the *intrinsic metric* of the variational fractal $K$, [23].

For every $x \in K$ and every $R > 0$, $B(x, R) = \{y \in K : d(x, y) < R\}$, are the *intrinsic balls* of $K$. Clearly, $B(x, R) = B_e(x, R^{1/\delta}) \cap K$. The diameter of $A \subset K$ for the intrinsic metric will be denoted by diam $A$, diam $A = (\text{diam}_e A)^{\delta}$.

THEOREM 4.1. *A variational fractal $K \equiv (K, \mu, E)$, endowed with its intrinsic metric, is a space of homogeneous type of dimension*

(4.7)
$$\nu = \frac{d_f}{\delta}.$$

*Moreover,*

(4.8)
$$\sigma = \frac{\nu - 2}{\nu}.$$

PROOF. The first part follows from Theorem 3.1, while (4.8) follows from Lemma 4.1.

The scaling laws for the mass and the energy *in the intrinsic metric* of $K$ can be stated now more precisely as follows.

THEOREM 4.2. *Let $\mu$ be the invariant measure, $B(x, R)$ the intrinsic balls of $K$. Then,*

$$(4.9) \qquad \mu\big(B(x, r)\big) \le M\alpha_1^{d_f}\mu\big(B(x, R)\big)\left(\frac{r}{R}\right)^{\nu}$$

*for every $x \in K$ and every $0 < r \le R \le \operatorname{diam} K$, moreover,*

$$(4.10) \qquad M^{-1}\alpha_1^{-d_f}\mu\big(B(x, R)\big)\left(\frac{r}{R}\right)^{\nu} \le \mu\big(B(x, r)\big)$$

*for every $x \in K$ and every $0 < r \le R$, where $M$ is the constant of Theorem 2.1.*

PROOF. This is an immediate consequence of Lemma 3.2.

THEOREM 4.3. *Let $E$ be the energy form of $K \equiv (K, \mu, E)$. Then, for every $n \ge 1$ we have*

$$(4.11) \qquad E[u] = \sum_{i_1, \ldots, i_n = 1}^{N} (\operatorname{diam} K_{i_1, \ldots, i_n} / \operatorname{diam} K)^{\nu-2} E[u \circ \psi_{i_1, \ldots, i_n}]$$

*for every $u \in D[E]$.*

PROOF. By iterating (4.2) along a finite sequence of indices $i_1, \ldots, i_n \in \{1, \ldots, N\}$, $n \ge 1$, we find

$$(4.12) \qquad E[u] = \sum_{i_1, \ldots, i_n = 1}^{N} \rho_{i_1} \cdots \rho_{i_n} E[u \circ \psi_{i_1, \ldots, i_n}].$$

We have $\rho_{i_1} \cdots \rho_{i_n} = \mu(K_{i_1})^{\sigma} \cdots \mu(K_{i_n})^{\sigma} = \alpha_{i_1}^{-d_f\sigma} \cdots \alpha_{i_n}^{-d_f\sigma}$ hence $\rho_{i_1} \cdots \rho_{i_n} = (\operatorname{diam} K_{i_1, \ldots, i_n} / \operatorname{diam} K)^{d_f\sigma/\delta}$. By (4.7), (4.8), this gives

$$(4.13) \qquad \rho_{i_1} \cdots \rho_{i_n} = (\operatorname{diam} K_{i_1, \ldots, i_n} / \operatorname{diam} K)^{\nu-2}.$$

By a well known representation theory of regular Dirichlet forms, [12], the form $E$ can be given the integral expression

$$(4.14) \qquad E(u, v) = \int_K \gamma(u, v)(dx) \qquad \forall u, v \in D[E]$$

where $\gamma(u, v)$ is a (signed) regular, Radon measure on $K$. The symmetric, bilinear form $\gamma = \gamma(u, v)$, $u, v \in D[E]$, is the *local energy measure* of $E$. We shall also use the notation $\gamma[u] = \gamma(u, u)$. For definition and properties of $\gamma$ we refer to [12], [22]. We only recall here that, for every given $u, v \in D[E]$, the restriction $\gamma(u, v) \llcorner A$ of the measure $\gamma(u, v)$ to any open subset $A$ of $K$ depends only on the restrictions to $A$ of $u$ and $v$. We now show, by a Fourier type argument for local Dirichlet forms (see [22]), that the local energy $\gamma$ inherits the invariance property of the total energy $E$ and that the scaling law (4.11) of $E$ is also satisfied by $\gamma$.

THEOREM 4.4. *Let $\gamma$ be the local measure of the invariant energy $E$ of a given variational fractal $K \equiv (K, \mu, E)$. Then, for every $n \geq 1$ we have*

$$
\begin{aligned}
(4.15) \quad & \int_K \varphi \gamma[u](dx) \\
& = \sum_{i_1,\dots,i_n=1}^{N} (\operatorname{diam} K_{i_1,\dots,i_n} / \operatorname{diam} K)^{\nu-2} \int_K \varphi \circ \psi_{i_1,\dots,i_n} \gamma[u \circ \psi_{i_1,\dots,i_n}](dx)
\end{aligned}
$$

*for every $u \in D[E]$ and for every $\varphi \in C(K)$.*

PROOF. By the regularity of the form, it suffices to prove (4.15) for every $u \in D[E] \cap C(K)$ and for every $\varphi \in D[E] \cap C(K)$, $\varphi \geq 0$. We introduce rapidly oscillating functions $\varphi \cos(tu)$, $\varphi \sin(tu)$, $t > 0$, in the identity (4.11). By summing up the two identities obtained and applying the Leibniz rule and the chain rule (see e.g. [22]), we get

$$
t^2 \int_K \varphi^2 \gamma[u](dx) + \int_K \gamma[\varphi](dx) = \sum_{i=1}^{N} \rho_i \left[ t^2 \int_K \varphi^2 \circ \psi_i \, \gamma[u \circ \psi_i](dx) + \int_K \gamma[\varphi \circ \psi_i](dx) \right]
$$

for every $t > 0$. By dividing by $t^2$ and letting $t \to \infty$ we then obtain

$$
(4.16) \qquad \int_K \varphi^2 \, \gamma[u](dx) = \sum_{i=1}^{N} \rho_i \int_K \varphi^2 \circ \psi_i \, \gamma[u \circ \psi_i](dx) .
$$

By the arbitrariness of $\varphi$ and by (4.13), this proves (4.15) for $n = 1$. By iterating (4.16) we conclude the proof.

