ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

NICOLAI V. KRYLOV

**Fully nonlinear second order elliptic equations : recent development**

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 25, nº 3-4 (1997), p. 569-595

<http://www.numdam.org/item?id=ASNSP_1997_4_25_3-4_569_0>

© Scuola Normale Superiore, Pisa, 1997, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
*Article numérisé dans le cadre du programme*
*Numérisation de documents anciens mathématiques*
http://www.numdam.org/

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXV (1997), pp. 569-595

# Fully Nonlinear Second Order Elliptic Equations: Recent Development

NICOLAI V. KRYLOV

**Abstract.** A short discussion of the history of the theory of fully nonlinear second-order elliptic equations is presented starting with the beginning of the century. Then an account of the explosion of results during the last decade is given. This explosion is based entirely on a generalization for nondivergence form linear operators of the celebrated De Giorgi result bearing an Hölder continuity. This is an extended version of a 1.5 hour talk at Mathfest, Burlington, Vermont, Aug 6-8, 1995.

**Mathematics Subject Classification (1991):** 35J60.

## 1. – Introduction

It seems Bernstein [8][10] in 1910 was the first to introduce general methods of solving nonlinear elliptic equations. These are: the method of continuity and the method of a priori estimates. He considered equations with two independent variables and showed that for proving the solvability of such equations it suffices to establish a priori estimates for absolute values of the first *two* derivatives of solutions. For Bernstein the equation

$$F(D^2u, Du, u, x) = 0 \quad x \in \Omega, \tag{1.1}$$

where $\Omega$ is a domain in $\mathbb{R}^d$, should be called elliptic if the matrix $(F_{u_{ij}})$ is definite. This definition is quite natural when the equation is linear with respect to second order derivatives, but is not of much help for fully nonlinear equations.

For example, before 1910 in 1903 Minkowski [65][03] proved existence and uniqueness of convex surface with prescribed Gaussian curvature in Euclidean space. He did not prove however that this surface is from class $C^2$.

Analytically, the Minkowski problem involves solving a highly nonlinear partial differential equation of Monge-Ampère type. An example of such an

This work was supported in part by NSF grant DMS-9302516.

equation, for a convex function $u(x) = u(x^1, \dots, x^d)$ defined in a domain in $\mathbb{R}^d$ is the following simplest Monge-Ampère equation

$$\det D^2u(x) = f(x), \tag{1.2}$$

where $f(x)$ is a given function. This equation is *not* elliptic in the sense of Bernstein.

It turns out that (1.2) and similar equations can be well understood even for non differentiable functions, so that one can investigate generalized solutions. This was done by Aleksandrov [1][58] in 1958 and led to some remarkable results for *linear equation* which 20 years later turned out to constitute the basis of the general theory of fully nonlinear elliptic equations. We mean the so-called Aleksandrov maximum principle and Aleksandrov estimates (see [2][61] and [3][63]).

The smoothness of Minkowski's two-dimensional generalized solution was studied by well-known mathematicians, such as Lewy [59][37], Miranda [66][39], Pogorelov [69][52], Nirenberg [67][53], Calabi [19][58], Bakel'man [6][65]. In 1971 that is 68 years after Minkowski's work, Pogorelov in [71][71] finally proved that the solution is indeed from class $C^2$ in multidimensional case. Nevertheless his proofs in [71][71] and [72][75] contained what looked like a vicious circle and only in 1977 Cheng and Yau [21][77] showed how to avoid it.

To prove the existence of solutions of equations like (1.2) by the methods known before 1981 was no easy task. It involved finding a priori estimates for solutions and their derivatives up to the *third* order. Big part of the work is based on differentiating (1.2) three times and on certain extremely cleverly organized manipulations invented by Calabi. After 1981 the approach to fully nonlinear equations changed dramatically. We discuss this in Section 3.

Until 1971 the theory of fully nonlinear elliptic equations only consisted of the theory of Monge-Ampère equations. In 1971 appeared the so-called Bellman equations which stemmed from the theory of controlled diffusion processes. The typical representative of Bellman's equations is the following:

$$\sup_{\beta\in\mathcal{A}}\{a^{ij}(\beta, u_x, u, x)u_{x^ix^j} + b(\beta, u_x, u, x)\} = 0, \tag{1.3}$$

where $\mathcal{A}$ is a set and $(a^{ij}) = (a^{ij})^* \geq 0$.

To non specialists equations in form (1.3) might look artificial. Indeed, equations which arise in other fields of mathematics look different. For example, in many natural geometrical applications there appear equations like the Monge-Ampère equations and prescribed curvature equations

$$\det(u_{xx}) = f(u_x, u, x), \tag{1.4}$$

or more general equations

$$H_m[u] = f(u_x, u, x), \tag{1.5}$$

where $H_m$ denotes the $m$-th symmetric polynomial of the curvature matrix of the graph of $u$. Equations of the form of the complex Monge-Ampère equation:

$$\det(u_{z\bar{z}}) = f(u_z, u, z), \tag{1.6}$$

are also very popular in the literature. These equations arise in complex geometry and complex analysis. The following equation

$$\operatorname{Im}\det(I + iD^2u) = 0, \tag{1.7}$$

where $I$ is the $d \times d$ unit matrix, comes from the theory of calibrated geometries (see Caffarelli, Nirenberg and Spruck [16][85]) in connection with absolutely volume-minimizing submanifolds of $\mathbb{R}^{2d}$. Its particular case (when $d = 3$)

$$\det D^2u = \Delta u + f$$

was considered earlier by Pogorelov [70][60] in two dimensions and by the author [42][83]. It is worth noting that although equations (1.4)-(1.7) look different from the Bellman equation (1.3), nevertheless each of them is equivalent to a corresponding Bellman equation of type (1.3). With regard to (1.4) and (1.6) this fact was known in 1971, which made the theory of Bellman's equations very attractive. Really general theory of fully nonlinear elliptic equations emerged from the theory of Bellman's equations. We will see later that even now the most general theory of fully nonlinear elliptic equations, in fact, reduces to the theory of Bellman's equations.

At early stages of the theory of Bellman's equations the only available methods were those of the theory of probability. It is remarkable that results obtained by these methods are sharp in many situations, and even now some of them, which admit purely analytic formulation, are only obtained by probabilistic means. The reader can make an acquaintance with the corresponding results starting with the book by Fleming and Sooner [27][93] and with references therein to which we only add Pragarauskas [73][82] and Krylov [46][89].

By the way for one of discontinuous controlled processes considered in [73][82] Bellman's equation takes the following form:

$$\sup_{h>0} \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - u(x) = f(x).$$

Until 1982 the probabilistic methods were the most powerful in the general theory of fully nonlinear elliptic equations. The situation changed dramatically in 1982 when Evans [25][82] and the author [40][82] proved the solvability in $C^{2+\alpha}$ of a broad class of Bellman's elliptic and parabolic equations. The proofs were based on the theory of *linear* equations and, in particular, on the fact that one can estimate the Hölder constant for solutions of linear equations with *measurable* coefficients. The latter fact was previously established in 1979 by Safonov and the author [52][79] (see also [53][80] and a beautiful exposition by

Safonov for elliptic case in [75]$^{80}$), as mentioned before on the sole basis of Aleksandrov's estimates. Remarkably enough, although the proofs in [53]$^{80}$ and [75]$^{80}$ are written in PDE terms, all underlying ideas are probabilistic, and perhaps this was the reason why people from PDEs did not succeed in obtaining the estimate before. After this the general PDE theory of fully nonlinear elliptic equations started up, and below we will give a report on major developments of this theory.

The article is organized as follows. In Section 2 we give a general notion of fully nonlinear elliptic equation and some existence theorems. Section 3 is devoted to results bearing on the *general* theory of fully nonlinear uniformly elliptic equations and Section 4 contains a discussion of results for the *general* theory of fully nonlinear degenerate elliptic equations. In Section 5 we speak about equations related to the Monge-Ampère equations. Section 6 contains a new (and probably the first) result on the rate of convergence of numerical approximations for fully nonlinear degenerate elliptic equations. We have already mentioned above that the modern theory of fully nonlinear elliptic equations is based entirely on some deep results from linear theory. Also many new brilliant ideas and techniques appeared. In Section 7 we present one of them which is Safonov's proof of the Hölder-Korn-Lichtenstein-Giraud estimate for the Laplacian. This proof was designed for nonlinear equations and turned out to be shorter and easier than usual ones even for the simplest linear equation. In my opinion his proof should be part of general mathematical education. Finally, not as brilliant and not a very popular technical idea is presented in Section 8. Exploiting this idea allowed the author to get some very general results on fully nonlinear degenerate elliptic equations. Also the idea is of a general character and might be of interest to mathematicians from other areas.

It is to be said that the literature on fully nonlinear elliptic equations is really immense, we present here a report on the only part of it which is close to interests of the author. In particular, we do not discuss concrete applications in which equations we discuss arose.

## 2. – A general notion of fully nonlinear elliptic equation and examples

Conceivably, the first question which arises when a theory starts is: what is the main object of investigation? Interestingly enough this was not the first question addressed in the case of the theory of fully nonlinear elliptic equations. The reason for this is that there were enough old problems regarding fully nonlinear equations which came up earlier and they were to be solved in the first place.

Now when the general theory is rather well developed, one may think how to make the field of its applications as wide as possible and the number of people who can use it as large as possible.

We have the following situation. From the one hand, a huge variety of results is available in the theory. On the other hand however, it turns out that if an inexperienced reader meets a fully nonlinear second order partial differential equation in his investigations and tries to get any information concerning its solvability from the literature, then almost certainly he fails to find what he needs, unless he considers an equation that is exactly one which had already been treated. The point is that in the general theory we treat only equations which satisfy certain conditions and while considering examples, we show how to *transform* the equations in these examples to other equations to which the theory is applicable. Therefore, from the point of view of applications the main question is how to describe in simple terms the most general situation when one can make an appropriate transformation. In other terms, one needs a general notion of fully nonlinear elliptic equation.

