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The Moduli Space of Special Lagrangian Submanifolds

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1. - Introduction

In their quest for examples of minimal submanifolds, Harvey and Lawson in 1982 [7] extended the well-known fact that a complex submanifold of a Kähler manifold is minimal to the more general context of *calibrated* submanifolds. One such class is that of special Lagrangian submanifolds of a Calabi-Yau manifold. New developments in the study of these have raised the question as to whether they should be accorded equal status with complex submanifolds.

The developments stem from two sources. The first is the deformation theory of R. C. McLean [8]. This shows that, given one compact special Lagrangian submanifold L, there is a local moduli space which is a manifold and whose tangent space at L is canonically identified with the space of harmonic 1-forms on L. The $\mathcal{L}^2$ inner product on harmonic forms then gives the moduli space a natural Riemannian metric. The second input is from the paper of Strominger, Yau and Zaslow [12] which studies the moduli space of special Lagrangian tori in the context of mirror symmetry.

This paper is in some sense a commentary on these two works, but it is provoked by the question: "What is the natural geometrical structure on the moduli space of special Lagrangian submanifolds in a Calabi-Yau manifold?" We know that a moduli space of complex submanifolds (when unobstructed) is a complex manifold. We shall show that the moduli space M of special Lagrangian submanifolds has the local structure of a Lagrangian submanifold, and we conjecture that it is "special" in an appropriate sense.

"A Lagrangian submanifold of what?" the reader may well ask. Recall that if V is a finite-dimensional real vector space, then the natural pairing with its dual space V^* defines a symplectic structure on $V \times V^*$. It also defines an indefinite metric. We shall show that there is a natural embedding of the local moduli space M as a Lagrangian submanifold in the product $H^1(L, \mathbf{R}) \times H^{n-1}(L, \mathbf{R})$ (where $n = \dim L$) of two dual vector spaces and that McLean's metric is the natural induced metric.

The symplectic manifold $V \times V^*$ can be thought of in two ways as a cotangent bundle: as either T^*V or T^*V^*. Thus the Lagrangian submanifold M is defined locally as the graph of the derivative of a function $\phi : V \to \mathbf{R}$ or ψ :

$V^* \to \mathbf{R}$. We show that this symmetry (which is really the Legendre transform) lies behind the viewpoint in [12], where it is viewed as a manifestation of mirror symmetry. This involves studying the structure of the moduli space of Lagrangian submanifolds together with flat line bundles. We show that there is a natural complex structure and Kähler metric on this space, and that this is a Calabi-Yau metric if the embedding of M above is special.

2. – Calabi-Yau manifolds

A *Calabi-Yau manifold* is a Kähler manifold of complex dimension n with a covariant constant holomorphic n-form. Equivalently it is a Riemannian manifold with holonomy contained in $SU(n)$.

It is convenient for our purposes to play down the role of the complex structure in describing such manifolds and to emphasise instead the role of three closed forms, satisfying certain algebraic identities (see [10]). We have the Kähler 2-form ω and the real and imaginary parts Ω_1 and Ω_2 of the covariant constant n-form. These satisfy some identities:

(i) ω is non-degenerate
(ii) $\Omega_1 + i\Omega_2$ is locally decomposable and non-vanishing
(iii) $\Omega_1 \wedge \omega = \Omega_2 \wedge \omega = 0$
(iv) $(\Omega_1 + i\Omega_2) \wedge (\Omega_1 - i\Omega_2) = \omega^n$ (respectively $i\omega^n$) if n is even (respectively odd)
(v) $d\omega = 0, \quad d\Omega_1 = 0, \quad d\Omega_2 = 0.$