From Theorem 4.4 we also obtain the following *change of variable* formula for $\gamma$:

THEOREM 4.5. *Let $K \equiv (K, \mu, E)$ be a given variational fractal, $\Gamma$ the boundary of $K$. Let $\gamma$ be the local measure of $E$ and let $u \in D[E]$. Then, for every $n \geq 1$ and every $i_1, \dots, i_n \in \{1, \dots, N\}$, we have*

$$
\begin{aligned}
(4.17) \quad & \int_{K_{i_1,\dots,i_n} - \Gamma_{i_1,\dots,i_n}} \varphi(y) \big( \gamma[u] \llcorner (K_{i_1,\dots,i_n} - \Gamma_{i_1,\dots,i_n}) \big)(dy) \\
& = (\operatorname{diam} K_{i_1,\dots,i_n} / \operatorname{diam} K)^{\nu-2} \\
& \qquad \cdot \int_{K-\Gamma} \varphi \circ \psi_{i_1,\dots,i_n}(x) \big( \gamma[u \circ \psi_{i_1,\dots,i_n}] \llcorner (K - \Gamma) \big)(dx)
\end{aligned}
$$

*for every $\varphi \in C(K)$ with $\operatorname{supp} \varphi \subset K_{i_1,\dots,i_n} - \Gamma_{i_1,\dots,i_n}$.*

PROOF. Let $i_1, \ldots, i_n \in \{1, \ldots, N\}$ be fixed and let $\varphi \in C(K)$ be such that $\operatorname{supp} \varphi \subset K_{i_1,\ldots,i_n} - \Gamma_{i_n,\ldots,i_n}$. Since $K_{i_1,\ldots,i_n} - \Gamma_{i_1,\ldots,i_n}$ is open in $K$, the restriction of the measure $\gamma[u]$ to $K_{i_1,\ldots,i_n} - \Gamma_{i_1,\ldots,i_n}$ depends only on the restriction of the function $u$ to $K_{i_1,\ldots,i_n} - \Gamma_{i_1,\ldots,i_n}$. Therefore, for such $\varphi \in C(K)$ we have

$$
(4.18) \quad
\begin{aligned}
\int_K \varphi \gamma[u](dy) &= \int_{K_{i_1,\ldots,i_n} - \Gamma_{i_1,\ldots,i_n}} \varphi(y) \gamma[u](dy) \\
&= \int_{K_{i_1,\ldots,i_n} - \Gamma_{i_1,\ldots,i_n}} \varphi(y) \big( \gamma[u] \llcorner (K_{i_1,\ldots,i_n} - \Gamma_{i_1,\ldots,i_n}) \big)(dy) .
\end{aligned}
$$

On the other hand, let us remark that for every $j_1, \ldots, j_n \in \{1, \ldots, N\}$ such that $(j_1, \ldots, j_n) \neq (i_1, \ldots, i_n)$, we have $(K_{i_1,\ldots,i_n} - \Gamma_{i_1,\ldots,i_n}) \cap K_{j_1,\ldots,j_n} = \emptyset$. Therefore, $\varphi \circ \psi_{j_1,\ldots,j_n} \equiv 0$ on $K$, whenever $(j_1, \ldots, j_n) \neq (i_1, \ldots, i_n)$. Thus,

$$
(4.19) \quad
\begin{aligned}
\sum_{j_1,\ldots,j_n=1}^{N} & (\operatorname{diam} K_{j_1,\ldots,j_n} / \operatorname{diam} K)^{\nu-2} \int_K \varphi \circ \psi_{j_1,\ldots,j_n} \, \gamma[u \circ \psi_{j_1,\ldots,j_n}](dx) \\
&= (\operatorname{diam} K_{i_1,\ldots,i_n} / \operatorname{diam} K)^{\nu-2} \int_K \varphi \circ \psi_{i_1,\ldots,i_n}(x) \, \gamma[u \circ \psi_{i_1,\ldots,i_n}](dx) \\
&= (\operatorname{diam} K_{i_1,\ldots,i_n} / \operatorname{diam} K)^{\nu-2} \cdot \\
&\qquad \cdot \int_{K-\Gamma} \varphi \circ \psi_{i_1,\ldots,i_n}(x) \big( \gamma[u \circ \psi_{i_1,\ldots,i_n}] \llcorner (K - \Gamma) \big)(dx)
\end{aligned}
$$

where, in the last equality, we have taken into account that $K - \Gamma$ is open in $K$ and $\operatorname{supp} \varphi \circ \psi_{i_1,\ldots,i_n} \subset K - \Gamma$. In order to get (4.17), it suffices now to replace both (4.18) and (4.19) into (4.15) of Theorem 4.4.

COROLLARY OF THEOREM 4.5. *Under the assumptions of Theorem 4.5, we have*

$$
(4.20) \quad
\begin{aligned}
\int_{K_{i_1,\ldots,i_n} - \Gamma_{i_1,\ldots,i_n}} & \big( \gamma[u] \llcorner (K_{i_1,\ldots,i_n} - \Gamma_{i_1,\ldots,i_n}) \big)(dy) \\
&= (\operatorname{diam} K_{i_1,\ldots,i_n} / \operatorname{diam} K)^{\nu-2} \int_{K-\Gamma} \big( \gamma[u \circ \psi_{i_1,\ldots,i_n}] \llcorner (K - \Gamma) \big)(dx) .
\end{aligned}
$$

PROOF. Since $\gamma[u]$ is a regular Radon measure on $K$, (4.20) follows from (4.17).

## 5. – Poincaré inequalities

In this section we come to the main goal of the paper: The proof of a family of *scaled Poincaré inequalities* on the intrinsic balls of a variational fractal. We thus consider a given variational fractal $K \equiv (K, E)$, in the sense of

Section 4. $\Gamma$ is the boundary of $K$, as defined in Section 2. As in the previous sections, $\mu$ is the invariant volume measure of $K$ and $\gamma$ the local measure of $E$. We now further assume that

(5.1)   the form $\int_{K-\Gamma} \gamma[u](dx)$, with domain $D[E]$, is closed in $L^2(K, \mu)$;

moreover, there exists a constant $c_P > 0$, such that the following Poincaré inequality holds:

$$(5.2) \qquad \int_K |u - u_K|^2 \, \mu(dx) \leq c_P \int_{K-\Gamma} \gamma[u] \llcorner (K - \Gamma)(dx)$$

for every $u \in D[E]$, where we use the notation $u_A = \int_A u \, \mu(dx)$ for $A \subset K$.

Note that the (closed) form (5.1) inherits from the initial form $E$ all properties that make it a strongly local, regular Dirichlet form in $L^2(K, \mu)$. Moreover, by the closed graph theorem, (5.1) is equivalent to to the condition

$$\int_K \gamma[u](dx) \leq c_0 \left( \int_{K-\Gamma} \gamma[u] \llcorner (K - \Gamma)(dx) + \int_K u^2 \, d\mu \right)$$

for every $u \in D[E]$. By cap $G$, where $G$ is a subset of $K$, we denote the capacity of $G$ with respect to the form (5.1), as defined e.g. in [12]. By $\tilde{u}$ we denote the quasi-continuous representative of $u \in D[E]$, which is defined quasi-everywhere on $K$. In the following, we shall write $\int_{K-\Gamma} \gamma[u](dx)$ in place of $\int_{K-\Gamma} \gamma[u] \llcorner (K - \Gamma)(dx)$, since the simplified notation has non-ambiguous meaning due to the local character of the measure $\gamma$. For every $u \in D[E]$ we have $u + c \in D[E]_{\text{loc}}$ and $\int_{K-\Gamma} \gamma[u + c](dx) = \int_{K-\Gamma} \gamma[u](dx)$ for every constant $c$.