Naturally, the type of equation should be defined only by the way of dependence of $F$ on $D^2u$, that is we call our equation (1.1) elliptic if for any $p \in \mathbb{R}^d, y \in \Omega$ and $z \in \mathbb{R}$ the following equation in $\Omega$ is elliptic: $F(D^2u(x), p, z, y) = 0$. Therefore, we have to concentrate on the case when $F$ depends only on the matrix of second order derivatives of $u$, in other words, we have to consider the equation

$$F(D^2u(x)) = 0 \quad x \in \Omega \,. \tag{2.8}$$

We assume of course that

$$\Gamma := \{(v_{ij}) : v_{ij} = v_{ji}, i, j = 1, \dots, d, F(v_{ij}) = 0\} \neq \emptyset \,.$$

Usually in the literature on nonlinear elliptic equations (see, for instance, [22][62], [29][83], [14][84], [15][85], [44][85]) one accepts the definition by Bernstein and equation (2.8) is called elliptic if the matrix $(F_{u_{ij}})$ is nonnegative (or nonpositive) for *all arguments*. As we have noticed above, this excludes at once even the simplest Monge-Ampère equation since for $F(u_{ij}) := \det(u_{ij})$ the matrix $(F_{u_{ij}})$ is definite if and only if the same is true for $(u_{ij})$.

An attempt to give a better definition is made in [6][65] where the equation is called elliptic *on a given solution* $u$ if at any point $x \in \Omega$ the matrix with entries $F_{u_{ij}}(D^2u(x))$ is nonnegative (or nonpositive). After that equation (2.8) is called elliptic in a given class $\mathcal{C}$ of functions (say, $\mathcal{C}$ is $C^2(\Omega)$ or the set of all smooth convex functions) if it is elliptic on any (if there is any) solution $u \in \mathcal{C}$. It is worth noting that only in rare cases we can take $\mathcal{C} = C^2(\Omega)$ in this definition. For instance, as we have seen above, this is not possible for the Monge-Ampère equation. However, the Monge-Ampère equation is elliptic on convex functions. But how to find an appropriate $\mathcal{C}$ for the following equations:

$$2u^2_{x^1x^1} + 5u_{x^1x^1}u_{x^2x^2} + 2u^2_{x^2x^2} = 1 \,, \tag{2.9}$$

$$2u^2_{x^1x^1} + 5u_{x^1x^1}u_{x^2x^2} + 2u^2_{x^2x^2} = -1 \,, \tag{2.10}$$

$$u_{x^1x^1}u_{x^2x^2} = 0 \,. \tag{2.11}$$

If we are only interested in definiteness of $(F_{u_{ij}})$, then as easy to check, equations (2.9), (2.10) are both elliptic in the same class of functions $\mathcal{C}$ defined as the set of all functions for which $\Delta u \geq 1/\sqrt{18}$. It turns out that in general the Dirichlet problem for (2.9) is solvable in this class and for (2.10) is not, and moreover the behavior of solutions of (2.10) is such that this equation should not be called elliptic at all.

Other flaws of the definition are also related to the fact that the objective is not only to give a definition of nonlinear elliptic equation, but to find such a definition which could do the job. For instance, usually we are interested in proving uniqueness, and usually we prove it via the maximum principle. In other words, if we are given two solutions $u_1, u_2$ of equation (2.8), then by proceeding as usual (cf., for instance, [22][62] Ch. 4, Section 6.2) for $v = u_1 - u_2$ we write

$$0 = F(D^2u_1(x)) - F(D^2u_2(x)) = a^{ij}(x)v_{x^ix^j}(x),$$

where

$$a^{ij} = \int_0^1 F_{u_{ij}}(tD^2u_1 + (1-t)D^2u_2)\,dt,$$

and we expect the matrix $a = (a^{ij})$ to be positive or negative. If we assume the above definition from [6][65], then we know that the matrices $(F_{u_{ij}})$ are say, positive *on* $u_1$ *and on* $u_2$, but generally speaking, $tu_1 + (1-t)u_2$ is not a solution and we do not know anything about definiteness of $a$. Actually, it may even happen that for one function $F$ the matrix $a$ is always positive, and for another function $F$, defining an equation *equivalent* to the initial equation (2.8), the corresponding matrix $a$ is neither positive nor negative. The point is that we can arbitrarily modify the function $F$ outside the set $\Gamma$, the only set where some properties of $F$ are given so far. By the way, this possibility of modifying nonlinear equations is the main reason for the radical difference between linear and nonlinear equations, since for the linear case the set $\Gamma$ is a hyperplane in the linear space

$$\mathbb{S}^d = \mathbb{R}^k \cap \{(u_{ij}) : u_{ij} = u_{ji},\ i, j = 1, \dots, d\},$$

where $k = d^2$, and there are not so many ways to represent a hyperplane as null set of a *linear* function.

One way to overcome the last difficulty is to accept the notion of elliptic convexity of $F$ from [6][65], that is to consider only $F$ such that for any two solutions (from the class $\mathcal{C}$) the matrix $a$ is positive. In this system of notions, given an equation, to decide if it is a "legal" elliptic equation, we first should guess in what class of functions we will look for solutions and then to modify (if it is possible at all) the function $F$, without changing the equation, in order to replace it with an elliptically convex $F$. For the Monge-Ampère equations appropriate modifications are

$$(\det D^2u)^{1/d} = f^{1/d}, \quad \log\det D^2u = \log f,$$
$$\min_{a=a^*\geq 0, \operatorname{tr} a=1}\{\operatorname{tr} aD^2u - (f\det a)^{1/d}d\} = 0.$$

Unfortunately, even after this other difficulties still remain. For instance, assume that at the very beginning we know the appropriate class of functions $\mathcal{C}$, and our $F$ is elliptically convex in this class. Assume that we even obtained a priori estimates for solutions of the equation. The question arises how to prove existence theorems.

Usually we introduce a parameter $t \in [0, 1]$ and we try to find functions $F_t$ continuous in $t$ such that $F_1 = F$ and $F_0$ defines an equation for which everything is known. After this we are trying to prove the same a priori estimates for solutions, *belonging to the same class* $\mathcal{C}$, of the equations corresponding to $F_t$ for all $t \in [0, 1]$, and then we apply some topological methods to get the solvability of the equation $F_t(D^2u) = 0$ for $t = 1$ from its solvability for $t = 0$. But on this way, in all interesting cases, we cannot afford to take $F_0$ linear since usually solutions of linear equations have no reasons to belong to $\mathcal{C}$. For instance, for the Monge-Ampère equation $\det D^2u = 1$ in a strictly convex domain $\Omega$ with boundary data on $\partial\Omega$, one of the right classes of solutions is the class of all convex functions. At the same time there is no linear equations for which all solutions with different boundary data are convex.

In a way, this cuts us off the linear theory and raises the obscure problem of finding "model" nonlinear function $F_0$ for any particular $F$. For professionals in the field this problem is not too hard, and many authors prefer to use model equations while treating concrete equations (see, for instance, Bakel'man [6][65], Caffarelli, Nirenberg and Spruck [14][84], Ivochkina [35][89]). But for a "ready-to-use" theory this "cut off" is highly undesirable since applications may advance equations different from those which have already been investigated. However, in the above system of notions we cannot avoid this difficulty unless we can either understand how to modify the method of continuity in the situation when the set $\mathcal{C}_t$ of solutions is evolving with $t$, or we can "hide" the set $\mathcal{C}$ by finding a function $\tilde{F}$ such that *any solution of* (2.8) *of class* $\mathcal{C}$ is a solution to the equation

$$\tilde{F}(D^2u(x)) = 0\,, \tag{2.12}$$

and vice versa, *any* solution of (2.12) is a solution of (2.8) *and belongs* to $\mathcal{C}$. Our definition is based on the latter possibility.

Following [50][95] we shall present a different approach to the notion of nonlinear elliptic equation. We shall give a method to decide if a given nonlinear equation is an elliptic one by *looking only at the equation* without using any information regarding the problem in which this equation appeared. After this we give a notion of admissible solutions of the equation and then we discuss the possibility of rewriting the equation with the help of elliptically convex functions $F$.

The most important concept in our approach is the notion of admissible solutions which shows the right class of functions in which to look for solutions. This notion is based on the notion of *elliptic branches* of the given equation, which turns out to be meaningful even for viscosity solutions of the *first* order nonlinear equations.

It is worth noting that in all cases known from the literature our class of admissible solutions coincides with the known ones. Also, our notion has many common features with similar notions or hypotheses from Caffarelli, Nirenberg and Spruck [16][85], Trudinger [84][90].

Our point of view is based on the observation that every individual equation (2.8) means and means only that for any $x \in \Omega$

$$(D^2 u(x)) \in \Gamma . \tag{2.13}$$

This point of view allows us to concentrate on properties of the set $\Gamma$ rather than occasional properties of numerous functions which define the same set $\Gamma$. Only properties of the set $\Gamma$ define the type of the equation.

Of course, we assume that $F$ is at least a continuous function, what implies that $\Gamma$ is a closed set in the linear space $\mathbb{S}^d$. We also keep the assumption that $\Gamma \neq \emptyset$. Finally, remember that $I$ is the unit $d \times d$ matrix.

DEFINITION 2.1. We say that a nonempty open (in $\mathbb{S}^d$) set $\Theta \neq \mathbb{S}^d$ is a (positive) *elliptic* set if

(a) $\Theta = \bar{\Theta} \setminus \partial(\bar{\Theta})$,

(b) for any $(u_{ij}) \in \partial\Theta$, $\xi \in \mathbb{R}^d$ it holds that $(u_{ij} + \xi^i \xi^j) \in \bar{\Theta}$.

DEFINITION 2.2. We say that equation (2.8) (or, more generally, equation (2.13) with any nonempty closed $\Gamma$) is an *elliptic* equation if there is an elliptic set $\Theta$ such that $\partial\Theta \subset \Gamma$. In this case we call the equation

$$(D^2 u(x)) \in \partial\Theta, \qquad x \in \Omega \tag{2.14}$$

an *elliptic branch of equation* (2.8) (*or* (2.13)) *defined by* $\Theta$.

Nonlinear equations may have many elliptic branches. For instance (2.9) has two and (2.11) has four elliptic branches.

DEFINITION 2.3. We say that an elliptic set $\Theta$ is *quasi nondegenerate* if for any $(u_{ij}) \in \partial\Theta$, $\xi \in \mathbb{R}^d \setminus \{0\}$ we have $(u_{ij} + \xi^i \xi^j) \in \Theta$.

