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On Homogenization of Solutions of Boundary Value Problems in Domains, Perforated Along Manifolds

M. LOBO – O. A. OLEINIK – M. E. PEREZ – T. A. SHAPOSHNIKOVA

This paper is dedicated to the memory of E. De Giorgi,
 a great mathematician of the XX century

1. In this paper the problem of homogenization of solutions to the Poisson equation in domains, perforated along a manifold, is considered with the Neumann boundary condition, with the Dirichlet boundary condition or the mixed condition on cavities. Some particular problems of this kind were considered in [1], [2]. The same method can be applied also to boundary value problems in domains, perforated in some subdomains. The short note on these results is published in [3]. We study here also the corresponding spectral problems. E. De Giorgi was one of the first mathematicians, who considered homogenization problems [4].

Let Ω be a bounded domain in R^n with a smooth boundary $\partial\Omega$ and γ be a manifold in $\overline{\Omega}$. Let P_j ($j = 1, \dots, N(\varepsilon)$) with $N(\varepsilon) \leq d_0\varepsilon^{1-n}$, $d_0 = \text{const}$ be a point such that $P_j \in \gamma$; ε is a small parameter. We denote by $G^j(a_\varepsilon^j)$ a domain which belongs to Ω , has a smooth boundary $\partial G^j(a_\varepsilon^j)$, $P_j \in \overline{G^j(a_\varepsilon^j)}$, the diameter of $G^j(a_\varepsilon^j)$ is a_ε^j and $a_\varepsilon^j \leq C_0\varepsilon$, $C_0 = \text{const} > 0$, $G^j(a_\varepsilon^j) \cap G^i(a_\varepsilon^i) = \emptyset$ for $i \neq j$. We consider all possible behavior of a_ε^j as $\varepsilon \rightarrow 0$.

We set

$$G_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} G^j(a_\varepsilon^j), \quad \Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad S'_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} \partial G^j(a_\varepsilon^j),$$

$$S_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} \partial G^j(a_\varepsilon^j) \cap \Omega, \quad \Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus S_\varepsilon.$$

We assume that $G^j(a_\varepsilon^j)$ are such that any function $u \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ can be extended on Ω as a function $\tilde{u} \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ in such a way that

$$(1) \quad \|\tilde{u}\|_{H_1(\Omega, \Gamma_\varepsilon)} \leq K_1 \|u\|_{H_1(\Omega_\varepsilon, \Gamma_\varepsilon)},$$

$$(2) \quad \|\nabla \tilde{u}\|_{L_2(\Omega)} \leq K_2 \|\nabla u\|_{L_2(\Omega_\varepsilon)},$$

where K_j here and in what follows are constants which do not depend on ε .

The space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ is defined as a closure of $C^\infty(\overline{\Omega_\varepsilon})$ -functions, which are equal to zero in a neighbourhood of Γ_ε , in the norm

$$\|u\|_{H_1(\Omega_\varepsilon, \Gamma_\varepsilon)} \equiv \left(\int_{\Omega_\varepsilon} (u^2 + |\nabla u|^2) dx \right)^{1/2}.$$

The cases, when it is possible, are considered in [5], [6].

In this domain Ω_ε we consider the boundary value problems for the equation

$$(3) \quad -\Delta u_\varepsilon = f$$

with the boundary conditions

$$(4) \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad S_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon,$$

or

$$(5) \quad \frac{\partial u_\varepsilon}{\partial \nu} + \beta u_\varepsilon = 0 \quad \text{on} \quad S_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon, \quad \beta(x) \geq \beta_0 = \text{const} > 0,$$

or

$$(6) \quad u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon,$$

where ν is an outward unit normal vector to S_ε .

2. Let us consider the problem (3), (4) (the Neumann condition on S_ε).

We define a weak solution of the problem (3), (4) as a function $u_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ which satisfies the integral identity

$$(7) \quad A_\varepsilon(u_\varepsilon, \varphi) \equiv \int_{\Omega_\varepsilon} (\nabla u_\varepsilon, \nabla \varphi) dx = \int_{\Omega_\varepsilon} f \varphi dx$$

for any function $\varphi \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$.

We also assume that for functions u from the space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ the Friedrichs inequality is valid:

$$(8) \quad \|u\|_{L_2(\Omega_\varepsilon)} \leq K_0 \|\nabla u\|_{L_2(\Omega_\varepsilon)}$$

with the constant K_0 , independent on ε . This inequality is satisfied if, for example, $\Gamma \subset \Gamma_\varepsilon$ and Γ is a smooth piece of $\partial\Omega$ with a positive measure on $\partial\Omega$, $\Gamma \cap \overline{G_\varepsilon} = \emptyset$ (see [5]).

From (7) and the Friedrichs inequality it follows that $\|u_\varepsilon\|_{H_1(\Omega_\varepsilon)} \leq K_3$.

Using the Riesz theorem, it is easy to prove that the problem (3), (4) has a unique weak solution in Ω_ε .

Let the function v_0 be a solution of the problem

$$(9) \quad -\Delta v = f \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega,$$

and f is a smooth function in $\bar{\Omega}$.

First we consider the case when $S'_\varepsilon \cap \partial\Omega = \emptyset$. Using the integral identity (7) for problem (3),(4) and the integral identity for the problem (9), we get

$$(10) \quad \int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - v_0)|^2 dx = \int_{G_\varepsilon} f(\tilde{u}_\varepsilon - v_0) dx + \int_{G_\varepsilon} (\nabla v_0, \nabla(\tilde{u}_\varepsilon - v_0)) dx,$$

where \tilde{u}_ε is an extension of u_ε in Ω , such that $\tilde{u}_\varepsilon - v_0$ satisfies (1),(2). From (10) we have

$$(11) \quad \begin{aligned} \|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)}^2 &\leq \max_{\Omega} |f| |G_\varepsilon|^{1/2} \|\tilde{u}_\varepsilon - v_0\|_{L_2(G_\varepsilon)} \\ &+ \max_{\Omega} |\nabla v_0| |G_\varepsilon|^{1/2} \|\nabla(\tilde{u}_\varepsilon - v_0)\|_{L_2(G_\varepsilon)}. \end{aligned}$$

Using inequalities (1), (2) for $\tilde{u}_\varepsilon - v_0$, and the Friedrichs inequality, we obtain from (11) that

$$\begin{aligned} K_4 \|\nabla(\tilde{u}_\varepsilon - v_0)\|_{L_2(\Omega)} &\leq \|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)} \leq K_5 \left(\max_{\Omega} |f| + \max_{\Omega} |\nabla v_0| \right) |G_\varepsilon|^{1/2} \\ &\leq K_6 |G_\varepsilon|^{1/2} \leq K_7 (\max_j a_\varepsilon^j)^{n/2} \varepsilon^{(1-n)/2}, \end{aligned}$$

where $|G|$ is the measure of the set G .