We now state a Poincaré inequality, that is standard in the classic case and also holds in the present framework (we omit the details):

LEMMA 5.1. *Under the assumptions* (5.1), (5.2), *for every constant* $\eta > 0$ *there exists a constant* $C = C(\eta)$, *such that*

$$(5.3) \qquad \int_K |u|^2 \, \mu(dx) \leq C \int_{K-\Gamma} \gamma[u](dx)$$

*for every* $u \in D[E]$, *such that* cap$\{x \in K : \tilde{u}(x) = 0\} \geq \eta$.

The following one is a scaled version of the inequality (5.2) to an arbitrary $n$-complex:

LEMMA 5.2. *For every* $i_1, \ldots, i_n$, $n \geq 0$, *we have*

$$(5.4) \qquad \begin{aligned} &\int_{K_{i_1 \ldots, i_n}} |u - u_{K_{i_1, \ldots, i_n}}|^2 \, \mu(dy) \\ &\leq c_P (\text{diam } K_{i_1, \ldots, i_n} / \text{diam } K)^2 \int_{K_{i_1, \ldots, i_n} - \Gamma_{i_1, \ldots, i_n}} \gamma[u](dx) \end{aligned}$$

*for every* $u \in D[E]$.

PROOF. By (5.2) we have

$$\int_{K_{i_1,\dots,i_n}} |\, u - u_{K_{i_1,\dots,i_n}}\,|^2 \, \mu(dy)$$

$$= (\mathrm{Lip}\ \psi_{i_1,\dots,i_n})^{d_f} \int_K |u \circ \psi_{i_1,\dots,i_n} - (u \circ \psi_{i_1,\dots,i_n})_K|^2 \mu(dx)$$

$$\leq c_{\mathrm{P}}\, \alpha_{i_1}^{-d_f} \cdots \alpha_{i_n}^{-d_f} \int_{K-\Gamma} \gamma[u \circ \psi_{i_1,\dots,i_n}](dx)$$

$$= c_{\mathrm{P}}(\mathrm{diam}\ K_{i_1,\dots,i_n}/\,\mathrm{diam}\ K)^{\nu} \int_{K-\Gamma} \gamma[u \circ \psi_{i_1,\dots,i_n}](dx)\,.$$

By the corollary of Theorem 4.5,

$$\int_{K-\Gamma} \big(\gamma[u \circ \psi_{i_1,\dots,i_n}]\big)(dx) = (\mathrm{diam}\ K_{i_1,\dots,i_n}/\,\mathrm{diam}\ K)^{2-\nu} \int_{K_{i_1,\dots,i_n}-\Gamma_{i_1,\dots,i_n}} \gamma[u](dy)\,,$$

therefore (5.4) follows.

If $K_{i_1,\dots,i_n}$, $K_{j_1,\dots,j_n}$, $(i_1,\dots,i_n) \neq (j_1,\dots,j_n)$ are distinct $n$-complexes, then by assumption (2.3), there exist two subsets $T$, $T' \subset \Gamma$, possibly empty, such that $K_{i_1,\dots,i_n} \cap K_{j_1,\dots,j_n} = \psi_{i_1,\dots,i_n}T = \psi_{j_1,\dots,j_n}T'$. We will consider below the case of two distinct complexes for which the previous condition holds with subsets $T$, $T'$ of positive capacity. More precisely, we give the following

DEFINITION 5.1. We say that two distinct complexes $K_{i_1,\dots,i_m}$, $K_{j_1,\dots,j_n}$, $m \geq 1$, $n \geq 1$, $(i_1,\dots,i_m) \neq (j_1,\dots,j_n) \in \{1,\dots,N\}$ are *connected* (in the capacity sense) if there exist two subsets $T$, $T' \subset \Gamma$, such that the following conditions hold

$$(5.5) \qquad K_{i_1,\dots,i_m} \cap K_{j_1,\dots,j_n} = \psi_{i_1,\dots,i_m}T = \psi_{j_1,\dots,j_n}T'\,,$$

$$(5.6) \qquad \mathrm{cap}\ T \geq k_0\,, \quad \mathrm{cap}\ T' \geq k_0\,,$$

$$(5.7) \quad k\,\mathrm{cap}\ S \leq \mathrm{cap}\ S' \leq (1/k)\,\mathrm{cap}\ S \quad \text{whenever} \quad S' = \psi_{j_1,\dots,j_n}^{-1} \circ \psi_{i_1,\dots,i_m}S,\ S \subset T,$$

for some constants $k_0 > 0$, $0 < k \leq 1$.

Clearly, it is not restrictive to assume that both (5.6), (5.7) hold with the same constant $k = k_0$, $(k \leq 1)$. We note that, because of condition (5.7), some weak form of symmetry, in the capacity sense, is enjoied by the pair of complexes.

LEMMA 5.3. *Let* $K_{i_1,\dots,i_m}$, $K_{j_1,\dots,j_n}$, $m \geq 1$, $n \geq 1$, $(i_1,\dots,i_m) \neq$ $(j_1,\dots,j_n) \in \{1,\dots,N\}$ *be two complexes, which are connected according to Definition 5.1 for some constant* $k > 0$, *and let* $Q = K_{i_1,\dots,i_m} \cup K_{j_1,\dots,j_n}$. *Then, there exists a constant* $c = c(c_P, k)$, *such that for every* $u \in D[E]$

$$
\begin{aligned}
(5.8) \quad & \int_Q |u - u_Q|^2\, \mu(dy) \\
& \leq c \left\{ \begin{aligned} & (\operatorname{diam} K_{i_1,\dots,i_m} / \operatorname{diam} K)^2 \int_{K_{i_1,\dots,i_m} - \Gamma_{i_1,\dots,i_m}} \gamma[u](dy) \\ & + (\operatorname{diam} K_{j_1,\dots,j_n} / \operatorname{diam} K)^2 \int_{K_{j_1,\dots,j_n} - \Gamma_{j_1,\dots,j_n}} \gamma[u](dy) \end{aligned} \right\}.
\end{aligned}
$$

PROOF. Let $u \in D[E]$. For every $\vartheta$, let

$$
\begin{aligned}
S_+(\vartheta) &= \{x \in T : \tilde{u} \circ \psi_{i_1,\dots,i_m}(x) - \vartheta \geq 0\}, \\
S_-(\vartheta) &= \{x \in T : \tilde{u} \circ \psi_{i_1,\dots,i_m}(x) - \vartheta \leq 0\}.
\end{aligned}
$$

Similarly, let

$$
\begin{aligned}
S'_+(\vartheta) &= \{x \in T' : \tilde{u} \circ \psi_{j_1,\dots,j_n}(x) - \vartheta \geq 0\}, \\
S'_-(\vartheta) &= \{x \in T' : \tilde{u} \circ \psi_{j_1,\dots,j_n}(x) - \vartheta \leq 0\}.
\end{aligned}
$$

We have

$$
(5.9) \quad S'_+(\vartheta) = \psi^{-1}_{j_1,\dots,j_n} \circ \psi_{i_1,\dots,i_m} S_+(\vartheta), \quad S'_-(\vartheta) = \psi^{-1}_{j_1,\dots,j_n} \circ \psi_{i_1,\dots,i_m} S_-(\vartheta).
$$