Given a number $\delta > 0$, we call an elliptic set $\Theta$ *$\delta$-nondegenerate* (or *uniformly elliptic*) if for any $w \in \partial\Theta$, $\xi \in \mathbb{R}^d$ we have

$$\operatorname{dist}(w + \xi\xi^*, \partial\Theta) \geq \delta |\xi|^2 .$$

If equation (2.14) is an elliptic branch of (2.8) (or (2.13)) and $\Theta$ is quasi nondegenerate ($\delta$-nondegenerate, uniformly elliptic), we call this branch and equation (2.8) (or (2.13)) itself quasi nondegenerate (respectively, $\delta$-nondegenerate, uniformly elliptic).

Notice that each of two elliptic branches of (2.9) is uniformly elliptic whereas all branches of (2.11) are degenerate.

DEFINITION 2.4. Given an elliptic equation (2.8) (or (2.13)), we say that a function $u$ is an *admissible solution* in $\Omega$ if $u$ is a solution in $\Omega$ of any elliptic branch of the equation (the branch should be the same in the whole of $\Omega$).

Note, for instance, that $u(x, y) = x^2 - y^2$ is not an admissible solution of the elliptic equation $u_{xx}^2 u_{yy}^2 = 16$.

The following theorem shows that equations written in somewhat unusual form (2.14) are actually the equations which one treats in the general theory of fully nonlinear elliptic equations. Exactly this theorem justifies our definition.

THEOREM 2.1. *Let $\Theta$ be an elliptic set and equation* (2.14) *be elliptic (for instance, be an elliptic branch of* (2.8)). *Define*

$$\tilde{F}(u_{ij}) = \operatorname{dist}((u_{ij}), \partial\Theta) \qquad \text{for} \quad (u_{ij}) \in \bar{\Theta}$$
$$\tilde{F}(u_{ij}) = -\operatorname{dist}((u_{ij}), \partial\Theta) \quad \text{for} \quad (u_{ij}) \in \mathbb{S}^d \setminus \Theta\,.$$

*Then*

$$w \in \partial\Theta \iff \tilde{F}(w) = 0\,,$$

*and in particular, equation* (2.12) *is equivalent to equation* (2.14). *Furthermore, for any $\xi \in \mathbb{R}^d$, $(u_{ij}) \in \mathbb{S}^d$*

$$0 \le \tilde{F}(u_{ij} + \xi^i \xi^j) - \tilde{F}(u_{ij}) \le |\xi|^2\,.$$

*Moreover, the function $\tilde{F}$ is elliptically convex in the sense that for any $(u_{ij})$, $(v_{ij}) \in \mathbb{S}^d$ the difference $\tilde{F}(u_{ij}) - \tilde{F}(v_{ij})$ can be represented as $a^{ij}(u_{ij} - v_{ij})$ with a nonnegative symmetric matrix $a$. Finally, if equation* (2.14) *is $\delta$-nondegenerate, then*

$$\delta|\xi|^2 \le \tilde{F}(u_{ij} + \xi^i \xi^j) - \tilde{F}(u_{ij})\,.$$

An immediate consequence of this theorem and of results from Crandall, Ishii and Lions [23][92] is the following

THEOREM 2.2. *Let $\Omega$ be a bounded smooth domain, and $\phi$ be a continuous function on $\partial\Omega$. Assume that equation* (2.8) *has a uniformly elliptic branch. Then this equation with the boundary condition $u = \phi$ on $\partial\Omega$ has an admissible viscosity solution $u \in C(\bar{\Omega})$. Moreover, every uniformly elliptic branch of* (2.8) *has its own unique admissible viscosity solution $u \in C(\bar{\Omega})$.*

One of the hardest and exciting open problems in the general theory of fully nonlinear elliptic equations concerns smoothness of solutions when neither $\Theta$ nor its complement is convex. If $d \ge 3$, *nothing* is known about boundedness or continuity of second order derivatives of solutions. For example, nothing is known about classical solvability of the Dirichlet problem for the following equation

$$2\Delta u + (u_{x^1 x^1})_+ + \ldots + (u_{x^k x^k})_+ - (u_{x^{k+1} x^{k+1}})_- - \ldots - (u_{x^d x^d})_- = f,$$

where $1 < k < d$. Theorem 2.2 only says that the Dirichlet problem is uniquely solvable in the class of viscosity solutions.

Note that in Theorem 2.1 the function $\tilde{F}$ is obviously concave if $\Theta$ is convex, and it is convex if the complement of $\Theta$ is convex. Graphs of convex or concave functions can be represented as envelopes of their tangent planes. Therefore equation (2.12) can be rewritten in the form of Bellman's equation (1.3). Actually, as easy to see even in general case equation (2.12) is equivalent to a Bellman equation, which contains sup and inf at the same time. If we combine this with results from [44][85], then we obtain

THEOREM 2.3. *Let $\Omega$ be a bounded domain of class $C^{2+\alpha}$ where $\alpha \in (0,1)$, and let $\phi \in C^{2+\alpha}(\mathbb{R}^d)$. Assume that equation* (2.8) *has a uniformly elliptic branch defined by a domain $\Theta$ such that either $\Theta$ or its complement is convex. Then this equation with the boundary condition $u = \phi$ on $\partial\Omega$ has an admissible solution $u \in C^{2+\beta}(\bar{\Omega})$, where $\beta \in (0,1)$. Moreover, the elliptic branch* (2.14) *with the given boundary condition has its own unique admissible solution $u \in C^{2+\beta}(\bar{\Omega})$.*

This theorem applies to equation (2.9) which has two uniformly elliptic branches.

General theory from [46][89] or [51][95] also implies the following theorem which can be restated in an obvious way for the case in which the complement of $\Theta$ is convex.

THEOREM 2.4. *Let $\Theta$ be an open convex set and let equation* (2.14) *be elliptic. Let $C$ be an open cone in $\mathbb{S}^d$ with vertex at the origin, and let $t_0$ be a number. Assume that $t_0 I + \Theta \subset C$, and that for any $w \in C$ we have $tw \in \Theta$ for all $t$ large enough. Let* $\operatorname{tr} w \geq 0$ *for any $w \in C$, and let $\Omega$ be a strictly convex domain of class $C^4$. Then for any $\phi \in C^4(\mathbb{R}^d)$ there is a unique function $u \in C(\bar{\Omega}) \cap C^{1,1}(\bar{\Omega})$ such that $u = \phi$ on $\partial\Omega$ and $D^2 u \in \partial\Theta$ (a.e.) in $\Omega$. If, in addition, equation* (2.14) *is quasi nondegenerate, then $u \in C^{2+\alpha}(\bar{\Omega})$ for an $\alpha \in (0,1)$.*

This is not the most general result from [46][89] or [51][95]. There we do not assume that $\Omega$ is convex or that the weak nondegeneracy condition: $\operatorname{tr} w \geq 0$ for any $w \in C$, is satisfied. The assertion then becomes more complicated. The general theorem applies to each of four elliptic branches of (2.11). For the branch $\max(u_{x^1x^1}, u_{x^2x^2}) = 0$ it says that for any $C^4$ function given on the boundary of a $C^4$ domain $\Omega \subset \mathbb{R}^2$ there exists a unique $C^{1,1}(\bar{\Omega})$ solution with this boundary value provided that the boundary of $\Omega$ is strictly convex (outward) at any point where the tangent line is parallel to one of the coordinate axis. We say more about these results in Section 4.

To conclude this section we mention a general method how to perturb an elliptic equation (2.8) in order to get a uniformly nondegenerate one. It suffices to take $\varepsilon > 0$ and consider

$$F(I\varepsilon\Delta u + D^2 u) = 0\,.$$

This method has been constantly in use in works by the author. Usually, in the literature other methods are applied. In connection with this it is worth noticing that the following "natural" perturbation $\varepsilon\Delta u + \det D^2 u = 1$ of the Monge-Ampère equation $\det D^2 u = 1$ is not elliptic at all unless $\varepsilon \leq 0$ !

## 3. – General convex fully nonlinear uniformly elliptic equations

Since works of Bernstein it was known that to prove the solvability of differential equations it suffices to obtain a priori estimates in appropriate classes of functions for solutions under the hypothesis that the solutions exist. One of ways to obtain estimates in $C^{2+\alpha}$ is to differentiate the equation once thus obtaining a quasilinear equation or a system of equations with respect to first derivatives and hope that there is $C^{1+\alpha}$-estimate for solutions of this new equation. This hope was destroyed by Safonov [77][87]. The example in [77][87] buried the hopes for an "easy" theory of fully nonlinear equations, and in a sense saved the work which has been done before for $F$'s either convex or concave with respect to the matrix $D^2u$ and sufficiently smooth in other variables. Below we only speak about such $F$'s.

In works by Evans [25][82] and the author [40][82] we obtained interior a priori estimates. Then in [41][83] the author published a priori estimates up to the boundary for the Dirichlet problem. After this many authors contributed to the general theory of uniformly elliptic equations. We will only mention works which played the most important role in the development of the general theory. We start with works by Evans [26][83], Trudinger [81][83], Krylov [43][84], Caffarelli, Kohn, Nirenberg and Spruck [15][85], Caffarelli, Nirenberg, Spruck [16][85].

The major results obtained before 1984 are summed up in the books by Gilbarg and Trudinger [29][83] and Krylov [44][85]. After the breakthrough made in the papers by Evans and the author usual technique allowed to develop a theory which contains the general theory of quasilinear elliptic and parabolic equations. In particular, the famous Ladyzhenskaya-Ural'tseva theorem has been generalized for fully nonlinear equations. It is to be mentioned that in all these works the data were assumed to be smoother than in linear theory, so that if equation (1.1) is just $\Delta u + f = 0$ and we want to get its solvability from the general theory of fully nonlinear equations, then we should assume that $f$ is smooth enough.

A major step forward in the general theory has been done by Safonov in 1984 who by using an entirely new technique proved the solvability in $C^{2+\alpha}$ for (1.1) under only natural smoothness assumptions on $F$ (see [76][84] and [78][88]). This is an extremely strong result which is sharp even for linear equations. What is even more surprising, Safonov's proof of $C^{2+\alpha}$ estimates for (1.1) goes the same way when the equation is linear, and in this very case it is much easier and shorter than the known proofs of these estimates for linear equations. We present this proof in Section 7.