Therefore,

$$(12) \quad \|u_\varepsilon - v_0\|_{H^1(\Omega_\varepsilon)}^2 \leq K_8 (\max_j a_\varepsilon^j)^n \varepsilon^{1-n}.$$

Let us consider the case, when $S'_\varepsilon \cap \partial\Omega \neq \emptyset$, and let $G^j(a_\varepsilon^j)$, $j = 1, \dots, M(\varepsilon)$, be such that $\overline{G^j(a_\varepsilon^j)} \cap \partial\Omega \neq \emptyset$, $M(\varepsilon) \leq d_1 \varepsilon^{-n+2}$, $d_1 = \text{const} > 0$, and $|\partial G^j(a_\varepsilon^j)| \leq d_2 (a_\varepsilon^j)^{n-1}$. We set

$$I_\varepsilon = \partial\Omega \cap \bigcup_{j=1}^{M(\varepsilon)} \overline{G^j(a_\varepsilon^j)}.$$

Since v_0 is a smooth function, we derive from the Green formula that

$$(13) \quad \int_{\Omega} (\nabla v_0, \nabla(\tilde{u}_\varepsilon - v_0)) dx = \int_{\Omega} f(\tilde{u}_\varepsilon - v_0) dx + \int_{I_\varepsilon} \frac{\partial v_0}{\partial \nu} (\tilde{u}_\varepsilon - v_0) ds.$$

From integral identity (7) with $\varphi = (u_\varepsilon - v_0)$ and (13), we obtain

$$(14) \quad \begin{aligned} \int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - v_0)|^2 dx &= \int_{G_\varepsilon} (\nabla v_0, \nabla(\tilde{u}_\varepsilon - v_0)) dx - \int_{G_\varepsilon} f(\tilde{u}_\varepsilon - v_0) dx \\ &- \int_{I_\varepsilon} \frac{\partial v_0}{\partial \nu} (\tilde{u}_\varepsilon - v_0) ds \equiv J^\varepsilon. \end{aligned}$$

Let us estimate J^ε . Using (1), (2), the Friedrichs inequality and the imbedding theorem, we get

$$\begin{aligned}
 \|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)}^2 &\leq K_9\{|G_\varepsilon|^{1/2}(\|\nabla(\tilde{u}_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)} \\
 &\quad + \|\tilde{u}_\varepsilon - v_0\|_{L_2(\Omega_\varepsilon)}) + \|\tilde{u}_\varepsilon - v_0\|_{L_2(\partial\Omega)}|I_\varepsilon|^{1/2}\} \\
 (15) \qquad \qquad \qquad &\leq K_{10}\{|G_\varepsilon|^{1/2} + |I_\varepsilon|^{1/2}\}\|\tilde{u}_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)} \\
 &\leq K_{11}(|G_\varepsilon|^{1/2} + |I_\varepsilon|^{1/2})\|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)}.
 \end{aligned}$$

Due to assumptions on $M(\varepsilon)$ and $\partial G^j(a_\varepsilon^j)$ we have

$$(16) \qquad \qquad \qquad |I_\varepsilon| \leq K_{12}(\max_j a_\varepsilon^j)^{n-1} \varepsilon^{2-n}.$$

From estimates (15), (16) it follows that

$$(17) \qquad \qquad \qquad \|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{13} \max_j (a_\varepsilon^j)^{n-1} \varepsilon^{2-n}.$$

Hence, we proved the following theorem.

THEOREM 1. *Let u_ε be a weak solution of the problem (3), (4), v_0 be a solution of the problem (9), $S'_\varepsilon \cap \partial\Omega = \emptyset$. Then*

$$\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{14}(\max_j a_\varepsilon^j)^n \varepsilon^{1-n}.$$

If $S'_\varepsilon \cap \partial\Omega \neq \emptyset$ and $M(\varepsilon) \leq d_1 \varepsilon^{2-n}$, $|\partial G^j(a_\varepsilon^j)| \leq d_2 (a_\varepsilon^j)^{n-1}$, then

$$\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{15} \max_j (a_\varepsilon^j)^{n-1} \varepsilon^{2-n}.$$

We note that in the proof of Theorem 1 the assumption that $P_j \in \gamma$ is not used.

3. Let us consider the problem (3), (5) (the mixed boundary condition). Let $\beta(x) \geq \beta_0 = \text{const} > 0$ and v_0 be a solution of problem (9). Then the function $w_\varepsilon = u_\varepsilon - v_0$ is a weak solution of the problem

$$\begin{aligned}
 -\Delta w_\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon, \quad w_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \\
 (18) \qquad \frac{\partial w_\varepsilon}{\partial \nu} + \beta(x)w_\varepsilon &= -\left(\frac{\partial v_0}{\partial \nu} + \beta(x)v_0\right) \quad \text{on } S_\varepsilon.
 \end{aligned}$$

From the integral identity for the problem (18) we have

$$(19) \qquad \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dx + \int_{S_\varepsilon} \beta(x)w_\varepsilon^2 ds = - \int_{S_\varepsilon} \left(\frac{\partial v_0}{\partial \nu} + \beta(x)v_0\right)w_\varepsilon ds.$$

For the right-hand side of (19) we get

$$\begin{aligned}
 & \left| \int_{S_\varepsilon} \left(\frac{\partial v_0}{\partial \nu} + \beta(x)v_0 \right) w_\varepsilon ds \right| \\
 (20) \quad & \leq \frac{1}{2} \beta_0 \int_{S_\varepsilon} w_\varepsilon^2 ds + K_{16} \int_{S_\varepsilon} \left(\frac{\partial v_0}{\partial \nu} + \beta(x)v_0 \right)^2 ds \\
 & \leq \frac{1}{2} \int_{S_\varepsilon} \beta(x) w_\varepsilon^2 ds + K_{16} \int_{S_\varepsilon} \left(\frac{\partial v_0}{\partial \nu} + \beta(x)v_0 \right)^2 ds.
 \end{aligned}$$

From inequalities (20) and (19) it follows that

$$(21) \quad \|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)}^2 \leq K_{17} \left\| \frac{\partial v_0}{\partial \nu} + \beta(x)v_0 \right\|_{L_2(S_\varepsilon)}^2.$$

Taking into account that $v_0(x)$ is a smooth function, we obtain

$$\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{18} |S_\varepsilon| \leq K_{19} (\max_j a_\varepsilon^j)^{n-1} \varepsilon^{1-n}.$$

From this estimate we derive the following theorem.

THEOREM 2. *Let $\beta(x) \geq \beta_0 = \text{const} > 0$, u_ε be a solution of the problem (3), (5), v_0 be a solution of the problem (9). Assume that $|\partial G^j(a_\varepsilon^j)| \leq K_{20}(a_\varepsilon^j)^{n-1}$, $\eta_\varepsilon \equiv (\max_j a_\varepsilon^j) \varepsilon^{-1} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then $\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)} \rightarrow 0$ as $\varepsilon \rightarrow 0$ and*

$$\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)}^2 \leq K_{21} \eta_\varepsilon^{n-1}.$$

In the proof of Theorem 2 we do not use that $P_j \in \gamma$.

4. Assume that γ is a domain in the hyperplane $\{x : x_1 = 0\}$ and $\gamma = \Omega \cap \{x : x_1 = 0\}$, $Q = \{x : -1/2 < x_j < 1/2, j = 1, \dots, n\}$, $G'_\varepsilon = \bigcup_{z \in Z} (a_\varepsilon G_0 + \varepsilon z)$, where G_0 is a smooth domain and $a_\varepsilon \leq C\varepsilon$, $C = \text{const}$, $\overline{G_0} \subset Q$, Z is the set of vectors with integer components. Let $a_\varepsilon \varepsilon^{-1} \rightarrow C_0 = \text{const} > 0$ as $\varepsilon \rightarrow 0$ and $C_0 \overline{G_0} \subset Q$.