We shall prove below that there exists a constant $\theta$, such that both inequalities below hold:

$$
(5.10) \quad \operatorname{cap} S_+(\theta) \geq (1/4) \operatorname{cap} T, \quad \operatorname{cap} S_-(\theta) \geq (1/4) \operatorname{cap} T.
$$

Once (5.10) has been proved, by (5.9) and assumption (5.7) it follows also, in addition, that both inequalities below hold:

$$
(5.11) \quad \operatorname{cap} S'_+(\theta) \geq (k/4) \operatorname{cap} T, \quad \operatorname{cap} S'_-(\theta) \geq (k/4) \operatorname{cap} T.
$$

Let us prove (5.10). Let

$$
M_0 = \operatorname{essinf}\{\tilde{u} \circ \psi_{i_1,\dots,i_m}(x) : x \in T\}, \quad M_1 = \operatorname{esssup}\{\tilde{u} \circ \psi_{i_1,\dots,i_m}(x) : x \in T\}.
$$

Let us introduce the set

$$
(5.12) \quad I = \{\vartheta \in (M_0, M_1) : \operatorname{cap} S_+(\vartheta) \geq (1/4) \operatorname{cap} T\}.
$$

We have $T \supset S_+(\vartheta_1) \supset S_+(\vartheta_2)$ if $\vartheta_1 \leq \vartheta_2$, therefore as $\vartheta_h \downarrow M_0$ the sets $S_+(\vartheta_h)$ form an increasing sequence of subsets of $T$, such that $S_+(\vartheta_h) \uparrow T$. This implies that the set (5.12) is not empty. Let

$$\theta_0 = \sup \{\vartheta : \vartheta \in I\}.$$

We have $\theta_0 < +\infty$. In fact, if $\theta_0 = +\infty$, then $\theta_0 = M_1 = +\infty$. As $\vartheta_h \uparrow M_1$, $\vartheta_h \in I$, the sets $S_+(\vartheta_h)$ form a decreasing sequence of subsets of $T$, such that $S_+(\vartheta_h) \downarrow \emptyset$. Therefore, $\mathrm{cap}\, S_+(\vartheta_h) \downarrow 0$. Then, since $\mathrm{cap}\, T > 0$, there exists some $\vartheta_h \in I$ such that $\mathrm{cap}\, S_+(\vartheta_h) < (1/4)\, \mathrm{cap}\, T$, hence a contradiction with the definition of $I$. Now let $\vartheta_h \uparrow \theta_0$, $\vartheta_h \in I$. Then, the sets $S_+(\vartheta_h)$ form a decreasing sequence of subsets of $T$, such that $S_+(\vartheta_h) \downarrow S_+(\theta_0)$. This implies that $\mathrm{cap}\, S_+(\theta_0) \geq (1/4)\, \mathrm{cap}\, T$. If $\theta_0 = M_1$, then $S_-(\theta_0) = S_-(M_1) = \{x \in T : \tilde{u} \circ \psi_{i_1,\dots,i_m}(x) - M_1 \leq 0\} = T$ up to a set of capacity zero, therefore $\mathrm{cap}\, S_-(\theta_0) \geq (1/4)\, \mathrm{cap}\, T$, and (5.10) has been proved. Now let $\theta_0 < M_1$. By the definition of $\theta_0$, for every $\varepsilon > 0$, $\theta_0 + \varepsilon < M_1$, we have $\mathrm{cap}\, S_+(\theta_0 + \varepsilon) < (1/4)\, \mathrm{cap}\, T$. As $\varepsilon_h \downarrow 0$, $S_+(\theta_0 + \varepsilon_h) \uparrow S_+(\theta_0)$, thus $\mathrm{cap}\, S_+(\theta_0) \leq (1/4)\, \mathrm{cap}\, T$. If also $\mathrm{cap}\, S_-(\theta_0) \leq (1/4)\, \mathrm{cap}\, T$, then – since $T = S_+(\theta_0) \cup S_-(\theta_0)$ – we would have $\mathrm{cap}\, T \leq \mathrm{cap}\, S_+(\theta_0) + \mathrm{cap}\, S_-(\theta_0) \leq (1/2)\, \mathrm{cap}\, T$, hence a contradiction because $\mathrm{cap}\, T > 0$. Therefore, we have $\mathrm{cap}\, S_-(\theta_0) \geq (1/4)\, \mathrm{cap}\, T$, thus (5.10) has been proved also in the present case. Therefore, we have shown that there exists a finite constant $\theta$, such that all inequalities in (5.10), (5.11) hold simultaneously, hence also, since $\mathrm{cap}\, T \geq k_0$, the following inequalities hold

$$(5.13) \qquad \mathrm{cap}\, S_+(\theta) \geq \eta, \qquad \mathrm{cap}\, S_-(\theta) \geq \eta,$$

$$(5.14) \qquad \mathrm{cap}\, S'_+(\theta) \geq \eta, \qquad \mathrm{cap}\, S'_-(\theta) \geq \eta,$$

where

$$(5.15) \qquad \eta = k_0 k/4 > 0.$$

This implies that the functions $(\tilde{u} \circ \psi_{i_1,\dots,i_m} - \theta)_+$, $(\tilde{u} \circ \psi_{i_1,\dots,i_m} - \theta)_-$, as well as the functions $(\tilde{u} \circ \psi_{j_1,\dots,j_n} - \theta)_+$, $(\tilde{u} \circ \psi_{j_1,\dots,j_n} - \theta)_-$, all vanish on subsets whose capacity is greater or equal to $\eta$, where $\eta$ is the constant (5.15). Since $\tilde{u} \circ \psi_{i_1,\dots,i_m} = u \circ \psi_{i_1,\dots,i_m}$ $\mu$-a.e. and $u \circ \psi_{i_1,\dots,i_m} \in D[E]$, then also $\tilde{u} \circ \psi_{i_1,\dots,i_m} \in D[E]$, and we may identify these two functions. Similarly, $\tilde{u} \circ \psi_{j_1,\dots,j_n} \equiv u \circ \psi_{j_1,\dots,j_n} \in D[E]$. We apply Lemma 5.1 to get

$$\int_K |u \circ \psi_{i_1,\dots,i_m} - \theta|^2 \, \mu(dx)$$

$$= \int_K |(u \circ \psi_{i_1,\dots,i_m} - \theta)_+|^2 \, \mu(dx) + \int_K |(u \circ \psi_{i_1,\dots,i_m} - \theta)_-|^2 \, \mu(dy)$$

$$\leq C \left\{ \int_{K-\Gamma} \gamma[u \circ \psi_{i_1,\dots,i_m} - \theta](dx) + \int_{K-\Gamma} \gamma[u \circ \psi_{i_1,\dots,i_m} - \theta](dx) \right\}$$

where we have taken into account the Markovianity of the form. Therefore,

$$(5.16) \qquad \int_K | u \circ \psi_{i_1,\dots,i_m} - \theta |^2 \, \mu(dx) \le 2C \int_{K-\Gamma} \gamma[u \circ \psi_{i_1,\dots,i_m}](dx),$$