The above mentioned works deal with the Hölder space theory of fully nonlinear elliptic equations. The first breakthrough in the Sobolev space theory has been done by Caffarelli [9][89] (also see the book by Caffarelli and Cabré [13][95]) for the elliptic case and Wang [98][92] for the parabolic equations. The works by Safonov, Caffarelli and Wang are remarkable in one more respect-they do not suppose that $F$ is convex or concave in $D^2u$. But in the general case they only show that to prove a priori estimates it suffices to prove the interior $C^2$-estimates for "harmonic" functions.

So far we were talking about the Dirichlet problem. Nonlinear oblique derivative problems were investigated as well. An example of such conditions is the following capillarity boundary condition

$$n \cdot Du = g(x,u)\sqrt{1+|Du|^2}\,.$$

The most relevant references here are Lions and Trudinger [63][85], Liberman and Trudinger [60][86], Anulova and Safonov [5][86]. Again as in the case of the Dirichlet problem, the results for fully nonlinear equations contain those for quasilinear equations.

## 4. – General degenerate fully nonlinear elliptic equation

Fully nonlinear degenerate elliptic equations arise in applications much more often than uniformly elliptic ones, though their investigation in classes $C^{2+\alpha}$ heavily relies on results from the theory of nonlinear *uniformly* elliptic equations.

There is a substantial difference in difficulties which arise when we are dealing with degenerate equations in the whole space or in bounded domains. The theory in the whole space has been developed mostly by probabilistic means and is understood to a very good extent. Many mathematicians contributed to the probabilistic version of the theory, between them are Lions and the author. A PDE counterpart of this theory can be found in [44][85]. Degenerate equations are important not only from the point of view of applications. The following degenerate Monge-Ampère equation

$$\det(D^2u - Iu)(x) = (f)^d_+(x) \quad x \in \mathbb{R}^d\,,$$

in a sense, even plays the main role in the theory of *uniformly* elliptic equations. By the way, as we have explained above the equation

$$\det(D^2u + I\varepsilon\Delta u - Iu)(x) = (f)^d_+(x) \quad x \in \mathbb{R}^d$$

is uniformly elliptic for any $\varepsilon > 0$.

The theory of nonlinear degenerate equations cannot be easier than the theory of linear ones. It is worth mentioning that even in the linear theory there are still very many unsolved problems. Probably the best references concerning the linear theory are Kohn, Nirenberg [38][67] and Oleinik, Radkevich [68][71].

After the book [44][85] the first general results on fully nonlinear degenerate equations in domains were obtained in 1985 by Caffarelli, Nirenberg and Spruck [16][85], where they considered equations like

$$f(\lambda(u)) = \psi\,, \tag{4.15}$$

where $\psi$ is a given function of $x$, $f$ is a symmetric function of $\lambda \in \mathbb{R}^d$ and $\lambda(u) = \lambda(u)(x)$ is a vector of eigenvalues of $D^2u(x)$. Equation (1.7) is a particular case of (4.15). In [18][88] the authors apply their theory to curvatures of the graph of $u$ instead of $\lambda(u)$ and prove the solvability of equations for starshaped surfaces. It is worth noticing that equations with curvatures are much more complicated than containing only $D^2u$. One can say that the latter are linearly fully nonlinear equations whereas the former are quasilinearly fully nonlinear equations. In both cases one deals with equation like (1.3) but in the case of linearly fully nonlinear theory one assumes that $a$ is independent of $Du, u$ and $b$ is linear at least with respect to $Du$.

A general theorem has been announced by the author in 1986 (see [45][86]) and proved in PDE terms in [48][93] and [51][95]. One of its versions is presented above in Section 2. This theorem allows one to consider a very large class of (linearly) fully nonlinear degenerate elliptic equations. Between them are the equations

$$u_t + \Delta u = 0, \quad \det D^2u = 0, \quad \det(u_{z\bar{z}}) = 0\,. \tag{4.16}$$

The first one is the heat equation which is a particular case of degenerate fully nonlinear elliptic equations. The general theorem implies that if $\Omega = \{(t, x) \in \mathbb{R}^d : |t|^2 + |x|^2 < r^2\}$ and $r < (3 - \sqrt{6})d$, then for any $C^4$-function $g$ on the boundary there exists a unique solution $u \in C^{1,1}(\bar{\Omega})$ such that $u = g$ on $\partial\Omega$. This result turned out to be unknown in the theory of linear degenerate equations. Also, in what concerns the restrictions on $r$ and $g$, it is sharp as Weinberger's example from [38][67] shows. For the second equation the theorem says that in any $C^4$ strictly convex domain $\Omega$ and any $C^4$-function $g$ the equation has a unique convex solution of class $C^{1,1}$ such that $u = g$ on $\partial\Omega$. The examples from Caffarelli, Nirenberg and Spruck [17][86] show that all the above conditions are necessary. In particular, an example by Urbas shows that generally the solution is not better than $C^{1,1}$ even for analytic boundary data in a ball. The same is true for the third equation only we have to speak about plurisubharmonic $u$ and strictly pseudoconvex domain $\Omega$. The example by Urbas admits an easy complexification.

One more example of applications of the general theorem from [48][93] and [51][95] is the following equation

$$P_m(D^2u) = \sum_{k=0}^{m-1} (l_k^+)^{m-k+1}(x) P_k(D^2u)\,, \tag{4.17}$$

where $P_k(A)$ are the $k$-th elementary symmetric polynomials of eigenvalues of the matrix $A$.

As in the case of linear theory, the theory of nonlinear degenerate equations is rather far from being well developed. One of very important questions is interior regularity of solutions. The point is that in many examples of equations of Monge-Ampère type even if one does not have solutions regular up to the boundary, the solutions are regular inside domains. The only *general* result known to the author is [49]93 where we prove interior $C^1$ estimates for equations slightly more general than (4.15). The two last equations in (4.16) as well as (4.17) and the equation $\Delta u = f$ present particular cases. Linear counterpart of results in [49]93 appeared earlier in [47]92.

We finish our discussion of general results by mentioning the work by Kutev [58]91 where he gives necessary and sufficient conditions for gradient estimates on the boundary for fully nonlinear degenerate equations.

## 5. – Special equations related to the Monge-Ampère equation

These were the most popular and therefore the best investigated equations. Nevertheless, only understanding of the general theory of nonlinear degenerate elliptic equations led to sharp results regarding even the Monge-Ampère equation let alone the third of the equations in (4.16) or (4.17).

### 5.1. – Global results

Ivochkina [34]83 in 1983 proved classical solvability of the Dirichlet problem in the class of convex functions for the Monge-Ampère equations which include the prescribed Gaussian curvature equation

$$\det D^2 u = H(u, x)(1 + |Du|^2)^{(d+2)/2}\,. \tag{5.18}$$

There are some restrictions on $C^2$-norm of the boundary data in [34]83. Urbas [89]84 extended her results and results from Trudinger and Urbas [86]83 and obtained the most general results concerning global smoothness of solutions.

Caffarelli, Nirenberg and Spruck [14]84 got a result which is, in a sense, more general than that of [34]83. They do not need the right-hand side of the equation be as specific as in [34]83, instead they assume that there exists a subsolution. On the other hand they use the computations by Calabi and therefore have extra smoothness assumptions.

Usually the Monge-Ampère equation is considered in convex domains. This is necessary if one wants to solve it for arbitrary smooth boundary data and $H > 0$. But for some data the equation can still be solvable. Surprisingly enough a sufficient condition for this to happen is existence of a subsolution. Trudinger [84]90 showed this for viscosity solutions.

Guan and Spruck [33]93 considered classical solutions and also established that the convexity of the domain can be replaced with existence of convex subsolution. Some of technical assumptions from this article are removed in Guan [31]94 and Guan and Li [32]94. Guan [30]94 extends this result to general nonlinear degenerate equations.

Sharp necessary conditions on $H$ for classical solvability of the Dirichlet problem for equation (5.18) and more general prescribed curvature equations in *nonconvex* domains are obtained in Trudinger and Urbas [86][83] and Trudinger [85][90]. They give a generalization of the classical result by Serrin related to quasilinear equations.

Ivochkina in [35][89] continues her research of equations like

$$P_m(\kappa(u)) = H(u, x)\,, \tag{5.19}$$

where $\kappa(u)$ is a vector of curvatures of the graph of $u$. Equation (5.19) looks like a particular case of equation (4.15) or equations from [18][88] but considering $\kappa(u)$ instead of $\lambda(u)$ and equations in domains instead of equations on spheres makes a dramatic difference.

For $m = 1$ equation (5.19) is the prescribed curvature equation and for $m = d$ the Monge-Ampère equation. For the mean curvatures of intermediate order the first breakthrough is due to Caffarelli, Nirenberg and Spruck [18][88] and Ivochkina [35][89] for the case of convex domains and zero boundary values. Ivochkina [36][91] extended her approach to embrace general boundary value and domains subject to natural geometric restrictions (also see Trudinger [85][90]). Trudinger [84][90] developed the viscosity solution approach to proving the existence of Lipschitz solutions for more general equations

$$\frac{P_m}{P_l}(\kappa(u)) = H(u, x)\,. \tag{5.20}$$

Lin and Trudinger [61][94] prove that solutions of (5.20) are indeed classical (when $H > 0$, so that in our classification the equation is quasi nondegenerate).

In [15][85] Caffarelli, Kohn, Nirenberg and Spruck not only develop the general theory but also apply it to the Dirichlet problem for complex Monge-Ampère equations of the type

$$\det(u_{z\bar{z}}) = f(Du, u, x)\,.$$

We have already mentioned above article [16][85] of these authors devoted to equation (1.7). In [17][86] they also treat the equation $\det D^2u = 0$.

### 5.2. – Local estimates

Urbas in [92][90] following some ideas of Pogorelov gave examples of degenerate equations with $m$th elementary symmetric polynomials whose solutions are not smooth (because the boundary data are not smooth). In these situations Trudinger in [84][90] showed existence and uniqueness of viscosity solutions.

Korevaar [39][87] proved interior gradient bounds for (5.19) if $H > 0$. Trudinger and Urbas [87][84] (also see [89][84]) proved interior second derivatives estimates for equations of Monge-Ampère type under the assumption that $\Omega$ and the boundary data are $C^{1,1}$. These equations include the equation $\det D^2u = 0$ as a particular case. Urbas [91][88] proved the same result for equations $\det D^2u = f$

with Lipschitz $f > 0$ under weaker assumption that $\Omega$ and the boundary data are of class $C^{1+\alpha}$ with $\alpha > 1 - 2/d$. This is an extremely strong result. By using a different technique Caffarelli [10][90] goes even further on by relaxing some of assumptions in [91][88] and obtaining $W^2_p$-estimates.