We set

$$\begin{aligned}
 & \Pi_\varepsilon = \Omega \cap \{x : |x_1| < \varepsilon/2\}, \Pi_\varepsilon^* = \Pi_\varepsilon \setminus \overline{G'_\varepsilon}, \gamma_\varepsilon^\pm = \Omega \cap \{x : x_1 = \pm \varepsilon/2\}, \\
 & \Omega_\varepsilon^+ = \Omega \cap \{x : x_1 > \varepsilon/2\}, \Omega_\varepsilon^- = \Omega \cap \{x : x_1 < -\varepsilon/2\}, G_\varepsilon = G'_\varepsilon \cap \Omega, \\
 (22) \quad & \Omega_\varepsilon = \Omega_\varepsilon^+ \cup \gamma_\varepsilon^+ \cup \Pi_\varepsilon^* \cup \gamma_\varepsilon^- \cup \Omega_\varepsilon^-, \\
 & S_\varepsilon = \partial G_\varepsilon \cap \Omega, \Gamma_\varepsilon = \partial \Omega_\varepsilon \setminus S_\varepsilon, l_\varepsilon = \overline{G_\varepsilon} \cap \partial \Omega.
 \end{aligned}$$

We assume that $\beta(x) = \beta_0 = \text{const} > 0$. Let v be a weak solution of the problem

$$(23) \quad \begin{aligned} -\Delta v &= f \quad \text{in } \Omega \setminus \gamma, \quad v = 0 \quad \text{on } \partial\Omega, \\ [v]|_\gamma &= 0, \quad \left[\frac{\partial v}{\partial x_1} \right] |_\gamma = \mu v|_\gamma, \end{aligned}$$

where $\mu = \beta_0 C_0^{n-1} |\partial G_0|$, $[\varphi]|_\gamma = \varphi(x_1 + 0, x') - \varphi(x_1 - 0, x')$, $(x_1, x') \in \gamma$, $x' = (x_2, \dots, x_n)$.

We assume that $|v(x)| < K_{22}$ for $x \in \overline{\Omega}$, $|\nabla v(x)| \leq K_{23}$ in $\Omega^+ = \Omega \cap \{x : x_1 > 0\}$ and $|\nabla v(x)| \leq K_{24}$ in $\Omega^- = \Omega \cap \{x : x_1 < 0\}$. From the integral identity for the problem (3), (5) we obtain

$$(24) \quad \begin{aligned} \int_{\Omega_\varepsilon} (\nabla u_\varepsilon, \nabla(u_\varepsilon - v)) dx + \beta_0 \int_{S_\varepsilon} (u_\varepsilon - v)^2 ds + \beta_0 \int_{S_\varepsilon} v(u_\varepsilon - v) ds \\ = \int_{\Omega_\varepsilon} f(u_\varepsilon - v) dx. \end{aligned}$$

Using the assumptions on $v(x)$ and the Green formula we get

$$(25) \quad \begin{aligned} \int_{\Omega} (\nabla v, \nabla(\tilde{u}_\varepsilon - v)) dx + \beta_0 C_0^{n-1} |\partial G_0| \int_{\gamma} v(\tilde{u}_\varepsilon - v) dx' \\ = \int_{\Omega} f(\tilde{u}_\varepsilon - v) dx + \int_{I_\varepsilon} \frac{\partial v}{\partial \nu} (\tilde{u}_\varepsilon - v) ds, \end{aligned}$$

where $I_\varepsilon = \partial\Omega \cap \overline{G_\varepsilon}$, $\tilde{u}_\varepsilon - v$ is an extension of $u_\varepsilon - v$ in Ω such that (1), (2) are satisfied.

From (24), (25) we derive

$$(26) \quad \begin{aligned} \int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - v)|^2 dx + \beta_0 \int_{S_\varepsilon} (u_\varepsilon - v)^2 ds \\ = \beta_0 \{ C_0^{n-1} |\partial G_0| \int_{\gamma} v(\tilde{u}_\varepsilon - v) dx' - \int_{S_\varepsilon} v(u_\varepsilon - v) ds \} + P_\varepsilon, \end{aligned}$$

where

$$(27) \quad P_\varepsilon \equiv - \int_{G_\varepsilon} f(\tilde{u}_\varepsilon - v) dx - \int_{I_\varepsilon} \frac{\partial v}{\partial \nu} (\tilde{u}_\varepsilon - v) ds.$$

LEMMA 1. Assume that $v \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ and $(a_\varepsilon \varepsilon^{-1} - C_0) \rightarrow 0$ as $\varepsilon \rightarrow 0$, Ω_ε is defined by (22). Then

$$(28) \quad \left| \int_{S_\varepsilon} v ds - C_0^{n-1} |\partial G_0| \int_{\gamma} v dx' \right| \leq K_{25} \{ \sqrt{\varepsilon} + |a_\varepsilon \varepsilon^{-1} - C_0| \} \|v\|_{H_1(\Omega)}.$$

PROOF. Consider the function $\theta_\varepsilon(y)$, $y = \varepsilon^{-1}x$, as a solution of the problem

$$(29) \quad \begin{cases} \Delta_y \theta_\varepsilon = 0 & y \in Q \setminus \overline{a_\varepsilon \varepsilon^{-1} G_0} = Y_\varepsilon, \\ \frac{\partial \theta_\varepsilon}{\partial \nu} = 1 & \text{on } a_\varepsilon \varepsilon^{-1} \partial G_0, \\ \frac{\partial \theta_\varepsilon}{\partial y_1} = (a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| & \text{on } \Sigma_0 = \partial Q \cap \{y : y_1 = -1/2\}, \\ \frac{\partial \theta_\varepsilon}{\partial y_1} = 0 & \text{on } \Sigma_1 = \partial Q \cap \{y : y_1 = 1/2\}. \\ \theta_\varepsilon(y) & \text{is 1-periodic in } y' = (y_2, \dots, y_n). \end{cases}$$

Since $|a_\varepsilon \varepsilon^{-1} \partial G_0| = (a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0|$, the problem (29) has a unique weak solution to within a constant. We define the constant in such a way that

$$\int_{Y_\varepsilon} \theta_\varepsilon(y) dy = 0.$$

From the integral identity for problem (29) it follows that

$$(30) \quad \|\nabla \theta_\varepsilon\|_{L_2(Y_\varepsilon)} \leq K_{26}.$$

Indeed,

$$(31) \quad \int_{Y_\varepsilon} |\nabla_y \theta_\varepsilon|^2 dy = -(a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| \int_{\Sigma_0} \theta_\varepsilon dy' + \int_{a_\varepsilon \varepsilon^{-1} \partial G_0} \theta_\varepsilon ds_y.$$

From the imbedding theorem and the Poincaré inequality it follows that

$$(32) \quad \|\theta_\varepsilon\|_{L_2(\Sigma_0)} \leq K_{27} \|\nabla_y \theta_\varepsilon\|_{L_2(Y_\varepsilon)}, \|\theta_\varepsilon\|_{L_2(a_\varepsilon \varepsilon^{-1} \partial G_0)} \leq K_{28} \|\nabla_y \theta_\varepsilon\|_{L_2(Y_\varepsilon)}.$$

The inequalities (31) and (32) imply (30). We set

$$P_j^\varepsilon = \frac{\partial \theta_\varepsilon}{\partial y_j}, \quad j = 1, \dots, n.$$

For the vector-function $P^\varepsilon(y) = (P_1^\varepsilon(y), \dots, P_n^\varepsilon(y))$ we have

$$\begin{aligned} \operatorname{div}_y P^\varepsilon &= 0, \quad \text{if } y \in Y_\varepsilon, \quad (P^\varepsilon, \nu) = 1 \quad \text{on } a_\varepsilon \varepsilon^{-1} \partial G_0, \\ (P^\varepsilon, \nu) &= -(a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| \quad \text{on } \Sigma_0 \quad \text{and} \quad (P^\varepsilon, \nu) = 0 \quad \text{on } \Sigma_1, \end{aligned}$$

where $\nu = (\nu_1, \dots, \nu_n)$ is a unit outward normal vector to ∂Y_ε . We denote by T_ε the set of cells of the form $(\varepsilon Q + \varepsilon z) \setminus (a_\varepsilon G_0 + \varepsilon z)$ which have a nonempty

intersection with Π_ε^* and $\partial\Omega$. Let $\Pi_\varepsilon^1 = \Pi_\varepsilon^* \cup T_\varepsilon$. We extend the function v on T_ε by setting $v = 0$ on $T_\varepsilon \setminus \Omega$. It is easy to see that

$$(33) \quad \begin{aligned} \int_{\Pi_\varepsilon^1} \operatorname{div}_x(P^\varepsilon(y)v)dx &= \int_{S_\varepsilon} (P^\varepsilon, v) v ds + \int_{\gamma_\varepsilon^-} (P^\varepsilon, v) v dx' \\ &= \int_{S_\varepsilon} v ds - (a_\varepsilon \varepsilon^{-1})^{(n-1)} |\partial G_0| \int_{\gamma_\varepsilon^-} v dx'. \end{aligned}$$