as well as

$$(5.17) \qquad \int_K | u \circ \psi_{j_1,\dots,j_n} - \theta |^2 \, \mu(dx) \le 2C \int_{K-\Gamma} \gamma[u \circ \psi_{j_1,\dots,j_n}](dx),$$

with the *same* constant $\theta$, and $C$ a constant depending only on $k$. We now scale down inequality (5.16) to $K_{i_1,\dots,i_m}$ and inequality (5.17) to $K_{j_1,\dots,j_n}$ by relying on (4.20). We find by (4.13)

$$\int_{K_{i_1,\dots,i_m}} | u - \theta |^2 \, \mu(dy) = \alpha_{i_1}^{-d_f} \cdots \alpha_{i_m}^{-d_f} \int_K | u \circ \psi_{i_1,\dots,i_m} - \theta |^2 \, \mu(dx)$$

$$\le 2C \alpha_{i_1}^{-d_f} \cdots \alpha_{i_m}^{-d_f} \int_{K-\Gamma} \gamma[u \circ \psi_{i_1,\dots,i_m}](dx)$$

$$= 2C \left( \alpha_{i_1}^{-d_f} \cdots \alpha_{i_m}^{-d_f} / \rho_{i_1} \cdots \rho_{i_m} \right) \int_{K_{i_1,\dots,i_m}-\Gamma_{i_1,\dots,i_m}} \gamma[u](dy)$$

therefore, by (4.6)

$$
\begin{aligned}
(5.18) \quad & \int_{K_{i_1,\dots,i_m}} | u - \theta |^2 \, \mu(dy) \\
& \qquad \le 2C (\operatorname{diam} K_{i_1,\dots,i_m} / \operatorname{diam} K)^2 \int_{K_{i_1,\dots,i_m}-\Gamma_{i_1,\dots,i_m}} \gamma[u](dy).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(5.19) \quad & \int_{K_{j_1,\dots,j_n}} | u - \theta |^2 \, \mu(dy) \\
& \qquad \le 2C (\operatorname{diam} K_{j_1,\dots,j_n} / \operatorname{diam} K)^2 \int_{K_{j_1,\dots,j_n}-\Gamma_{j_1,\dots,j_n}} \gamma[u](dy),
\end{aligned}
$$

with the same constant $\theta$. Now let $Q = K_{i_1,\dots,i_m} \cup K_{j_1,\dots,j_n}$ and $u_Q = \int_Q u \, \mu(dy)$. Then,

$$
\begin{aligned}
\int_Q | u - u_Q |^2 \, u(dy) &\le \int_Q | u - \theta |^2 \, \mu(dy) \\
&\le \int_{K_{i_1,\dots,i_m}} | u - \theta |^2 \, \mu(dy) + \int_{K_{j_1,\dots,j_n}} | u - \theta |^2 \, \mu(dy).
\end{aligned}
$$

Therefore, (5.8) follows from (5.18) and (5.19) and the proof is concluded.

The following lemma, that permits to extend Poincaré inequality accross two contiguous sets that overlap on a set of positive measure, is standard. For sake of brevity we omit the proof.

LEMMA 5.4. *Let $Q_1$, $Q_2$, be two subsets of $K$, such that $\mu(Q_1 \cap Q_2) > 0$. Then,*

$$(5.20) \quad \int_{Q_1 \cup Q_2} |u - u_{Q_1 \cup Q_2}|^2 \mu(dx) \leq 4 \frac{\mu(Q_1 \cup Q_2)}{\mu(Q_1 \cap Q_2)} \max_{i=1,2} \int_{Q_i} |u - u_{Q_i}|^2 \mu(dx).$$

By iterating Lemma 5.4, we get

LEMMA 5.5. *Let $Q_1, \ldots, Q_m$ be $m \geq 2$ subsets of $K$, such that $\mu(Q_s \cap Q_{s+1}) > 0$ for every $s = 1, \ldots, m - 1$. Let $Q = Q_1 \cup \ldots \cup Q_m$. Then,*

$$(5.21) \quad \int_Q |u - u_Q|^2 \mu(dx) \leq \left[ 4 \frac{\mu(Q)}{\min\limits_{s=1,\ldots,m-1} \mu(Q_s \cap Q_{s+1})} \right]^{m-1} \max_{s=1,\ldots,m} \int_{Q_s} |u - u_{Q_s}|^2 \mu(dx).$$

From Lemma 5.5 we get easily:

LEMMA 5.6. *Let $K_{i_1^s,\ldots,i_{n_s}^s}$, $s = 1, \ldots, M$, $M \geq 3$, be given $n_s$-complexes, $n_s \geq 1$, $i_1, \ldots, i_{n_s} \in \{1, \ldots, N\}$. Let $Q = \bigcup_{s=1}^M K_{i_1^s,\ldots,i_{n_s}^s}$ and, for each $s = 1, \ldots, M - 1$, let $Q_s = K_{i_1^s,\ldots,i_{n_s}^s} \cup K_{i_1^{s+1},\ldots,i_{n_{s+1}}^{s+1}}$. Then*

$$(5.22) \quad \int_Q |u - u_Q|^2 \mu(dx) \leq \left[ 4 \frac{\mu(Q)}{\min\limits_{s=1,\ldots,M-2} \mu(K_{i_1^{s+1},\ldots,i_{n_{s+1}}^{s+1}})} \right]^{M-2} \max_{s=1,\ldots,M-1} \int_{Q_s} |u - u_{Q_s}|^2 \mu(dx).$$

PROOF. For every $s = 1, \ldots, M - 1$ we have

$$(5.23) \quad \mu(Q_s \cap Q_{s+1}) \geq \mu(K_{i_1^{s+1},\ldots,i_{n_{s+1}}^{s+1}}) > 0.$$

Therefore, we can apply Lemma 5.5 to the sets $Q_s$, $s = 1, \ldots, m$, $m = M - 1$, and we obtain (5.22).

DEFINITION 5.2. We say that a finite sequence $K_{i_1^s,\ldots,i_{n_s}^s}$, $s = 1, \ldots, m$, $m \geq 2$ of $n_s$-complexes, $n_s \geq 1$, $i_1^s, \ldots, i_{n_s}^s \in \{1, \ldots, N\}$ for every $s = 1, \ldots, m$, with $(i_1^s, \ldots, i_{n_s}^s) \neq (i_1^{s+1}, \ldots, i_{n_{s+1}}^{s+1})$ for every $s = 1, \ldots, m - 1$, is a *k-capacitory chain*, where $0 < k \leq 1$ is a given constant, if each pair $K_{i_1^s,\ldots,i_{n_s}^s}$, $K_{i_1^{s+1},\ldots,i_{n_{s+1}}^{s+1}}$, $s = 1, \ldots, m - 1$, is connected in the capacity sense, according to Definition 5.1, with constant $k$.