In order for solutions of the prescribed Gauss curvature equations to take boundary values some conditions should be satisfied (see Trudinger and Urbas [86][83] and Trudinger [85][90]). Urbas in [91][88], [94][91] and [95][91] shows that even without these conditions the function $u$ or the surface $z = u(x)$ are sufficiently regular. Kutev [57][91] investigated classical solvability of the Monge-Ampère equations in nonuniformly convex domains.

### 5.3. – Other boundary conditions

The second boundary-value problem for Monge-Ampère equations was first posed and solved for $d = 2$ by Pogorelov in 1960 in a generalized sense. In analytical terms the problem reduces to a Monge-Ampère equation with nonlinear boundary condition on $Du$ which reflects the fact that the gradient of $u$ maps a given domain $\Omega$ onto another given domain $\Omega_1$. If we describe the boundary of $\Omega_1$ as null set of a function $h$, then the boundary condition for $u$ is $h(Du) = 0$.

The classical regularity of solutions for $d=2$ is established in Delanoë [24][91]. More general problems were considered in Urbas [97][93] for $d = 2$ and in Caffarelli [11][92] and [12][92] for any $d$. Lions, Trudinger and Urbas [64][86] considered the Neumann boundary value problem for the Monge-Ampère equations.

### 5.4. – Parabolic equations

There are two very interesting articles by Wang and Wang [99][92] and [100][93] where the authors prove existence of generalized and smooth solutions of the following parabolic Monge-Ampère equation

$$u_t \det D^2 u = f$$

in $(0, T) \times \Omega$ where $\Omega$ is a smooth strictly convex domain. If [44][85] we considered the case in which $\Omega$ is a ball. As in the case of elliptic equations the equation

$$(u_t - u) \det(D^2 u - uI) = f$$

considered in the whole space played a crucial role in the general theory of fully nonlinear parabolic equations.

For the equations which arose in investigations of flows of surfaces by various kinds of curvatures see also Andrews [4][93], Gerhardt [28][90], Tso [88][85], Urbas [93][90] and Urbas [96][91].

### 5.5. – Viscosity solutions and approximations

A notion of weak solution, called *viscosity* solution, was introduced for first order equations by Crandall and Lions and extended to degenerate elliptic, second order equations by Lions (see [23][92]). The breakthrough in the issue of uniqueness of viscosity solutions made by Jensen [37][88] led to numerous results providing unique solutions to boundary value problems for nonlinear differential equations without any convexity assumption.

The uniqueness allows one to construct various methods of approximating the solutions. All known in the literature methods of proving the convergence are based on uniqueness.

For example, Barles and Souganidis in [7][91] prove a general convergence theorem assuming the uniqueness. In works by Kuo and Trudinger [54][92] and [55][93] one can find a different approach to approximating solutions based on finite-difference schemes. These and previous works by Kuo and Trudinger are pioneering in providing a large variety of discrete versions of basic results from the theory of linear and fully nonlinear equations.

When the equation is a Bellman's equation, there is a probabilistic interpretation of solutions. One uses this interpretation and the theory of weak convergence of probability measures to prove the convergence of numerical approximations. This direction is investigated mainly by Kushner and his collaborators. Kushner and Dupuis [56][92] give an account of the results obtained in this direction. Pragarauskas [74][83] considered more general situation of integro-differential Bellman equations.

If the Bellman equation is uniformly nondegenerate, then one knows that solutions are of class $C^{2+\alpha}$, and one can estimate the rate of convergence of numerical approximations in the same way as in linear theory. However, in the general case absence of results providing sufficient smoothness of solutions prevents from estimating rates of convergence. It seems that our result in Section 6 is the first result for degenerate fully nonlinear equations.

In the general case without convexity even for uniformly nondegenerate equations from Tru dinger [82][88] (also see Section 5.2 in [44]) we know only that the gradient of solutions is Hölder continuous, and from Trudinger [83][89] that the solutions are twice differentiable only almost everywhere. This information is not sufficient to characterize solutions uniquely and does not provide enough information to estimate the rate of convergence of approximations by known means.

### 5.6. – Infinite dimensions

The notion of viscosity solution allows one to investigate fully nonlinear equations in infinite dimensions. We shall only mention works by Lions [62][89], Tataru [79][92], [80][92] and Cannarsa, Gozzi and Soner [20][95].

## 6. – Convergence of numerical approximations

Let $\mathcal{A}$ be a set, $d \geq 1$ an integer and assume that on $\mathcal{A}$ we are given some functions $a(\beta)$, $b(\beta)$, $c(\beta)$ taking values in the set of symmetric nonnegative

$d \times d$ matrices, in $\mathbb{R}^d$ and in $\mathbb{R}$ respectively. Also assume that on $\mathcal{A} \times \mathbb{R}^d$ we are given a real-valued function $f(\beta, x)$.

Recall that if $\alpha$ is a number in $(0, 1)$, then by $C^\alpha = C^\alpha(\mathbb{R}^d)$ we denote the space of all functions $u = u(x)$ given on $\mathbb{R}^d$ such that $|u|_\alpha < \infty$, where

$$|u|_\alpha = \sup |u| + [u]_\alpha, \quad [u]_\alpha = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Also, for functions having second order derivatives we set

$$[u]_{2+\alpha} = \sum_{i,j=1}^{d} [D_{ij}u]_\alpha ,$$

and we denote $C^{2+\alpha} = C^{2+\alpha}(\mathbb{R}^d)$ the set of all functions $u$ for which $|u|_{2+\alpha} := \sup |u| + [u]_{2+\alpha} < \infty$.

ASSUMPTION 6.1. For some numbers $K \geq 1, \kappa, \alpha \in (0, 1)$, for any $\beta \in \mathcal{A}$ we have

$$\operatorname{tr} a(\beta) + |b(\beta)| + |c(\beta)| + |f(\beta, \cdot)|_\alpha \leq K, \quad c(\beta) \leq -\kappa .$$

Define

$$L^\beta u(x) = a^{ij}(\beta) u_{x^i x^j}(x) + b^i(\beta) u_{x^i}(x) ,$$
$$F(u_{ij}, u_i, u, x) = \sup_{\beta \in \mathcal{A}} \{a^{ij}(\beta) u_{ij} + b^i(\beta) u_i + c(\beta) u + f(\beta, x)\} .$$

We will be concerned with the following Bellman equation

$$F(u_{x^i x^j}(x), u_{x^i}(x), u(x), x) = 0 \quad x \in \mathbb{R}^d . \tag{6.21}$$

It is well known that under the above conditions there is a unique bounded viscosity solution of (6.21). This viscosity solution coincides with the probabilistic solution (see, for instance, [27][93]). We denote this solution by $v$.

Next for any $h \in (0, 1)$ and $\beta \in \mathcal{A}$ let a linear bounded operator $L_h^\beta$ : $\mathcal{B}(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$ and a number $p_h^\beta > 0$ be defined.

ASSUMPTION 6.2. (i) for any $u \in C^{2+\alpha}(\mathbb{R}^d)$ we have

$$|L_h^\beta u - L^\beta u| \leq K h^\alpha |u|_{2+\alpha} , \quad L_h^\beta 1 = 0,$$

(ii) the operator $u \to L_h^\beta u + p_h^\beta u$ maps nonnegative functions into nonnegative functions and

$$0 < \inf_\beta p_h^\beta \leq \sup_\beta p_h^\beta < \infty .$$

(iii) The operators $L_h^\beta$ are translation invariant: $L_h^\beta[u(\cdot + y)] = [L_h^\beta u](\cdot + y)$ for any $y \in \mathbb{R}^d$, $u \in \mathcal{B}(\mathbb{R}^d)$.

LEMMA 6.1. *For any $h \in (0, 1)$ there exists a unique bounded solution $v_h$ of the equation*

$$\sup_{\beta \in \mathcal{A}} \{L_h^\beta v_h(x) + c(\beta) v_h(x) + f(\beta, x)\} = 0 \quad x \in \mathbb{R}^d .$$

A new (and to the best of author's knowledge, the first general) result on the rate of convergence is the following theorem.

THEOREM 6.2. *There is a constant $N = N(d, K, \alpha, \kappa)$ such that for all $x \in \mathbb{R}^d$ and $h \in (0, 1)$*

$$|v(x) - v_h(x)| \leq N h^{\alpha^3/(2+2\alpha)^2} .$$

## 7. – Safonov's proof of the basic a priori estimate

The author believes that this proof should be part of general knowledge for mathematicians even remotely concerned with the theory of partial differential equations.

We will use several simple and known facts. First of them is the maximum principle, from which one derives that if $u$ is sufficiently smooth and $\Delta u = f$ in the ball $B_\rho$ of radius $\rho$ centered at the origin and $u = 0$ on $\partial B_\rho$, then $|u(x)| \leq (\rho^2 - |x|^2) \max |f|$ and $|u| \leq \rho^2 \max |f|$ in $B_\rho$.

Also, if a smooth function $g$ is given on $\partial B_\rho$, one can solve Laplace's equation $\Delta u = 0$ in $B_\rho$ with boundary condition $u = g$ on $\partial B_\rho$. This solution can be represented by Poisson's integral formula from which one sees that, say

$$|D_{ijk} u(0)| \leq N \rho^{-3} |g|_{B_\rho} \quad (|g|_{B_\rho} := \max_{B_\rho} |g|) ,$$

where $N$ is a constant independent of $g, \rho$.

Further, for $k = 0, 2$ denote by $\mathcal{P}_k$ the set of all polynomials of $x \in \mathbb{R}^d$ of degree at most $k$. It is easy to understand and prove that Hölder continuous functions can be characterized by the rate of their approximation by polynomials. For example, if

$$[u]'_{k+\alpha} = \sup_{x,\rho} \frac{1}{\rho^{k+\alpha}} \inf_{p \in \mathcal{P}_k} |u - p|_{x+B_\rho} ,$$

then for a constant $N$ depending only on $d, \alpha$ and any $u \in C^{k+\alpha}$ we have

$$[u]'_{k+\alpha} \leq N[u]_{k+\alpha}, \quad [u]_{k+\alpha} \leq N[u]'_{k+\alpha} .$$

With these facts at hand we can prove the following basic a priori estimate.