From (33) it follows that

$$(34) \quad \begin{aligned} \left| \int_{S_\varepsilon} v ds - C_0^{n-1} |\partial G_0| \int_\gamma v dx' \right| &\leq \left| \int_{\Pi_\varepsilon^1} \operatorname{div}_x(P^\varepsilon(y)v) dx \right| \\ &+ \left| (a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| \int_{\gamma_\varepsilon^-} v dx' - C_0^{n-1} |\partial G_0| \int_\gamma v dx' \right|, \end{aligned}$$

Let us estimate the right-hand side of (34). We have

$$(35) \quad I_1^\varepsilon \equiv \left| \int_{\Pi_\varepsilon^1} \operatorname{div}_x(P^\varepsilon(y)v) dx \right| \leq \int_{\Pi_\varepsilon^1} |\nabla v| |P^\varepsilon(y)| dx.$$

Therefore,

$$(36) \quad I_1^\varepsilon \leq \left(\int_{\Pi_\varepsilon^1} |\nabla_y \theta|^2 dx \right)^{1/2} \|v\|_{H_1(\Omega)}.$$

It is easy to see that

$$(37) \quad \|\nabla_y \theta\|_{L_2(\varepsilon Y_\varepsilon)}^2 \leq K_{29} \varepsilon^n.$$

Since Π_ε^1 can contain sets of the form $\varepsilon Y_\varepsilon + \varepsilon z$ no more than $a_1 \varepsilon^{1-n}$, $a_1 = \text{const} > 0$, from (36) and (37) we derive that

$$(38) \quad I_1^\varepsilon \leq K_{30} \sqrt{\varepsilon} \|v\|_{H_1(\Omega)}.$$

In order to estimate the second term in the right-hand side of (34) we use the continuity of functions from $H_1(\Omega)$ on hyperplanes in L_2 - norm. We have

$$(39) \quad \begin{aligned} I_2^\varepsilon &\equiv |(a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| \int_{\gamma_\varepsilon^-} v dx' - C_0^{n-1} |\partial G_0| \int_\gamma v dx'| \\ &\leq K_{31} \left\{ (a_\varepsilon \varepsilon^{-1})^{n-1} \left| \int_{\gamma_\varepsilon^-} v dx' - \int_\gamma v dx' \right| + |(a_\varepsilon \varepsilon^{-1})^{n-1} - C_0^{n-1}| \int_\gamma |v| dx' \right\} \\ &\leq K_{32} \{ \sqrt{\varepsilon} \|v\|_{H_1(\Omega)} + |a_\varepsilon \varepsilon^{-1} - C_0| \|v\|_{H_1(\Omega)} \} \\ &\leq K_{33} \{ \sqrt{\varepsilon} + |a_\varepsilon \varepsilon^{-1} - C_0| \} \|v\|_{H_1(\Omega)}. \end{aligned}$$

From (38) and (39) it follows that (28) is valid. Lemma 1 is proved.

In order to prove Theorem 3 we note that from (26), (27) it follows that

$$(40) \quad \begin{aligned} \|\nabla(u_\varepsilon - v)\|_{L_2(\Omega_\varepsilon)}^2 &\leq K_{34} (\sqrt{\varepsilon} + |a_\varepsilon \varepsilon^{-1} - C_0| \\ &+ |G_\varepsilon|^{1/2} + |l_\varepsilon|^{1/2}) \|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)}. \end{aligned}$$

We assume that $|l_\varepsilon| \leq d_3 \varepsilon$, $|G_\varepsilon| \leq d_4 \varepsilon$. Then we have from (40) and the Friedrichs inequality that

$$\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)} \leq K_{35} \{ \sqrt{\varepsilon} + |a_\varepsilon \varepsilon^{-1} - C_0| \}.$$

Hence, we have the following theorem

THEOREM 3. *Let u_ε be a solution of the problem (3), (5), the domain Ω_ε be defined by (22), v be a solution of problem (23), $a_\varepsilon \varepsilon^{-1} \rightarrow C_0 = \text{const} > 0$ as $\varepsilon \rightarrow 0$, $|I_\varepsilon| = |\overline{G}_\varepsilon \cap \partial\Omega| \leq d_3\varepsilon$. Then*

$$\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)} \leq K_{36}(\sqrt{\varepsilon} + |a_\varepsilon \varepsilon^{-1} - C_0|).$$

5. Consider now the problem (3), (6) (the Dirichlet boundary condition on cavities). We study the behavior of solutions of the problem

$$-\Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon$$

as $\varepsilon \rightarrow 0$.

Let us define the function φ_ε^j ($j = 1, \dots, N(\varepsilon)$) for $n \geq 3$, setting $\varphi_\varepsilon^j \equiv 0$ for $|x - P_j| \leq a_\varepsilon^j$, $\varphi_\varepsilon^j \equiv \frac{1}{\ln c_0} \ln\left(\frac{|x - P_j|}{a_\varepsilon^j}\right)$ for $a_\varepsilon^j \leq |x - P_j| \leq c_0 a_\varepsilon^j$, $\varphi_\varepsilon^j \equiv 1$ for $|x - P_j| > c_0 a_\varepsilon^j$. For $n = 2$ we set $\varphi_\varepsilon^j = \varphi\left(\frac{|\ln|x - P_j||}{|\ln c_0 a_\varepsilon^j|}\right)$, where $\varphi(\xi) = 1$ for $\xi \leq 1/2$, $\varphi = 0$ for $\xi \geq 1$, $0 \leq \varphi \leq 1$, $\varphi \in C^\infty(\mathbb{R}^n)$, c_0 is a constant, $c_0 > 0$, $c_0 a_\varepsilon^j \leq 1$.

We pose

$$\psi_\varepsilon(x) = \sum_{j=1}^{N(\varepsilon)} \psi_\varepsilon^j(x),$$

where $\psi_\varepsilon^j(x) = 1 - \varphi_\varepsilon^j(x)$ and $w_\varepsilon = v\psi_\varepsilon - v_\varepsilon$, v_ε is a solution of the problem

$$\Delta v_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \quad v_\varepsilon = v \quad \text{on } S_\varepsilon, \quad v_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon,$$

v is a solution of the problem (9).