LEMMA 5.7. *Let $K_{i_1^s,\ldots,i_{n_s}^s}$, $s = 1,\ldots,M$, $M \geq 3$, $n_s \geq 1$, $i_1^s,\ldots,i_{n_s}^s \in \{1,\ldots,N\}$ for every $s = 1,\ldots,M$, $(i_1^s,\ldots,i_{n_s}^s) \neq (i_1^{s+1},\ldots,i_{n_{s+1}}^{s+1})$, for every $s = 1,\ldots,M-1$, be a k-capacitory chain, $0 < k \leq 1$, according to Definition 5.2. Then, there exists a constant $c = c(c_P, k)$, such that, if $Q = \bigcup_{s=1}^{M} K_{i_1^s,\ldots,i_{n_s}^s}$, for every $u \in D[E]$ we have*

$$\int_Q |u - u_Q|^2 \mu(dy)$$

$$\leq c \left[ 4 \frac{\mu(Q)}{\min\limits_{s=2,\ldots,M-1} \mu(K_{i_1^s,\ldots,i_{n_s}^s})} \right]^{M-2} \max_{s=1,\ldots,M} \left( \frac{\operatorname{diam} K_{i_1^s,\ldots,i_{n_s}^s}}{\operatorname{diam} K} \right)^2 \int_{K_{i_1^s,\ldots,i_{n_s}^s} - \Gamma_{i_1^s,\ldots,i_{n_s}^s}} \gamma[u](dy).$$

PROOF. By the preceding lemma, the inequality (5.22) holds, with $Q_s = K_{i_1^s,\ldots,i_{n_s}^s} \cup K_{i_1^{s+1},\ldots,i_{n_{s+1}}^{s+1}}$ for every $s = 1,\ldots,M-1$. Moreover, by Lemma 5.3, for each $s = 1,\ldots,M-1$, we have

$$\int_{Q_s} |u - u_{Q_s}|^2 \mu(dy) \leq c \max_{\ell=s,s+1} (\operatorname{diam} K_{i_1^\ell,\ldots,i_{n_\ell}^\ell} / \operatorname{diam} K)^2 \int_{K_{i_1^\ell,\ldots,i_{n_\ell}^\ell} - \Gamma_{i_1^\ell,\ldots,i_{n_\ell}^\ell}} \gamma[u](dy),$$

with $c = c(c_P, k)$. By taking this inequality into account, we get from (5.22)

$$\int_Q |u - u_Q|^2 \mu(dy)$$

$$\leq \left[ 4 \frac{\mu(Q)}{\min\limits_{s=1,\ldots,M-2} \mu(K_{i_1^{s+1},\ldots,i_{n_{s+1}}^{s+1}})} \right]^{M-2} \max_{s=1,\ldots,M-1} \int_{Q_s} |u - u_{Q_s}|^2 \mu(dx)$$

$$\leq c \left[ 4 \frac{\mu(Q)}{\min\limits_{s=1,\ldots,M-2} \mu(K_{i_1^{s+1},\ldots,i_{n_{s+1}}^{s+1}})} \right]^{M-2} \max_{s=1,\ldots,M-1} \max_{\ell=s,s+1} \left( \frac{\operatorname{diam} K_{i_1^\ell,\ldots,i_{n_\ell}^\ell}}{\operatorname{diam} K} \right)^2$$

$$\cdot \int_{K_{i_1^\ell,\ldots,i_{n_\ell}^\ell} - \Gamma_{i_1^\ell,\ldots,i_{n_\ell}^\ell}} \gamma[u](dy)$$

$$\leq c \left[ 4 \frac{\mu(Q)}{\min\limits_{s=2,\ldots,M-1} \mu(K_{i_1^s,\ldots,i_{n_s}^s})} \right]^{M-2} \max_{s=1,\ldots,M} (\operatorname{diam} K_{i_1^s,\ldots,i_{n_s}^s} / \operatorname{diam} K)^2$$

$$\cdot \int_{K_{i_1^s,\ldots,i_{n_s}^s} - \Gamma_{i_1^s,\ldots,i_{n_s}^s}} \gamma[u](dy)$$

and this proves the lemma.

We now have all the tools we need to prove our scaled Poincaré inequalities. In addition to assumptions (5.1), (5.2), we now suppose that the variational fractal $K \equiv (K, E)$ is connected in the capacity sense, according to the following definition:

DEFINITION 5.3. We say that $K \equiv (K, E)$ is *connected (in the capacity sense)* if there exists a constant $0 < k \leq 1$, such that for every $x \in K$ and every $0 < R \leq \text{diam}_e K$ the (finite) family $G \equiv G_{x,R}$ in Theorem 2.1, suitably indexed, is a $k$-capacitory chain according to Definition 5.2.

We recall from Theorem 2.1 that the family $G_{x,R}$ has at most $M$ elements and $M$ is independent of $x$ and $R$. The cardinality of $G_{x,R}$ will be denoted below by $M' \equiv M'_{x,R}$, hence $M' \leq M$.

THEOREM 5.1. *Let $K \equiv (K, \mu, E)$ be a variational fractal, according to Definition 4.1, which satisfies (5.1), (5.2) and is connected in the capacity sense, according to Definition 5.3, for some constant $0 < k \leq 1$. Then, there exist two constants $C > 0$ and $q \geq 1$, such that the following inequalities hold:*

$$(5.24) \qquad \int_{B(x,r)} |u - \bar{u}_{B(x,r)}|^2 \mu(dx) \leq C \left( \frac{r}{\text{diam } K} \right)^2 \int_{B(x,qr)} \gamma[u](dx)$$

*for every $u \in D[E]$ and every $0 < r \leq \text{diam } K$. Moreover, $C = C_1 C_2$, where $C_1 = (4M\alpha_1^{d_f})^M$, where $M$ is the constant of Theorem 2.1, $C_2 = c(c_P, k)$, and $q = 2^\delta$.*

PROOF. By the regularity of the form $E$, it suffices to prove the inequality by assuming $u \in D[E] \cap C(K)$. Let $x \in K$, $0 < r \leq \text{diam } K$. By Theorem 2.1, we have

$$B(x, r) = K \cap B_e(x, r^{1/\delta}) \subset \bigcup_G K_{i_1,\dots,i_m} \cap B_e(x, r^{1/\delta})$$

where the family $G \equiv G_{x,R}$ has $M' \leq M$ elements. By the connectedness condition it is not restrictive to assume, up to renumbering, that the sets $K_{i_1^l,\dots,i_m^l}$ in $G$ form a $k$-capacitory chain, that is, each pair of two successive $K_{i_1^s,\dots,i_n^s}$, $K_{i_1^{s+1},\dots,i_{n_{s+1}}^{s+1}}$, $s = 1,\dots,M' - 1$ is connected in the capacity sense, according to Definition 5.1. Moreover, to simplify notation, we write below $M$ in place of $M'$. Let

$$Q = \bigcup_{s=1}^{M} K_{i_1^s,\dots,i_{n_s}^s}.$$

If $M \geq 3$, by Lemma 5.7 we have

$$\int_Q |u - u_Q|^2 \mu(dy)$$

$$\leq c \left[ 4 \frac{\mu(Q)}{\min\limits_{s=2,\ldots,M-1} \mu(K_{i_1^s,\ldots,i_{n_s}^s})} \right]^{M-2} \max_{s=1,\ldots,M} \left( \frac{\operatorname{diam} K_{i_1^s,\ldots,i_{n_s}^s}}{\operatorname{diam} K} \right)^2$$

$$\cdot \int_{K_{i_1^s,\ldots,i_{n_s}^s} - \Gamma_{i_1^s,\ldots,i_{n_s}^s}} \gamma[u](dy)$$