THEOREM 7.1. *Let $0 < \alpha < 1$. Then there exists a constant $N = N(d, \alpha)$ such that for any $u \in C^{2+\alpha}(\mathbb{R}^d)$ we have*

$$[u]_{2+\alpha} \leq N[\Delta u]_\alpha . \tag{7.1}$$

PROOF. Denote $f = \Delta u$, take a constant $K \geq 1$ to be specified later and take $\rho > 0$. Also denote by $T_0^2 u$ the second-order Taylor polynomial of $u$ at 0 and let $h$ be a unique solution of $\Delta h = 0$ in $B_{(K+1)\rho}$ with boundary data $h = u - T_0^2 u$ on $\partial B_{(K+1)\rho}$. The function

$$w(x) = u(x) - T_0^2 u(x) - h(x)$$

satisfies $\Delta w(x) = f(x) - f(0)$ in $B_{(K+1)\rho}$ and $w = 0$ on $\partial B_{(K+1)\rho}$. Observe that by definition $|f - f(0)| \leq (K+1)^\alpha \rho^\alpha [f]_\alpha$ in $B_{(K+1)\rho}$, so that

$$|w|_{B_\rho} \leq N(K+1)^2 \rho^2 \max_{B_{(K+1)\rho}} |f - f(0)| \leq N(K+1)^{2+\alpha} \rho^{2+\alpha} [f]_\alpha .$$

By Taylor's formula

$$|h - T_0^2 h|_{B_\rho} \leq N\rho^3 \max_{i,j,k} |D_{ijk} h|_{B_\rho} ,$$

and since about any point in $B_\rho$ there is a ball of radius $K\rho$ belonging to $B_{(K+1)\rho}$, we have

$$\begin{aligned} |D_{ijk} h|_{B_\rho} &\leq N(K\rho)^{-3} |h|_{B_{(K+1)\rho}}, \quad |h - T_0^2 h|_{0;B_\rho} \\ &\leq NK^{-3} |h|_{B_{(K+1)\rho}} = NK^{-3} |h|_{\partial B_{(K+1)\rho}} = NK^{-3} |u - T_0^2 u|_{\partial B_{(K+1)\rho}} . \end{aligned}$$

By Taylor's formula the last norm is less than $N(K+1)^{2+\alpha}\rho^{2+\alpha}[u]_{2+\alpha}$. Therefore,

$$|h - T_0^2 h|_{B_\rho} \leq NK^{-3}(K+1)^{2+\alpha} \rho^{2+\alpha} [u]_{2+\alpha} ,$$

$$\begin{aligned} |u - T_0^2 u - T_0^2 h|_{B_\rho} &\leq |w|_{B_\rho} + |h - T_0^2 h|_{B_\rho} \\ &\leq N\rho^{2+\alpha} \{(K+1)^{2+\alpha}[f]_\alpha + K^{-3}(K+1)^{2+\alpha}[u]_{2+\alpha}\} . \end{aligned}$$

In particular,

$$\frac{1}{\rho^{2+\alpha}} \inf_{p \in \mathcal{P}_2} |u - p|_{B_\rho} \leq N(K+1)^{2+\alpha}[f]_\alpha + NK^{-3}(K+1)^{2+\alpha}[u]_{2+\alpha} .$$

Since we can consider balls centered at any point in the same way, we get

$$[u]'_{2+\alpha} \leq N(K+1)^{2+\alpha}[f]_\alpha + NK^{-3}(K+1)^{2+\alpha}[u]_{2+\alpha} ,$$

which by the equivalence of seminorms $[\cdot]'$ and $[\cdot]$ implies that

$$[u]_{2+\alpha} \leq N_1(K+1)^{2+\alpha}[f]_\alpha + N_1 K^{-3}(K+1)^{2+\alpha}[u]_{2+\alpha} .$$

To finish the proof of (7.1) it remains only to take $K$ so large that $N_1 K^{-3}(K+1)^{2+\alpha} \leq 1/2$. The theorem is proved.

## 8. – Four coordinates

In this section we outline a method of reducing the Dirichlet problem in a domain to equations on a manifold without boundary. For simplicity we demonstrate this method on the example of Laplace's equation.

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$. Moreover, assume that there is a function $\psi \in C^2(\mathbb{R}^d)$ such that $\Omega = \{\psi > 0\}$ and $\Delta\psi \le -1$ in $\Omega$ and $|\psi_x| > 0$ on $\partial\Omega$. Consider the following problem

$$\Delta u + f = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{8.2}$$

where $f$ is a given function. In view of the boundary condition it is natural to look for a solution in the form $u = v\psi$. Then from (8.2) we get the following equation for $v$

$$\psi\Delta v + 2\psi_{x^i} v_{x^i} + v\Delta\psi + f = 0 \quad \text{in} \quad \Omega. \tag{8.3}$$

Usually in order to have uniqueness for solutions of an equation one needs to add boundary conditions. However, no boundary conditions are needed for (8.3). Indeed, the difference between two solutions satisfies $\psi\Delta w + 2\psi_{x^i} w_{x^i} + w\Delta\psi = 0$, that is $\Delta(\psi w) = 0$ in $\Omega$ and $\psi w = 0$ on $\partial\Omega$, hence $\psi w = 0$ and $w = 0$ in $\Omega$.

In a sense the boundary disappears. Nevertheless, equation (8.3) cannot be treated by methods known for the whole space. The reason for this is that those methods work, roughly speaking, only when the matrix of coefficients of second order derivatives can be represented as square of a smooth matrix and, in addition, the coefficient of the unknown function is negative and bounded away from zero.

Since even the first derivatives of $\sqrt{\psi}$ are unbounded near $\partial\Omega$ the natural idea appeared to take a surface "above" $\Omega$ and lift all objects to this surface in such a way that $\sqrt{\psi}$ become a smooth function. For example, we can take the surface in $\mathbb{R}^{d+1}$ defined by the equation $r^2 = \psi(x)$, $r \in [0,\infty)$, $x \in \Omega$. Then $\sqrt{\psi} = r$ is a very good function of $r$. To lift the equation onto the surface we have to find a function $w(x, r)$ such that $w(x, \sqrt{\psi(x)}) = v(x)$. The latter equation makes it possible to express derivatives of $v$ through derivatives of $w$. For instance, for $r^2 = \psi(x)$ one has

$$v_{x^i} = \Big(\frac{\partial}{\partial x^i} + \frac{1}{2\sqrt{\psi}}\psi_{x^i}\frac{\partial}{\partial r}\Big)w = w_{x^i} + \frac{1}{2r}\psi_{x^i} w_r\,,$$

$$\begin{aligned} v_{x^i x^j} &= \Big(\frac{\partial}{\partial x^i} + \frac{1}{2\sqrt{\psi}}\psi_{x^i}\frac{\partial}{\partial r}\Big)\Big(\frac{\partial}{\partial x^j} + \frac{1}{2\sqrt{\psi}}\psi_{x^j}\frac{\partial}{\partial r}\Big) \\ &= w_{x^i x^j} + \frac{1}{2r}\psi_{x^i} w_{x^j r} + \frac{1}{2r}\psi_{x^j} w_{x^i r} + \frac{1}{4r^2}\psi_{x^i}\psi_{x^j}\Big(w_{rr} - \frac{1}{r}w_r\Big) \\ &\quad + \frac{1}{2r}\psi_{x^i x^j} w_r\,. \end{aligned}$$

By plugging this formulas in (8.3) we get the equation

$$(8.4)\quad r^2\Delta w + r\psi_{x^i} w_{x^i r} + \frac{1}{4}|\psi_x|^2\Big(w_{rr} + \frac{3}{r}w_r\Big) + \frac{r}{2}w_r\Delta\psi + 2\psi_{x^i}w_{x^i} + w\Delta\psi + f = 0,$$

which $w$ has to satisfy on the surface $r^2 = \psi(x)$. It turns out that although (8.4) involves derivatives with respect to all coordinates $x$ and $r$, it is an equation *on the surface*. In other words, the equation defines $w$ uniquely on the surface. Of course, this property follows at once from the fact that if $w$ satisfies (8.4) on the surface $r^2 = \psi(x)$, then $w(x, \sqrt{\psi(x)})$ satisfies (8.3) in $\Omega$.

On the first sight we have not gained to much. Indeed, while trying to make derivatives of $\sqrt{\psi}$ bounded we got unbounded coefficients in equation (8.4) itself. However, one can remember that $w_{rr} + (3/r)w_r$ is the radial part of four-dimensional Laplacian, so that if $r$ is interpreted as $[(x^{d+1})^2 + \ldots + (x^{d+4})^2]^{1/2}$, then

$$w_{rr} + \frac{3}{r}w_r = \sum_{\nu=d+1}^{d+4} w_{x^\nu x^\nu}\,.$$

Now the equation $r^2 = \psi(x)$ describes a surface in $\mathbb{R}^{d+4}$ and on this surface the function $w(x^1, \ldots, x^{d+4})$ which is spherically symmetric with respect to $x^{d+1}, \ldots, x^{d+4}$ satisfies the equation

$$\sum_{\nu=d+1}^{d+4}(x^i)^2\Delta w + \sum_{\nu=d+1}^{d+4}\sum_{i=1}^{d} x^\nu\psi_{x^i}w_{x^i x^\nu} + \frac{1}{4}|\psi_x|^2\sum_{\nu=d+1}^{d+4} w_{x^\nu x^\nu}$$
$$+\frac{1}{2}\Delta\psi\sum_{\nu=d+1}^{d+4} x^\nu w_{x^\nu} + 2\sum_{i=1}^{d}\psi_{x^i}w_{x^i} + w\Delta\psi + f = 0\,.$$

This equation has smooth coefficients, the condition on the matrix of coefficients can be easily checked out and there is no boundary.

Of course, an appeal to this equation in investigating (8.2) looks weird, but for many degenerate fully nonlinear equations this method yielded the best results.

## REFERENCES


[1] A. D. ALEKSANDROV, *Dirichlet's problem for the equation* $\mathrm{Det}\,\|z_{ij}\| = \varphi(z_1, \ldots, z_n, z, x_1, \ldots, x_n)$, Vestnik Leningrad Univ., Vol. 13, No. 1 (1958), 5-24 in Russian.

[2] A. D. ALEKSANDROV, *Research into maximum principle. VI*, Izvestiya Vysshikh Uchebnykh Zavedenii, Ser. Math. No. 1 (1961), 3-20 in Russian.