These conditions (together with a positivity condition) we now show serve to characterize Calabi-Yau manifolds. Firstly if $\Omega^c = \Omega_1 + i\Omega_2$ is locally decomposable as $\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n$, then take the subbundle Λ of $T^*M \otimes \mathbf{C}$ spanned by $\theta_1, \ldots, \theta_n$. By (iv) and the fact that $\omega^n \neq 0$, we have

$$\theta_1 \wedge \ldots \wedge \theta_n \wedge \bar{\theta}_1 \wedge \ldots \wedge \bar{\theta}_n \neq 0$$

and so $T^*M = \Lambda + \overline{\Lambda}$ and we have an almost-complex structure. In this description a 1-form θ is of type $(1,0)$ if and only if $\Omega^c \wedge \theta = 0$. Since from (v) $d\Omega_1 = d\Omega_2 = 0$ this means that $\Omega^c \wedge d\theta = 0$. Writing

$$d\theta = \sum a_{ij}\theta_i \wedge \theta_j + \sum b_{ij}\theta_i \wedge \bar{\theta}_j + \sum c_{ij}\bar{\theta}_i \wedge \bar{\theta}_j \tag{1}$$

we see that $c_{ij} = 0$. Thus the ideal generated by Λ is closed under exterior differentiation, and by the Newlander-Nirenberg theorem the structure is integrable.

Similarly, applying the decomposition of 2-forms (1) to ω, (iii) implies that the $(0,2)$ component vanishes, and since ω is real, it is of type $(1,1)$. It is closed by (v), so if the hermitian form so defined is positive definite, then we have a Kähler metric.

Since Ω^c is closed and of type $(n, 0)$ it is a non-vanishing holomorphic section s of the canonical bundle. Relative to the trivialization s, the hermitian connection has connection form given by $\partial \log(\|s\|^2)$. But property (iv) implies that it has constant length, so the connection form vanishes and $s = \Omega^c$ is covariant constant.

3. – Special Lagrangian submanifolds

A submanifold L of a symplectic manifold X is Lagrangian if ω restricts to zero on L and $\dim X = 2 \dim L$. A submanifold of a Calabi-Yau manifold is *special Lagrangian* if in addition $\Omega = \Omega_1$ restricts to zero on L. This condition involves only two out of the three forms, and in many respects what we shall be doing is to treat them both —the 2-form ω and the n-form Ω— on the same footing.

REMARKS.

1. We could relax the definition a little since Ω^c is a *chosen* holomorphic n-form: any constant multiple of Ω^c would also be covariant constant, so under some circumstances we may need to say that L is special Lagrangian if, for some non-zero $c_1, c_2 \in \mathbf{R}$, $c_1\Omega_1 + c_2\Omega_2 = 0$.
2. On a special Lagrangian submanifold L, the n-form Ω_2 restricts to a non-vanishing form, so in particular L is always oriented.

Examples of special Lagrangian submanifolds are difficult to find, and so far consist of three types:

- Complex Lagrangian submanifolds of hyperkähler manifolds
- Fixed points of a real structure on a Calabi-Yau manifold
- Explicit examples for non-compact Calabi-Yau manifolds

The hyperkähler examples arise easily. In this case we have $n = 2k$ and three Kähler forms $\omega_1, \omega_2, \omega_3$ corresponding to the three complex structures I, J, K of the hyperkähler manifold. With respect to the complex structure I the form $\omega^c = (\omega_2 + i\omega_3)$ is a holomorphic symplectic form. If L is a complex Lagrangian submanifold (i.e. L is a complex submanifold and ω^c vanishes on L), then the real and imaginary parts of this, ω_2 and ω_3, vanish on L. Thus $\omega = \omega_2$ vanishes and if k is odd (respectively even), the real (respectively imaginary) part of $\Omega^c = (\omega_3 + i\omega_1)^k$ vanishes. Using the complex structure J instead of I, we see that L is special Lagrangian. For examples here, we can take any holomorphic curve in a K3 surface S, or its symmetric product in the Hilbert scheme $S^{[m]}$, which is hyperkähler from [1].

If X is a Calabi-Yau manifold with a real structure (an antiholomorphic involution σ) for which $\sigma^*\omega = -\omega$ and $\sigma^*\Omega = -\Omega$, then the fixed point set (the set of *real* points of X) is easily seen to be a special Lagrangian submanifold L.