It is evident that w_ε is a weak solution of the problem

$$(41) \quad \Delta w_\varepsilon = \Delta(v\psi_\varepsilon) \quad \text{in } \Omega_\varepsilon, \quad w_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Let us estimate w_ε and its derivatives. From the integral identity for the problem (41) it follows that

$$(42) \quad \begin{aligned} \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dx &\leq K_{37} \int_{\Omega_\varepsilon} |\nabla(v\psi_\varepsilon)|^2 dx \leq K_{38} \int_{\Omega_\varepsilon} (\psi_\varepsilon^2 + |\nabla\psi_\varepsilon|^2) dx \\ &\leq K_{39} \left\{ \max_j (a_\varepsilon^j)^n \varepsilon^{-n+1} + \sum_{j=1}^{N(\varepsilon)} \int_{a_\varepsilon^j}^{c_0 a_\varepsilon^j} r^{-3+n} dr \right\} \\ &\leq K_{40} \max_j (a_\varepsilon^j)^{n-2} \varepsilon^{1-n}, \end{aligned}$$

for $n \geq 3$. For $n = 2$ we have

$$\begin{aligned}
 \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dx &\leq K_{41} \int_{\Omega_\varepsilon} (\psi_\varepsilon^2 + |\nabla \psi_\varepsilon|^2) dx \\
 (43) \quad &\leq K_{42} \left(\max_j a_\varepsilon^j \varepsilon^{-1} + \sum_{j=1}^{N(\varepsilon)} |\ln c_0 a_\varepsilon^j|^{-2} \int_{c_0 a_\varepsilon^j}^{(c_0 a_\varepsilon^j)^{1/2}} r^{-1} dr \right) \\
 &\leq K_{43} \max_j |\ln c_0 a_\varepsilon^j|^{-1} \varepsilon^{-1}.
 \end{aligned}$$

From (42), (43), the relation $v_\varepsilon = v - u_\varepsilon$, estimates for $|\nabla(v\psi_\varepsilon)|$ and the Friedrichs inequality it follows that

$$(44) \quad \|u_\varepsilon - v\|_{H^1(\Omega_\varepsilon)}^2 \leq K_{44} (\max_j a_\varepsilon^j)^{n-2} \varepsilon^{1-n} \quad \text{for } n \geq 3,$$

$$(45) \quad \|u_\varepsilon - v\|_{H^1(\Omega_\varepsilon)}^2 \leq K_{45} (\max_j |\ln a_\varepsilon^j|)^{-1} \varepsilon^{-1} \quad \text{for } n = 2.$$

The inequalities (44), (45) imply the following theorem.

THEOREM 4. *Assume that*

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \max_j (a_\varepsilon^j)^{n-2} \varepsilon^{1-n} &= 0, \quad \text{if } n \geq 3, \\
 \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \max_j |\ln a_\varepsilon^j|^{-1} &= 0, \quad \text{if } n = 2,
 \end{aligned}$$

v is a solution of the problem (9), u_ε is a solution of problem (3), (6). Then the estimates (44), (45) are valid.

Here as in Theorems 1 and 2 we do not use that $P_j \in \gamma$.

6. Let us consider the problem (3), (6), when

$$(46) \quad \begin{cases} \lim_{\varepsilon \rightarrow 0} a_\varepsilon^{n-2} \varepsilon^{1-n} = C_1 = \text{const} > 0, & \text{if } n \geq 3, \\ \lim_{\varepsilon \rightarrow 0} (\varepsilon |\ln a_\varepsilon|)^{-1} = C_2 = \text{const} > 0, & \text{if } n = 2. \end{cases}$$

We assume that $G_0 = \{x : |x| < a\}$, $Q = \{x : -1/2 < x_j < 1/2, j = 1, \dots, n\}$, $a_\varepsilon^j = a_\varepsilon$, Ω_ε has the form given by (22). In this case the limit problem for (3), (6) is

$$(47) \quad \begin{aligned}
 -\Delta v &= f \quad \text{in } \Omega^- \cup \Omega^+, \\
 [v] \Big|_\gamma &= 0, \quad \left[\frac{\partial v}{\partial x_1} \right] \Big|_\gamma = \mu_1 v \Big|_\gamma, \quad v = 0 \quad \text{on } \partial\Omega,
 \end{aligned}$$

where $\Omega^- = \Omega \cap \{x : x_1 < 0\}$, $\Omega^+ = \Omega \cap \{x : x_1 > 0\}$, $\mu_1 = (n-2)a^{n-2}\omega(n)C_1$, if $n \geq 3$, $\mu_1 = 2\pi C_2$, if $n = 2$, $\omega(n)$ is the area of the unit sphere in R^n .

We assume that for a solution of the problem (47) the following estimates are valid

$$|v(x)| \leq A_1 \quad \text{for } x \in \overline{\Omega}, \quad |\nabla v(x)| \leq A_2 \quad \text{for } x \in \overline{\Omega}^-$$

$$\text{and } |\nabla v(x)| \leq A_3 \quad \text{for } x \in \overline{\Omega}^+.$$

Consider w_ε^j as a solution of the problem

$$(48) \quad \begin{aligned} \Delta w_\varepsilon^j &= 0 \quad \text{in } T_{b\varepsilon}^j \setminus T_{aa\varepsilon}^j, \quad w_\varepsilon^j = 0 \quad \text{on } \partial T_{aa\varepsilon}^j, \\ w_\varepsilon^j &= 1 \quad \text{on } \partial T_{b\varepsilon}^j, \quad \varepsilon/2 > b\varepsilon > aa\varepsilon, \quad b = \text{const} > 0, \end{aligned}$$

where T_s^j is the ball with the center P_j and with the radius s . It is easy to see that the solution of the problem (48) has the form:

$$w_\varepsilon^j = \frac{r^{2-n} - (a_\varepsilon a)^{2-n}}{(b\varepsilon)^{2-n} - (a_\varepsilon a)^{2-n}}, \quad \text{if } n \geq 3,$$

$$w_\varepsilon^j = \left(\ln \frac{r}{a_\varepsilon a} \right) \left(\ln \frac{b\varepsilon}{a_\varepsilon a} \right)^{-1}, \quad \text{if } n = 2,$$

where $r = |x - P_j|$. We introduce the function w_ε such that $w_\varepsilon = w_\varepsilon^j$ for $T_{b\varepsilon}^j \setminus T_{aa\varepsilon}^j$, $w_\varepsilon = 0$ for $x \in T_{aa\varepsilon}^j$, $j = 1, \dots, N(\varepsilon)$, $w_\varepsilon = 1$ for $x \in \mathbb{R}^n \setminus \sum_{j=1}^{N(\varepsilon)} T_{b\varepsilon}^j$.

Using the Green formula for functions w_ε and $\varphi \in H_1(\Omega, G_\varepsilon \cup \partial\Omega)$ we obtain

$$(49) \quad \begin{aligned} \sum_{j=1}^n \int_{\Omega} \frac{\partial w_\varepsilon}{\partial x_j} \frac{\partial \varphi}{\partial x_j} dx &= \sum_{i=1}^{N(\varepsilon)} \int_{T_{b\varepsilon}^i \setminus T_{aa\varepsilon}^i} (\nabla w_\varepsilon^i, \nabla \varphi) dx \\ &= - \sum_{i=1}^{N(\varepsilon)} \int_{T_{b\varepsilon}^i \setminus T_{aa\varepsilon}^i} (\Delta w_\varepsilon^i) \varphi dx + \sum_{i=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^i} \frac{\partial w_\varepsilon^i}{\partial \nu} \varphi ds \\ &= \sum_{i=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^i} \frac{\partial w_\varepsilon^i}{\partial \nu} \varphi ds. \end{aligned}$$

Since

$$\frac{\partial w_\varepsilon^i}{\partial \nu} = \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1 - (a^{-1}b)^{2-n}(a_\varepsilon \varepsilon^{-1})^{n-2}} \quad \text{for } x \in \partial T_{b\varepsilon}^j, \quad n \geq 3,$$

$$\frac{\partial w_\varepsilon^j}{\partial \nu} = - \frac{1}{b\varepsilon \ln a_\varepsilon} \frac{1}{\left(1 - \frac{\ln b\varepsilon}{\ln a_\varepsilon} + \frac{\ln a}{\ln a_\varepsilon} \right)} \quad \text{for } x \in \partial T_{b\varepsilon}^j, \quad n = 2,$$

we derive from (49) that

$$(50) \quad \int_{\Omega} (\nabla w_\varepsilon, \nabla \varphi) dx = \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1 - (a^{-1}b)^{2-n}(a_\varepsilon \varepsilon^{-1})^{n-2}} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} \varphi ds,$$

if $n \geq 3$, and

$$(51) \quad \int_{\Omega} (\nabla w_{\varepsilon}, \nabla \varphi) dx = \frac{1}{b\varepsilon |\ln a_{\varepsilon}|} \frac{1}{\left(1 - \frac{\ln b\varepsilon}{\ln a_{\varepsilon}} + \frac{\ln a}{\ln a_{\varepsilon}}\right)} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} \varphi ds,$$

if $n = 2$.