[3] A. D. ALEKSANDROV, *Uniqueness conditions and estimates for solutions of the Dirichlet problem*, Vestnik Leningrad. Univ., Vol. 18, No. 3 (1963), 5-29. English translation in Amer. Math. Soc. Transl. (2) Vol. 68 (1968), 89-119.

[4] B. ANDREWS, Evolving Convex Hypersurfaces, Thesis, Australian Nat. University, 1993.

[5] S. V. ANULOVA – M. V. SAFONOV, *Control of a diffusion process in a region with fixed reflection on the boundary*, pp. 1-15 in "Statistics and Controlled Stoch. Processes", Steklov Seminars 1985-86, Vol. 2, Optimization Software Inc., New York, 1989.

[6] I. YA. BAKEL'MAN, "Geometricheskie Metody Reshenia Ellipticheskikh Uravnenni", Nauka, Moscow, 1965 in Russian.

[7] G. BARLES – P. E. SOUGANIDIS, *Convergence of approximation schemes for fully nonlinear second order equations*, Asymp. Anal., Vol. 4, No. 3 (1991), 271-283.

[8] S. BERNSTEIN, *Sur la généralization du problèms de Dirichlet. (Deuxième partie)*, Math. Annalen., Vol. 69 (1910), 82-136.

[9] L. A. CAFFARELL, *Interior a priori estimates for solutions of fully nonlinear equations*, Annals of Math.**130** No. 1 (1989), 189-213.

[10] L. A. CAFFARELLI, *Interior $W^{2,p}$-estimates for solutions of the Monge-Ampère equation*, Annals of Math. **131** No. 1 (1990), 135-150.

[11] L. A. CAFFARELLI, *The regularity of mappings with convex potentials*, J. Amer. Math. Soc. Vol. 5 (1992), 99-104.

[12] L. A. CAFFARELLI, *Boundary regularity of maps with convex potentials*, Comm. Pure Appl. Math. Vol. 45 (1992), 1141-1151.

[13] L. A. CAFFARELLI – X. CABRÉ , "Fully Nonlinear Elliptic Equations", Colloquium Publications Vol. 43, American Math. Soc., Providence, RI, 1995.

[14] L. CAFFARELLI – L. NIRENBERG – J. SPRUCK, *The Dirichlet problem for nonlinear second order elliptic equations, I. Monge-Ampère equations*, Comm. Pure Appl. Math. Vol. 37 (1984), 369-402.

[15] L. CAFFARELLI – J. J. KOHN – L. NIRENBERG – J. SPRUCK, *The Dirichlet problem for nonlinear second order elliptic equations, II. Complex Monge-Ampère, and uniformly elliptic, equations*, Comm. Pure Appl. Math. Vol. 38 (1985), 209-252.

[16] L. CAFFARELLI – L. NIRENBERG – J. SPRUCK, *The Dirichlet problem for nonlinear second order elliptic equations, III. Functions of the eigenvalues of the Hessian*, Acta Math., Vol. 155, No. 3-4 (1985), 261-301.

[17] L. CAFFARELLI – L. NIRENBERG – J. SPRUCK, *The Dirichlet problem for the degenerate Monge-Ampère equation*, Revista Mat. Iberoamericana, Vol. 86, No. 1, 2 (1986), 19-27.

[18] L. CAFFARELLI – L. NIRENBERG – J. SPRUCK, *Nonlinear second order equations, IV. The Dirichlet problem for Weingarten surfaces*, Comm. Pure Appl. Math. Vol. 41 (1988), 47-70.

[19] E. CALABI, *Improper affine hypersurfaces of convex type and a generalization of a theorem by K. Jorgens*, Michigan. Math. J. Vol. 5 (1958), 2.

[20] P. CANNARSA – F. GOZZI – H. T. SONER, *A dynamic programming approach to nonlinear boundary control problems of parabolic type*, preprint.

[21] S. Y. CHENG – S. T. YAU, *On the regularity of the Monge-Ampère equation* $\det(\partial^2 u/\partial x^i \partial x^j) = F(x,u)$, Comm. Pure Appl. Math. Vol. 30 (1977), 41-68.

[22] R. COURANT – D. HILBERT, "Methods of Mathematical Physics", Vol. 2, Interscience Publishers, NY, 1962.

[23] M. G. CRANDALL – H. ISHII – P. L. LIONS, *User's guide to viscosity solutions of second order partial differential equations*, Bulletin Amer. Math. Soc. Vol. 27, No. 1 (1992), 1-67.

[24] P. DELANOË P., *Classical solvability in dimension two of the second boundary-value problem associated with the Monge-Ampère operator*, Ann. Inst. Henri Poincaré, Vol. 8, No. 5 (1991), 443-457.

[25] L. C. EVANS, *Classical solutions of fully nonlinear convex, second order elliptic equations*, Comm. Pure Appl. Math., Vol. 25 (1982), 333-363.

[26] L. C. EVANS, *Classical solutions of the Hamilton-Jacobi-Bellman equation for uniformly elliptic operators*, Trans. Amer. Math. Soc., Vol. 275 (1983), 245-255.

[27] W. FLEMING – M. SOONER, "Controlled Markov Processes and Viscosity Solutions", Springer Verlag, 1993.

[28] C. GERHARDT, *Flow of nonconvex hypersurfaces into spheres*, J. Diff. Geom. Vol. 32 (1990), 299-314.

[29] D. GILBARG – N. S. TRUDINGER, "Elliptic Partial Differential Equations of Second Order", 2nd ed., Springer Verlag, Berlin, 1983.

[30] B. GUAN, *The Dirichlet problem for a class of fully nonlinear elliptic equations*, Comm. in PDE, Vol. 19, No. 3-4 (1994), 399-416.

[31] B. GUAN, *The Dirichlet Problem for Monge-Ampère Equations in non-Convex domains*, preprint.

[32] B. GUAN – Y. Y. LI, *Monge-Ampère Equations on Riemannian Manifolds*, preprint.

[33] B. GUAN – J. SPRUCK, *Boundary value problem on $S^n$ for surfaces of constant Gauss curvature*, Annals of Math., Vol. 138 (1993), 601-624.

[34] N. M. IVOCHKINA, *Classical solvability of the Dirichlet problem for the Monge-Ampère equation*, Zapiski Nauchn. Sem. LOMI AN SSSR, Vol. 131 (1983), 72-79 in Russian. English translation in J. Soviet Math. Vol. 30, No. 4 (1985), 2287-2292.

[35] N. M. IVOCHKINA, *Solution of the Dirichlet problem for curvature equation of order $m$*, Math. Sbornik, Vol. 180 No. 7 (1989), 867-887 in Russian. English translation in Math.USSR-Sb., Vol. 67, No. 2 (1990), 317-339.

[36] N. M. IVOCHKINA, *The Dirichlet problem for the equations of curvature of order $m$*, English translation in Leningrad Math. J. Vol. 2 No. 3 (1991), 631-654.

[37] B. JENSEN, *The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations*, Arch. Rational Mach. Anal. Vol. 101 No. 1 (1988), 1-27.

[38] J. J. KOHN – L. NIRENBERG, *Degenerate elliptic-parabolic equations of second order*, Comm. Pure and Appl. Math. Vol. 20, No. 4 (1967), 797-872.

[39] N. J. KOREVAAR, *A priori interior gradient bounds for solutions to elliptic Weingarten equations*, Ann. Inst. Henri Poincaré, Analyse non lineaire, Vol. 4 No. 5 (1987), 405-421.

[40] N. V. KRYLOV, *Boundedly nonhomogeneous elliptic and parabolic equations*, Izvestija Akad. Nauk SSSR, ser. mat. Vol. 46 No. 3 (1982), 487-523. English translation in Math. USSR Izvestija, Vol. 20 No. 3 (1983), 459-492.

[41] N. V. KRYLOV, *Boundedly nonhomogeneous elliptic and parabolic equations in a domain*, Izvestija Akad. Nauk SSSR, ser. mat., Vol. 47 No. 1 (1983), 75-108. English translation in Math. USSR Izvestija Vol. 22 No. 1 (1984), 67-97.

[42] N. V. KRYLOV, *On degenerate nonlinear elliptic equations II*, Matematicheski Sbornik, Vol. 121 No. 2 (1983), 211-232 in Russian. English translation in Math. USSR Sbornik Vol. 49 No. 1 (1984), 207-228.

[43] N. V. KRYLOV, *On estimates for the derivatives of solutions of nonlinear parabolic equations*, Doklady Acad. Nauk SSSR, Vol. 274 No. 1 (1984), 23-26. English translation in Soviet Math. Dokl. Vol. 29 No. 1 (1984), 14-17.

[44] N. V.KRYLOV, "Nonlinear Elliptic and Parabolic Equations of Second Order", Nauka, Moscow, 1985 in Russian. English translation: Reidel, Dordrecht, 1987.

[45] N. V. KRYLOV, *Some new results in the theory of nonlinear elliptic and parabolic equations*, Proceed. of Intern. Congr. of Math., Berkeley Vol. 2 (1986), 1101-1109.

[46] N. V. KRYLOV, *Smoothness of the value function for a controlled diffusion process in a domain*, Izvestija Akad. Nauk SSSR, ser. mat. Vol. 53 No. 1 (1989), 66-96 in Russian. English translation in Math. USSR Izvestija Vol. 34, No. 1 (1990), 65-96.

[47] N. V. KRYLOV, *On first quasiderivatives of solutions of Itô's stochastic equations*, Izvestija Akad. Nauk SSSR, ser. mat. Vol. 56 No. 2 (1992), 398-426 in Russian. English translation in Math. USSR Izvestija Vol. 40 (1993).

[48] N. V. KRYLOV, "Lectures on Fully Nonlinear Second Order Elliptic Equations", Vorlesungsreihe No. 29, Rheinische Friedrich-Wilhelms-Universität, Sonderforschungsbereich 256, Bonn, 1993, 85.

[49] N. V. KRYLOV, *Interior first order derivative estimates for solutions of nonlinear degenerate elliptic equations with constant coefficients*, Comm. in PDE, Vol. 18 No. 1-2 (1993), 1-40.

[50] N. V. KRYLOV, *On the general notion of fully nonlinear second order elliptic equation*, Trans. Amer. Math. Soc. Vol. 347 No. 3, (1995), 857-895.