All Calabi-Yau metrics on compact manifolds are produced by the existence theorem of Yau. In the non-compact case, Stenzel [11] has some concrete examples. In particular T^*S^n (with the complex structure of an affine quadric) has a complete Calabi-Yau metric for which the zero section is special Lagrangian. When $n = 2$ this is the hyperkähler Eguchi-Hanson metric.

4. – Deformations of special Lagrangian submanifolds

R. C. McLean has studied deformations of special Lagrangian submanifolds. His main result is

THEOREM 1 [8]. *A normal vector field V to a compact special Lagrangian submanifold L is the deformation vector field to a normal deformation through special Lagrangian submanifolds if and only if the corresponding* 1*-form* (IV)* *on L is harmonic. There are no obstructions to extending a first order deformation to an actual deformation and the tangent space to such deformations can be identified through the cohomology class of the harmonic form with $H^1(L, \mathbf{R})$.*

Let us briefly see how the tangent space to the (local) moduli space M is identified with the space of harmonic 1-forms. Consider a 1-parameter family L_t of Lagrangian submanifolds as a smooth map $f : \mathcal{L} \to X$ of the manifold $\mathcal{L} = L \times U$ to X where $U \subset \mathbf{R}$ is an interval and $f(L, t) = L_t$. Since each L_t is Lagrangian, $f^*\omega$ restricts to zero on each fibre of $p : \mathcal{L} \to U$ so we can find a 1-form $\tilde{\theta}$ on $\mathcal{L}$ such that

$$f^*\omega = dt \wedge \tilde{\theta}\,.$$

The restriction θ of $\tilde{\theta}$ to each fibre $L \times \{t\}$ is independent of the choice of $\tilde{\theta}$, and since $d\omega = 0$, it follows that

$$d\theta = 0\,.$$

Similarly, since L_t is *special* Lagrangian, the n-form Ω vanishes on each fibre, so that

$$f^*\Omega = dt \wedge \tilde{\varphi}$$

and since $d\Omega = 0$ we have $d\varphi = 0$. Using the induced metric on L_t one can show that

$$\varphi = *\theta$$

so that θ is the required harmonic form.

A more invariant way of seeing this is to take a section of the normal bundle of L_t, since this is what an infinitesimal variation canonically describes. Take a representative vector field V on X and form the interior product $\iota(V)\omega$. Since ω vanishes on L_t, the restriction of $\iota(V)\omega$ to L_t is a 1-form which is independent of the choice of V. Now $df(\partial/\partial t)$ is naturally a section of the normal bundle of $L_t \subset X$ and θ is then the corresponding 1-form.

Suppose now we take local coordinates $t_1, \dots, t_m$ on the moduli space M of deformations of $L = L_0$. Here of course, from McLean, we know that $m = b_1(L) = \dim H^1(L)$. For each tangent vector $\partial/\partial t_j$ we define as above a corresponding closed 1-form θ_j on L_t for each $t \in M$:

$$\iota(\partial/\partial t_j)\omega = \theta_j$$

(with a slight abuse of notation).

Let $A_1, \dots, A_m$ be a basis for $H_1(L, \mathbf{Z})$ (modulo torsion), then we can evaluate the closed form θ_j on the homology class A_i to obtain a period matrix λ_{ij} which is a function on the moduli space:

$$\lambda_{ij} = \int_{A_i} \theta_j\,.$$

Since by McLean's theorem, the harmonic forms θ_j are linearly independent, it follows that λ_{ij} is invertible. We can now be explicit about the identification of the tangent space to M with the cohomology group $H^1(L, \mathbf{R})$. Let $\alpha_1, \dots, \alpha_m \in H^1(L, \mathbf{Z})$ be the basis dual to $A_1, \dots, A_m$. It follows that

$$\partial/\partial t_j \mapsto [\iota(\partial/\partial t_j)\omega] = \sum \lambda_{ij}\alpha_i \tag{2}$$

identifies T_tM with $H^1(L, \mathbf{R})$.

We now investigate further properties of the period matrix λ.