If $M = 2$, by Lemma 5.3 we have

$$\int_Q |u - u_Q|^2 \mu(dy) \leq c \max_{s=1,2} \left( \frac{\operatorname{diam} K_{i_1^s,\ldots,i_{n_s}^s}}{\operatorname{diam} K} \right)^2 \int_{K_{i_1^s,\ldots,i_{n_s}^s} - \Gamma_{i_1^s,\ldots,i_{n_s}^s}} \gamma[u](dy) .$$

On the other hand, if $M = 1$ we have by Lemma 5.2

$$\int_Q |u - u_Q|^2 \mu(dy) \leq c_P (\operatorname{diam} K_{i_1^1,\ldots,i_{m_1}^1} / \operatorname{diam} K)^2 \int_{K_{i_1^1,\ldots,i_{m_1}^1} - \Gamma_{i_1^1,\ldots,i_{m_1}^1}} \gamma[u](dx) .$$

Therefore, in all cases $M \geq 1$, we have

$$\int_Q |u - u_Q|^2 \mu(dy)$$

$$\leq c \left[ 4 \frac{\mu(Q)}{\min\limits_{s=1,\ldots,M} \mu(K_{i_1^s,\ldots,i_{n_s}^s})} \right]^M \cdot$$

$$\cdot \max_{s=1,\ldots,M} \left( \frac{\operatorname{diam} K_{i_1^s,\ldots,i_{n_s}^s}}{\operatorname{diam} K} \right)^2 \int_{K_{i_1^s,\ldots,i_{n_s}^s} - \Gamma_{i_1^s,\ldots,i_{n_s}^s}} \gamma[u](dy)$$

where $c = c(c_P, k)$. We now recall that for every $s = 1, \ldots, M$ we have

$$K_{i_1^s,\ldots,i_{n_s}^s} \cap B_e(x, r^{1/\delta}) \neq \emptyset$$

$$\alpha_1^{-1} r^{1/\delta} < \alpha_{i_1^s}^{-1}, \ldots, \alpha_{i_{n_s}^s}^{-1} \operatorname{diam}_e K \leq r^{1/\delta},$$

$$\operatorname{diam}_e K_{i_1^s,\ldots,i_{n_s}^s} = \alpha_{i_1^s}^{-1}, \ldots, \alpha_{i_{n_s}^s}^{-1} \operatorname{diam}_e K .$$

Therefore, $K_{i_1^s,\ldots,i_{n_s}^s} \subset K \cap B_e(x, 2r^{1/\delta}) = B(x, 2^\delta r)$ for every $s = 1, \ldots, M$. It follows

$$\int_{B(x,r)} |u(y) - u_{B(x,r)}|^2 \mu(dy)$$

$$\leq \int_{B(x,r)} |u(y) - u_Q|^2 \mu(dy)$$

$$\leq \int_Q |u(y) - u_Q|^2 \mu(dy)$$

$$\leq c \left[ 4 \frac{\mu(Q)}{\min_{s=1,\ldots,M} \mu(K_{i_1^s,\ldots,i_{n_s}^s})} \right]^M \cdot$$

$$\cdot \max_{s=1,\ldots,M} \left( \frac{\text{diam } K_{i_1^s,\ldots,i_{n_s}^s}}{\text{diam } K} \right)^2 \int_{K_{i_1^s,\ldots,i_{n_s}^s} - \Gamma_{i_1^s,\ldots,i_{n_s}^s}} \gamma[u](dy).$$

We have

$$\mu(Q) \leq \sum_{s=1}^M \mu(K_{i_1^s,\ldots,i_{n_s}^s}) = \sum_{s=1}^M \alpha_{i_1^s}^{-d_f}, \ldots, \alpha_{i_{n_s}^s}^{-d_f} \leq M \left( \frac{r^{1/\delta}}{\text{diam}_e K} \right)^{d_f},$$

$$\mu(K_{i_1^s,\ldots,i_{n_s}^s}) = \alpha_{i_1^s}^{-d_f}, \ldots, \alpha_{i_{n_s}^s}^{-d_f} > \alpha_1^{-d_f} \left( \frac{r^{1/\delta}}{\text{diam}_e K} \right)^{d_f}$$

for every $s = 1, \ldots, M$. Therefore,

$$\left[ 4 \frac{\mu(Q)}{\min_{s=1,\ldots,M} \mu(K_{i_1^s,\ldots,i_{n_s}^s})} \right]^M \leq [4M\alpha_1^{d_f}]^M.$$

Moreover,

$$\max_{s=1,\ldots,M} \left( \frac{\text{diam } K_{i_1^s,\ldots,i_{n_s}^s}}{\text{diam } K} \right)^2 \leq \max_{s=1,\ldots,M} (\alpha_{i_1^s}^{-1}, \ldots, \alpha_{i_{n_s}^s}^{-1})^{2\delta} \leq \left( \frac{r}{\text{diam } K} \right)^2.$$

Furthermore, $\int_{K_{i_1^s,\ldots,i_{n_s}^s} - \Gamma_{i_1^s,\ldots,i_{n_s}^s}} \gamma[u](dy) \leq \int_{B(x,2^\delta r)} \gamma[u](dy)$ for every $s = 1, \ldots, M$. Thus,

$$\int_{B(x,r)} |u(y) - u_{B(x,r)}|^2 \mu(dy) \leq c(4M\alpha_1^{d_f})^M \left( \frac{r}{\text{diam } K} \right)^2 \int_{B(x,2^\delta r)} \gamma[u](dy)$$

with $c = c(c_P, k)$ and this concludes the proof of the theorem.

## 6. – Imbeddings

As mentioned in the Introduction, the Poincaré inequalities of Theorem 5.1 allow us to apply the imbedding theory developed in [6], [7]. We summarize below some of the main inequalities obtained. As in Theorem 5.1, $K = (K, \mu, E)$ is a variational fractal satisfying (5.1), (5.2) and connected in the capacity sense for some constant $0 < k \le 1$. In particular, $K$ with the intrinsic metric of $E$ is a space of homogeneous type of dimension $\nu$.

From the family of scaled Poincaré inequalities of Theorem 5.1 and from the results in [6],[7], we obtain several important inequalities on the intrinsic balls of $K$. For general variational fractals, these inequalities were first stated in [23]. Below by $c$ we denote a constant that depends only on the constants $M$, $N$, $\alpha$, $\nu$ and on the constants $c_P$ and $k$.