[51] N. V. KRYLOV, *A theorem on degenerate elliptic Bellman equations in bounded domains*, Differential and Integr. Eq. Vol. 8 No. 5 (1995), 961-980.

[52] N. V. KRYLOV – M. V. SAFONOV, *An estimate of the probability that a diffusion process hits a set of positive measure*, Doklady Acad. Nauk SSSR Vol. 245 No. 1 (1979), 18-20. English translation in Soviet Math. Dokl. Vol. 20 No. 2 (1979), 253-255.

[53] N. V. KRYLOV – M. V. SAFONOV, *A certain property of solutions of parabolic equations with measurable coefficients*, Izvestija Akad. Nauk SSSR, ser. mat. Vol. 44 No. 1 (1980), 161-175. English translation in Math. USSR Izvestija, Vol. 16, No. 1 (1981), 151-164.

[54] H. J. KUO – N. S. TRUDINGER, *Discrete methods for fully nonlinear elliptic equations*, SIAM J. on Numer. Anal. Vol. 29 (1992), 123-135.

[55] H. J. KUO – N. S. TRUDINGER, *On the discrete maximum principle for parabolic difference operators*, Math. Modelling and Numer. Anal. Vol. 27 No. 6 (1993), 719-737.

[56] H. J. KUSHNER – P. G. DUPUIS, "Numerical Methods for Stochastic Control Problems in Continuous Time", Springer-Verlag, 1992.

[57] N. KUTEV, *On the solvability of Monge-Ampère type equations in non-uniformly convex domains*, Math. Z. Vol. 208 (1991), 167-176.

[58] N. KUTEV, *On the solvability of Dirichlet's problem for a class of nonlinear elliptic and parabolic equations*, International Conference on Differential Equations, Barselona, 26-31 August 1991, Eds: C. Perelló, C. Simó and J. Solà-Morales, World Scientific, Singapore-New Jersey-London-Hong Kong Vol. 2, 666-670.

[59] H. LEWY, *A priori limitations for solutions of Monge-Ampère equations I, II*, Trans. Amer. Math. Soc. Vol. 37 (1935), 417-434; Vol. 42 (1937), 365-374.

[60] G. M. LIBERMAN – N. S. TRUDINGER, *Nonlinear oblique boundary value problems for nonlinear elliptic equations*, Trans. Amer. Math. Soc. Vol. 295 No. 2 (1986), 509-546.

[61] M. LIN – N. S. TRUDINGER, *The Dirichlet Problem for the Prescribed Curvature Quotient Equations*, preprint, 1994.

[62] P. L. LIONS, *Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part III. Uniqueness of viscosity solutions of general second order equations*, J. Funct. Anal. Vol. 86 (1989), 1-18.

[63] P. L. LIONS – N. S. TRUDINGER, *Neumann problems for uniformly elliptic Bellman equations*, Math. Z., Vol. 191 (1985), 1-15.

[64] P. L. LIONS – N. S. TRUDINGER – J. I. E. URBAS, *The Neumann problem for equations of Monge-Ampère type*, Comm. Pure Appl. Math. Vol. 39 (1986), 539-563.

[65] H. MINKOWSKI, *Volumen und Oberfläche*, Math. Ann. Vol. 57 (1903).

[66] C. MIRANDA, *Su un problema di Minkowski*, Rend. Sem. Mat. Roma Vol. 3 (1939).

[67] L. NIRENBERG, *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math. Vol. 6 (1953).

[68] O. A. OLEINIK – E. V. RADKEVICH, "Second Order Equations with Nonnegative Characteristic form", Itogi Nauki, Mat. Analiz 1969, VINITI, Moscow, 1971 in Russian. English translation: Amer. Math. Soc., Plenum Press, Providence R.I., 1973.

[69] A. V. POGORELOV, *Regularity of convex surfaces with prescribed Gaussian curvature*, Matem. Sbornik Vol. 31, 1 (1952), 88-103 in Russian.

[70] A. V. POGORELOV, *Monge-Ampère equations of elliptic type*, Izd-vo Kharkovskogo Un-ta, Kharkov, 1960 in Russian. English translation: Noordhoff, Groningen, 1964.

[71] A. V. POGORELOV, *The Dirichlet problem for the n-dimensional analogue of the Monge-Ampère equation*, Dokl. Akad. Nauk SSSR Vol. 201 No. 4 (1971), 790-793 in Russian. English translation in Soviet Math. Dokl. Vol. 12 (1971), 1727-1731.

[72] A. V. POGORELOV, "The Minkowski Multidimensional Problem", Nauka, Moscow, 1975 in Russian. English translation: J. Wiley, New York, 1978.

[73] G. PRAGARAUSKAS, *Uniqueness of the solution of the Bellman equation in the case of general controlled random processes*, Lit. Mat. Sbornik Vol. 22 No. 2 (1982), 137-156 in Russian. English translation in Lithuanian Math. J. Vol. 22 No. 2 (1982), 160-168.

[74] G. PRAGARAUSKAS, *Approximation of controlled solutions of Itô equations by controlled Markov chains*, Lit. Mat. Sbornik Vol. 23 No. 1 (1983), 175-188 in Russian. English translation in Lithuanian Math. J. Vol. 23 No. 1 (1983), 98-108.

[75] M. V. SAFONOV, *Harnack inequality for elliptic equations and the Hölder property of their solutions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) Vol. 96 (1980), 272-287 in Russian. English translation in J. Soviet Math. Vol. 21 No. 5 (1983), 851-863.

[76] M. V. SAFONOV, *On the classical solution of Bellman's elliptic equations*, Dokl. Akad. Nauk SSSR, Vol. 278 No. 4(1984), 810-813 in Russian. English translation in Soviet Math. Dokl. Vol. 30 No. 2 (1984), 482-485.

[77] M. V. SAFONOV, *Unimprovability of estimates of Hölder constants for solutions of linear elliptic equations with measurable coefficients*, Matem. Sbornik Vol. 132 No. 2 (1987), 275-288 in Russian. English translation in Math. USSR Sbornik Vol. 60 No. 1 (1988), 269-281.

[78] M. V. SAFONOV, *On the classical solution of nonlinear elliptic equations of second order*, Izvestija Akad. Nauk SSSR, ser. mat. Vol. 137 No. 2 (1988), 184-201 in Russian. English translation in Math. USSR Izvestija Vol. 33 No. 3 (1989), 597-612.

[79] D. TATARU, *Viscosity solutions for Hamilton-Jacobi equations with unbounded nonlinear terms*, J. Math. Anal. Appl. Vol. 163 (1992), 345-392.

[80] D. TATARU, *Viscosity solutions for the dynamic programming equations*, Applied Math. and Optimiz. Vol. 25 (1992), 109-126.

[81] N. S. TRUDINGER, *Fully nonlinear, uniformly elliptic equations under natural structure conditions*, Trans. Amer. Math. Soc. Vol. 278 No. 2 (1983), 751-769.

[82] N. S. TRUDINGER, *Hölder gradient estimates for fully nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh, Sect. A Vol. 108 (1988), 57-65.

[83] N. S. TRUDINGER, *On the twice differentiability of viscosity solutions of nonlinear elliptic equations*, Bull. Austral. Math. Soc. Vol. 39 (1989), 443-447.

[84] N. S. TRUDINGER, *The Dirichlet problem for the prescribed curvature equations*, Arch. Rat. Mech. Anal. Vol. 111 No. 2 (1990), 153-179.

[85] N. S. TRUDINGER, *A priori bounds and necessary conditions for solvability of prescribed curvature equations*, Manuscripta Nath. Vol. 67 (1990), 99-112.

[86] N. S. TRUDINGER – J. I. E. URBAS, *The Dirichlet problem for the equation of prescribed Gauss curvature*, Bull. Aust. Math. Soc. Vol. 28 (1983), 217-231.

[87] N. S. TRUDINGER – J. I. E. URBAS, *On second derivatives estimates for equations of Monge-Ampère type*, Bull. Austral. Math. Soc. Vol. 30 (1984), 321-334.

[88] K. TSO, *Deforming a hypersurface by its Gauss-Kroneker curvature*, Comm. Pure and Appl. Math Vol. 38 (1985), 867-882.

[89] J. I. E. URBAS, " Elliptic Equations of Monge-Ampère Type", PhD thesis, Australian Nat. University, 1984.

[90] J. I. E. URBAS, *Global Hölder estimates for equations of Monge-Ampère type*, Invent. Math. Vol. 91 (1988), 1-29.

[91] J. I. E. URBAS, *Regularity of generalized solutions of Monge-Ampère equations*, Math. Zeit. Vol. 197 (1988), 365-393.

[92] J. I. E. URBAS, *On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations*, Indiana Univ. Math. J. Vol. 39 No. 4 (1990), 355-382.

[93] J. I. E. URBAS, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Zeit. Vol. 205 (1990), 355-372.

[94] J. I. E. URBAS, *Regularity of almost extremal solutions of Monge-Ampère equations*, Proc. Royal Soc. Edinburgh A Vol. 117 (1991), 21-29.

[95] J. I. E. URBAS, *Boundary regularity for solutions of the equation of prescribed Gauss curvature*, Ann. Inst. Henri Poincaré, Analyse non lineaire Vol. 8 No. 5 (1991), 499-522.

[96] J. I. E. URBAS, *An expansion of convex hypersurfaces*, J. Diff. Geom. Vol. 33 (1991), 91-125; corrections to, ibid. Vol 35, 763-765.

[97] J. I. E. URBAS, *Nonlinear Oblique Boundary Value Problems for two Dimensional Curvature Equations*, preprint.

[98] WANG LIHE, *On the regularity theory of fully nonlinear parabolic equations I-III*, Comm. Pure Appl. Math. Vol. 45 (1992), 27-76, 141-178, 255-262.

[99] WANG ROUHUAI – WANG GUANGLIE, *On existence, uniqueness and regularity of viscosity solutions for the first initial boundary value problems to parabolic Monge-Ampère equation*, Northeastern Math. J. Vol. 8 No. 4 (1992), 417-446.

[100] WANG ROUHUAI – WANG GUANGLIE, *The geometric measure theoretical characterization of viscosity solutions to parabolic Monge-Ampère type equation*, J. Partial Diff. Eqs. Vol. 6 No. 3 (1993), 237-254.


University of Minnesota
127 Vincent Hall
Minneapolis, MN, 55455 USA
krylov@@math.umn.edu