Proposition 1. *The 1-forms $\xi_i = \sum \lambda_{ij} dt_j$ on M are closed.*

Proof. We represent the full local family of deformations by a map $f : \mathcal{M} \to X$ where $\mathcal{M} \cong L \times M$ with projection $p : \mathcal{M} \to M$. Choose smoothly in each fibre of p a circle representing A_i to give an $n+1$-manifold $\mathcal{M}_i \subseteq \mathcal{M}$ fibering over M. Define the 1-form ξ on M by

$$\xi = p_* f^* \omega\,.$$

The push-down map p_* (integration over the fibres) takes closed forms to closed forms, and since $d\omega = 0$, $df^*\omega = 0$ and so $d\xi = 0$.

Now in local coordinates $\omega = \sum_j dt_j \wedge \tilde{\theta}_j$ and $\tilde{\theta}_j$ restricts to θ_j on each fibre. Since θ_j is closed, integration over the fibres of $\mathcal{M}_i$ is just evaluation on the homology class A_i. Thus $\xi_i = \xi$ and ξ_i is closed.

From this proposition, we can find on M local functions $u_1, \dots, u_m$, well-defined up to the addition of a constant, such that

$$du_i = \xi_i = \sum_j \lambda_{ij} dt_j\,. \tag{3}$$

Since λ_{ij} is invertible, $u_1, \dots, u_m$ are local coordinates on M. More invariantly, we have a coordinate chart

$$u : M \to H^1(L, \mathbf{R}) \tag{4}$$

defined by $u(t) = \sum_i u_i \alpha_i$ which is independent of the choice of basis, and is well-defined up to a translation.

Clearly, we should follow our even-handed policy with respect to ω and Ω and enact the same procedure for Ω. Thus, the basis $\alpha_1, \dots, \alpha_m$ defines a basis $B_1, \dots, B_m$ of $H_{n-1}(L, \mathbf{Z})$ and we form a period matrix μ_{ij}:

$$\mu_{ij} = \int_{B_i} \varphi_j \, .$$

In a similar fashion we find local coordinates $v_1, \dots, v_m$ on M such that

$$dv_i = \sum_j \mu_{ij} dt_j \tag{5}$$

and an invariantly defined map

$$v : M \to H^{n-1}(L, \mathbf{R}) \tag{6}$$

given, using the basis $\beta_1, \dots, \beta_m$ of $H^{n-1}(L, \mathbf{R})$ dual to $B_1, \dots, B_m$ by $v(t) = \sum_i v_i \beta_i$.

We obtain from u and v a map

$$F : M \to H^1(L, \mathbf{R}) \times H^{n-1}(L, \mathbf{R})$$

defined by $F(t) = (u(t), v(t))$.

Let us see now how this fits in with the natural $\mathcal{L}^2$ metric on M. Note that since L is oriented, $H^1(L)$ and $H^{n-1}(L)$ are canonically dual. For any vector space V there is a natural indefinite symmetric form on $V \oplus V^*$ defined by

$$B((v, \alpha), (v, \alpha)) = \langle v, \alpha \rangle \, .$$

Thus $H^1(L) \times H^{n-1}(L)$ has a natural flat indefinite metric G.

PROPOSITION 2. *The $\mathcal{L}^2$ metric g on M is F^*G.*

PROOF. From (2), we have

$$dF(\partial/\partial t_j) = \left(\sum_i \lambda_{ij} \alpha_i, \sum_i \mu_{ij} \beta_i \right) \, .$$

Thus

$$\begin{aligned} F^*G\left(\sum_j a_j \partial/\partial t_j, \sum_j a_j \partial/\partial t_j\right) &= \sum_{i,j,k,l} a_j a_k \lambda_{ij} \mu_{lk} \langle \alpha_i, \beta_l \rangle \\ &= \sum_{i,j,k,l} a_j a_k \lambda_{ij} \mu_{lk} \int_L \alpha_i \wedge \beta_l \, . \end{aligned} \tag{7}$$

But

$$\int_L \left(\sum_i a_i \theta_i \right) \wedge * \left(\sum_i a_i \theta_i \right) = \int_L \sum_{j,k} a_j a_k \theta_j \wedge \varphi_k$$

and using $\theta_j = \sum_i \lambda_{ij} \alpha_i, \varphi_k = \sum_i \mu_{ik} \beta_i$ this is the same as (7).