Applying Lemma 1 proved in Section 4 we get

$$(52) \quad \left| \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} \varphi ds - b^{n-1} \omega(n) \int_{\gamma} \varphi dx' \right| \leq K_{46} \sqrt{\varepsilon} \|\varphi\|_{H_1(\Omega)}.$$

Let us note that for a solution of problem (47) we have the integral identity

$$(53) \quad \int_{\Omega} (\nabla v, \nabla \varphi) dx + (n-2) a^{n-2} \omega(n) C_1 \int_{\gamma} v \varphi dx' = \int_{\Omega} f \varphi dx,$$

if $n \geq 3$, and

$$(54) \quad \int_{\Omega} (\nabla v, \nabla \varphi) dx + 2\pi C_2 \int_{\gamma} v \varphi dx_2 = \int_{\Omega} f \varphi dx,$$

in $n = 2$, $\varphi \in H_1(\Omega, \partial\Omega)$. From (50), (53) for $n \geq 3$ it follows that

$$(55) \quad \begin{aligned} \int_{\Omega} (\nabla(\tilde{u}_{\varepsilon} - w_{\varepsilon}v), \nabla \tilde{\varphi}) dx &= \int_{\Omega} (\nabla \tilde{u}_{\varepsilon}, \nabla \tilde{\varphi}) dx - \int_{\Omega} (\nabla(w_{\varepsilon}v), \nabla \tilde{\varphi}) dx \\ &= \int_{\Omega_{\varepsilon}} f \tilde{\varphi} dx - \int_{\Omega} (\nabla w_{\varepsilon}, v \nabla \tilde{\varphi}) dx - \int_{\Omega} w_{\varepsilon} (\nabla v, \nabla \tilde{\varphi}) dx \\ &= \int_{\Omega} f \tilde{\varphi} dx - \int_{\Omega} (\nabla w_{\varepsilon}, \nabla(v \tilde{\varphi})) dx + \int_{\Omega} (\nabla w_{\varepsilon}, \nabla v) \tilde{\varphi} dx \\ &\quad - \int_{\Omega} (\nabla v, \nabla \tilde{\varphi}) dx + \int_{\Omega} (1 - w_{\varepsilon}) (\nabla v, \nabla \tilde{\varphi}) dx \\ &= \int_{\Omega} f \tilde{\varphi} dx - \frac{(n-2)(b\varepsilon)^{1-n} (a_{\varepsilon}a)^{n-2}}{1 - (a^{-1}b)^{2-n} (a_{\varepsilon}\varepsilon^{-1})^{n-2}} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v \tilde{\varphi} ds \\ &\quad - \int_{\Omega} f \tilde{\varphi} dx + (n-2) a^{n-2} \omega(n) C_1 \int_{\gamma} v \tilde{\varphi} dx' \\ &\quad + \int_{\Omega} (1 - w_{\varepsilon}) (\nabla v, \nabla \tilde{\varphi}) dx + \int_{\Omega} (\nabla w_{\varepsilon}, \nabla v) \tilde{\varphi} dx \\ &= \left\{ (n-2) a^{n-2} \omega(n) C_1 \int_{\gamma} v \tilde{\varphi} dx' \right. \\ &\quad \left. - \frac{(n-2)(b\varepsilon)^{1-n} (a_{\varepsilon}a)^{n-2}}{1 - (a^{-1}b)^{2-n} (a_{\varepsilon}\varepsilon^{-1})^{n-2}} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v \tilde{\varphi} ds \right\} \\ &\quad + \int_{\Omega} (1 - w_{\varepsilon}) (\nabla v, \nabla \tilde{\varphi}) dx + \int_{\Omega} (\nabla w_{\varepsilon}, \nabla v) \tilde{\varphi} dx, \end{aligned}$$

where $\tilde{\varphi} = \varphi$ for $x \in \Omega_\varepsilon$ and $\tilde{\varphi} = 0$ for $x \in \Omega \setminus \Omega_\varepsilon$, $\varphi \in H_1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$.

In a similar way we get for $n = 2$

$$(56) \quad \int_{\Omega} (\nabla(\tilde{u}_\varepsilon - w_\varepsilon v), \nabla\tilde{\varphi})dx = \left\{ 2\pi C_2 \int_{\gamma} v\tilde{\varphi}dx_2 - \frac{1}{b\varepsilon|\ln a_\varepsilon|} \frac{1}{\left(1 - \frac{\ln b\varepsilon}{\ln a_\varepsilon} + \frac{\ln a}{\ln a_\varepsilon}\right)} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds \right\} + \int_{\Omega} (1 - w_\varepsilon)(\nabla v, \nabla\tilde{\varphi})dx + \int_{\Omega} (\nabla w_\varepsilon, \nabla v)\tilde{\varphi}dx.$$

Let us estimate two last integrals in the right-hand side of (55) and (56). We denote them by $J_\varepsilon(\varphi)$. Using Lemma 1, we get

$$(57) \quad \left| \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1 - (a^{-1}b)^{2-n}(a_\varepsilon \varepsilon^{-1})^{n-2}} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds - (n-2)a^{n-2}\omega(n)C_1 \int_{\gamma} v\tilde{\varphi}dx' \right| \leq K_{47} \left\{ \left| \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds - b^{n-1}\omega(n) \int_{\gamma} \tilde{\varphi}vdx' \right| + \int_{\gamma} |v\tilde{\varphi}|dx' |a_\varepsilon^{n-2}\varepsilon^{1-n} - C_1| + \alpha(\varepsilon) \int_{\gamma} v\tilde{\varphi}dx' \right\} \leq K_{48} \{ \sqrt{\varepsilon} + |a_\varepsilon^{n-2}\varepsilon^{1-n} - C_1| \} \|\tilde{\varphi}\|_{H_1(\Omega)},$$

for $n \geq 3$ and $\alpha(\varepsilon) \leq c_1\varepsilon$, and

$$(58) \quad \left| 2\pi C_2 \int_{\gamma} v\tilde{\varphi}dx_2 - \frac{1}{b\varepsilon|\ln a_\varepsilon|} \frac{1}{(1 - \alpha(\varepsilon))} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds \right| \leq K_{49} \{ \sqrt{\varepsilon} + |(\varepsilon|\ln a_\varepsilon|)^{-1} - C_2| \} \|\tilde{\varphi}\|_{H_1(\Omega)},$$

for $n = 2$, $\alpha(\varepsilon) = \frac{\ln b\varepsilon}{\ln a_\varepsilon} - \frac{\ln a}{\ln a_\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

For $J_\varepsilon(\tilde{\varphi})$ we have estimates:

$$(59) \quad \left| J_\varepsilon(\tilde{\varphi}) \right| \leq K_{50} \{ \|w_\varepsilon - 1\|_{L_2(\Omega)} \|\tilde{\varphi}\|_{H_1(\Omega)} + \sqrt{\varepsilon} \|\nabla w_\varepsilon\|_{L_2(\Omega)} \|\tilde{\varphi}\|_{L_2(\gamma)} \} \leq K_{51} \sqrt{\varepsilon} \|\tilde{\varphi}\|_{H_1(\Omega)}.$$

Taking $\tilde{\varphi} = \tilde{u}_\varepsilon - w_\varepsilon v$ in (55) and (56), we obtain

$$(60) \quad \|u_\varepsilon - w_\varepsilon v\|_{H_1(\Omega_\varepsilon)} \leq K_{52} \{ \sqrt{\varepsilon} + |C_1 - a_\varepsilon^{n-2}\varepsilon^{1-n}| \},$$

if $n \geq 3$, and

$$(61) \quad \|u_\varepsilon - w_\varepsilon v\|_{H_1(\Omega_\varepsilon)} \leq K_{53} \{\sqrt{\varepsilon} + |C_2 - (\varepsilon |\ln a_\varepsilon|)^{-1}|\},$$

if $n = 2$.