### 6.1. – Nash inequalities

THEOREM 6.1. *Let $\nu > 0$ be the intrinsic dimension of $K$. There exists a constant $c > 0$, such that for every $x \in K$ and every $R > 0$,*

$$\|u\|_{L^2(B(x,R),\mu)}^{2+4/\nu}$$

$$(6.1) \qquad \le c\mu(B(x, R))^{-2/\nu} \left[ R^2 \int_{B(x,c_T(q+1)R)} \gamma[u](dx) + \int_{B(x,R)} |u|^2 dm \right]$$

$$\cdot \left[ \int_{B(x,R)} |u| dm \right]^{4/\nu}$$

From Nash inequality, following [9], we can derive an estimate of the behaviour of the semigroup associated with the form $E$. In fact, if we assume that the semigroup associated with $E$, with domain the closure of $C_0(K - \Gamma)$ in $D[E]$, is a Feller semigroup – by Theorem 6.3 below, this is always the case if $\nu < 2$ – and we denote by $p_t(x, y)$ its transition function, then we can deduce from (6.1) – as in [9] – the on-diagonal estimate $p_t(x, x) \le ct^{-\nu/2}$ for every $t \in (0, +\infty)$.

### 6.2. – Sobolev inequalities

THEOREM 6.2. *Let $\nu > 2$. Then, there exists a constant $c > 0$, such that for every $x \in K$ and every $R > 0$, we have*

$$\left( \frac{1}{\mu(B(x, R))} \int_{B(x,R)} |u|^{2^*} \mu(dx) \right)^{1/2^*}$$

$$(6.2) \qquad \le c \left[ \frac{R^2}{\mu(B(x, R))} \int_{B(c,c_T(q+1)R)} \gamma[u](dx) \right.$$

$$\left. + \frac{1}{\mu(B(x, R))} \int_{B(x,R)} |u|^2 \mu(dx) \right]^{1/2}$$

*with $2^* = 2\nu/(\nu - 2)$.*

### 6.3. – Morrey inequalities

THEOREM 6.3. *Let $\nu < 2$. Then, for every $x \in K$ and every $R > 0$,*

$$(6.3) \qquad |u(y) - u(z)| \leq c \left[ \frac{R^2}{\mu(B(x,R))} \int_{B(x,qR)} \gamma[u](dx) \right]^{1/2} d(y,z)^{1-\nu/2},$$

$$(6.4) \qquad \begin{aligned} &|u(y)| \\ &\leq c \left[ \frac{R^2}{\mu(B(x,R))} \int_{B(x,qR)} \gamma[u](dx) + \frac{1}{\mu(B(x,R))} \int_{B(x,R)} |u|^2 \mu(dx) \right]^{1/2}, \end{aligned}$$

*for all $y, z \in B(x, R)$.*

As a consequence of the intrinsic Morrey's imbedding $D[E] \subset C^\beta$, $\beta = 1 - \nu/2$, we obtain the Euclidean imbedding $D[E] \subset C^{\beta_e}_{\text{eucl}}$, where now $C^{\beta_e}_{\text{eucl}}$ is the space of Hölder continuous functions with Hölder exponent $\beta_e = \delta\beta$ in the Euclidean metric of $K$. We shall come back on this point in Section 7.1 below.

We notice also that the case $\nu = 2$ leads to *John-Nirenberg inequalities* and related exponential integrability properties, see [5], [6].

## 7. – Examples

We conclude with a few examples of variational fractals to which our theory applies, by pointing out that they include both *fractal* and *Euclidean* classic examples.

1. $K$ a general *nested fractal* in $\mathbb{R}^D$, $D \geq 2$, and $E$ the "standard" form constructed – as a diffusion – by Lindstrøm, [21], and – as a Dirichlet form – by Kusuoka, [19], and Fukushima, [11]. Now $\alpha_i = \alpha \ \forall i = 1, \ldots, N$, $d_f = \log_\alpha N$, $\rho_i = \rho > 1$, $\delta_i = \delta = (1/2) \log_\alpha(\rho N)$, $\nu = 2 \log N / \log(\rho N) < 2$. In this class we find the *Sierpinski gasket*, [14], [18], [3], [16], [13], where $N = D + 1$, $\alpha = 2$, $\rho = (N+2)/N$, $d_f = \log_2 N$, $\delta = \log_4(N+2)$, $\nu = 2 \log N / \log(N+2)$, as well as other well known "finitely ramified" fractals like the *curve of von Koch*, the *snowflake*, etc. For the two-dimensional gasket ($D = 2$), we have, for example, $d_f = \log 3 / \log 2$, $\rho = 5/3$, $\delta = \log 5 / \log 4$, $\nu = \log 9 / \log 5$, $\beta = \log(5/3) / \log 5$, $\beta_e = \log(5/3) / \log 4$. In particular, as a consequence of Morrey's estimate (6.3), we find the Euclidean estimate $|u(y) - u(z)| \leq c[E[u]]^{1/2} |x - y|^{\log(5/3)/\log 4}$, first obtained by direct calculations by Kozlov, [17]. Notice also that the homogeneous dimension $\nu$ equals the so called *spectral dimension $d_s$* that governs the Weyl's asymptotic of the eigenvalues, as first

computed by [1], [27] (for a review of the physics literature see *e.g.* [8]). Therefore, we reach a remarkable conclusion. Both spectral asymptotics and Morrey-Sobolev imbeddings are governed by the *same* parameter: the homogeneous dimension $\nu$ of $K$, which expresses the *polynomial growth* of the volume of the balls $B(x, R)$ of the *intrinsic* metric $d$. The metric $d$ has a remarkable "focussing effect" also on the standard diffusion in $K$ – the "Brownian motion" – associated with the form $E$. In fact, if expressed in the metric $d$, this fractal Brownian motion acquires the "correct" space-time scaling as the Euclidean Brownian motion in all dimensions, namely $E^x d(X_t, x)^2 = t$. This has to be compared with the anomalous *subdiffusive* Euclidean scaling $E^x |X_t - x|^2 = t^{2/d_w}$, where $d_w = \ln(\rho N)/\ln \alpha = \log 5/\log 2 > 2$ is the so-called *path dimension* of the gasket, [1], [3]. The variational metric on the Sierpinski gasket was previously introduced in [6], [26].

**2.** $K$ the *Sierpinski carpet* in $\mathbb{R}^2$, $E$ the "standard" form constructed by Barlow-Bass, [2] and Kusuoka-Zhou [20]. Now $N = 3^2 - 1 = 8$, $\alpha = 3$, $d_f = \log_3 8$ and $1 < \rho_i = \rho$. This is not a nested fractal – it is not "finitely ramified" – however, since $\rho > 1$, it has intrinsic dimension $\nu < 2$ as any nested fractal.

**3.** $K$ is the $D$-dimensional coordinate cube $K = [0, 1]^D$ of $\mathbb{R}^D$, $D \geq 1$, and $E$ is the usual Dirichlet integral in $K$, with domain the Sobolev space $H^1(\overset{\circ}{K})$. This is a simple example in which the energy scaling (4.2) is satisfied with a constant $\rho \leq 1$, if $D \geq 2$, as well as with $\rho > 1$, if $D = 1$. In fact, $K = [0, 1]^D$ is easily seen to be a selfsimilar fractal of dimension $d_f = D$ with respect to any set of $N = \alpha^D$ coordinate similitudes with $\alpha$ any given integer $\alpha \geq 2$. The Dirichlet integral satisfies (4.2) with $\rho = \alpha^{2-D}$. Note that in this example the three basic scaling factors $N$, $\alpha$, $\rho$ are related by the identity $\rho N = \alpha^2$, hence $\delta = 1$ and $\nu = d_f = D$.

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