5. – Symplectic aspects

We have seen that the function F embeds the moduli space of special Lagrangian submanifolds of X which are deformations of L as a submanifold of $H^1(L) \times H^{n-1}(L)$. A vector space of the form $V \oplus V^*$ also has a natural symplectic form w defined by

$$w((v,\alpha),(v',\alpha')) = \langle v,\alpha'\rangle - \langle v',\alpha\rangle$$

so that $H^1(L) \times H^{n-1}(L)$ may be considered as a symplectic manifold. We shall now show the following:

THEOREM 2. *The map F embeds M in $H^1(L) \times H^{n-1}(L)$ as a Lagrangian submanifold.*

PROOF. We need to use the algebraic identity (iii) in Section 2 relating ω and Ω on X:

$$\omega \wedge \Omega = 0\,.$$

Let Y and Z be two vector fields, then taking interior products with this identity, we obtain

$$0 = (\iota(Z)\iota(Y)\omega) \wedge \Omega - \iota(Y)\omega \wedge \iota(Z)\Omega + \iota(Z)\omega \wedge \iota(Y)\Omega + \omega \wedge (\iota(Z)\iota(Y)\Omega)$$

and restricting to a special Lagrangian submanifold L, since ω and Ω vanish, we have

$$\iota(Y)\omega \wedge \iota(Z)\Omega = \iota(Z)\omega \wedge \iota(Y)\Omega\,.$$

Now for Y and Z use vector fields extending $\partial/\partial t_i$ and $\partial/\partial t_j$, and we then obtain on L

$$\theta_i \wedge \varphi_j = \theta_j \wedge \varphi_i\,.$$

Thus, integrating,

$$\int_L \theta_i \wedge \varphi_j = \int_L \theta_j \wedge \varphi_i$$

and so using $\theta_j = \sum_i \lambda_{ij}\alpha_i, \varphi_k = \sum_i \mu_{ik}\beta_i$,

$$\sum_i \lambda_{ik}\mu_{ij} = \sum_i \lambda_{ij}\mu_{ik} \tag{8}$$

From the definitions of the coordinates u and v in (3) and (5) we have

$$\lambda_{ij} = \frac{\partial u_i}{\partial t_j}, \quad \mu_{ij} = \frac{\partial v_i}{\partial t_j}$$

so that (8) becomes

$$\sum_i \frac{\partial u_i}{\partial t_k}\frac{\partial v_i}{\partial t_j} = \sum_i \frac{\partial u_i}{\partial t_j}\frac{\partial v_i}{\partial t_k}\,.$$

But this says precisely that

$$F^*\left(\sum_i du_i \wedge dv_i\right) = 0.$$

It is well-known that a Lagrangian submanifold of the cotangent bundle T^*N of a manifold for which the projection to N is a local diffeomorphism is locally defined as the image of a section $d\phi : N \to T^*N$ for some function $\phi : N \to \mathbf{R}$. Thus, as a consequence of the theorem, taking $N = H^1(L)$, we can write

$$v_j = \frac{\partial \phi}{\partial u_j} \tag{9}$$

for some function $\phi(u_1, \ldots, u_m)$. From Proposition 2 the natural metric on M can be written in the coordinates $u_1, \ldots, u_m$ as

$$g = F^*G = \sum_i du_i\, dv_i = \sum_{i,j} \frac{\partial^2 \phi}{\partial u_i \partial u_j} du_i\, du_j \tag{10}$$

Equally, we can take $N = H^{n-1}(L)$ and find a function $\psi(v_1, \ldots, v_m)$ to represent the metric in a similar form:

$$g = \sum_{ij} \frac{\partial^2 \psi}{\partial v_i \partial v_j} dv_i\, dv_j .$$

The two functions ϕ, ψ are related by the classical Legendre transform.