From (60) and (61) we derive

$$(62) \quad \|u_\varepsilon - v\|_{L_2(\Omega_\varepsilon)} \leq K_{54} \{\sqrt{\varepsilon} + |C_1 - a_\varepsilon^{n-2} \varepsilon^{1-n}|\},$$

if $n \geq 3$, and

$$(63) \quad \|u_\varepsilon - v\|_{L_2(\Omega_\varepsilon)} \leq K_{55} \{\sqrt{\varepsilon} + |C_2 - (\varepsilon |\ln a_\varepsilon|)^{-1}|\},$$

if $n = 2$.

Thus we have proved the following theorem.

THEOREM 5. *Let conditions (46) be satisfied. Assume that Ω_ε has the form (22), u_ε is a solution of the problem (3), (6), v is a solution of the problem (47). Then estimates (62), (63) hold.*

7. Consider the problem (3), (6) under conditions:

$$(64) \quad a_\varepsilon^{2-n} \varepsilon^{n-1} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, n \geq 3,$$

$$(65) \quad \varepsilon |\ln a_\varepsilon| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, n = 2.$$

We assume that $a_\varepsilon^j = a_\varepsilon$, the domain Ω_ε has the form (22). In this case the limit problem for the problem (3), (6) is:

$$(66) \quad -\Delta v^- = f \text{ in } \Omega^-, v^- = 0 \text{ on } \partial\Omega^-,$$

$$(67) \quad -\Delta v^+ = f \text{ in } \Omega^+, v^+ = 0 \text{ on } \partial\Omega^+.$$

We will use the following Lemma 2, proved in [6].

LEMMA 2. *Let $u \in H_1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$. Then*

$$(68) \quad \|u\|_{L_2(\Pi_\varepsilon)} \leq K_{56} a_\varepsilon^{(2-n)/2} \varepsilon^{n/2} \|\nabla u\|_{L_2(\Pi_\varepsilon)},$$

if $n \geq 3$, and

$$(69) \quad \|u\|_{L_2(\Pi_\varepsilon)} \leq K_{57} \varepsilon \sqrt{|\ln a_\varepsilon|} \|\nabla u\|_{L_2(\Pi_\varepsilon)},$$

if $n = 2$.

From the integral identity for the problem (3), (6) we derive

$$(70) \quad \|u_\varepsilon\|_{H_1(\Omega_\varepsilon)} \leq K_{58}.$$

Therefore, from (68)-(70) we get

$$(71) \quad \frac{1}{|\Pi_\varepsilon|} \int_{\Pi_\varepsilon} \tilde{u}_\varepsilon^2 dx \leq K_{59} a_\varepsilon^{2-n} \varepsilon^{n-1},$$

if $n \geq 3$, and

$$(72) \quad \frac{1}{|\Pi_\varepsilon|} \int_{\Pi_\varepsilon} \tilde{u}_\varepsilon^2 dx \leq K_{60} \varepsilon |\ln a_\varepsilon|,$$

if $n = 2$. Here $\tilde{u}_\varepsilon = u_\varepsilon$ for $x \in \Omega_\varepsilon$ and $\tilde{u}_\varepsilon = 0$ for $x \in \Omega \setminus \Omega_\varepsilon$.

It is easy to prove that

$$(73) \quad \int_{\gamma_\varepsilon^\pm} u^2(x_1, x') dx' \leq K_{61} \left(\frac{1}{|\Pi_\varepsilon|} \int_{\Pi_\varepsilon} u^2(x) dx + \varepsilon \|\nabla u\|_{L_2(\Pi_\varepsilon)}^2 \right),$$

From this inequality and (71), (72), (73) it follows that

$$(74) \quad \|\tilde{u}_\varepsilon\|_{L_2(\gamma_\varepsilon^\pm)}^2 \leq K_{62} \{\varepsilon + a_\varepsilon^{2-n} \varepsilon^{n-1}\}$$

for $n \geq 3$, and

$$(75) \quad \|\tilde{u}_\varepsilon\|_{L_2(\gamma_\varepsilon^\pm)}^2 \leq K_{63} \{\varepsilon + \varepsilon |\ln a_\varepsilon|\}$$

for $n = 2$.

We set $w_\varepsilon^- = u_\varepsilon - v^-$, $w_\varepsilon^+ = u_\varepsilon - v^+$, $\Omega_\varepsilon^+ = \Omega \cap \{x : x_1 > \varepsilon/2\}$, $\Omega_\varepsilon^- = \Omega_\varepsilon \cap \{x : x_1 < -\varepsilon/2\}$. The functions w_ε^\pm are weak solutions of the problems:

$$\Delta w_\varepsilon^\pm = 0 \quad \text{in } \Omega_\varepsilon^\pm, \quad w_\varepsilon^\pm = u_\varepsilon - v^\pm \quad \text{on } \gamma_\varepsilon^\pm, \quad w_\varepsilon^\pm = 0 \quad \text{on } \partial\Omega_\varepsilon^\pm \setminus \gamma_\varepsilon^\pm.$$

From the inequality (see [7], ch 4, sec. 1)

$$\|w_\varepsilon^\pm\|_{L_2(\Omega_\varepsilon^\pm)} \leq K_{64} \{\|u_\varepsilon\|_{L_2(\gamma_\varepsilon^\pm)} + \|v^\pm\|_{L_2(\gamma_\varepsilon^\pm)}\},$$

and (71), (72) we have estimates

$$(76) \quad \|\tilde{u}_\varepsilon - v^\pm\|_{L_2(\Omega^\pm)}^2 \leq K_{65} a_\varepsilon^{2-n} \varepsilon^{n-1},$$

for $n \geq 3$, and

$$(77) \quad \|\tilde{u}_\varepsilon - v^\pm\|_{L_2(\Omega^\pm)}^2 \leq K_{66} \varepsilon |\ln a_\varepsilon|,$$

for $n = 2$.

From the estimates (76), (77) we get the following theorem.

THEOREM 6. *Let Ω_ε be a domain of the form (22), $a_\varepsilon^{2-n} \varepsilon^{n-1} \rightarrow 0$ as $\varepsilon \rightarrow 0$, $n \geq 3$, and $\varepsilon |\ln a_\varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0$, $n = 2$, $a_\varepsilon^j = a_\varepsilon$. Let u_ε be a solution of the Dirichlet problem (3), (6), v^\pm be solutions of problems (66), (67). Then estimates (76), (77) are valid.*

The Dirichlet problems in domains with holes were considered in [8].

8. Using Theorems 1-6, proved above, we can get theorems about the spectrum of corresponding eigenvalue problems. We apply here the theorem about the spectrum of a sequence of singularly perturbed operators proved in [5].