Remark. Metrics of the above form are said to be of *Hessian type*. V. Ruuska characterized them in [9] as those metrics admitting an abelian Lie algebra of gradient vector fields, the local action being simply transitive.

Given that M parametrizes *special* Lagrangian submanifolds, it would seem reasonable to seek an analogue of the special condition which M might inherit from the embedding F. Now the generators of $\Lambda^m V$ and $\Lambda^m V^*$ define two constant m-forms W_1 and W_2 on the $2m$-dimensional manifold $V \times V^*$. We could say that a Lagrangian submanifold of $V \times V^*$ is *special* if a linear combination of these forms vanishes, in addition to the symplectic form w. With this set-up we have:

Proposition 3. *The map F embeds M as a special Lagrangian submanifold if and only if any of the following equivalent statements holds:*

- *ϕ satisfies the Monge-Ampère equation* $\det(\partial^2\phi/\partial u_i \partial u_j) = c$
- *ψ satisfies the Monge-Ampère equation* $\det(\partial^2\psi/\partial v_i \partial v_j) = c^{-1}$
- *The volume of the torus $H^1(L_t, \mathbf{R}/\mathbf{Z})$ is independent of $t \in M$*
- *The volume of the torus $H^{n-1}(L_t, \mathbf{R}/\mathbf{Z})$ is independent of $t \in M$.*

PROOF. For the first part, note that, using the coordinates $u_1, \dots, u_m$, the m-form $c_1 W_1 + c_2 W_2$ vanishes on $F(M)$ if and only if

$$c_1 du_1 \wedge \dots \wedge du_m + c_2 \det(\partial^2\phi/\partial u_i \partial u_j) du_1 \wedge \dots \wedge du_m = 0$$

which gives

$$\det(\partial^2\phi/\partial u_i \partial u_j) = -c_1/c_2 = c\,.$$

Interchanging the roles of V and V^* gives the second statement.

To determine the volume of the torus $H^1(L_t, \mathbf{R}/\mathbf{Z})$, we take a basis $a_1, \dots, a_m$ of harmonic 1-forms, normalized by

$$\int_{A_i} a_j = \delta_{ij}$$

and then the volume is $\sqrt{\det(a_i, a_j)}$ using the inner product on harmonic forms. Now from the definition of λ_{ij}, the normalized harmonic forms are

$$a_j = \sum_k (\lambda^{-1})_{kj} \theta_k$$

and the inner product

$$(\theta_j, \theta_k) = \int_L \theta_j \wedge *\theta_k = \sum_i \lambda_{ij} \mu_{ik}\,.$$

Thus the volume is

$$\sqrt{\det(\mu\lambda^{-1})}\,.$$

Now in the coordinates $t_1, \dots, t_m$ the form $c_1 W_1 + c_2 W_2$ restricted to $F(M)$ is

$$(c_1 \det\lambda + c_2 \det\mu) dt_1 \wedge \dots \wedge dt_m$$

and this vanishes if and only if $\det(\mu\lambda^{-1}) = -c_1/c_2$. The final statement follows in similar way. The volume in this case is $\sqrt{\det(\lambda\mu^{-1})}$

REMARKS.

1. The relationship between pairs of solutions to the Monge-Ampère equations related by the Legendre transform is well-documented (see [2]).

2. On any special Lagrangian submanifold the volume form is the restriction of Ω_2, and Ω_2 is *closed* in X, so the cohomology class of the volume form is independent of t. Thus the 1-dimensional torus $H^n(L, \mathbf{R}/\mathbf{Z})$ has constant volume.

3. In the case where X is hyperkähler and L is complex Lagrangian with respect to the complex structure I, then the flat metric on $H^1(L, \mathbf{R}/\mathbf{Z})$ is Kähler and its volume is essentially the Liouville volume of the Kähler form. But the symplectic form on the torus is cohomologically determined: if $[\omega_1] \in H^2(L, \mathbf{R})$

is the cohomology class of the I-Kähler form of X, then for $\alpha, \beta \in H^1(L, \mathbf{R})$ the skew form is given by

$$\langle \alpha, \beta \rangle [\omega_1]^k = \alpha \wedge \beta \wedge [\omega_1]^{k-1} .$$

Since this is entirely cohomological, it is independent of t.

4. Another geometrical interpretation of the structure on M is as an affine hypersurface $x_{m+1} = \phi(x_1, \dots, x_m)$. The Legendre transform then corresponds to the dual hypersurface of tangent planes, and a solution to the Monge-Ampère equation describes a *parabolic affine hypersphere* ([4], [2]).

6. – Kähler metrics

The approach of Strominger, Yau and Zaslow takes the moduli space not just of special Lagrangian submanifolds, but of submanifolds together with flat unitary line bundles ("supersymmetric cycles"). Since a flat line bundle on L is classified by an element of $H^1(L, \mathbf{R}/\mathbf{Z})$, then by homotopy invariance (we are working locally or on a simply connected space) this augmented moduli space can be taken to be

$$M^c = M \times H^1(L, \mathbf{R}/\mathbf{Z}) .$$

The tangent space T_m at a point of M^c is thus canonically

$$T_m \cong H^1(L, \mathbf{R}) \oplus H^1(L, \mathbf{R}) \cong H^1(L, \mathbf{R}) \otimes \mathbf{C} .$$

This is a complex vector space, so M^c has an almost complex structure. Moreover, for any real vector space V, a positive definite inner product on V defines a hermitian form on $V \otimes \mathbf{C}$, so M^c has a hermitian metric. We then have:

PROPOSITION 4. *The almost complex structure I on M^c is integrable and the inner product on $H^1(L, \mathbf{R})$ defines a Kähler metric on M^c.*

PROOF. Use the basis $\alpha_1, \dots, \alpha_m$ of $H^1(L, \mathbf{R})$ to give coordinates $x_1, \dots, x_m$ on the universal covering of the torus $H^1(L, \mathbf{R}/\mathbf{Z})$. Then $(t_1, \dots, t_m, x_1, \dots, x_m)$ are local coordinates for M^c and from (2) the almost complex structure is defined by

$$I(\partial/\partial t_j) = \sum_i \lambda_{ij} \partial/\partial x_i$$

$$I\left(\sum_i \lambda_{ij} \partial/\partial x_i\right) = -\partial/\partial t_j .$$

If we define the complex vector fields

$$X_j = \partial/\partial t_j - iI\partial/\partial t_j = \partial/\partial t_j - i \sum \lambda_{jk} \partial/\partial x_j$$

then these satisfy $IX_j = iX_j$ and so form a basis for the $(1,0)$ vector fields. The forms θ_j defined by

$$\theta_j = \sum \lambda_{jk} dt_k - i dx_j$$

annihilate the X_j and thus form a basis of the (0, 1)-forms. But from (3)

$$\theta_j = d(u_j - ix_j)$$

so that $w_j = u_j + ix_j$ are complex coordinates, and the complex structure is integrable.

The 2-form $\tilde{\omega}$ for the Hermitian metric is defined by

$$\tilde{\omega}(\partial/\partial t_j, \partial/\partial x_k) = g(\partial/\partial t_j, I\partial/\partial x_k)$$

and from the definition of I,

$$\tilde{\omega}(\partial/\partial t_j, \partial/\partial x_k) = -\sum_l \lambda_{lk}^{-1} g_{jl} .$$

But from Proposition 2 the metric is F^*G, so in the local coordinates $t_1, \dots, t_m$,

$$g_{ij} = \sum_k \frac{\partial u_k}{\partial t_i} \frac{\partial v_k}{\partial t_j} = \sum_k \lambda_{ki} \mu_{kj}$$

(note that symmetry follows from (8)). Thus,

$$\tilde{\omega} = -\sum_{j,k} \mu_{kj} dt_j \wedge dx_k = -\sum_k dv_k \wedge dx_k$$

from (5). This is clearly closed, so the metric is Kählerian.