THEOREM 7. *Consider the eigenvalue problem*

$$(78) \quad \Delta u_\varepsilon^k + \lambda_\varepsilon^k u_\varepsilon^k = 0 \quad \text{in } \Omega_\varepsilon,$$

$$(79) \quad \frac{\partial u_\varepsilon^k}{\partial \nu} = 0 \quad \text{on } S_\varepsilon, \quad u_\varepsilon^k = 0 \quad \text{on } \Gamma_\varepsilon,$$

and the eigenvalue problem

$$(80) \quad \Delta v^k + \lambda^k v^k = 0 \quad \text{in } \Omega, \quad v^k = 0 \quad \text{on } \partial\Omega.$$

Assume that $S'_\varepsilon \cap \partial\Omega = \emptyset$. Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_1 \left(\max_j a_\varepsilon^j \right)^n \varepsilon^{1-n}.$$

If $S'_\varepsilon \cap \partial\Omega \neq \emptyset$ and $M(\varepsilon) \leq d_1 \varepsilon^{2-n}$, then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_2 \left(\max_j a_\varepsilon^j \right)^{n-1} \varepsilon^{2-n}.$$

where $\lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \dots$ is a nondecreasing sequence of eigenvalues to problem (78), (79) and $\lambda^1 \leq \lambda^2 \leq \dots$ is a nondecreasing sequence of eigenvalues to problem (80) and every eigenvalue is counted as many times as its multiplicity.

Here and in what follows constants C_j do not depend on ε . This Theorem is a consequence of Theorem 1.

We note that in Theorem 1, from which Theorem 7 follows, it is not necessary to assume that P_j belongs to γ . We use only the fact, that the number $N(\varepsilon)$ of P_j is such that $N(\varepsilon) \leq d_0 \varepsilon^{1-n}$ and $a_\varepsilon^j \leq C_0 \varepsilon$.

THEOREM 8. *Consider the eigenvalue problem*

$$(81) \quad \Delta u_\varepsilon^k + \lambda_\varepsilon^k u_\varepsilon^k = 0 \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u_\varepsilon^k}{\partial \nu} + \beta(x) u_\varepsilon^k = 0 \quad \text{on } S_\varepsilon, \quad u_\varepsilon^k = 0 \quad \text{on } \Gamma_\varepsilon,$$

and the eigenvalue problem

$$(82) \quad \Delta v^k + \lambda^k v^k = 0 \quad \text{in } \Omega, \quad v^k = 0 \quad \text{on } \partial\Omega.$$

Assume that $|\partial G^j(a_\varepsilon^j)| \leq C_3 (a_\varepsilon^j)^{n-1}$, $\max_j a_\varepsilon^j \varepsilon^{-1} \rightarrow 0$ as $\varepsilon \rightarrow 0$, $\beta(x) \geq \beta_0 = \text{const} > 0$. Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_4 \left(\left(\max_j a_\varepsilon^j \right) \varepsilon^{-1} \right)^{n-1}.$$

Here λ_ε^k and λ^k are ordering in the same way as in Theorem 7.

Now let us consider the case when Ω_ε has the structure, given by (22), and the eigenvalues of the problem (81).

THEOREM 9 (Critical case). *Let u_ε^k be an eigenfunction of the problem (81) and u^k be an eigenfunction of the problem*

$$\Delta v^k + \lambda^k v^k = 0 \text{ in } \Omega \setminus \gamma, v^k = 0 \text{ on } \partial\Omega,$$

$$[v^k] \Big|_\gamma = 0, \left[\frac{\partial v^k}{\partial x_1} \right] \Big|_\gamma = \mu v^k,$$

where $\beta(x) = \beta_0 = \text{const} > 0$, $\mu = \beta_0 c_0^{n-1} |\partial \overline{G}_0|$, $a_\varepsilon \varepsilon^{-1} \rightarrow c_0 = \text{const} > 0$ as $\varepsilon \rightarrow 0$, $|\overline{G}_\varepsilon \cap \partial\Omega| \leq C_4 \varepsilon$. Let Ω_ε have the structure, defined by (22). Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_5(\varepsilon + |a_\varepsilon \varepsilon^{-1} - c_0|^2).$$

This Theorem is a consequence of Theorem 3.

THEOREM 10. *Consider the Dirichlet eigenvalue problem*

$$(83) \quad \Delta u_\varepsilon^k + \lambda_\varepsilon^k u_\varepsilon^k = 0 \text{ in } \Omega_\varepsilon, u_\varepsilon^k = 0 \text{ on } \partial\Omega_\varepsilon,$$

and the Dirichlet eigenvalue problem

$$\Delta v^k + \lambda^k v^k = 0 \text{ in } \Omega, v^k = 0 \text{ on } \partial\Omega.$$

Assume that

$$\max_j (a_\varepsilon^j)^{n-2} \varepsilon^{1-n} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ for } n \geq 3,$$

$$\max_j (|\ln a_\varepsilon| \varepsilon)^{-1} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ for } n = 2.$$

Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_8 (\max_j a_\varepsilon)^{n-2} \varepsilon^{1-n} \text{ for } n \geq 3,$$

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_9 (\max_j |\ln a_\varepsilon| \varepsilon)^{-1} \text{ for } n = 2.$$

This Theorem is a consequence of Theorem 4.

THEOREM 11 (Critical case). *Let Ω_ε be given by (22), $G_0 = \{x : |x| < a, 0 < a < 1/2\}$, $a_\varepsilon^j = a_\varepsilon$, $Q = \{x : -1/2 < x_i < 1/2, i = 1, \dots, n\}$. Assume that $a_\varepsilon^{n-2} \varepsilon^{1-n} \rightarrow c_0 = \text{const} > 0$ as $\varepsilon \rightarrow 0$ for $n \geq 3$ and $(\varepsilon |\ln a_\varepsilon|)^{-1} \rightarrow c_1 = \text{const} > 0$ as $\varepsilon \rightarrow 0$ for $n = 2$. Let u_ε^k be a solution of the eigenvalue problem (83) and v^k be a solution of the eigenvalue problem*

$$\Delta v^k + \lambda^k v^k = 0 \text{ in } \Omega,$$

$$[v^k] \Big|_\gamma = 0, \left[\frac{\partial v^k}{\partial x_1} \right] \Big|_\gamma = \mu_1 v^k \Big|_\gamma, v^k = 0 \text{ on } \partial\Omega,$$

where $\mu_1 = (n - 2)a^{n-2} \omega_n c_0$ for $n \geq 3$ and $\mu_1 = 2\pi c_1$ for $n = 2$.

Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_{10}(\varepsilon + |c_0 - a_\varepsilon^{n-2}\varepsilon^{1-n}|^2) \quad \text{for } n \geq 3,$$

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_{11}\{\varepsilon + |c_1 - (\varepsilon|\ln a_\varepsilon|)^{-1}|^2\} \quad \text{for } n = 2.$$

This theorem is a consequence of Theorem 5.

THEOREM 12. *Let u_ε^k be a solution of the eigenvalue problem (83) and let us consider two eigenvalue problems:*

$$\Delta v_+^k + \lambda_+^k v_+^k = 0 \quad \text{in } \Omega^+ = \Omega \cap \{x : x_1 > 0\}, \quad v_+^k = 0 \quad \text{on } \partial\Omega^+,$$

$$\Delta v_-^k + \lambda_-^k v_-^k = 0 \quad \text{in } \Omega^- = \Omega \cap \{x : x_1 < 0\}, \quad v_-^k = 0 \quad \text{on } \partial\Omega^-.$$

Let $\{\lambda^k\}$ be a sequence, which is the set $\{\lambda_-^k\} \cup \{\lambda_+^k\}$, ordered as a nondecreasing sequence and every eigenvalue is counted as many time as its multiplicity. Assume that $a_\varepsilon^{2-n}\varepsilon^{n-1} \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $n \geq 3$ and $\varepsilon|\ln a_\varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $n = 2$. Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_{12}a_\varepsilon^{2-n}\varepsilon^{n-1} \quad \text{for } n \geq 3,$$

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_{13}\varepsilon|\ln a_\varepsilon| \quad \text{for } n = 2.$$

We note that in Theorems 1-12 the Laplace operator can be substituted by any elliptic second order selfadjoint operator.

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