REMARK. Since $v_k = \partial\phi/\partial u_k$, we can also write

$$\begin{aligned}\tilde{\omega} &= -\sum_{j,k} (\partial^2\phi/\partial u_j \partial u_k) du_j \wedge dx_k \\ &= (2i)^{-1} \partial\bar{\partial}\phi\end{aligned}$$

so that $\phi/2$ is a Kähler potential for this metric. Such metrics, where the potential depends only on the real part of the complex variables, were considered by Calabi in [3].

We have seen that the pulled-back metric F^*G defines a Kähler metric on M^c. If we pull back the constant m-form $F^*W_1 = du_1 \wedge \dots \wedge du_m$, then this defines directly a complex m-form

$$\tilde{\Omega}^c = d(u_1 + ix_1) \wedge \dots d(u_m + ix_m) = dw_1 \wedge \dots \wedge dw_m$$

which is clearly non-vanishing and holomorphic. Using this, we have:

PROPOSITION 5. *The holomorphic m-form $\widetilde{\Omega}^c$ has constant length with respect to the Kähler metric if and only if any of the equivalent conditions of Proposition* 3 *hold.*

PROOF. First note that

$$dw_j \wedge d\bar{w}_j = \left(\sum_k \lambda_{jk}\, dt_k + i\, dx_j\right) \wedge \left(\sum_k \lambda_{jk}\, dt_k - i\, dx_j\right) = 2i\, dx_j \wedge \sum_j \lambda_{jk}\, dt_k\,.$$

Thus

$$dw_1 \wedge \dots dw_m \wedge d\bar{w}_1 \dots \wedge d\bar{w}_m = (2i)^m (\det\lambda) dx_1 \wedge \dots \wedge dx_m \wedge dt_1 \dots \wedge dt_m\,.$$

But

$$\tilde{\omega}^m = \left(-\sum \mu_{kj}\, dt_j \wedge dx_k\right)^m = (-1)^{m(m+1)/2} (\det\mu)\, dx_1 \wedge \dots \wedge dx_m \wedge dt_1 \dots \wedge dt_m\,.$$

Thus $\widetilde{\Omega}^c$ has constant length iff $\det\mu$ is a constant multiple of $\det\lambda$. But from the proof of Proposition 3, this is equivalent to the volume of the torus being constant.

Note that we could equally have argued using the Monge-Ampère equation for the Kähler potential.

We have thus seen that if F maps M to a *special* Lagrangian submanifold of $H^1(L) \times H^{n-1}(L)$, the complex manifold M^c has a natural *Calabi-Yau* metric.

REMARKS.

1. It is not hard to see that the tori $H^1(L, \mathbf{R}/\mathbf{Z}) \times \{t\}$ in M^c are special Lagrangian with respect to the natural Kähler metric and the holomorphic form $i^m\widetilde{\Omega}^c$. Since the first Betti number of this torus is $m = \dim M$, the family parametrized by $t \in M$ is complete by McLean's result, and so we can repeat the process to find another Kähler manifold. The reader may easily verify that the roles of (λ, u_i, ϕ) and (μ, v_i, ψ) are interchanged. In [12], one begins with a Calabi-Yau manifold with a family of special Lagrangian tori, and produces its "mirror" M^c in the above sense. Performing the process a second time one obtains some sort of approximation to the first manifold. The metric defined here, however, even when it is Calabi-Yau, will hardly ever extend to a *compact* manifold, since it has non-trivial Killing fields $\partial/\partial x_i$ – by Bochner's original Weitzenböck argument, zero Ricci tensor would imply that these are covariant constant.

2. The simplest case of the above process consists of considering elliptic curves in a hyperkähler 4-manifold (a 2-dimensional Calabi-Yau manifold). Thus $m = 2$ and we obtain a 4-dimensional hyperkähler metric on M^c. The existence of two Killing fields shows that it must be produced from the Gibbons-Hawking ansatz [6] using a harmonic function of two variables. From the above arguments, this means that the 2-dimensional Monge-Ampère equation can be reduced to Laplace's equation in two variables. In fact, as the reader will find

in [5], this is classically known. In the same way curves of genus g in (for example) a K3 surface generate a solution to the $2g$-dimensional Monge-Ampère equation.

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